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THE ZARISKI TOPOLOGY ON THE GRADED SECOND SPECTRUM OF A GRADED MODULE

SAIF SALAM AND KHALDOUN AL-ZOUBI*

ABSTRACT. Let R be a G -graded ring and M be a G -graded R -module. The graded second spectrum of M , denoted by $Spec_G^s(M)$, is the set of all graded second submodules of M . In this paper, we define a topology on $Spec_G^s(M)$ which is analogous to that for $Spec_G(R)$, and investigate several topological properties of this topology.

1. INTRODUCTION AND PRELIMINARIES

Let G be a multiplicative group with identity e and R be a commutative ring with identity 1. Then R is called a G -graded ring if there exist additive subgroups R_g of R indexed by the elements $g \in G$ such that $R = \bigoplus_{g \in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. The elements of R_g are called homogeneous elements of degree g . The homogeneous elements of the ring R are denoted by $h(R)$, i.e. $h(R) = \bigcup_{g \in G} R_g$. In particular, R_e is a subring of R and $1 \in R_e$.

If $x \in R$, then x can be written uniquely as $\sum_{g \in G} x_g$, where x_g is the component of x in R_g . Let R be a G -graded ring and I be an ideal of R . Then I is called a G -graded ideal of R , denoted by $I \triangleleft_G R$, if $I = \bigoplus_{g \in G} (I \cap R_g)$. A subring H of R is said to be graded subring if $H = \bigoplus_{g \in G} (H \cap R_g)$ (see [9]).

A proper graded ideal J of R is said to be graded maximal if there is no graded ideal K of R such that $J \subsetneq K \subsetneq R$. We will denote the set of all graded maximal ideals of R by $Max_G(R)$. The graded prime spectrum of R is the set of all graded prime ideals of R and it is denoted by $Spec_G(R)$. It has been proved that $Spec_G(R)$ is a topological space whose closed sets are $V_G^R(I) = \{p \in Spec_G(R) \mid I \subseteq p\}$ for each graded ideal I of R (see, for example, [10–12]).

Let R be a G -graded ring and M be a left R -module. Then M is said to be a G -graded R -module if $M = \bigoplus_{g \in G} M_g$ with $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$, where M_g is an additive subgroup of M for all $g \in G$. The elements of M_g are called homogeneous of degree g . The set of all homogeneous elements

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of M is denoted by $h(M)$, i.e. $h(M) = \bigcup_{g \in G} M_g$. If $m \in M$, then m can be written uniquely as $\sum_{g \in G} m_g$, where m_g is the component of m in M_g . Let $M = \bigoplus_{g \in G} M_g$ be a G -graded R -module and N be a submodule of M . For $g \in G$, let $N_g = N \cap M_g$. Then N is called a G -graded submodule of M , denoted by $N \leq_G M$, if $N = \bigoplus_{g \in G} N_g$. In this case, N_g is called the g -component of N for $g \in G$. Also, M/N becomes G -graded R -module with g -component $(M/N)_g = (M_g + N)/N$ for $g \in G$. Moreover, M_g is an R_e -module and N_g is an R_e -submodule of M_g for $g \in G$, see [9].

Let M be a G -graded R -module. Let I be a G -graded ideal of R and N be a G -graded submodule of M . Then $\text{Ann}_R(N) = \{r \in R \mid rN = \{0\}\}$ is a G -graded ideal of R . In addition, $\text{Ann}_M(I) = \{m \in M \mid Im = \{0\}\}$ and $(N :_M I) = \{m \in M \mid Im \subseteq N\}$ are G -graded submodules of M . In particular, $\text{Ann}_M(I)$ becomes a G -graded R/I -module by $(\text{Ann}_M(I))_g = \text{Ann}_M(I) \cap M_g$ for all $g \in G$.

A G -graded R -module M is called a comultiplication graded R -module if every $N \leq_G M$ has the form $\text{Ann}_M(I)$ for some $I \triangleleft_G R$ (see, for example, [2, 6]). It is clear that if N is a graded submodule of a comultiplication graded module M , then $N = \text{Ann}_M(\text{Ann}_R(N))$. A G -graded R -module M is said to be graded Artinian if it satisfies the descending chain condition for graded submodules, see [9].

Let M and M' be graded R -modules. An R -module homomorphism $f : M \rightarrow M'$ is called graded homomorphism if $f(M_g) \subseteq M'_g$ for all $g \in G$ (see [9]). A non-zero graded submodule S of M is said to be graded second if for each $r \in h(R)$, the graded R -homomorphism $f : S \rightarrow S$ defined by $f(x) = rx$ is either surjective or zero. In other words, S is a graded second submodule of M if $rS = S$ or $rS = 0$ for every $r \in h(R)$. This implies that $p = \text{Ann}_R(S)$ is a graded prime ideal of R and S is called a p -graded second submodule. The graded second spectrum of M , denoted by $\text{Spec}_G^s(M)$, is the set of all graded second submodules of M . For $p \in \text{Spec}_G(R)$, $p\text{-Spec}_G^s(M)$ denotes the collection of all p -graded second submodules, i.e. $p\text{-Spec}_G^s(M) = \{S \in \text{Spec}_G^s(M) \mid \text{Ann}_R(S) = p\}$. Graded second submodules of graded modules have been studied by various authors (see, for example, [5, 8, 13]).

A non-zero graded submodule N of a graded R -module M is said to be graded large if for every non-zero graded submodule L of M , we have $N \cap L \neq 0$. A graded R -module M is said to be graded simple if $\{0\}$ and M are its only graded submodules. A non-zero graded submodule N of a graded R -module M is called graded minimal if it contains no other non-zero graded submodules of M . Equivalently, N is a non-zero graded submodule of M which is a graded simple module. The set of all graded minimal submodules of a G -graded R -module M will be denoted by $\text{Min}_G(M)$. The graded socle of a G -graded R -module M , denoted by $\text{Soc}_G^R(M)$, is defined to be the sum

of all graded minimal submodules of M . If there are no graded minimal submodules of M , we put $Soc_G^R(M) = \{0\}$, see [9]. A graded R -module M is said to be graded cocyclic if $Soc_G^R(M)$ is a graded large and a graded simple submodule of M , see [6].

The Zariski-topology on some graded spectrums of graded rings and graded modules has been studied by several authors (see, for example, [3, 4, 10, 11, 14]).

In this paper, we introduce the notion of the G -cotop module which is analogous to the G -top module in [10] and the primary G -top module in [14]. For this, we define the variety of any graded submodule N of a G -graded R -module M by $V_G^{s*}(N) = \{S \in Spec_G^s(M) \mid S \subseteq N\}$ and we set $\Omega^{s*}(M) = \{V_G^{s*}(N) \mid N \leq_G M\}$. Then M is called a G -cotop module if $\Omega^{s*}(M)$ is closed under finite union. In this case, $\Omega^{s*}(M)$ generates a topology. We call it the quasi-Zariski topology on $Spec_G^s(M)$. After this, we define another variety for any $N \leq_G M$ by $V_G^s(N) = \{S \in Spec_G^s(M) \mid Ann_R(N) \subseteq Ann_R(S)\}$. Then the collection $\Omega^s(M) = \{V_G^s(N) \mid N \leq_G M\}$ satisfies the axioms for closed sets of a topology on $Spec_G^s(M)$ and we call this topology the Zariski topology on $Spec_G^s(M)$. By [13], the map $\phi : Spec_G^s(M) \rightarrow Spec_G(R/Ann_R(M))$ defined by $\phi(S) = Ann_R(S)/Ann_R(M)$ is called the natural map of $Spec_G^s(M)$. We show that ϕ is continuous and we investigate some conditions under which ϕ is injective, open and closed. In addition, using this map, we give some relations between the properties of $Spec_G^s(M)$ and $Spec_G(R/Ann_R(M))$ such as the connectedness. Furthermore, we find a base for the Zariski-topology on $Spec_G^s(M)$ and we investigate $Spec_G^s(M)$ with the Zariski-topology from the viewpoint of being quasi-compact, irreducible, T_0 -space and T_1 -space.

Throughout this paper, G is a multiplicative group and every ring is commutative with identity. Let M be a G -graded R -module. For any graded ideal I of R containing $Ann_R(M)$, $\bar{I}$ and $\bar{R}$ will denote $I/Ann_R(M)$ and $R/Ann_R(M)$, respectively. In addition, we denote $r + Ann_R(M)$ by $\bar{r}$ for any $r \in R$.

2. TOPOLOGIES ON $Spec_G^s(M)$

In this section, we first introduce the notion of the G -cotop modules and discover some properties of such modules. Then, we define a variety of any graded submodule of a graded module and we use it to define the Zariski-topology on $Spec_G^s(M)$.

Proposition 2.1. *Let M be a G -graded R -module. For any G -graded submodule N of M , we define the variety of N by $V_G^{s*}(N) = \{S \in Spec_G^s(M) \mid S \subseteq N\}$. Then the following hold:*

- (1) $V_G^{s*}(M) = Spec_G^s(M)$ and $V_G^{s*}(0) = \emptyset$.
- (2) $\bigcap_{i \in I} V_G^{s*}(N_i) = V_G^{s*}(\bigcap_{i \in I} N_i)$ for any index set I and any family of graded submodules $\{N_i\}_{i \in I}$.

(3) $V_G^{s*}(N) \cup V_G^{s*}(N') \subseteq V_G^{s*}(N + N')$ for any $N, N' \leq_G M$.

Proof. The proof is straightforward. $\square$

By Proposition 2.1, the collection $\Omega^{s*}(M) = \{V_G^{s*}(N) \mid N \leq_G M\}$ satisfies the axioms for closed sets of a topology on $\text{Spec}_G^s(M)$ if and only if $\Omega^{s*}(M)$ is closed under finite union. In this case, we call M a G -cotop module and the generated topology, τ_G^{s*} , the quasi-Zariski topology on $\text{Spec}_G^s(M)$.

Theorem 2.2. *Every comultiplication G -graded R -module M is a G -cotop module.*

Proof. Let $N, N' \leq_G M$ and $S \in V_G^{s*}(N + N')$. It is enough to prove that $S \in V_G^{s*}(N) \cup V_G^{s*}(N')$. Assume that $S \not\subseteq N$ and let $s \in S$. Since $S \not\subseteq N$ and M is a comultiplication graded module, then $\exists a \in S$ such that $a \notin N = \text{Ann}_M(\text{Ann}_R(N))$ and thus $\exists r \in h(R) \cap \text{Ann}_R(N)$ such that $ra \neq 0$. Since $S \in \text{Spec}_G^s(M)$ and $rS \neq 0$, then $rS = S$ and hence $s = rm$ for some $m \in S$. But $S \subseteq N + N'$. This follows that $m = n + n'$ for some $n \in N$ and $n' \in N'$. So $s = r(n + n') = rn'$ as $r \in \text{Ann}_R(N)$ and so $s \in N'$. This implies that $S \subseteq N$ or $S \subseteq N'$. Therefore $S \in V_G^{s*}(N) \cup V_G^{s*}(N')$. $\square$

Recall that an R -module M is called secondless if it has no second submodules. On the other hand, a G -graded R -module M is said to be graded secondless if it has no graded second submodules (i.e. $\text{Spec}_G^s(M) = \emptyset$). For a graded submodule N of a G -graded R -module M , the graded second socle (or graded second radical) of N is defined as the sum of all graded second submodules of M contained in N and it is denoted by $\text{soc}_G(N)$ (or $\text{sec}_G(N)$). If N does not contain any graded second submodule, we put $\text{soc}_G(N) = \{0\}$. Moreover, N is called a graded socle submodule if $N \neq 0$ and $\text{soc}_G(N) = N$ (see [13]).

Remark 2.3. If a G -graded R -module M is graded secondless, then M is trivially a G -cotop module.

Lemma 2.4. *Let M be a G -graded R -module. Let N be a G -graded R -submodule of M and p be a G -graded ideal of R . Then the following hold:*

- (1) N is a graded second R -submodule of M if and only if N is a graded second $\overline{R}$ -submodule of M .
- (2) If S is a G -graded second R/p -submodule of $\text{Ann}_M(p)$, then S is a G -graded second $\overline{R}$ -submodule of $\text{Ann}_M(p)$.
- (3) $V_G^{s*}(N) = V_G^{s*}(\text{soc}_G(N))$.
- (4) If $p \in \text{Spec}_G^s(R)$ and $N' \in p\text{-Spec}_G^s(M)$, then $N' \subseteq N$ or $(N + N')/N \in p\text{-Spec}_G^s(M/N)$.

Proof. (1) follows from [8, Proposition 2.4] and (2) is obvious.

(3) Since $\text{soc}_G(N) \subseteq N$, then $V_G^{s*}(\text{soc}_G(N)) \subseteq V_G^{s*}(N)$. Let $S \in V_G^{s*}(N)$. Then $S \in \text{Spec}_G^s(M)$ and $S \subseteq N$. So $S = \text{soc}_G(S) \subseteq \text{soc}_G(N)$ and thus $S \in V_G^{s*}(\text{soc}_G(N))$.

(4) Assume that $N' \not\subseteq N$. Then $(N + N')/N \in \text{Spec}_G^s(M/N)$. It is clear that

$p \subseteq \text{Ann}_R((N+N')/N)$. Let $r \in \text{Ann}_R((N+N')/N) \cap h(R)$. By hypothesis, $rN' = 0$ or $rN' = N'$. If $rN' = N'$, then $N' \subseteq N$ which is a contradiction. Thus $r \in \text{Ann}_R(N') = p$. Therefore $\text{Ann}_R((N+N')/N) = p$. $\square$

Proposition 2.5. *Let M be a G -graded R -module. Then the following hold:*

- (1) *If M_g is a secondless R_e -module for every $g \in G$, then M is a graded secondless R -module. The converse is true if R is trivially graded by G .*
- (2) *If $N \leq_G M$ such that both N and M/N are graded secondless R -modules, then M is a graded secondless R -module.*

Proof. (1) Suppose that M is not a graded secondless R -module. Then $\exists S \in \text{Spec}_G^s(M)$. Since $S \neq 0$, then $\exists g \in G$ such that $S_g \neq 0$. By [8, Theorem 2.3], S_g is a second R_e -submodule of M_g , a contradiction. Conversely, assume by way of contradiction that $\exists t \in G$ such that M_t is not a secondless R_e -module which implies that there is a second R_e -submodule H of M_g . Since R is trivially graded by G , then it is easy to see that $H \in \text{Spec}_G^s(M)$ which is also a contradiction.

(2) Suppose that M is not graded secondless. Then $\exists S \in \text{Spec}_G^s(M)$. Let $p = \text{Ann}_R(S)$. So we get $S \in p\text{-Spec}_G^s(M)$. Since M/N is graded secondless, then $(N+S)/N \notin p\text{-Spec}_G^s(M)$ and hence $S \subseteq N$ by Lemma 2.4(4). Therefore $S \in \text{Spec}_G^s(N) = \emptyset$, a contradiction. $\square$

If we take the ring of integers $R = \mathbb{Z}$ as a $\mathbb{Z}_2$ -graded $\mathbb{Z}$ -module by $R_0 = R$ and $R_1 = \{0\}$, then R_0 and R_1 are secondless R_0 -module which follows that $\mathbb{Z}$ is a $\mathbb{Z}_2$ -graded secondless $\mathbb{Z}$ -module by Proposition 2.5.

A graded second submodule S of a graded R -module M is said to be graded extraordinary if whenever N_1 and N_2 are graded socle submodules of M with $S \subseteq N_1 + N_2$, then $S \subseteq N_1$ or $S \subseteq N_2$.

Theorem 2.6. *Let M be a G -graded R -module. Then the following statements are equivalent:*

- (1) *M is a G -cotop module.*
- (2) *Every graded second submodule of M is graded extraordinary.*
- (3) *$V_G^{s*}(N) \cup V_G^{s*}(N') = V_G^{s*}(N+N')$ for any graded socle submodules N, N' of M .*

Proof. If M is graded secondless, then the result is trivially true. Suppose that M is not graded secondless.

(1) $\Rightarrow$ (2): Let $S \in \text{Spec}_G^s(M)$ and let N and L be graded socle submodules of M such that $S \subseteq N + L$. By hypothesis, $V_G^{s*}(N) \cup V_G^{s*}(L) = V_G^{s*}(H)$ for some $H \leq_G M$. Now there is an index set I such that $N = \sum_{i \in I} S_i$,

where $S_i \in \text{Spec}_G^s(M)$ and $S_i \subseteq N$ for every $i \in I$. For each $i \in I$, $S_i \in V_G^{s*}(N) \subseteq V_G^{s*}(H)$ and hence $S_i \subseteq H$ which implies that $N \subseteq H$. Similarly $L \subseteq H$. Thus $N + L \subseteq H$. So $V_G^{s*}(N) \cup V_G^{s*}(L) \subseteq V_G^{s*}(N+L) \subseteq V_G^{s*}(H) = V_G^{s*}(N) \cup V_G^{s*}(L)$ and so $V_G^{s*}(N+L) = V_G^{s*}(N) \cup V_G^{s*}(L)$. But $S \in V_G^{s*}(N+L)$. Therefore $S \subseteq N$ or $S \subseteq L$, as desired.

(2) $\Rightarrow$ (3): It is clear.

(3) $\Rightarrow$ (1): Let $W, W' \leq_G M$. By Lemma 2.4(3), we have $V_G^{s*}(W) \cup V_G^{s*}(W') = V_G^{s*}(\text{soc}_G(W)) \cup V_G^{s*}(\text{soc}_G(W'))$. If $\text{soc}_G(W) = 0$ or $\text{soc}_G(W') = 0$, then the result is trivially true. If not, then $\text{soc}_G(W)$ and $\text{soc}_G(W')$ are graded socle submodules of M . By hypothesis, $V_G^{s*}(W) \cup V_G^{s*}(W') = V_G^{s*}(\text{soc}_G(W) + \text{soc}_G(W'))$. Therefore M is a G -cotop module. $\square$

Corollary 2.7. (1) Every graded submodule N of a G -cotop module M is a G -cotop module.

- (2) M is a G -cotop R -module if and only if M is a G -cotop $\bar{R}$ -module.
- (3) Let M be a G -graded R -module and $p \triangleleft_G R$. If M is a G -cotop R -module, then $\text{Ann}_M(p)$ is a G -cotop R/p -module.
- (4) Let H be a graded subring of a graded ring R and let M be a G -graded R -module such that it is a G -cotop as graded H -module. Then M is a G -cotop R -module.
- (5) Let $f : M \rightarrow M'$ be a G -graded module isomorphism of G -graded R -modules. If M is a G -cotop module, then M' is a G -cotop module.

Proof. (1) Let $S \in \text{Spec}_G^s(N)$ and L_1, L_2 be graded socle submodules of N such that $S \subseteq L_1 + L_2$. Clearly, $\text{Spec}_G^s(N) \subseteq \text{Spec}_G^s(M)$. So $S \in \text{Spec}_G^s(M)$. Now it is easy to see that L_1, L_2 are graded socle submodules of M . Since M is a G -cotop module, then $S \subseteq L_1$ or $S \subseteq L_2$ by Theorem 2.6. Therefore N is a G -cotop module by Theorem 2.6 again.

(2) If M is a graded secondless R -module, then M is a graded secondless $\bar{R}$ -module by Lemma 2.4(1). Thus the result is clearly true by Remark 2.3. Suppose that M is not a graded secondless R -module. Then by Lemma 2.4(1) and Theorem 2.6, we have M is a G -cotop module $\Leftrightarrow$ Every graded second R -submodule of M is graded extraordinary $\Leftrightarrow$ Every graded second $\bar{R}$ -submodule of M is graded extraordinary $\Leftrightarrow M$ is a G -cotop $\bar{R}$ -module.

(3) Suppose that $\text{Ann}_M(p)$ is not a graded secondless R/p -module. Let S be a graded second R/p -submodule of $\text{Ann}_M(p)$ and N, K be graded socle R/p -submodules of $\text{Ann}_M(p)$ with $S \subseteq N + K$. By Lemma 2.4(2), S is a graded second $\bar{R}$ -submodule of $\text{Ann}_M(p)$. Also, it is easy to see that N, K are graded socle $\bar{R}$ -submodules of $\text{Ann}_M(p)$. But M is a G -cotop R -module. So $\text{Ann}_M(p)$ is a G -cotop R -module by (1) and hence it is a G -cotop $\bar{R}$ -module by (2). This implies that $S \subseteq N$ or $S \subseteq K$ by Theorem 2.6. Therefore $\text{Ann}_M(p)$ is a G -cotop R/p -module by Theorem 2.6 again.

(4) Suppose that M is not a graded secondless R -module. Let S be a graded second R -submodule of M and W_1, W_2 be graded socle R -submodules of M such that $S \subseteq W_1 + W_2$. It is straightforward to see that S is a graded second H -submodule of M and W_1, W_2 are graded socle H -submodules of M . But M is a G -cotop H -module. By Theorem 2.6, $S \subseteq W_1$ or $S \subseteq W_2$. Therefore M is a G -cotop R -module.

(5) Suppose that M' is not graded secondless. Let $S' \in \text{Spec}_G^s(M')$ and N'_1, N'_2 be graded socle submodules of M' such that $S' \subseteq N'_1 + N'_2$. Since f is a graded module isomorphism, then $f^{-1}(S') \in \text{Spec}_G^s(M)$ and $f^{-1}(N'_1), f^{-1}(N'_2)$

are graded socle submodules of M . Now $f^{-1}(S') \subseteq f^{-1}(N'_1 + N'_2) = f^{-1}(N'_1) + f^{-1}(N'_2)$. But M is a G -cotop module. Hence $f^{-1}(S') \subseteq f^{-1}(N'_1)$ or $f^{-1}(S') \subseteq f^{-1}(N'_2)$. By the surjectivity of f , we obtain $S' \subseteq N'_1$ or $S' \subseteq N'_2$. Therefore M' is a G -cotop module. $\square$

A graded ring F is called a graded field if every non-zero homogeneous element of F is unit. As usual, the set of all units of a graded ring R will be denoted by $U(R)$. A graded R -module M is said to be graded cyclic if there exists $m \in h(M)$ such that $M = Rm$, see [9].

Lemma 2.8. *Let M be a G -graded R -module. Then the following hold:*

- (1) *If R is a graded field, then every non-zero graded submodule of M is graded second.*
- (2) *If $A \subseteq h(M)$ is a base for M such that $|A| > 1$, then there exist non-zero graded submodules N_1 and N_2 of M such that $N_1 \not\subseteq N_2$ and $N_2 \not\subseteq N_1$.*
- (3) *If R is a graded field and M is a non-zero graded cyclic R -module, then M is a graded simple R -module.*

Proof. (1) Let N be a non-zero graded submodule of M . Let $r \in h(R)$. If $r = 0$, then $rN = 0$. If $r \neq 0$, then $r \in U(R)$. So for any $n \in N$, we have $n = r(r^{-1}n) \in rN$ and thus $rN = 0$ or $rN = N$. Therefore $N \in \text{Spec}_G^s(M)$.

(2) Assume by way of contradiction that for any non-zero graded submodules N_1 and N_2 of M , we have $N_1 \subseteq N_2$ or $N_2 \subseteq N_1$. By hypothesis, there exist $m_1, m_2 \in A$ such that $m_1 \neq m_2$. It is clear that $m_1, m_2 \neq 0$ as A is a base for M . Since $A \subseteq h(M)$, then Rm_1 and Rm_2 are non-zero graded submodules of M . Hence $Rm_1 \subseteq Rm_2$ or $Rm_2 \subseteq Rm_1$. If $Rm_1 \subseteq Rm_2$, then $m_1 = rm_2$ for some $r \in R$ and thus $m_1 + (-r)m_2 = 0$. Since A is a linearly independent set, then $r = 0$ and thus $m_1 = 0$, a contradiction. Similarly, if $Rm_2 \subseteq Rm_1$, we also have a contradiction.

(3) By hypothesis, $M = Rm$ for some $m \in h(M)$. Let $x \in M$ and $m' \in h(M) - \{0\}$. Then $x = rm$ and $m' = cm$ for some $r, c \in R$. Now $m' \in M_g$ and $m \in M_h$ for some $g, h \in G$. So $m' = c_{gh^{-1}}m$. Let $a = c_{gh^{-1}}$. Since $m' \neq 0$, then $a \neq 0$ and thus $a \in U(R)$. This implies that $m = a^{-1}m'$ and hence $x = r(a^{-1}m') \in Rm'$. So we obtain $M = Rm'$. This means that M is a non-zero graded cyclic module generated by any non-zero homogeneous element. Therefore M is a graded simple R -module. $\square$

Corollary 2.9. *Let M be a G -graded module over a graded field F . If $A \subseteq h(M)$ is a base for M , then M is a G -cotop module if and only if $|A| \leq 1$.*

Proof. $\Rightarrow$: Suppose that $|A| > 1$. By Lemma 2.8(2), there exist non-zero graded submodules N_1 and N_2 of M such that $N_1 \not\subseteq N_2$ and $N_2 \not\subseteq N_1$. Also $N_1, N_2, N_1 + N_2 \in \text{Spec}_G^s(M)$ by Lemma 2.8(1). Thus $N_1 = \text{soc}_G(N_1)$ and $N_2 = \text{soc}_G(N_2)$. Since M is a G -cotop module and $N_1 + N_2 \subseteq N_1 + N_2$, then $N_1 \subseteq N_2$ or $N_2 \subseteq N_1$ by Theorem 2.6, a contradiction.

$\Leftarrow$: If $M = 0$, then M is a graded secondless module and hence it a G -cotop module by Remark 2.3. If $M \neq 0$, then $|A| = 1$ and thus M is a graded simple F -module by Lemma 2.8(3). Therefore M is a G -cotop module. $\square$

The following is an example on a non G -cotop module.

Example 2.10. Let F be any field. Take $G = Z_2$ and $M = F \times F$. Then F is a G -graded field by $F_0 = F$ and $F_1 = \{0\}$. Also M is a G -graded F -module by $M_0 = F \times \{0\}$ and $M_1 = \{0\} \times F$. Moreover $M_0, M_1 \leq_G M$. Since F is a G -graded field, then $M \in \text{Spec}_G^s(M)$ and M_0, M_1 are graded socle submodules of M by Lemma 2.8(1). Now $M \subseteq M_0 + M_1$. But $M \not\subseteq M_0$ and $M \not\subseteq M_1$. Therefore M is not a G -cotop module by Theorem 2.6.

Let M be a G -cotop module and $Y \subseteq \text{Spec}_G^s(M)$. The closure of Y in $\text{Spec}_G^s(M)$ for the quasi-Zariski topology will be expressed by $Cl(Y)$.

Lemma 2.11. *Let M be a G -cotop module and $Y \subseteq \text{Spec}_G^s(M)$. Let $T(Y) = \sum_{S \in Y} S$. Then $Cl(Y) = V_G^{s*}(T(Y))$. In particular, if $M \in Y$, then Y is dense in $\text{Spec}_G^s(M)$, that is, $Cl(Y) = \text{Spec}_G^s(M)$.*

Proof. It is clear that $Y \subseteq V_G^{s*}(T(Y))$. Let $V_G^{s*}(N)$ be any closed set containing Y , where $N \leq_G M$. It is enough to show that $V_G^{s*}(T(Y)) \subseteq V_G^{s*}(N)$. Let $S \in V_G^{s*}(T(Y))$. Then $S \subseteq T(Y)$. Note that for any $S' \in Y$, we have $S' \subseteq N$ and hence $T(Y) \subseteq N$. This implies that $S \subseteq N$ and thus $S \in V_G^{s*}(N)$. Therefore $Cl(Y) = V_G^{s*}(T(Y))$. Now if $M \in Y$, then $Cl(Y) = V_G^{s*}(T(Y)) = V_G^{s*}(M) = \text{Spec}_G^s(M)$. $\square$

Corollary 2.12. *Let M be a G -cotop module. Then $\text{Spec}_G^s(M)$ is a T_0 -space.*

Proof. Let $S_1, S_2 \in \text{Spec}_G^s(M)$ with $S_1 \neq S_2$. If $Cl(\{S_1\}) = Cl(\{S_2\})$, then $V_G^{s*}(S_1) = V_G^{s*}(S_2)$ by Lemma 2.11. This implies that $S_1 = S_2$, a contradiction. Thus $Cl(\{S_1\}) \neq Cl(\{S_2\})$. This means that the closures of any two distinct points of $\text{Spec}_G^s(M)$ are distinct. Therefore $\text{Spec}_G^s(M)$ is a T_0 -space. $\square$

Lemma 2.13. *Let M be a G -graded R -module. If $N \leq_G M$ and $I \triangleleft_G R$, then $V_G^{s*}(N) \cup V_G^{s*}(\text{Ann}_M(I)) = V_G^{s*}((N :_M I))$.*

Proof. It is clear that $V_G^{s*}(N) \cup V_G^{s*}(\text{Ann}_M(I)) \subseteq V_G^{s*}((N :_M I))$. Let $S \in V_G^{s*}((N :_M I))$. Suppose that $S \not\subseteq N$ and it is enough to show that $S \subseteq \text{Ann}_M(I)$. Let $s \in S$ and $i \in I$. Now $S \subseteq (N :_M I)$. So $iS \subseteq N$. Since $S \in \text{Spec}_G^s(M)$, then $iS = \{0\}$ and hence $is = 0$, as desired. $\square$

Corollary 2.14. *Let M be a G -graded R -module. If $I, J \triangleleft_G R$, then $V_G^{s*}(\text{Ann}_M(I)) \cup V_G^{s*}(\text{Ann}_M(J)) = V_G^{s*}(\text{Ann}_M(IJ))$.*

Proof. Clearly, $(\text{Ann}_M(I) :_M J) = \text{Ann}_M(IJ)$. By Lemma 2.13, $V_G^{s*}(\text{Ann}_M(I)) \cup V_G^{s*}(\text{Ann}_M(J)) = V_G^{s*}((\text{Ann}_M(I) :_M J)) = V_G^{s*}(\text{Ann}_M(IJ))$. $\square$

Let M be a G -graded R -module. By Corollary 2.14, the family $\Omega^{s'} = \{V_G^{s'}(Ann_M(I)) \mid I \triangleleft_G R\}$ always induce a topology on $Spec_G^s(M)$ and it is denoted by $\tau_G^{s'}$. Now, the proof of the following proposition is obvious.

Proposition 2.15. *Let M be a G -graded R -module. If M is a G -cotop module, then the quasi-Zariski topology τ_G^{s*} on $Spec_G^s(M)$ is finer than $\tau_G^{s'}$.*

Now, we define another variety for a graded submodule N of a G -graded R -module M . We set $V_G^s(N) = \{S \in Spec_G^s(M) \mid Ann_R(N) \subseteq Ann_R(S)\}$. We state some properties of this variety in the following theorem to construct the Zariski-topology on $Spec_G^s(M)$.

Theorem 2.16. *Let M be a G -graded R -module. Then the following hold:*

- (1) $V_G^s(M) = Spec_G^s(M)$ and $V_G^s(0) = \emptyset$.
- (2) $\bigcap_{i \in I} V_G^s(N_i) = V_G^s(\bigcap_{i \in I} Ann_M(Ann_R(N_i)))$, for any family $\{N_i\}_{i \in I}$ of graded submodules.
- (3) $V_G^s(N) \cup V_G^s(N') = V_G^s(N + N')$ for any graded submodules N, N' of M .

Proof. (1) It is clear.

(2) Let $S \in \bigcap_{i \in I} V_G^s(N_i)$. For all $i \in I$, we have $Ann_R(N_i) \subseteq Ann_R(S)$ and

thus $Ann_M(Ann_R(S)) \subseteq Ann_M(Ann_R(N_i))$. Hence $Ann_M(Ann_R(S)) \subseteq \bigcap_{i \in I} Ann_M(Ann_R(N_i))$. It follows that $Ann_R(\bigcap_{i \in I} Ann_M(Ann_R(N_i))) \subseteq Ann_R(Ann_M(Ann_R(S))) = Ann_R(S)$. So $S \in V_G^s(\bigcap_{i \in I} Ann_M(Ann_R(N_i)))$ and so $\bigcap_{i \in I} V_G^s(N_i) \subseteq V_G^s(\bigcap_{i \in I} Ann_M(Ann_R(N_i)))$.

Conversely, let $S \in V_G^s(\bigcap_{i \in I} Ann_M(Ann_R(N_i)))$. Then $Ann_R(\bigcap_{i \in I} Ann_M(Ann_R(N_i))) \subseteq Ann_R(S)$. For all $i \in I$, we have $\bigcap_{i \in I} Ann_M(Ann_R(N_i)) \subseteq Ann_M(Ann_R(N_i))$ and hence $Ann_R(N_i) = Ann_R(Ann_M(Ann_R(N_i))) \subseteq Ann_R(\bigcap_{i \in I} Ann_M(Ann_R(N_i))) \subseteq Ann_R(S)$. Therefore $S \in \bigcap_{i \in I} V_G^s(N_i)$.

(3) Since $N, N' \subseteq N + N'$, then $V_G^s(N) \subseteq V_G^s(N + N')$ and $V_G^s(N') \subseteq V_G^s(N + N')$ and thus $V_G^s(N) \cup V_G^s(N') \subseteq V_G^s(N + N')$. For the reverse inclusion, let $S \in V_G^s(N + N')$. Then $Ann_R(N + N') \subseteq Ann_R(S)$. So we get $Ann_R(N)Ann_R(N') \subseteq Ann_R(S) \in Spec_G(R)$. By [11, Proposition 1.2], we have $Ann_R(N) \subseteq Ann_R(S)$ or $Ann_R(N') \subseteq Ann_R(S)$. Therefore $S \in V_G^s(N) \cup V_G^s(N')$. $\square$

In view of Theorem 2.16, the collection $\Omega^s(M) = \{V_G^s(N) \mid N \leq_G M\}$ satisfies the axioms for closed sets of a topology on $Spec_G^s(M)$. This topology is called the Zariski-topology on $Spec_G^s(M)$ and it is denoted by τ_G^s .

In the following lemma, we state some relations between the varieties $V_G^{s*}(N)$ and $V_G^s(N)$ for any graded submodule N of a G -graded R -module M .

Lemma 2.17. *Suppose that N and N' are graded submodules of a G -graded R -module M and that I is a G -graded ideal of R . Then the following hold:*

- (1) If $\text{Ann}_R(N) = \text{Ann}_R(N')$, then $V_G^s(N) = V_G^s(N')$. The converse is also true if $N, N' \in \text{Spec}_G^s(M)$.
- (2) $V_G^s(N) = \bigcup_{p \in V_G^R(\text{Ann}_R(N))} p\text{-Spec}_G^s(M)$
- (3) $V_G^s(N) = V_G^s(\text{Ann}_M(\text{Ann}_R(N))) = V_G^{s*}(\text{Ann}_M(\text{Ann}_R(N)))$. In particular, $V_G^s(\text{Ann}_M(I)) = V_G^{s*}(\text{Ann}_M(I))$.

Proof. The proof is straightforward. $\square$

Theorem 2.18. *For any G -graded R -module M , the Zariski-topology τ_G^s on $\text{Spec}_G^s(M)$ is identical with $\tau_G^{s'}$. Moreover, if M is a G -cotop module, then the quasi-Zariski topology τ_G^{s*} on $\text{Spec}_G^s(M)$ is finer than the Zariski-topology τ_G^s .*

Proof. It is clear by Lemma 2.17(3). $\square$

Proposition 2.19. *Let M be a G -graded module over a graded field F . If $A \subseteq h(M)$ is a base for M such that $|A| \leq 1$, then M is a G -cotop module and $\tau_G^s = \tau_G^{s*} =$ the trivial topology on $\text{Spec}_G^s(M)$.*

Proof. Since $|A| \leq 1$, then M is a G -cotop module by Corollary 2.9. It is clear that if $|A| = 0$, then $\tau_G^s = \tau_G^{s*} =$ the trivial topology on $\text{Spec}_G^s(M)$. If $|A| = 1$, then $M = Fm$ for some $m \in A$. If $m = 0$, then $M = 0$ and the result is trivially true. Suppose that $m \neq 0$. By Lemma 2.8(3), M is a graded simple F -module. Now let $U \in \tau_G^{s*}$. Then $U = \text{Spec}_G^s(M) - V_G^{s*}(N)$ for some $N \leq_G M$. But $N = 0$ or $N = M$. This implies that $\tau_G^{s*} = \{\emptyset, \text{Spec}_G^s(M)\}$. By Theorem 2.18, $\tau_G^s \subseteq \tau_G^{s*} = \{\emptyset, \text{Spec}_G^s(M)\} \subseteq \tau_G^s$. Therefore $\tau_G^s = \tau_G^{s*} =$ the trivial topology on $\text{Spec}_G^s(M)$. $\square$

3. RELATING $\text{Spec}_G^s(M)$ AND $\text{Spec}_G(\bar{R})$

Let M be a G -graded R -module. From now on, we assume that $\text{Spec}_G^s(M)$ is a non-empty set and is equipped the Zariski-topology. We also consider the natural map ϕ of $\text{Spec}_G^s(M)$ as described in the introduction. In this section, we provide some relations between $\text{Spec}_G^s(M)$ and $\text{Spec}_G(\bar{R})$ for any G -graded R -module M .

Proposition 3.1. *Let M be a G -graded R -module. Then $\phi^{-1}(V_G^{\bar{R}}(\bar{I})) = V_G^s(\text{Ann}_M(I))$, for every graded ideal I of R containing $\text{Ann}_R(M)$, and therefore the natural map ϕ of $\text{Spec}_G^s(M)$ is continuous.*

Proof. For any $S \in \text{Spec}_G^s(M)$, we have $S \in \phi^{-1}(V_G^{\bar{R}}(\bar{I})) \Leftrightarrow \phi(S) = \overline{\text{Ann}_R(S)} \in V_G^{\bar{R}}(\bar{I}) \Leftrightarrow \bar{I} \subseteq \overline{\text{Ann}_R(S)} \Leftrightarrow I \subseteq \text{Ann}_R(S) \Leftrightarrow S \subseteq \text{Ann}_M(I) \Leftrightarrow \text{Ann}_R(\text{Ann}_M(I)) \subseteq \text{Ann}_R(S) \Leftrightarrow S \in V_G^s(\text{Ann}_M(I))$. Therefore $\phi^{-1}(V_G^{\bar{R}}(\bar{I})) = V_G^s(\text{Ann}_M(I))$. $\square$

Proposition 3.2. *The following are equivalent for any G -graded R -module M :*

- (1) If whenever $S_1, S_2 \in \text{Spec}_G^s(M)$ with $V_G^s(S_1) = V_G^s(S_2)$, then $S_1 = S_2$.
- (2) $|p\text{-Spec}_G^s(M)| \leq 1$ for every $p \in \text{Spec}_G(R)$.
- (3) The natural map ϕ is injective.

Proof. (1) $\Rightarrow$ (2): Let $p \in \text{Spec}_G(R)$ and $S_1, S_2 \in p\text{-Spec}_G^s(M)$. Then $S_1, S_2 \in \text{Spec}_G^s(M)$ and $\text{Ann}_R(S_1) = \text{Ann}_R(S_2) = p$. By Lemma 2.17(1), we have $V_G^s(S_1) = V_G^s(S_2)$ and hence $S_1 = S_2$ by the assumption (1).

(2) $\Rightarrow$ (3): Assume that $\phi(S_1) = \phi(S_2)$, where $S_1, S_2 \in \text{Spec}_G^s(M)$. Then $\text{Ann}_R(S_1) = \text{Ann}_R(S_2)$. Let $p = \text{Ann}_R(S_1) \in \text{Spec}_G(R)$. Then we get $S_1, S_2 \in p\text{-Spec}_G^s(M)$ and by hypothesis, we obtain $S_1 = S_2$.

(3) $\Rightarrow$ (1): Let $S_1, S_2 \in \text{Spec}_G^s(M)$ with $V_G^s(S_1) = V_G^s(S_2)$. By Lemma 2.17(1), we have $\text{Ann}_R(S_1) = \text{Ann}_R(S_2)$. So $\phi(S_1) = \phi(S_2)$. Since ϕ is injective, then $S_1 = S_2$. $\square$

Proposition 3.3. *Let M be a G -graded R -module. If M is a comultiplication graded module, then the natural map ϕ of $\text{Spec}_G^s(M)$ is injective.*

Proof. Assume that $\phi(S_1) = \phi(S_2)$, where $S_1, S_2 \in \text{Spec}_G^s(M)$. Then $\text{Ann}_R(S_1) = \text{Ann}_R(S_2)$. Since M is a comultiplication graded module, then $S_1 = \text{Ann}_M(\text{Ann}_R(S_1)) = \text{Ann}_M(\text{Ann}_R(S_2)) = S_2$. $\square$

Let M be a G -graded R -module. Recall that a proper graded submodule N of M is said to be graded completely irreducible if $N = \bigcap_{i \in I} N_i$, where $\{N_i\}_{i \in I}$ is a family of graded submodules of M , implies that $N = N_i$ for some $i \in I$. By [1, Theorem 2.1], every proper graded submodule of M is an intersection of graded completely irreducible submodules of M . Therefore, if $M \neq 0$, then the intersection of all graded completely irreducible submodules of M is zero.

Let M and M' be G -graded R -modules. By $M \cong_G M'$, we mean that there exists a G -graded module isomorphism $f : M \rightarrow M'$.

Lemma 3.4. *Let M, M' be G -graded R -modules and $p \triangleleft_G R$. Then the following hold:*

- (1) If S is a graded second R/p -submodule of $\text{Ann}_M(p)$, then S is a graded second R -submodule of M .
- (2) If M is graded cocyclic and $M \cong_G M'$, then M' is graded cocyclic.
- (3) If M is a graded cocyclic R -module and N is a non-zero graded submodule of M , then N is a graded cocyclic R -module.

Proof. (1) By Lemma 2.4(2), S is a graded second $\overline{R}$ -submodule of $\text{Ann}_M(p)$. So S is a graded second R -submodule of $\text{Ann}_M(p)$ by Lemma 2.4(1). Therefore S is a graded second R -submodule of M .

(2) Since $M \cong_G M'$, then there exists a G -graded module isomorphism $f : M \rightarrow M'$. By hypothesis, $\text{Soc}_G^R(M)$ is a graded large and a graded simple submodule of M . Thus $\text{Soc}_G^R(M') = f(\text{Soc}_G^R(M))$ is a graded large and a graded simple submodule of M' . Therefore M' is a graded cocyclic

R -module.

(3) Since $N \subseteq M$, then $Soc_G^R(N) \subseteq Soc_G^R(M)$. By hypothesis, $Soc_G^R(M)$ is a graded simple submodule of M . So $Soc_G^R(N) = 0$ or $Soc_G^R(N) = Soc_G^R(M)$. Since $N \neq 0$ and $Soc_G^R(M)$ is a graded large submodule of M , then $Soc_G^R(M) \cap N \neq 0$. Now, it is easy to see that $Soc_G^R(M) \cap N$ is a graded simple submodule of N . This follows that $Soc_G^R(N) \neq 0$ and thus $Soc_G^R(N) = Soc_G^R(M)$. But $Soc_G^R(N) \subseteq N$. So $Soc_G^R(N)$ is a graded large and a graded simple submodule of N , as needed. $\square$

The following is a result of Proposition 3.2 and Lemma 3.4.

Corollary 3.5. *Let M be a G -graded R -module. If the natural map ϕ is injective, then $Ann_M(p)$ is a graded cocyclic module for every $p \in Max_G(R)$ with $Ann_M(p) \neq 0$.*

Proof. Let $p \in Max_G(R)$ such that $Ann_M(p) \neq 0$. Then $Ann_M(p)$ is a graded module over the graded field R/p which follows that $Ann_M(p)$ is a graded second R/p -submodule of $Ann_M(p)$ by Lemma 2.8(1). Thus $Ann_M(p)$ is a graded second R -submodule of M by Lemma 3.4(1). Since $p \subseteq Ann_R(Ann_M(p))$ and $Ann_M(p) \neq 0$, then $Ann_R(Ann_M(p)) = p$. Note that there exists a graded completely irreducible submodule C of M such that $Ann_M(p) \not\subseteq C$. If not, then $Ann_M(p) \subseteq \bigcap H = 0$, where the intersection runs over all graded completely irreducible submodules of M , a contradiction. Now, if $Ann_M(p) \cap C \neq 0$, then $Ann_R(Ann_M(p) \cap C) \neq R$. But $p = Ann_R(Ann_M(p)) \subseteq Ann_R(Ann_M(p) \cap C)$. So $Ann_R(Ann_M(p) \cap C) = p = Ann_R(Ann_M(p))$. Also, it is easy to see that $Ann_M(p) \cap C \in Spec_G^s(M)$. Since ϕ is injective, then $|p-Spec_G^s(M)| \leq 1$ by Proposition 3.2 and thus $Ann_M(p) = Ann_M(p) \cap C$. This implies that $Ann_M(p) \subseteq C$, a contradiction. So we get $Ann_M(p) \cap C = 0$ and hence $Ann_M(p) \cong_G (C + Ann_M(p))/C \subseteq M/C$. By [13, Lemma 2.10], M/C is a graded cocyclic module and thus $Ann_M(p)$ is graded cocyclic by Lemma 3.4(2), (3). $\square$

Corollary 3.6. *If M is a comultiplication graded R -module, then $Ann_M(p)$ is a graded cocyclic R -module for every $p \in Max_G(R)$ with $Ann_M(p) \neq 0$.*

Proof. By Corollary 3.5 and Proposition 3.3. $\square$

Theorem 3.7. *Let M be a G -graded R -module. If the natural map ϕ of $Spec_G^s(M)$ is surjective, then ϕ is both open and closed; more precisely, for every $N \leq_G M$, $\phi(V_G^s(N)) = V_G^{\bar{R}}(\overline{Ann_R(N)})$ and $\phi(Spec_G^s(M) - V_G^s(N)) = Spec_G(\bar{R}) - V_G^{\bar{R}}(\overline{Ann_R(N)})$.*

Proof. By Proposition 3.1, we have $\phi^{-1}(V_G^{\bar{R}}(\bar{I})) = V_G^s(Ann_M(I))$ for every $I \triangleleft_G R$ containing $Ann_R(M)$ and thus $\phi^{-1}(V_G^{\bar{R}}(\overline{Ann_R(N)})) = V_G^s(Ann_M(Ann_R(N))) = V_G^s(N)$. So $\phi(V_G^s(N)) = V_G^{\bar{R}}(\overline{Ann_R(N)})$ as ϕ is surjective. For the second part, note that $Spec_G^s(M) - V_G^s(N) = Spec_G^s(M) - \phi^{-1}(V_G^{\bar{R}}(\overline{Ann_R(N)})) = \phi^{-1}(Spec_G(\bar{R}) - V_G^{\bar{R}}(\overline{Ann_R(N)}))$. This implies that $\phi(Spec_G^s(M) - V_G^s(N)) = Spec_G(\bar{R}) - V_G^{\bar{R}}(\overline{Ann_R(N)})$. $\square$

Corollary 3.8. *Let M be a G -graded R -module. Then the natural map ϕ of $\text{Spec}_G^s(M)$ is bijective if and only if ϕ is homeomorphic.*

Let R be a G -graded ring and $I \triangleleft_G R$. The set $D_r = \text{Spec}_G(R) - V_G^R(rR) = \text{Spec}_G(R) - V_G^R(r)$ is open in $\text{Spec}_G(R)$ for each $r \in h(R)$ and the family $\{D_r \mid r \in h(R)\}$ forms a base for the Zariski topology on $\text{Spec}_G(R)$ (see [10, Theorem 2.3]). The radical of I , the graded radical of I , the nilradical of I and the graded nilradical of I will be denoted by $\sqrt{I}$, $Gr(I)$, $N(R)$ and $N_G(R)$, respectively. For details concerning the graded radical and the graded nilradical of a graded ideal of a graded ring, see [11].

Now, we need the following lemma to give a simple characterization for the connectedness of $\text{Spec}_G^s(M)$.

Lemma 3.9. *Let R be a G -graded ring. Then $\text{Spec}_G(R)$ is disconnected if and only if R contains idempotent element other than 0 and 1.*

Proof. $\Rightarrow$: Suppose that $\text{Spec}_G(R)$ is disconnected. Then $\text{Spec}_G(R) = V_G^R(I) \cup V_G^R(J)$ for some $I, J \triangleleft_G R$ such that $V_G^R(I) \cap V_G^R(J) = \emptyset$ and $V_G^R(I), V_G^R(J) \neq \emptyset$. By [10, Proposition 2.1(2)], $V_G^R(I + J) = V_G^R(I) \cap V_G^R(J) = \emptyset$ and hence $I + J = R$. So there exist $a \in I$ and $b \in J$ such that $a + b = 1$ which follows that $1 = 1_e = (a + b)_e = a_e + b_e$ with $a_e \in I$ and $b_e \in J$. Also by [10, Proposition 2.1(3)], we have $\text{Spec}_G(R) = V_G^R(IJ)$. This implies that for every $p \in \text{Spec}_G(R)$, we have $IJ \subseteq p$ and thus $a_e b_e \in N_G(R)$ by [11, Proposition 2.5]. Since $a_e b_e \in h(R)$, then $a_e b_e \in N(R)$ and hence there exists a positive integer m such that $(a_e b_e)^m = 0$. Note that $V_G^R(Ra_e^m + Rb_e^m) = \emptyset$. If not, then $\exists p \in \text{Spec}_G(R)$ such that $a_e^m, b_e^m \in p$ which follows that $a_e, b_e \in \sqrt{p}$. But $a_e, b_e \in h(R)$. So we get $a_e, b_e \in Gr(p) = p$ by [11, Proposition 2.4(5)]. Thus $1 = a_e + b_e \in p$, a contradiction. Therefore $Ra_e^m + Rb_e^m = R$ which implies that $1 = ra_e^m + r'b_e^m$ for some $r, r' \in R$. Note that $(ra_e^m)^2 = (ra_e^m)(1 - r'b_e^m) = ra_e^m - rr'(a_e b_e)^m = ra_e^m$. If $ra_e^m = 0$, then $b_e^m \in U(R)$ and thus $D_{b_e^m} = \text{Spec}_G(R)$ by [11, Proposition 3.6]. So we obtain $V_G^R(Rb_e^m) = \emptyset$ and so $V_G^R(J) \subseteq V_G^R(Rb_e^m) = \emptyset$, a contradiction. Similarly, if $ra_e^m = 1$, then we have $V_G^R(I) = \emptyset$, a contradiction. Therefore, ra_e^m is an idempotent element other than 0 and 1.

$\Leftarrow$: Suppose that x is an idempotent element of R other than 0 and 1. By [15, Lemma 4.4], $x \in R_e$ and thus $1 - x \in R_e$ as $1 \in R_e$. By [10, Theorem 2.3(2)], $D_x \cap D_{1-x} = D_{x(1-x)} = D_0 = \emptyset$. Now for any $p \in \text{Spec}_G(R)$, we have $x \notin p$ or $1 - x \notin p$ which follows that $p \in D_x \cup D_{1-x}$ and thus $\text{Spec}_G(R) = D_x \cup D_{1-x}$. Note that if $D_x = \emptyset$, then $x \in N(R)$ by [11, Proposition 3.6(2)]. So $x^n = 0$ for some positive integer n . But x is idempotent which follows that $x = 0$, a contradiction. Thus $D_x \neq \emptyset$. Similarly, if $D_{1-x} = \emptyset$, then we obtain $x = 1$, a contradiction. Hence $D_{1-x} \neq \emptyset$. Now $\text{Spec}_G(R)$ can be written as a union two non-empty disjoint open sets which means that $\text{Spec}_G(R)$ is disconnected. $\square$

The following is a result for Proposition 3.1, Theorem 3.7 and Lemma 3.9.

Corollary 3.10. *Let M be a G -graded R -module such that the natural map ϕ of $\text{Spec}_G^s(M)$ is surjective. Then the following statements are equivalent:*

- (1) $\text{Spec}_G^s(M)$ is connected.
- (2) $\text{Spec}_G(\overline{R})$ is connected.
- (3) The only idempotent elements of $\overline{R}$ are $\overline{0}$ and $\overline{1}$.

Proof. (1) $\Rightarrow$ (2): Since ϕ is continuous surjective map and $\text{Spec}_G^s(M)$ is connected, then $\text{Spec}_G(\overline{R})$ is connected.

(2) $\Rightarrow$ (1): Assume by way of contradiction that $\text{Spec}_G^s(M)$ is disconnected. Then there exists a clopen set U in $\text{Spec}_G^s(M)$ such that $U \neq \emptyset$ and $U \neq \text{Spec}_G^s(M)$. Since ϕ is surjective, then $\phi(U)$ is clopen in $\text{Spec}_G(\overline{R})$ by Theorem 3.7. This follows that $\phi(U) = \emptyset$ or $\phi(U) = \text{Spec}_G(\overline{R})$. Since U is open in $\text{Spec}_G^s(M)$, then $U = \text{Spec}_G^s(M) - V_G^s(N)$ for some $N \leq_G M$. Now if $\phi(U) = \text{Spec}_G(\overline{R})$, then $\text{Spec}_G(\overline{R}) = \phi(U) = \text{Spec}_G(\overline{R}) - V_G^{\overline{R}}(\overline{\text{Ann}_R(N)})$ by Theorem 3.7, again. Thus $V_G^{\overline{R}}(\overline{\text{Ann}_R(N)}) = \emptyset$ which implies that $\emptyset = \phi^{-1}(V_G^{\overline{R}}(\overline{\text{Ann}_R(N)})) = V_G^s(\text{Ann}_M(\text{Ann}_R(N))) = V_G^s(N)$ by Proposition 3.1 and Lemma 2.17(3). So $U = \text{Spec}_G^s(M)$, a contradiction. If $\phi(U) = \emptyset$, then $U \subseteq \phi^{-1}(\phi(U)) = \emptyset$. It follows that $U = \emptyset$ which is also a contradiction. Therefore $\text{Spec}_G^s(M)$ is connected.

The equivalence of (2) and (3) follows from Lemma 3.9. $\square$

Corollary 3.11. *Let M be a G -graded R -module such that the natural map ϕ of $\text{Spec}_G^s(M)$ is surjective. If either R is a graded local ring or $\text{Ann}_R(M) \in \text{Spec}_G(R)$, then both $\text{Spec}_G^s(M)$ and $\text{Spec}_G(\overline{R})$ are connected.*

Proof. First, assume that R is a graded local ring. Then the only idempotent elements of $\overline{R}$ are $\overline{0}$ and $\overline{1}$ (see the proof of [15, Lemma 4.4]). Now the result follows from Corollary 3.10. Secondly, assume that $\text{Ann}_R(M) \in \text{Spec}_G(R)$ and let r be idempotent element of R . Then $\bar{r} = \bar{r}^2$ and hence $r(r-1) \in \text{Ann}_R(M)$. Again, by [15, Lemma 4.4], we have $r \in R_e$ and thus $r-1 \in R_e$. This follows that $r \in \text{Ann}_R(M)$ or $r-1 \in \text{Ann}_R(M)$. Therefore $\bar{r} = \overline{0}$ or $\bar{r} = \overline{1}$. By Corollary 3.10, $\text{Spec}_G^s(M)$ and $\text{Spec}_G(\overline{R})$ are connected. $\square$

Proposition 3.12. *Let M and M' be G -graded R -modules. If $f : M \rightarrow M'$ is a graded module monomorphism, then the mapping $\pi : \text{Spec}_G^s(M) \rightarrow \text{Spec}_G^s(M')$ by $\pi(S) = f(S)$ is continuous.*

Proof. It is easy to see that π is well-defined. Now for any $S \in \text{Spec}_G^s(M)$ and any closed set $V_G^s(N')$ in $\text{Spec}_G^s(M')$, where $N' \leq_G M'$, we have $S \in \pi^{-1}(V_G^s(N')) = \pi^{-1}(V_G^{s*}(\text{Ann}_{M'}(\text{Ann}_R(N')))) \Leftrightarrow \pi(S) = f(S) \subseteq \text{Ann}_{M'}(\text{Ann}_R(N')) \Leftrightarrow S = f^{-1}(f(S)) \subseteq f^{-1}(\text{Ann}_{M'}(\text{Ann}_R(N'))) = \text{Ann}_M(\text{Ann}_R(N')) \Leftrightarrow S \in V_G^{s*}(\text{Ann}_M(\text{Ann}_R(N'))) = V_G^s(\text{Ann}_M(\text{Ann}_R(N')))$. So $\pi^{-1}(V_G^s(N')) = V_G^s(\text{Ann}_M(\text{Ann}_R(N')))$ and so π is continuous. $\square$

Let M be a G -graded R -module. Now, we write $X_r^s = \text{Spec}_G^s(M) - V_G^s(rM)$ for each $r \in h(R)$ and prove that $B = \{X_r^s \mid r \in h(R)\}$ is a base for the Zariski topology on $\text{Spec}_G^s(M)$, which is similar to that on $\text{Spec}_G(R)$.

Proposition 3.13. *For any G -graded R -module M , the set $B = \{X_r^s \mid r \in h(R)\}$ forms a base for the Zariski topology on $\text{Spec}_G^s(M)$.*

Proof. Let $U = \text{Spec}_G^s(M) - V_G^s(N)$ be an open set in $\text{Spec}_G^s(M)$, where $N \leq_G M$. Let $S \in U$ and it is enough to find an element $r \in h(R)$ such that $S \in X_r^s \subseteq U$. Since $S \in U$, then $\text{Ann}_R(N) \not\subseteq \text{Ann}_R(S)$ and hence there exists $t \in R$ and $g \in G$ such that $t_g \in \text{Ann}_R(N) - \text{Ann}_R(S)$. Take $r = t_g \in h(R)$. If $S \notin X_r^s$, then $rR \subseteq \text{Ann}_R(\text{Ann}_M(r)) \subseteq \text{Ann}_R(S)$ and thus $r \in \text{Ann}_R(S)$, a contradiction. This implies that $S \in X_r^s$. Since $r \in \text{Ann}_R(N)$, then $\text{Ann}_M(\text{Ann}_R(N)) \subseteq \text{Ann}_M(r)$. Now let $S' \in X_r^s$. Then we have $\text{Ann}_R(\text{Ann}_M(r)) \not\subseteq \text{Ann}_R(S')$. If $S' \notin U$, then $\text{Ann}_R(N) \subseteq \text{Ann}_R(S')$ which follows that $\text{Ann}_M(\text{Ann}_R(S')) \subseteq \text{Ann}_M(\text{Ann}_R(N)) \subseteq \text{Ann}_M(r)$. So $\text{Ann}_R(\text{Ann}_M(r)) \subseteq \text{Ann}_R(\text{Ann}_M(\text{Ann}_R(S'))) = \text{Ann}_R(S')$, a contradiction. This means that $S' \in U$ and hence $S \in X_r^s \subseteq U$, as desired. $\square$

Proposition 3.14. *Let M be a G -graded R -module and $r \in h(R)$. Then:*

- (1) $\phi^{-1}(D_{\bar{r}}) = X_r^s$.
- (2) $\phi(X_r^s) \subseteq D_{\bar{r}}$. If the natural map ϕ is surjective, then the equality holds.
- (3) $X_a^s \cap X_b^s = X_{ab}^s$, for any $a, b \in h(R)$.
- (4) If $r \in N(R)$, then $X_r^s = \emptyset$.
- (5) If $r \in U(R)$, then $X_r^s = X_{r^{-1}}^s = \text{Spec}_G^s(M)$.

Proof. (1) $\phi^{-1}(D_{\bar{r}}) = \phi^{-1}(\text{Spec}_G(\bar{R}) - V_G^{\bar{R}}(\bar{r}\bar{R})) = \text{Spec}_G^s(M) - \phi^{-1}(V_G^{\bar{R}}(\bar{r}\bar{R})) = \text{Spec}_G^s(M) - V_G^s(\text{Ann}_M(r)) = X_r^s$ by Proposition 3.1.

(2) Trivial.

(3) By [10, Theorem 2.3(2)], we have $D_{\bar{a}} \cap D_{\bar{b}} = D_{\overline{ab}}$ and thus $X_a^s \cap X_b^s = \phi^{-1}(D_{\bar{a}}) \cap \phi^{-1}(D_{\bar{b}}) = \phi^{-1}(D_{\overline{ab}}) = X_{ab}^s$.

(4) Suppose that $r \in N(R)$. By [11, Proposition 3.6(2)], we have $D_r = \emptyset$ and hence $D_{\bar{r}} = \emptyset$. This follows that $X_r^s = \phi^{-1}(D_{\bar{r}}) = \emptyset$ by (1).

(5) Assume that $r \in U(R)$. Then $D_r = \text{Spec}_G(R)$ by [11, Proposition 3.6(3)]. This implies that $D_{\bar{r}} = \text{Spec}_G(\bar{R})$ and thus $X_r^s = \phi^{-1}(D_{\bar{r}}) = \phi^{-1}(\text{Spec}_G(\bar{R})) = \text{Spec}_G^s(M)$ by (1). Now, since $r \in U(R) \cap h(R)$, then $r^{-1} \in U(R) \cap h(R)$. Therefore $X_{r^{-1}}^s = \text{Spec}_G^s(M) = X_r^s$. $\square$

Let M be a G -graded R -module. By [10, Theorem 2.3(4)], each D_r is quasi-compact and hence $D_1 = \text{Spec}_G(R)$ is quasi-compact. In the following theorem, we give a similar result for $\text{Spec}_G^s(M)$ and its basic open sets.

Theorem 3.15. *Let M be a G -graded R -module. If the natural map ϕ is surjective, then the open set X_r^s is quasi compact for every $r \in h(R)$. Specifically, $\text{Spec}_G^s(M)$ is quasi compact.*

Proof. Let $r \in h(R)$ and $\zeta = \{X_t^s \mid t \in \Delta\}$ be a basic open cover for X_r^s , where Δ is an index set subset of $h(R)$. Then $X_r^s \subseteq \bigcup_{t \in \Delta} X_t^s$ and hence $D_{\bar{r}} = \phi(X_r^s) \subseteq \bigcup_{t \in \Delta} \phi(X_t^s) = \bigcup_{t \in \Delta} D_{\bar{t}}$ by Proposition 3.14(1). So $\bar{\zeta} = \{D_{\bar{t}} \mid t \in \Delta\}$

is a basic open cover for the quasi compact set $D_{\bar{r}}$. Thus it has a finite subcover $\bar{\zeta} = \{D_{\bar{t}_i} \mid i = 1, \dots, n\}$, where $t_i \in \Delta$ for any $i = 1, \dots, n$. Now $X_r^s = \phi^{-1}(D_{\bar{r}}) \subseteq \bigcup_{i=1}^n \phi^{-1}(D_{\bar{t}_i}) = \bigcup_{i=1}^n X_{t_i}^s$ and this follows that $\zeta = \{X_{t_i}^s \mid i = 1, \dots, n\} \subseteq \zeta$ is a finite subcover for X_r^s . Therefore X_r^s is quasi compact. Now $\text{Spec}_G^s(M) = X_1^s$. So $\text{Spec}_G^s(M)$ is quasi compact. $\square$

4. IRREDUCIBILITY IN $\text{Spec}_G^s(M)$

Let M be a G -graded R -module and Y be a subset of $\text{Spec}_G^s(M)$. We will denote the closure of Y in $\text{Spec}_G^s(M)$ by $Cl(Y)$ and the summation $\sum_{S \in Y} S$ by $T(Y)$. If Z is a subset of $\text{Spec}_G(R)$, then the intersection of all members of Z will be expressed by $\xi(Y)$.

Proposition 4.1. *Let M be a G -graded R -module and $Y \subseteq \text{Spec}_G^s(M)$. Then $Cl(Y) = V_G^s(T(Y))$. In particular, Y is closed if and only if $V_G^s(T(Y)) = Y$.*

Proof. It is easy to see that $Y \subseteq V_G^s(T(Y))$. Let $V_G^s(N)$ be any closed set in $\text{Spec}_G^s(M)$ such that $Y \subseteq V_G^s(N)$ and it is enough to show that $V_G^s(T(Y)) \subseteq V_G^s(N)$. So let $S' \in V_G^s(T(Y))$. Then $\text{Ann}_R(T(Y)) \subseteq \text{Ann}_R(S')$. Note that for any $Q \in Y$, we have $\text{Ann}_R(N) \subseteq \text{Ann}_R(Q)$ and thus $\text{Ann}_R(N) \subseteq \bigcap_{Q \in Y} \text{Ann}_R(Q) = \text{Ann}_R(T(Y)) \subseteq \text{Ann}_R(S')$. Hence $S' \in V_G^s(N)$. Therefore $V_G^s(T(Y))$ is the smallest closed set containing Y and this means that $Cl(Y) = V_G^s(T(Y))$. $\square$

Recall that a topological space X is irreducible if for any decomposition $X = F_1 \cup F_2$ with closed subsets F_i of X with $i = 1, 2$, we have $F_1 = X$ or $F_2 = X$. A subset X' of X is irreducible if it is an irreducible topological space with the induced topology. An irreducible component of a topological space is a maximal irreducible subset of it. Let X be a topological space. It is easy to see that every singleton subset of X is irreducible. Furthermore, a subset Y of X is irreducible if and only if its closure is irreducible (see [7]).

Theorem 4.2. *Let M be a G -graded R -module. If $S \in \text{Spec}_G^s(M)$, then $Cl(\{S\}) = V_G^s(S)$ and $V_G^s(S)$ is an irreducible closed subset of $\text{Spec}_G^s(M)$.*

Proof. By Proposition 4.1, $Cl(\{S\}) = V_G^s(T(\{S\})) = V_G^s(S)$. Since $\{S\}$ is irreducible, then $Cl(\{S\}) = V_G^s(S)$ is irreducible, as desired. $\square$

Proposition 4.3. *Let M be a G -graded R -module and $Y \subseteq \text{Spec}_G^s(M)$. Then :*

- (1) *If $T(Y) \in \text{Spec}_G^s(M)$, then Y is irreducible.*
- (2) *If Y is irreducible, then $\Upsilon = \{\text{Ann}_R(S) \mid S \in Y\}$ is an irreducible subset of $\text{Spec}_G(R)$, i.e., $\xi(\Upsilon) = \text{Ann}_R(T(Y))$ is a graded prime ideal of R .*

Proof. (1) Assume that $T(Y) \in \text{Spec}_G^s(M)$. By Proposition 4.1 and Theorem 4.2, $V_G^s(T(Y)) = Cl(Y)$ is irreducible. Therefore Y is irreducible.

(2) Assume that Y is irreducible. Then $\phi(Y) = Y'$ is an irreducible subset of $\text{Spec}_G(\bar{R})$ since ϕ is continuous by Proposition 3.1. Note that $\xi(Y') = \xi(\phi(Y)) = \bigcap_{S \in Y} \overline{Ann_R(S)} = \overline{\bigcap_{S \in Y} Ann_R(S)} = \overline{Ann_R(T(Y))}$ and thus $\xi(Y') = \overline{Ann_R(T(Y))} \in \text{Spec}_G(\bar{R})$ by [14, Lemma 4.3]. This implies that $\xi(Y) = Ann_R(T(Y)) \in \text{Spec}_G(R)$ and thus Y is an irreducible subset of $\text{Spec}_G(R)$ by [14, Lemma 4.3] again. $\square$

Corollary 4.4. *Let M be a G -graded R -module. Assume that $p\text{-Spec}_G^s(M) \neq \emptyset$ for some $p \in \text{Spec}_G(R)$. Then the following hold:*

- (1) $p\text{-Spec}_G^s(M)$ is irreducible.
- (2) If $p \in \text{Max}_G(R)$, then $p\text{-Spec}_G^s(M)$ is an irreducible closed subset of $\text{Spec}_G^s(M)$.

Proof. By [5, Proposition 2.4(a)], we have $T(p\text{-Spec}_G^s(M)) \in p\text{-Spec}_G^s(M)$. So $p\text{-Spec}_G^s(M)$ is irreducible subset of $\text{Spec}_G^s(M)$ by Proposition 4.3(1). Now it is easy to see that $Ann_M(p) \neq 0$. So if $p \in \text{Max}_G(R)$, then $Ann_R(Ann_M(p)) = p$. By Lemma 2.17(2), we obtain $V_G^s(Ann_M(p)) = \bigcup_{q \in V_G^R(Ann_R(Ann_M(p)))} q\text{-Spec}_G^s(M) = \bigcup_{q \in V_G^R(p)} q\text{-Spec}_G^s(M) = \bigcup_{q \in \{p\}} q\text{-Spec}_G^s(M) = p\text{-Spec}_G^s(M)$. Therefore $p\text{-Spec}_G^s(M) = V_G^s(Ann_M(p))$ and this follows that $p\text{-Spec}_G^s(M)$ is an irreducible closed subset of $\text{Spec}_G^s(M)$. $\square$

Proposition 4.5. *Let M be a G -graded R -module and $Y \subseteq \text{Spec}_G^s(M)$ such that $Ann_R(T(Y)) = p$ is a graded prime ideal of R . If $p\text{-Spec}_G^s(M) \neq \emptyset$, then Y is irreducible.*

Proof. Let $S \in p\text{-Spec}_G^s(M)$. By Lemma 2.17(1) and Proposition 4.1, we have $V_G^s(S) = V_G^s(T(Y)) = Cl(Y)$. So $Cl(Y)$ is irreducible by Theorem 4.2 and thus Y is irreducible. $\square$

Theorem 4.6. *Let M be a G -graded R -module and $Y \subseteq \text{Spec}_G^s(M)$. If the natural map ϕ is surjective, then Y is an irreducible closed subset of $\text{Spec}_G^s(M)$ if and only if $Y = V_G^s(S)$ for some $S \in \text{Spec}_G^s(M)$.*

Proof. Assume that Y is an irreducible closed subset of $\text{Spec}_G^s(M)$. Then $Ann_R(T(Y)) \in \text{Spec}_G(R)$ by Proposition 4.3(2). Thus $\overline{Ann_R(T(Y))} \in \text{Spec}_G(\bar{R})$. Since ϕ is surjective, then $Ann_R(S) = Ann_R(T(Y))$ for some $S \in \text{Spec}_G^s(M)$. By Lemma 2.17(1) and Proposition 4.1, $V_G^s(S) = V_G^s(T(Y)) = Cl(Y) = Y$. Conversely, if $Y = V_G^s(S)$ for some $S \in \text{Spec}_G^s(M)$, then Y is closed and it is irreducible by Theorem 4.2. $\square$

Corollary 4.7. *Let M be a G -graded R -module and $S \in \text{Spec}_G^s(M)$. If $\overline{Ann_R(S)}$ is a minimal graded prime ideal of $\bar{R}$, then $V_G^s(S)$ is an irreducible component of $\text{Spec}_G^s(M)$. The converse is true if the natural map ϕ is surjective.*

Proof. By Theorem 4.2, $V_G^s(S)$ is irreducible and it is enough to show that it is a maximal irreducible. Let Y be irreducible subset of $\text{Spec}_G^s(M)$ with $V_G^s(S) \subseteq Y$ and it remains to prove $Y = V_G^s(S)$. Since $S \in V_G^s(S) \subseteq Y$, then $S \in Y$ and thus $\overline{\text{Ann}_R(T(Y))} \subseteq \overline{\text{Ann}_R(S)}$. By Proposition 4.3(2), $\overline{\text{Ann}_R(T(Y))} \in \text{Spec}_G(\overline{R})$. But $\overline{\text{Ann}_R(S)}$ is a minimal graded prime ideal of $\overline{R}$. So $\overline{\text{Ann}_R(T(Y))} = \overline{\text{Ann}_R(S)}$ and this follows that $V_G^{\overline{R}}(\overline{\text{Ann}_R(T(Y))}) = V_G^{\overline{R}}(\overline{\text{Ann}_R(S)})$. Thus $V_G^s(T(Y)) = \phi^{-1}(V_G^{\overline{R}}(\overline{\text{Ann}_R(T(Y))})) = \phi^{-1}(V_G^{\overline{R}}(\overline{\text{Ann}_R(S)})) = V_G^s(S)$ by Lemma 2.17(3) and Proposition 3.1. Since $Y \subseteq V_G^s(T(Y))$, then $Y \subseteq V_G^s(S)$ and hence $Y = V_G^s(S)$. Conversely, assume that ϕ is surjective. Let $\overline{p} \in \text{Spec}_G(\overline{R})$ with $\overline{p} \subseteq \overline{\text{Ann}_R(S)}$ and it is enough to show that $\overline{p} = \overline{\text{Ann}_R(S)}$. By the surjectivity of ϕ , there exists $S' \in \text{Spec}_G^s(M)$ such that $\text{Ann}_R(S') = p$. Then $\text{Ann}_R(S') \subseteq \text{Ann}_R(S)$. So we get $V_G^s(S) \subseteq V_G^s(S')$. Since $V_G^s(S)$ is irreducible component and $V_G^s(S')$ is irreducible by Theorem 4.2, then $V_G^s(S) = V_G^s(S')$. By Lemma 2.17(1), we have $\text{Ann}_R(S) = \text{Ann}_R(S') = p$. Therefore $\overline{p} = \overline{\text{Ann}_R(S)}$, as needed. $\square$

Theorem 4.8. *Let M be a G -graded R -module. Then the following statements are equivalent for any $S_1, S_2 \in \text{Spec}_G^s(M)$:*

- (1) $\text{Spec}_G^s(M)$ is a T_0 -space.
- (2) If $V_G^s(S_1) = V_G^s(S_2)$, then $S_1 = S_2$.
- (3) $|p\text{-Spec}_G^s(M)| \leq 1$, for every $p \in \text{Spec}_G(R)$.
- (4) The natural map ϕ is injective.

Proof. The equivalence of (2), (3) and (4) follows from Proposition 3.2. Also (1) $\Leftrightarrow$ (2) follows from Theorem 4.2 and the fact that a topological space is a T_0 -space if the closures of distinct points are distinct. $\square$

Proposition 4.9. *Let M be a G -graded module over a graded field F . If $\text{Spec}_G^s(M)$ is a T_0 -space, then $|A| \leq 1$ for any base A of M with $A \subseteq h(M)$.*

Proof. Suppose that there exists a base A of M such that $A \subseteq h(M)$ and $|A| > 1$. By Lemma 2.8(1),(2), there exist $N_1, N_2 \in \text{Spec}_G^s(M)$ with $N_1 \neq N_2$. Since F is a G -graded field, then $r \in U(F)$ for every $r \in h(F) - \{0\}$ and hence $B = \{X_r \mid r \in h(F)\} = \{\text{Spec}_G^s(M), \emptyset\}$ by Proposition 3.14(5). This follows that τ_G^s is the trivial topology on $\text{Spec}_G^s(M)$. Now it is easy to see that $\text{Spec}_G^s(M)$ is not a T_0 -space, a contradiction. $\square$

Recall that a topological space X is called a T_1 -space if every singleton subset of X is closed. Now, we need the following lemma to investigate $\text{Spec}_G^s(M)$ with the Zariski topology from the viewpoint of being a T_1 -space.

Lemma 4.10. *Let M be a G -graded R -module. If Q is a graded minimal submodule of M , then $\text{Ann}_R(Q)$ is a graded maximal ideal of R .*

Proof. Since $Q \neq 0$, then $\text{Ann}_R(Q) \neq R$. Let $J \triangleleft_G R$ such that $\text{Ann}_R(Q) \subseteq J \subseteq R$ and $\text{Ann}_R(Q) \neq J$. Then $\exists t \in J - \text{Ann}_R(Q)$. So $\exists m \in h(M) \cap Q$ such that $tm \neq 0$. Note that Jm is a non-zero graded submodule of M . But $Jm \subseteq Q$. So $Jm = Q$ as Q is a graded minimal submodule. Thus $m = jm$

for some $j \in J$. This implies that $1 - j \in \text{Ann}_R(m) \subseteq \text{Ann}_R(Jm) = \text{Ann}_R(Q) \subseteq J$. Hence $1 \in J$. Therefore $J = R$. $\square$

Theorem 4.11. *Let M be a G -graded R -module such that every graded second submodule of M contains a graded minimal submodule and let $S \in \text{Spec}_G^s(M)$. Then the set $\{S\}$ is closed in $\text{Spec}_G^s(M)$ if and only if S is a graded minimal submodule of M and $p\text{-Spec}_G^s(M) = \{S\}$, where $p = \text{Ann}_R(S)$.*

Proof. Suppose that $\{S\}$ is closed in $\text{Spec}_G^s(M)$. Then $Cl(\{S\}) = \{S\}$ and hence $V_G^s(S) = \{S\}$ by Theorem 4.2. Now $\exists W \in \text{Min}_G(M)$ such that $W \subseteq S$. So we get $\text{Ann}_R(S) \subseteq \text{Ann}_R(W)$, which implies that $W \in V_G^s(S) = \{S\}$. Hence $W = S$ and this means that $S \in \text{Min}_G(M)$. Also it is clear that $p\text{-Spec}_G^s(M) = \{S\}$. Conversely, let $H \in Cl(\{S\})$. Then $H \in V_G^s(S)$ and hence $\text{Ann}_R(S) \subseteq \text{Ann}_R(H)$. By Lemma 4.10, we have $\text{Ann}_R(S) \in \text{Max}_G(R)$. This follows that $\text{Ann}_R(S) = \text{Ann}_R(H)$. Since $p\text{-Spec}_G^s(M) = \{S\}$, then $H = S$. Therefore $Cl(\{S\}) = \{S\}$, as desired. $\square$

Corollary 4.12. *Let M be a G -graded R -module such that every graded second submodule contains a graded minimal submodule. Then $\text{Spec}_G^s(M)$ is a T_1 -space if and only if $\text{Min}_G(M) = \text{Spec}_G^s(M)$.*

Proof. Assume that $\text{Spec}_G(M)$ is a T_1 -space. By [5, Proposition 2.3(c)], $\text{Min}_G(M) \subseteq \text{Spec}_G^s(M)$. Let $S \in \text{Spec}_G^s(M)$. Since $\text{Spec}_G^s(M)$ is a T_1 -space, then $\{S\}$ is closed in $\text{Spec}_G^s(M)$. By Theorem 4.11, we have $S \in \text{Min}_G(M)$. For the converse, assume that $\text{Min}_G(M) = \text{Spec}_G^s(M)$. Let $W \in \text{Spec}_G^s(M)$ and it is enough to show that $\{W\}$ is closed in $\text{Spec}_G^s(M)$. Let $p = \text{Ann}_R(W)$ and $H \in p\text{-Spec}_G^s(M)$. By [5, Proposition 2.4(a)], we have $H + W \in p\text{-Spec}_G^s(M)$. This follows that $H + W \in \text{Min}_G(M)$. But $H, W \subseteq H + W$. So $H = W$ and so $p\text{-Spec}_G^s(M) = \{W\}$. By Theorem 4.11, $\{W\}$ is closed in $\text{Spec}_G^s(M)$. $\square$

Let M be a G -graded R -module and $S \in \text{Spec}_G^s(M)$. We say that S is a maximal graded second submodule if it is not properly contained in another graded second submodule of M (see [8]).

Proposition 4.13. *Let M be a G -graded R -module. Then the following hold:*

- (1) *If $\text{Spec}_G^s(M)$ is a T_1 -space, then every graded second submodule of M is a maximal graded second.*
- (2) *If M is a graded Artinian module, then $\text{Spec}_G^s(M)$ is a T_1 -space if and only if $\text{Spec}_G^s(M)$ is discrete.*

Proof. (1) Let $S \in \text{Spec}_G^s(M)$ and we show that it is maximal graded second. So let $S' \in \text{Spec}_G^s(M)$ with $S \subseteq S'$. By hypothesis, $\{S\} = Cl(\{S\}) \subseteq Cl(\{S'\}) = \{S'\}$ and thus $S = S'$.

(2) If $M = 0$, then the result is trivially true. Suppose that $M \neq 0$. Clearly, if $\text{Spec}_G^s(M)$ is discrete, then it is a T_1 -space. Conversely, assume that

$\text{Spec}_G^s(M)$ is a T_1 -space. Let $\text{Max}_G^s(M)$ be the set of all maximal graded second submodules of M . By (1), $\text{Spec}_G^s(M) \subseteq \text{Max}_G^s(M)$. But by [8, Theorem 2.11], $\text{Max}_G^s(M)$ is a finite set and hence $\text{Spec}_G^s(M)$ is finite. Now, since $\text{Spec}_G^s(M)$ is a T_1 -space, then it is discrete with respect to the Zariski topology. $\square$

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