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The ground states for the non-cooperative autonomous systems involving the fractional Laplacian

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Abstract

The aim of this paper is to study the the following non-cooperative autonomous systems involving the fractional Laplacian

$$\begin{cases} (-\Delta)^s u + \lambda u = g(v), & \text{in } \mathbb{R}^N, \\ (-\Delta)^s v + \lambda v = f(u), & \text{in } \mathbb{R}^N, \end{cases}$$

where $s \in (0, 1)$, $N > 2s$, $\lambda > 0$, $(-\Delta)^s$ is the fractional Laplacian and f and g are power-type nonlinearities having super-linear and subcritical growth at infinity. We establish the existence of the ground states of the system through variational methods. The variational frame associated to the system is strongly indefinite, which is different from the one of the single equation case and the one of a cooperative type. Furthermore, some properties of the solutions such as regularity, symmetry and decay are also discussed.

Mathematics Subject Classifications (2020): 35Q40, 49J35, 34A08, 47J30.

Keywords: Fractional Laplacian; Ground states; Strongly indefinite; Symmetry; Decay estimates.

1 Introduction and main results

In these last years a great deal of work has devoted to the study of the weak solutions for the following fractional Schrödinger systems

$$\begin{cases} (-\Delta)^s u + a(x)u = f(u, v), & \text{in } \mathbb{R}^N, \\ (-\Delta)^s v + a(x)v = g(u, v), & \text{in } \mathbb{R}^N, \end{cases} \quad (1.1)$$

where $s \in (0, 1)$ with $N > 2s$ and $a(x)$, f, g satisfying appropriate conditions in order to use a variational method.

The fractional Schrödinger equations are formulated by Laskin [1], they are functional equations of fractional quantum mechanics. Equations involving the fractional Laplacian have attracted much attention in recent years, they appear in several areas such as optimization,

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finance, phase transitions, stratified materials, crystal dislocation, flame propagation, conservation laws, ultra-relativistic limits of quantum mechanics, materials science, and water waves, see for instance [2-4] for an introduction to these topics and their applications. Different from the classical Laplace operator, the analytical methods for elliptic PDEs cannot be directly applied to (1.1) since the operator $(-\Delta)^s$ is nonlocal. In [5], Caffaralli and Silvestre gave a new formulation of the fractional Laplacian through Dirichlet-Neumann maps. This is extensively used in the recent literature since it allows to transform nonlocal problems to local ones, which permits to use variational methods. For example, for the single nonlocal problems, this is, $u = v, f = g$ in (1.1), there have been many results on the existence which were studied using the idea of the s -harmonic extension, we refer to [6-16] for the superlinear and subcritical or critical case and to [17-18] for the asymptotically linear case.

In the case of the standard Laplacian operator ($s = 1$, local case), the existence of solution for the Schrödinger systems have been studied, and relatively complete methods have been formed. However, for the autonomous fractional Schrödinger systems, there are only some literature on the existence of positive solutions, for example, see [19-21].

In [19], X.M. He, Marco Squassina and W.M. Zou concerned with the multiplicity of positive solutions for the following elliptic system involving the fractional Laplacian

$$\begin{cases} (-\Delta)^s u = \lambda |u|^{q-2} u + \frac{2\alpha}{\alpha+\beta} |u|^{\alpha-2} u |v|^\beta, & \text{in } \Omega, \\ (-\Delta)^s v = \mu |v|^{q-2} v + \frac{2\beta}{\alpha+\beta} |u|^\alpha u |v|^{\beta-2} v, & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega. \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $\lambda, \mu > 0, 1 < q < 2$ and $\alpha > 1, \beta > 1$ satisfy $\alpha + \beta = 2_s^* = \frac{2N}{N-2s}, s \in (0, 1)$. They proved that the system admits at least two positive solutions when the pair of parameters (λ, μ) belongs to a suitable subset of $\mathbb{R}^2$, with the help of the Nehari manifold.

Edir Junior Ferreira Leite and Marcos Montenegro [20] studied the following strongly coupled systems

$$\begin{cases} (-\Delta)^s u = v^p & \text{in } \Omega, \\ (-\Delta)^t v = u^q & \text{in } \Omega, \end{cases} \quad (1.2)$$

in non-variational form involving fractional Laplace operators, where Ω is a smooth bounded open subset of $\mathbb{R}^N, n \geq 2, 0 < s, t < 1, p, q > 0$. They proved Liouville type theorems and by mean of the blow-up method, established a priori bounds of positive solutions. By using those latter, they then derived the existence of positive solutions through topological methods.

Alexander Quaas, Aliang Xia [21] studied the nonexistence of solutions for fractional elliptic problems (1.2) in the case of $s = t$ via a monotonicity result which obtained by the method of moving planes with an improved Aleksandrow-Bakelman-Pucci type estimate for the fractional Laplacian in unbounded domain.

In line with the above works, it worth mentioning that the nonlinearities are cooperative type in [19], hence the energy functions corresponding to them can be proved to have mountain pass structure and Nehari manifold arguments can be used. However, another question arises: for the following more general non-cooperative type fractional systems:

$$\begin{cases} (-\Delta)^s u + \lambda u = g(v), & \text{in } \mathbb{R}^N, \\ (-\Delta)^s v + \lambda v = f(u), & \text{in } \mathbb{R}^N, \end{cases} \quad (1.3)$$

where $\lambda > 0$ is any constant, whether the ground states and their properties can be obtained? Answering this question constitutes the goal of this paper.

As for the non-cooperative type system (1.2), the existence of solutions was derived through some topological methods in [20-21], but it is not known whether this solution is a least energy solution.

In this paper, motivated by the references mentioned above, we focus our attention on the non-cooperative type fractional systems (1.3) with more general power-type nonlinearities having super-linear and subcritical growth at infinity. The corresponding energy functional is strongly indefinite, that is, the quadratic part of the energy functional has no longer a positive sign, the problems become rather complicated, mathematically. The main purpose of this paper is to obtain the existence of ground states of the system (1.3) through variational method, which are different from the ones of [20] and [21]. Furthermore, regularity and symmetry of solutions are also discussed. By constructing suitable comparison functions based on the Bessel Kernel, we find out that the ground states of (1.3) have a power type decay at infinity.

Since we are interested in positive solutions, we assume the continuous functions f, g satisfy the following conditions:

$$(H_1) \quad f(0) = g(0) = f'(0) = g'(0) = 0, \quad f(t) = g(t) = 0 \text{ for } t \leq 0;$$

$$(H_2) \quad \text{there exist real numbers } l_1, l_2 > 0 \text{ and } p, q > 2 \text{ such that } \frac{1}{p} + \frac{1}{q} > \frac{N-2s}{N} \text{ and}$$

$$\lim_{|t| \rightarrow \infty} \frac{f'(t)}{|t|^{p-2}} = l_1, \quad \lim_{|t| \rightarrow \infty} \frac{g'(t)}{|t|^{q-2}} = l_2;$$

(H₃) there exists $\delta > 0$ such that $0 < (1 + \delta)f(t)t \leq f'(t)t^2$ for every $t \in \mathbb{R}$ and similarly for g ;

(H₄) for every $\mu > 0$ there exists $C_\mu > 0$ such that

$$|f(u)v| + |g(v)u| \leq \mu(u^2 + v^2) + C_\mu(f(u)u + g(v)v), \quad u, v \in \mathbb{R}.$$

(H₅) $f'(t), g'(t)$ are non-decreasing for $t \geq 0$.

Similar assumptions have been introduced in [22]. When $s = 1$, the system (1.3) gives back the classical Schrödinger system. It has been studied by Djairo G. De Figueiredo and Jianfu Yang in [23].

The main result of this paper is stated as follows:

Theorem 1.1. *Assume f, g satisfy $(H_1) - (H_4)$, $s \in (0, 1)$, then*

(i) *(Existence) the nonlocal system (1.3) has at least one positive solution $(u, v) \in H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N)$ which indeed is a least energy solution;*

(ii) *(Regularity) If $s \in [\frac{1}{4}, \frac{1}{2})$ for $N = 1$, $s \in [\frac{N}{2(N+1)}, 1)$ for $N \geq 2$, then $u, v \in L^r(\mathbb{R}^N) \cap C^{0,\mu}(\mathbb{R}^N)$ for all $r \in [2, +\infty)$ and some $\mu \in (0, 1)$. Moreover, $|u(x)| \rightarrow 0, |v(x)| \rightarrow 0$ as $|x| \rightarrow \infty$;*

(iii) *(Decay estimates) there exist some constants $0 < C_1 \leq C_2$ such that*

$$\frac{C_1}{|x|^{N+2s}} \leq u(x), v(x) \leq \frac{C_2}{|x|^{N+2s}},$$

for all $|x| \geq R$, where $R > 0$ is an appropriate constant;

(iv) *(Symmetry) If we further assume that f, g satisfy (H_5) , then all positive solutions of (1.3) are radially symmetric with respect to the origin, and $u'(r), v'(r) < 0$ for $r = |x|$.*

A typical example of functions verifying the assumption $(H_1) - (H_5)$ is given by $f(t) = l_1|t|^{p-2}t, g(t) = l_2|t|^{q-2}t$ with $l_1, l_2 > 0$ and $p, q > 2$ such that $\frac{1}{p} + \frac{1}{q} > \frac{N-2s}{N}$.

Remark 1.2. (i) Although we have a variational problem, the functional associated to it is strongly indefinite, that is, compared to the single equation case and cooperative type systems, the quadratic part of the energy functional has no longer a positive sign, and so we have to recourse to the “Indefinite Functional Theorem” introduced by Benci and Rabinowitz in [24] which is an extension of both the mountain-pass theorem and the saddle point theorem.

(ii) Theorem 1.1 is a counterpart, precisely a generalization of the main results obtained in [23,25], where the authors considered the case when $s = 1$. Compared with the operator $-\Delta$, which is local, main difficulty of studying of the system (1.3) in the non-local character of the involved operator $(-\Delta)^s$. To overcome this difficulty we use the Caffarelli-Silvestre extension method. This allows us to apply variational techniques to these kinds of problems. However, we would like to point out the ideas as in [23], Liouville-type theorems, classical blow-up arguments in [25] and the moving planes method in [26] are not suitable absolutely for our situation because of emergence of nonlocal operators, some estimates and analysis are more delicate.

(iii) The assumptions on p and q : $p, q > 2$ and $\frac{1}{p} + \frac{1}{q} > \frac{N-2s}{N}$ are natural to the system (1.3), which are more general than the ones: $2 < p, q < 2_s^* := \frac{2N}{N-2s}$. But, the associated functional may not be well defined in the space $H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N)$ under the assumptions, because it may happen that say $p < 2_s^* < q$, here 2_s^* denotes the fractional critical Sobolev exponent. However, as explained in Sect.4, we only have to prove Theorem 1.1 in the case $2 < p = q < 2_s^*$.

In fact, given $n \in \mathbb{N}$, we can define the truncated functions,

$$g_n(t) = \begin{cases} g(t), & t \leq n, \\ A_n t^{p-1} + B_n, & t > n, \end{cases}$$

where the coefficients are chosen in such a way that g_n is C^1 . Thus, in view of (H_2) , we see that $A_n = (\frac{l_2}{p-1} + o(1)) \cdot n^{q-p}$, $B_n = (\frac{l_2(p-q)}{(p-1)(q-1)} + o(1)) \cdot n^{q-1}$. In Sect.4, we consider the truncated problem and obtain the existence of the solutions (u_n, v_n) to the corresponding system. Then we show that $\|u_n\|_\infty, \|v_n\|_\infty \leq C$ for some $C > 0$ independent of n , therefore they solve the original problem (1.3) if n is taken sufficiently large. Thus, in Sect.2-3, we assume that $2 < p = q < 2_s^*$.

This paper is organized as follows. In Sect.2, we review certain notations related to the fractional Laplacian and describe the appropriate functional setting for the system (1.3). Sect.3 is devoted to studying the autonomous system (1.3) and giving the proof of main results. In Sect.4, we will show that the solutions to the truncated problem are bounded in $L^\infty(\mathbb{R}^N)$, for this, some Liouville-type theorems need be established.

Notations Here we list some notations which will be used throughout the paper.

- The symbol $\mathbb{R}_+^{N+1}$ denotes the upper half-space $\{(x, y) : x \in \mathbb{R}^N, y > 0\}$.
- The letters $C, C_i, i = 0, 1, 2, \dots$, will be repeatedly used to denote various positive constants whose exact values are irrelevant.
- For $k \in \mathbb{N}$, we will denote by $B_k(x_0, r)$ the ball $\{x \in \mathbb{R}^k : |x - x_0| < r\}$ for each $x_0 \in \mathbb{R}^k$ and $r > 0$. $B_{N+1}^+(x_0, r) := B_{N+1}(x_0, r) \cap \mathbb{R}_+^{N+1}$.

2 Preliminaries

In this section, we collect some preliminary results for the fractional Laplacian. Recall that for $s \in (0, 1)$, the fractional space $H^s(\mathbb{R}^N)$ is defined by

$$H^s(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : \|u\|_{H^s} := \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))^2}{|x - y|^{N+2s}} dx dy < +\infty\},$$

equipped with the norm

$$\|u\|_{H^s(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))^2}{|x - y|^{N+2s}} dx dy \right)^{\frac{1}{2}}.$$

The fractional Laplacian $(-\Delta)^s$ of a smooth function $u : \mathbb{R}^N \rightarrow \mathbb{R}$ is defined by

$$\mathcal{F}((-\Delta)^s u)(\xi) = |\xi|^{2s} \mathcal{F}(u)(\xi), \xi \in \mathbb{R}^N,$$

where $\mathcal{F}$ denotes the Fourier transform, that is

$$\mathcal{F}(\omega)(\xi) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{-i\xi \cdot x} \omega(x) dx := \hat{\omega}(\xi),$$

for function ω in the Schwartz class. Also, $(-\Delta)^s u$ can be equivalently represented as

$$(-\Delta)^s u = -\frac{1}{2} C(N, s) \int_{\mathbb{R}^N} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{N+2s}} dy,$$

for all $x \in \mathbb{R}^N$, where

$$C(N, s) = \left(\int_{\mathbb{R}^N} \frac{1 - \cos \xi_1}{|\xi|^{N+2s}} d\xi \right)^{-1}, \xi = (\xi_1, \xi_2, \dots, \xi_N).$$

Also, we have from [4] that

$$\|(-\Delta)^s u\|_{L^2(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |\xi|^{2s} |\mathcal{F}(u)|^2 d\xi = \frac{1}{2} C(N, s) \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))^2}{|x - y|^{N+2s}} dx dy,$$

for all $u \in H^s(\mathbb{R}^N)$. For $N > 2s$, we also know that, for any $p \in [2, 2_s^*]$, there exists $C_p > 0$ such that

$$\|u\|_{L^p(\mathbb{R}^N)} \leq C_p \|u\|_{H^s(\mathbb{R}^N)}, \text{ for all } u \in H^s(\mathbb{R}^N).$$

To deal with the nonlocal system (1.3), we will use a method due to Caffarilli and Silvestre in [5] to study a corresponding extension problem, which allows us to investigate the system (1.3) by studying a local problem via classical variational methods. Recall that for $u \in H^s(\mathbb{R}^N)$, the solution $U \in X^s(\mathbb{R}_+^{N+1})$ of

$$\begin{cases} -\operatorname{div}(y^{1-2s} \nabla U) = 0, & \text{in } \mathbb{R}_+^{N+1}, \\ U = u, & \text{on } \mathbb{R}^N \times \{0\}, \end{cases} \quad (2.1)$$

is called s -harmonic extension $U = E_s(u)$ of u . In [12] it is proved that

$$\lim_{y \rightarrow 0^+} y^{1-2s} \frac{\partial U}{\partial y}(x, y) = -\frac{1}{k_s} (-\Delta)^s u(x),$$

where k_s is a normalization constant.

Remarking (2.1), we introduce the function space $X^s(\mathbb{R}_+^{N+1})$ that is defined as the completion of $C_0^\infty(\mathbb{R}_+^{N+1})$ with respect to the norm

$$\|U\|_{X^s} = (k_s \int_{\mathbb{R}_+^{N+1}} y^{1-2s} |\nabla U|^2 dx dy)^{\frac{1}{2}}.$$

It is a Hilbert space endowed with the inner product

$$\langle U, V \rangle = k_s \int_{\mathbb{R}_+^{N+1}} y^{1-2s} \langle \nabla U, \nabla V \rangle dx dy, \text{ for } U, V \in X^s(\mathbb{R}_+^{N+1}).$$

With the constant k_s , we have the extension operator to be an isometry between $H^s(\mathbb{R}^N)$ and $X^s(\mathbb{R}_+^{N+1})$. That is

$$\|U\|_{X^s}^2 = \|u\|_{H^s}^2 = \|(-\Delta)^{\frac{s}{2}} u\|_2^2.$$

On the other hand, for a function $U \in X^s(\mathbb{R}_+^{N+1})$, we will denote its trace on $\mathbb{R}^N \times \{0\}$ as $Tr(U)$. This trace operator is also well defined and it satisfies

$$\|Tr(U)\|_{H^s} \leq \|U\|_{X^s}.$$

For convenience, we will use the following notations:

$$\begin{aligned} L_s w &:= -\operatorname{div}(y^{1-2s} \nabla w), \\ \partial_\nu w &:= -k_s \left(\lim_{y \rightarrow 0^+} y^{1-2s} \frac{\partial w}{\partial y}(x, y) \right), \text{ for } x \in \mathbb{R}^N. \end{aligned}$$

With the above extension (2.1), we can reformulate our problem (1.3) as

$$\begin{cases} L_s U = L_s V = 0 & \text{in } \mathbb{R}_+^{N+1}, \\ \partial_\nu U = g(V) - \lambda U & \text{in } \mathbb{R}^N \times \{0\}, \\ \partial_\nu V = f(U) - \lambda V & \text{in } \mathbb{R}^N \times \{0\}, \\ U = u, V = v & \text{on } \mathbb{R}^N \times \{0\}. \end{cases} \quad (2.2)$$

We are looking for a positive solution (U, V) in the Hilbert space $E := X^s(\mathbb{R}_+^{N+1}) \times X^s(\mathbb{R}_+^{N+1})$, endowed with the norm

$$\|(U, V)\|_E^2 = \|U\|_{X^s}^2 + \|V\|_{X^s}^2,$$

Consider the Euler-Lagrange function associated to (2.2) given by

$$\begin{aligned} J_\lambda(U, V) &= k_s \int_{\mathbb{R}_+^{N+1}} y^{1-2s} \langle \nabla U, \nabla V \rangle dx dy + \lambda \int_{\mathbb{R}^N \times \{0\}} UV dx \\ &\quad - \int_{\mathbb{R}^N \times \{0\}} F(U) dx - \int_{\mathbb{R}^N \times \{0\}} G(V) dx. \end{aligned}$$

where $F(t) := \int_0^t f(\xi) d\xi$, $G(t) := \int_0^t g(\xi) d\xi$, which is C^2 well defined over the Hilbert space E when $2 < p = q < 2_s^*$, and moreover, the critical points of J_λ correspond to the weak solutions of (2.2). If (U, V) is a solution of (2.2), then the trace $(u, v) = (Tr(U), Tr(V)) = (U(x, 0), V(x, 0))$ is a solution of (1.3). The converse is also true. Therefore, both formulations are equivalent. Moreover, we have that $X^s(\mathbb{R}_+^{N+1})$ is local compactly embedding in $L^2(\mathbb{R}_+^{N+1}, y^{1-2s})$, the weight Lebesgue space endowed with the norm

$$\|U\|_{L^2(\mathbb{R}_+^{N+1}, y^{1-2s})} = \left(\int_{\mathbb{R}_+^{N+1}} y^{1-2s} U^2 dx dy \right)^{\frac{1}{2}}.$$

For the proof of the regularity, we utilize the Sobolev inequality on weighted spaces which appeared in [27].

Proposition 2.1. *Let Ω be an open bounded set in $\mathbb{R}_+^{N+1}$. Then there exists a constant $C = C(N, s, \Omega) > 0$ such that*

$$\left(\int_{\Omega} y^{1-2s} |U(x, y)|^{\frac{2(N+1)}{N}} dx dy \right)^{\frac{N}{2(N+1)}} \leq C \left(\int_{\Omega} y^{1-2s} |\nabla U(x, y)|^2 dx dy \right)^{\frac{1}{2}}$$

holds for any function U whose support is contained in Ω whenever the right-hand side is well-defined.

What as follows, we recall that the definitions of the relative Morse index and solutions having finite index.

Let E be a real Hilbert space, for a closed subspace of $V \subset E$, we denote by P_V the orthogonal projection onto V and by $V^\perp$ the orthogonal complement of V . Following [28] and [29], we say that the closed subspaces V, W of E are commensurable if $P_{V^\perp}P_W$ and $P_{W^\perp}P_V$ are compact operators.

If V and W are commensurable, the relative dimension of W with respect to V is defined as

$$\dim_V W = \dim(W \cap V^\perp) - \dim(W^\perp \cap V).$$

Commensurability guarantees that both terms in the above formula are finite.

Definition 2.2. The relative Morse index of a critical point (u, v) of a functional I with respect to the splitting $E = E^+ \oplus E^-$ can be defined as the integer

$$m(u, v) = \dim_{E^-} [\text{negative eigenspace of } I''(u, v)].$$

We will also borrow the definition of solutions having finite index as defined in [30].

Definition 2.3. Let (U, V) is a weak solution of (2.2), we say that $m(U, V) < +\infty$ if there exists $R_0 > 0$ with the property that for every $\phi \in C_0^\infty(\mathbb{R}_+^{N+1})$ such that $\phi = 1$ in $B_{N+1}^+(0, 2R_0) \setminus B_{N+1}^+(0, R_0)$ and $\text{Supp} \phi \subset B_{N+1}^+(0, 3.5R_0) \setminus B_{N+1}^+(0, 0.5R_0)$, it holds that

$$\begin{aligned} J_\lambda''(U, V)(\phi, \phi)(\phi, \phi) &= 2\|\phi\|_{X^s}^2 - \int_{\mathbb{R}^N \times \{0\}} f'(U)\phi^2(x, 0)dx \\ &\quad - \int_{\mathbb{R}^N \times \{0\}} g'(V)\phi^2(x, 0)dx \geq 0. \end{aligned} \quad (2.3)$$

3 The proof of Theorem 1.1

3.1 The existence of weak solutions

In this subsection, we prove Theorem 1.1 on the existence of weak solutions of the system (1.3), in view of hypothesis $2 < p = q < 2_s^*$, we work with the space $E := X^s(\mathbb{R}_+^{N+1}) \times X^s(\mathbb{R}_+^{N+1})$. So we consider the functional $J_\lambda : E \rightarrow \mathbb{R}^1$, defined by

$$\begin{aligned} J_\lambda(U, V) &= k_s \int_{\mathbb{R}_+^{N+1}} y^{1-2s} \langle \nabla U, \nabla V \rangle dx dy + \lambda \int_{\mathbb{R}^N \times \{0\}} UV dx \\ &\quad - \int_{\mathbb{R}^N \times \{0\}} F(U) dx - \int_{\mathbb{R}^N \times \{0\}} G(V) dx, \end{aligned}$$

J_λ is a C^2 functional and

$$\begin{aligned} J_\lambda'(U, V)(\Phi, \Psi) &= k_s \int_{\mathbb{R}_+^{N+1}} y^{1-2s} [\langle \nabla U, \nabla \Psi \rangle + \langle \nabla \Phi, \nabla V \rangle] dx dy \\ &\quad + \lambda \int_{\mathbb{R}^N \times \{0\}} [U\Psi + \Phi V] dx - \int_{\mathbb{R}^N \times \{0\}} f(U)\Phi dx \\ &\quad - \int_{\mathbb{R}^N \times \{0\}} g(V)\Psi dx, \end{aligned}$$

for any $\Psi, \Phi \in X^s(\mathbb{R}_+^{N+1})$. So, the critical points of J_λ satisfy the equations

$$k_s \int_{\mathbb{R}_+^{N+1}} y^{1-2s} \langle \nabla U, \nabla \Psi \rangle dx dy + \lambda \int_{\mathbb{R}^N \times \{0\}} U\Psi dx - \int_{\mathbb{R}^N \times \{0\}} g(V)\Psi dx = 0, \quad (3.1)$$

and

$$k_s \int_{\mathbb{R}_+^{N+1}} y^{1-2s} \langle \nabla \Phi, \nabla V \rangle dx dy + \lambda \int_{\mathbb{R}^N \times \{0\}} \Phi V dx - \int_{\mathbb{R}^N \times \{0\}} f(U)\Phi dx = 0, \quad (3.2)$$

for any $\Psi, \Phi \in X^s(\mathbb{R}_+^{N+1})$. Equations (3.1)-(3.2) are the weak formulation of (2.2).

It can be observed that the following orthogonal splitting holds $E = E^- \oplus E^+$, where $E^\pm := \{(\Phi, \pm\Phi) : \Phi \in X^s(\mathbb{R}_+^{N+1})\}$ (Since for any $(U, V) \in E$, $(U, V) = (\frac{U+V}{2}, \frac{U+V}{2}) + (\frac{U-V}{2}, \frac{V-U}{2})$). So that, denoting by Q_λ the quadratic term of the energy functional J_λ , namely

$$Q_\lambda(U, V) = k_s \int_{\mathbb{R}_+^{N+1}} y^{1-2s} \langle \nabla U, \nabla V \rangle dx dy + \lambda \int_{\mathbb{R}^N \times \{0\}} UV dx$$

We have that Q_λ is positive definite (resp, negative definite) in E^+ (resp, in E^-). Therefore, J_λ is a indefinite functional, we have to refer the ‘‘Indefinite functional theorem’’ introduced by Benci and Rabinowitz in [22] to obtain a nontrivial critical point of J_λ .

The following Lemma will play a significant role in the sequel whose proof is similar with [31] Lemma 2.1, so we omit it.

Lemma 3.1. *Let (U_n, V_n) be a $(PS)_c$ sequence for the functional J_λ , namely*

$$J_\lambda(U_n, V_n) \rightarrow c \in \mathbb{R}^+,$$

$$\mu_n := \sup |J'_\lambda(U_n, V_n)(\Phi, \Psi)|, \Phi, \Psi \in X^s(\mathbb{R}_+^{N+1}), \|\Phi\|_{X^s} + \|\Psi\|_{X^s} \leq 1 \rightarrow 0,$$

Then (U_n, V_n) is bounded in E and

$$\sup_{E^- \oplus \mathbb{R}^+(U_n, V_n)} J_\lambda = J_\lambda(U_n, V_n) + O(\mu_n^2).$$

The proof of Theorem 1.1(i). By the assumptions $(H_1) - (H_3)$, it is easy to check that the energy function J_λ possesses the linking structure, that is, $J_\lambda \leq 0$ in E^- , $J_\lambda \geq \rho > 0$ in $E^+ \cap \partial B_{N+1}^+(0, r)$, for some small $r > 0, \rho > 0$; moreover, if $r > 0$ is sufficiently large and $e = (e_1, e_2) \in E, e_1 > 0, e_2 > 0$, then

$$\sup_{(E^- \oplus \mathbb{R}^+ e) \cap \partial B_{N+1}^+(0, r)} J_\lambda \leq 0.$$

Then, according to ‘‘Indefinite functional theorem’’[22], J_λ has a $(PS)_c$ sequence $\{(U_n, V_n)\} \subset E$, where $0 < \rho \leq c \leq \sup_{E^- \oplus \mathbb{R}^+ e} J_\lambda$, using Lemma 3.1, (U_n, V_n) are bounded in E and may assume $(U_n, V_n) \rightharpoonup (U, V)$ as $n \rightarrow \infty$, then clearly $J'_\lambda(U, V) = 0$. Next we need to show that there exists a non-trivial critical point. For this purpose, by concentration compact principle [26], it is possible to find a sequence $\{x_n\} \subset \mathbb{R}^N$ and some constants $R > 0$ and $\beta > 0$ such that

$$\int_{B_N(x_n, R)} u_n^2 dx > \beta, \int_{B_N(x_n, R)} v_n^2 dx > \beta, \text{ for any } n \in \mathbb{N}.$$

Indeed, assuming the contrary, we have

$$u_n(x) \rightarrow 0, v_n(x) \rightarrow 0 \text{ in } L^p(\mathbb{R}^N), \quad (2 \leq p < 2_s^*).$$

But then, for large n and some constants $a > 0$ and $C_1, C_2 > 0$, we have

$$\begin{aligned} 2c + o(1) &= 2J_\lambda(U_n, V_n) - J'_\lambda(U_n, V_n)(U_n, V_n) \\ &= \int_{\mathbb{R}^N} [f(u_n)u_n - 2F(u_n) + g(v_n)v_n - 2G(v_n)] dx \\ &\leq \int_{\mathbb{R}^N} C_1(|u_n|^2 + |v_n|^2) + C_2(|u_n|^p + |v_n|^p) dx \end{aligned}$$

proving a contradiction, since $c > 0$.

Now we define $\widetilde{U}_n(x, y) = U_n(x + x_n, y)$, $\widetilde{V}_n(x, y) = V_n(x + x_n, y)$, then $(\widetilde{U}_n, \widetilde{V}_n) \rightharpoonup (U_0, V_0) \neq (0, 0)$ is a non-trivial critical point of J_λ .

Let $c(\lambda) := \inf\{J_\lambda(U, V) : (U, V) \neq (0, 0), J'_\lambda(U, V) = 0\}$, using the standard arguments, the infimum is actually a minimum and it follows that J_λ admits a ground state critical level $c(\lambda)$. It follows the proof of Lemma 3.1 in [32], the map $\lambda \rightarrow c(\lambda)$ is continuous and increasing and $\lim_{\lambda \rightarrow \infty} c(\lambda) = +\infty$. This completes the proof of Theorem 1.1(i). $\square$

3.2 The regularity and decay estimates of weak solutions

In this subsection, we prove Theorem 1.1(ii)(iii) stated in the Introduction.

The proof of Theorem 1.1(ii). Let (U, V) is the positive solution obtained in Theorem 1.1(i). Choose a smooth function $\eta \in C_0^\infty(\mathbb{R}_+^{N+1}, [0, 1])$ supported on $B_{N+1}^+(0, 2R) \subset \mathbb{R}_+^{N+1}$ satisfying $\eta = 1$ on $B_{N+1}^+(0, R)$ and $|\nabla \eta|^2 \leq |\eta|$. Multiplying the both side of the first equality of (2.2) by $\eta^2 U^\beta$, here $\beta = 1 + \frac{2}{N}$, we discover that

$$\begin{aligned} & k_s \int_{\mathbb{R}_+^{N+1}} y^{1-2s} \langle \nabla U, \nabla(\eta^2 U^\beta) \rangle dx dy + \lambda \int_{\mathbb{R}^N \times \{0\}} U^{\beta+1} \eta^2 dx \\ &= \int_{\mathbb{R}^N \times \{0\}} g(V) \eta^2 U^\beta dx. \end{aligned} \quad (3.3)$$

For easy reference, let us denote the three above integrals by I_1, I_2 and I_3 , in the order they appear. We now estimate I_1 , employing the Young's inequality: $ab \leq \frac{a^2}{2\delta} + \frac{\delta b^2}{2}$ for $\delta = \frac{\beta}{2}$ to get

$$\begin{aligned} I_1 &= k_s \int_{\mathbb{R}_+^{N+1}} y^{1-2s} \langle \nabla U, \nabla(\eta^2 U^\beta) \rangle dx dy \\ &= k_s \int_{\mathbb{R}_+^{N+1}} (\beta y^{1-2s} \eta^2 U^{\beta-1} |\nabla U|^2 + 2y^{1-2s} U^\beta \eta(\nabla U) \cdot (\nabla \eta)) dx dy \\ &\geq \frac{\beta}{2} k_s \int_{\mathbb{R}_+^{N+1}} y^{1-2s} |U^{\frac{\beta-1}{2}} \eta(\nabla U)|^2 dx dy \\ &\quad - \frac{2}{\beta} k_s \int_{\mathbb{R}_+^{N+1}} y^{1-2s} |U^{\frac{\beta+1}{2}} (\nabla \eta)|^2 dx dy. \end{aligned} \quad (3.4)$$

On the other hand, applying the identity

$$\nabla(U^{\frac{\beta+1}{2}} \eta) = \frac{\beta+1}{2} U^{\frac{\beta-1}{2}} \eta \cdot (\nabla U) + U^{\frac{\beta+1}{2}} (\nabla \eta),$$

we obtain

$$2\left(\frac{\beta+1}{2}\right)^2 |U^{\frac{\beta-1}{2}} \eta(\nabla U)|^2 + 2|U^{\frac{\beta+1}{2}} \nabla \eta|^2 \geq |\nabla(U^{\frac{\beta+1}{2}} \eta)|^2.$$

This gives

$$|U^{\frac{\beta-1}{2}} \eta(\nabla U)|^2 \geq \frac{2}{(\beta+1)^2} [|\nabla(U^{\frac{\beta+1}{2}} \eta)|^2 - 2|U^{\frac{\beta+1}{2}} \nabla \eta|^2].$$

Combing this with (3.3) (3.4), $I_1 \leq I_3$ and using the Sobolev trace inequality, we deduce that

$$\begin{aligned} I_3 &= \int_{\mathbb{R}^N \times \{0\}} g(V) \eta^2 U^\beta dx \\ &\geq k_s \frac{\beta}{(\beta+1)^2} \int_{\mathbb{R}_+^{N+1}} y^{1-2s} |\nabla(U^{\frac{\beta+1}{2}} \eta)|^2 dx dy \\ &\quad - \left(\frac{2}{\beta} + \frac{2\beta}{(\beta+1)^2}\right) k_s \int_{\mathbb{R}_+^{N+1}} y^{1-2s} |U^{\frac{\beta+1}{2}} \nabla \eta|^2 dx dy \\ &\geq \frac{C \cdot \beta k_s}{(\beta+1)^2} \left(\int_{\mathbb{R}^N \times \{0\}} (U^{\frac{\beta+1}{2}} \eta)^{\frac{2N}{N-2s}} dx \right)^{\frac{N-2s}{N}} \\ &\quad - \left(\frac{2}{\beta} + \frac{2\beta}{(\beta+1)^2}\right) k_s \int_{\mathbb{R}_+^{N+1}} y^{1-2s} |U^{\frac{\beta+1}{2}} \nabla \eta|^2 dx dy. \end{aligned} \quad (3.5)$$

Now we compute I_3 . Given $\varepsilon > 0$, it follows from the assumptions $(H_1)(H_2)$ that there is a constant c_ε such that

$$|g(v)| \leq \varepsilon |v|^{\frac{N+2s}{N-2s}} + c_\varepsilon |v|,$$

which we use to estimate I_3 ,

$$\begin{aligned} I_3 &\leq \varepsilon \int_{\mathbb{R}^N \times \{0\}} V^{\frac{N+2s}{N-2s}} \eta^2 U^\beta dx + c_\varepsilon \int_{\mathbb{R}^N \times \{0\}} V \eta^2 U^\beta dx \\ &:= \varepsilon I_4 + c_\varepsilon I_5. \end{aligned} \quad (3.6)$$

By the Hölder's and Young's inequalities, we obtain

$$I_4 \leq (\int_{\mathbb{R}^N \times \{0\}} (\eta^2 U^{\beta+1})^{\frac{N}{N-2s}} dx)^{\frac{\beta}{\beta+1} \cdot \frac{N-2s}{N}} \cdot (\int_{\mathbb{R}^N \times \{0\}} V^{\frac{2N}{N-2s}} dx)^{\frac{2s}{N}} \cdot (\int_{\mathbb{R}^N \times \{0\}} (\eta^2 V^{\beta+1})^{\frac{N}{N-2s}} dx)^{\frac{1}{\beta+1} \cdot \frac{N-2s}{N}}, \quad (3.7)$$

and

$$\begin{aligned} I_5 &\leq (\int_{\mathbb{R}^N \times \{0\}} \eta^2 U^{\beta+1} dx)^{\frac{\beta}{\beta+1}} \cdot (\int_{\mathbb{R}^N \times \{0\}} \eta^2 V^{\beta+1} dx)^{\frac{1}{\beta+1}} \\ &\leq C \int_{\mathbb{R}^N \times \{0\}} \eta^2 (U^{\beta+1} + V^{\beta+1}) dx. \end{aligned} \quad (3.8)$$

Let

$$\begin{aligned} A &:= (\int_{\mathbb{R}^N \times \{0\}} (\eta^2 U^{\beta+1})^{\frac{N}{N-2s}} dx)^{\frac{N-2s}{N}} \\ B &:= (\int_{\mathbb{R}^N \times \{0\}} (\eta^2 V^{\beta+1})^{\frac{N}{N-2s}} dx)^{\frac{N-2s}{N}}. \end{aligned} \quad (3.9)$$

It follows from (3.5)-(3.8) that

$$\begin{aligned} A &\leq \varepsilon C A^{\frac{\beta}{\beta+1}} B^{\frac{1}{\beta+1}} + c_\varepsilon \int_{\mathbb{R}^N \times \{0\}} \eta^2 (U^{\beta+1} + V^{\beta+1}) dx \\ &\quad + C k_s \int_{\mathbb{R}_+^{N+1}} y^{1-2s} U^{\beta+1} |\nabla \eta|^2 dx dy. \end{aligned}$$

Similarly, we also obtain that

$$\begin{aligned} B &\leq \varepsilon C_1 B^{\frac{\beta}{\beta+1}} A^{\frac{1}{\beta+1}} + c_\varepsilon \int_{\mathbb{R}^N \times \{0\}} \eta^2 (U^{\beta+1} + V^{\beta+1}) dx \\ &\quad + C_2 k_s \int_{\mathbb{R}_+^{N+1}} y^{1-2s} V^{\beta+1} |\nabla \eta|^2 dx dy. \end{aligned}$$

We start iteration with $\beta = 1 + \frac{2}{N}$ and use the condition on the parameter s and Proposition 2.1, we derive that

$$\int_{\mathbb{R}^N \times \{0\}} \eta^2 (U^{\beta+1} + V^{\beta+1}) dx < +\infty,$$

and

$$\begin{aligned} &k_s \int_{\mathbb{R}_+^{N+1}} y^{1-2s} U^{\beta+1} |\nabla \eta|^2 dx dy \\ &\leq (\int_{R \leq |x|^2 + y^2 \leq 2R} y^{1-2s} |\nabla U|^2 dx dy)^{\frac{N+1}{N}} \\ &< +\infty. \end{aligned}$$

Therefore,

$$A \leq \varepsilon C A^{\frac{\beta}{\beta+1}} B^{\frac{1}{\beta+1}} + c_\varepsilon, \quad (3.10)$$

$$B \leq \varepsilon C B^{\frac{\beta}{\beta+1}} A^{\frac{1}{\beta+1}} + c_\varepsilon. \quad (3.11)$$

Multiplying (3.10) by (3.11) we obtain

$$AB \leq \varepsilon^2 CAB + \varepsilon c_\varepsilon A^{\frac{\beta}{\beta+1}} B^{\frac{1}{\beta+1}} + \varepsilon c_\varepsilon A^{\frac{1}{\beta+1}} B^{\frac{\beta}{\beta+1}} + c_\varepsilon.$$

For small fixed $\varepsilon > 0$, we get

$$AB \leq \varepsilon c_\varepsilon A^{\frac{\beta}{\beta+1}} B^{\frac{1}{\beta+1}} + \varepsilon c_\varepsilon A^{\frac{1}{\beta+1}} B^{\frac{\beta}{\beta+1}} + c_\varepsilon.$$

Which implies that $A < +\infty, B < +\infty$, letting $R \rightarrow +\infty$ and it follows from (3.9) that

$$\int_{\mathbb{R}^N} u^{(\beta+1) \cdot \frac{N}{N-2s}} dx < +\infty \quad \text{and} \quad \int_{\mathbb{R}^N} v^{(\beta+1) \cdot \frac{N}{N-2s}} dx < +\infty.$$

Repeating this procedure we see that $u, v \in L^r$ for $r = (\beta + 1)(\frac{N}{N-2s})^k, k = 1, 2, \dots$. Using the Riesz-Thorin interpolation theorem, we conclude that $u, v \in L^r$ for all $r \geq 2$. Moreover, by Sobolev embedding [33], there exists $0 < \alpha < 1$ such that $W^{s,r}(\mathbb{R}^N) \hookrightarrow C^{0,\alpha}(\mathbb{R}^N)$ (for r large enough), we deduce that $u, v \in C^{0,\alpha}(\mathbb{R}^N)$. Finally, the fact that $u, v \in L^r(\mathbb{R}^N) \cap C^{0,\alpha}(\mathbb{R}^N)$ implies that $u(x) \rightarrow 0, v(x) \rightarrow 0$ as $|x| \rightarrow \infty$, completing the proof. $\square$

Next, we study the decay behavior of positive solutions of (1.3). Before starting to give the proof, let us consider for $h \in L^2(\mathbb{R}^N)$ the equation

$$(-\Delta)^s \phi + \phi = h \text{ in } \mathbb{R}^N.$$

Then in terms of Fourier transform, this problem, for $\phi \in L^2$, reads

$$(|\xi|^{2s} + 1)\widehat{\phi} = \widehat{h},$$

and has a unique solution $\phi \in H^s(\mathbb{R}^N)$ given by the convolution

$$\phi(x) = \mathcal{K} * h = \int_{\mathbb{R}^N} \mathcal{K}(x-z)h(z)dx dy,$$

where $\mathcal{K}$ is the fundamental solution of $(-\Delta)^s + 1$ or called the Bessel kernel,

$$\widehat{\mathcal{K}(\xi)} = \frac{1}{|\xi|^{2s} + 1}.$$

Let us recall the main properties of the kernel $\mathcal{K}$ that are stated for instance in [33], which are useful in what follows.

We have that

(i) $\mathcal{K}$ is positive, radially symmetric and smooth in $\mathbb{R}^N \setminus \{0\}$. Moreover, it is non-increasing as a function of $r = |x|$;

(ii) For appropriate positive constants C_1 and C_2 ,

$$\mathcal{K}(x) \leq \frac{C_1}{|x|^{N+2s}} \text{ if } |x| \geq 1 \text{ and } \mathcal{K}(x) \leq \frac{C_2}{|x|^{N-2s}} \text{ if } |x| \leq 1;$$

(iii) There is a constant $C > 0$ such that

$$|\nabla \mathcal{K}(x)| \leq \frac{C}{|x|^{N+1+2s}}, |D^2 \mathcal{K}(x)| \leq \frac{C}{|x|^{N+2+2s}}, \text{ if } |x| \geq 1;$$

(iv) If $q \geq 1$ and $t \in (N - 2s - \frac{N}{q}, N + 2s - \frac{N}{q})$, then $|x|^t \mathcal{K}(x) \in L^q(\mathbb{R}^N)$;

(v) If $q \in [1, \frac{N}{N-2s})$, then $\mathcal{K}(x) \in L^q(\mathbb{R}^N)$;

(vi) $|x|^{N+2s} \mathcal{K}(x) \in L^\infty(\mathbb{R}^N)$.

The proof of Theorem 1.1(iii). Firstly, we recall the following Claims in [33] that:

There are continuous functions w_1, w_2 in $\mathbb{R}^N$ satisfying, resp.

$$(-\Delta)^s w_1(x) + \lambda w_1(x) = 0, \text{ if } |x| > 1 \text{ and } w_1(x) \geq \frac{C_1}{|x|^{N+2s}},$$

for some $C_1 > 0$;

$$(-\Delta)^s w_2(x) + \frac{\lambda}{2} w_2(x) = 0, \text{ if } |x| > 1 \text{ and } 0 < w_2(x) \leq \frac{C_2}{|x|^{N+2s}},$$

for some $C_2 > 0$.

By $u(x), v(x) \rightarrow 0$ as $|x| \rightarrow \infty$ and the condition (H_1) , we conclude that there is a large $R_1 > 0$ such that

$$(-\Delta)^s(u+v) + \frac{\lambda}{2}(u+v) = g(v) + f(u) - \frac{\lambda}{2}(u+v) \leq 0 \text{ in } B_{R_1}^c.$$

Moreover, by the continuity of solution (u, v) and w_2 , there exists $C > 0$ such that

$$u(x) + v(x) - Cw_2(x) \leq 0 \text{ in } \overline{B_{R_1}}.$$

Therefore,

$$(-\Delta)^s(u + v - Cw_2) + \frac{\lambda}{2}(u + v - Cw_2) \leq 0 \text{ in } B_{R_1}^c.$$

Using comparison arguments, we get that

$$u + v \leq Cw_2 \leq \frac{C_2}{|x|^{N+2s}} \text{ for } |x| \geq R_1.$$

Since (u, v) is a positive solution, then

$$u(x) \leq \frac{C}{|x|^{N+2s}}, v(x) \leq \frac{C}{|x|^{N+2s}}, \text{ for } |x| \geq R_1.$$

On the other hand, by the continuity of (u, v) and w_1 , there exist constants $C_2, C_3 > 0$ such that, resp,

$$\begin{aligned} u(x) - C_2w_1(x) &\geq 0 \text{ in } \overline{B_1}, \\ v(x) - C_3w_1(x) &\geq 0 \text{ in } \overline{B_1}, \end{aligned}$$

which imply that

$$\begin{aligned} (-\Delta)^s(u - C_2w_1) + \lambda(u - C_2w_1) &\geq 0 \text{ in } \overline{B_1}^c, \\ (-\Delta)^s(v - C_3w_1) + \lambda(v - C_3w_1) &\geq 0 \text{ in } \overline{B_1}^c. \end{aligned}$$

By the similar comparison arguments, we conclude the second inequality. $\square$

3.3 The radial symmetry of positive solutions

In this subsection, we show that the positive solutions $(u(x), v(x))$ of the system (1.3) are radial symmetry. Here we give the proof based on the moving planes method as developed recently in [33], where the radial symmetry and monotonicity properties of the kernel $\mathcal{K}$ play a key role. The approach is different from the usual moving planes technique originated in [26] for the case $s = 1$.

We consider, initially, the planes parallel to the plane $x_1 = 0$. For each real α , we define

$$\Sigma_\alpha := \{x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N : x_1 > \alpha\}, T_\alpha = \partial\Sigma_\alpha,$$

$x^\alpha = (2\alpha - x_1, x_2, \dots, x_N)$ is the reflection of x on the plane T_α .

Let us define in $\overline{\Sigma_\alpha}$ the following functions

$$\begin{aligned} u_\alpha(x) &= u(x^\alpha), v_\alpha(x) = v(x^\alpha), \\ U_\alpha(x) &= u_\alpha(x) - u(x), V_\alpha(x) = v_\alpha(x) - v(x). \end{aligned}$$

Then

$$(-\Delta)^s U_\alpha(x) + U_\alpha(x) = g(v_\alpha) - g(v), \quad (3.12)$$

$$(-\Delta)^s V_\alpha(x) + V_\alpha(x) = f(u_\alpha) - f(u). \quad (3.13)$$

By the mean value theorem

$$\begin{aligned} g(v_\alpha) - g(v) &= g'(\psi(x; \alpha))V_\alpha(x), \\ f(u_\alpha) - f(u) &= f'(\varphi(x; \alpha))U_\alpha(x), \end{aligned}$$

where $\psi(x; \alpha)$ is a real function between $v(x^\alpha)$ and $v(x)$, the similar to $\varphi(x; \alpha)$. By the assumption (H_3) , we see that $g'(\psi(x; \alpha)) \geq 0, f'(\varphi(x; \alpha)) \geq 0$.

Lemma 3.2. *We have*

$$\begin{aligned} U_\alpha(x) &= \int_{\Sigma_\alpha} (\mathcal{K}(x - \xi) - \mathcal{K}(x^\alpha - \xi))(g(v_\alpha(\xi)) - g(v(\xi)))d\xi, \\ V_\alpha(x) &= \int_{\Sigma_\alpha} (\mathcal{K}(x - \xi) - \mathcal{K}(x^\alpha - \xi))(f(u_\alpha(\xi)) - f(u(\xi)))d\xi. \end{aligned}$$

Proof. It follows from the variable substitution and the radially symmetry of $\mathcal{K}$ that,

$$\begin{aligned} u(x) &= \mathcal{K} * g(v) = \int_{\mathbb{R}^N} \mathcal{K}(x - \xi)g(v(\xi))d\xi \\ &= \int_{\Sigma_\alpha} \mathcal{K}(x - \xi)g(v(\xi))d\xi + \int_{\Sigma_\alpha^c} \mathcal{K}(x - \xi)g(v(\xi))d\xi \\ &= \int_{\Sigma_\alpha} \mathcal{K}(x - \xi)g(v(\xi))d\xi + \int_{\Sigma_\alpha} \mathcal{K}(x^\alpha - \xi)g(v_\alpha(\xi))d\xi, \end{aligned}$$

and

$$\begin{aligned} u_\alpha(x) &= \mathcal{K} * g(v) = \int_{\mathbb{R}^N} \mathcal{K}(x^\alpha - \xi)g(v(\xi))d\xi \\ &= \int_{\Sigma_\alpha} \mathcal{K}(x^\alpha - \xi)g(v(\xi))d\xi + \int_{\Sigma_\alpha} \mathcal{K}(x - \xi)g(v_\alpha(\xi))d\xi. \end{aligned}$$

Therefore,

$$U_\alpha(x) = u_\alpha(x) - u(x) = \int_{\Sigma_\alpha} (\mathcal{K}(x - \xi) - \mathcal{K}(x^\alpha - \xi))(g(v_\alpha(\xi)) - g(v(\xi)))d\xi.$$

The similar to the computation of $V_\alpha(x)$. $\square$

Proposition 3.3. *There exists $\alpha^* < +\infty$ such that for all $\alpha \leq \alpha^*$, $U_\alpha(x) \leq 0$, $V_\alpha(x) \leq 0$ for all $x \in \Sigma_\alpha$.*

Proof. Suppose, by contradiction that, for all α , we have that $U_\alpha(x) > 0$ for some $x \in \Sigma_\alpha$ or $V_\alpha(x) > 0$ for some $x \in \Sigma_\alpha$.

Assume $v(x) < v_\alpha(x)$, consequently

$$v(x) \leq \psi(x; \alpha) \leq v_\alpha(x).$$

Using the conditions (H_1) (H_2) (H_5) , we derive that there exist $C > 0, \tau > 0$ such that

$$g'(\psi(x; \lambda)) \leq g'(v_\alpha(x)) \leq C|v_\alpha(x)|^\tau.$$

Denote by $\Sigma_1 := \{x \in \Sigma_\alpha \mid V_\alpha(x) > 0\}$. Next we claim $\Sigma_1 = \emptyset$.

Indeed, firstly Σ_1 is bounded, since V_α decay to zero at infinity. If $x \in \Sigma_1$, using the fact that $|\xi - x^\alpha| \geq |\xi - x|$ in $\xi \in \Sigma_\alpha$, $\mathcal{K}$ is decreasing, f is increasing and Lemma 3.3, we have

$$\begin{aligned} U_\alpha(x) &= \int_{\Sigma_1} (\mathcal{K}(x - \xi) - \mathcal{K}(x^\alpha - \xi))(g(v_\alpha(\xi)) - g(v(\xi)))d\xi \\ &\quad + \int_{\Sigma_\alpha \setminus \Sigma_1} (\mathcal{K}(x - \xi) - \mathcal{K}(x^\alpha - \xi))(g(v_\alpha(\xi)) - g(v(\xi)))d\xi \\ &\leq \int_{\Sigma_1} (\mathcal{K}(x - \xi) - \mathcal{K}(x^\alpha - \xi)) \cdot (g(v_\alpha) - g(v))d\xi \\ &\leq \int_{\Sigma_1} \mathcal{K}(x - \xi)(g(v_\alpha(\xi)) - g(v(\xi)))d\xi \\ &\leq \int_{\Sigma_1} \mathcal{K}(x - \xi)g'(\psi(x; \alpha))V_\alpha(\xi)d\xi \\ &\leq C \int_{\Sigma_1} \mathcal{K}(x - \xi)|v_\alpha(\xi)|^\tau \cdot |V_\alpha(\xi)|d\xi. \end{aligned}$$

Thus, by Lemma 5.2 in [33] for $q = m$ and $r = \frac{1}{2}m$ with m large, such that $m > \frac{N}{s}$ and $m\tau \geq 2$, we obtain

$$\|U_\alpha(x)\|_{L^m(\Sigma_1)} \leq C\|v_\alpha^\tau V_\alpha\|_{L^{\frac{m}{2}}(\Sigma_1)}.$$

Using the Hölder inequality, we get

$$\|U_\alpha(x)\|_{L^m(\Sigma_1)} \leq C\|v_\alpha\|_{L^{m\tau}(\Sigma_1)}^\tau \|V_\alpha\|_{L^m(\Sigma_1)}.$$

Then, for sufficiently small $\varepsilon > 0$, there is α large enough (negative) such that $\|v_\alpha\|_{L^{m\tau}(\Sigma_1)}^\tau < \varepsilon$, which conclude together with $u, v \in L^r(r \geq 2)$ that

$$\|U_\alpha(x)\|_{L^m(\Sigma_1)} \leq M\varepsilon. \quad (3.14)$$

Consequently $\Sigma_1 = \emptyset$. If not, we have from (3.14) that as $\alpha \rightarrow -\infty, \varepsilon \rightarrow 0$,

$$\int_{\Sigma_1} |u(x)|^m dx \rightarrow 0.$$

However, assume that there is $x_0 \in \Sigma_1$ such that $V_\alpha(x_0) > 0$, which implies from (3.12) that $u(x_0) > 0$. By the continuity of u , there is a small neighborhood $B_N(x_0, \delta) \subset \Sigma_1$ such that $\int_{B_N(x_0, \delta)} |u(x)|^m dx > \sigma$, we come to a contradiction.

If we assume $u(x) < u_\alpha(x)$, let $\Sigma_2 := \{x \in \Sigma_\alpha \mid U_\alpha(x) > 0\}$, similarly, $\Sigma_2 = \emptyset$.

In conclusion, there exists $\alpha^* < +\infty$ such that for all $\alpha \leq \alpha^*$, $U_\alpha(x) \leq 0$ and $V_\alpha(x) \leq 0$ for all $x \in \Sigma_\alpha$. $\square$

Now Proposition 3.4 allows us to define $\alpha_0 = \sup\{\alpha : U_\alpha(x) \leq 0, V_\alpha(x) \leq 0, \text{ for all } x \in \Sigma_\alpha\}$.

Proposition 3.4. *We have $U_{\alpha_0}(x) = 0, V_{\alpha_0}(x) = 0$ for all $x \in \Sigma_{\alpha_0}$.*

Proof. By the continuity, we see that $U_{\alpha_0}(x) \leq 0$ and $V_{\alpha_0}(x) \leq 0$ for all $x \in \Sigma_{\alpha_0}$. It follows from (3.12) and (3.13) that $U_{\alpha_0} = 0$ in Σ_{α_0} if and only if $V_{\alpha_0} = 0$. So, by contradiction, that $U_{\alpha_0} \neq 0$ and thus also $V_{\alpha_0} \neq 0$. Observe from (3.12) and (3.13) that

$$\begin{cases} (-\Delta)^s U_{\alpha_0} + U_{\alpha_0} \leq 0 & \text{in } \Sigma_{\alpha_0}, \\ (-\Delta)^s V_{\alpha_0} + V_{\alpha_0} \leq 0 & \text{in } \Sigma_{\alpha_0}. \end{cases}$$

So by the maximum principle $U_{\alpha_0}, V_{\alpha_0} < 0$ in Σ_{α_0} .

Now, take a sequence $\alpha_k \rightarrow \alpha_0$ with $\alpha_k > \alpha_0$. By the definition of α_0 , for each α_k , there is a point x_k or $y_k \in \Sigma_{\alpha_k}$ such that $U_{\alpha_k}(x_k) > 0$ or $V_{\alpha_k}(y_k) > 0$. The sequence $\{x_k\}$ and $\{y_k\}$ are bounded, since U_{α_k} and V_{α_k} decay to zero at infinity, so we may assume that $x_k \rightarrow \bar{x}$ with $\bar{x} \in \Sigma_{\alpha_0}$ or $y_k \rightarrow \bar{y}$ with $\bar{y} \in \Sigma_{\alpha_0}$. By the continuity, we have

$$u(\bar{x}) \leq u_{\alpha_0}(\bar{x}) \text{ or } v(\bar{y}) \leq v_{\alpha_0}(\bar{y}),$$

which contradict with $U_{\alpha_0}(\bar{x}) < 0$ and $V_{\alpha_0}(\bar{y}) < 0$. $\square$

The proof of Theorem 1.1(iv). By the translation, we may say that $\alpha_0 = 0$. Thus, we have that u, v are symmetric about the x_1 -axis, i.e. $u(x_1, x') = u(-x_1, x'), v(x_1, x') = v(-x_1, x')$. Using the same approach in any direction implies that u, v are radially symmetric. $\square$

4 The case $p \neq q$

In Sect.3, we have proved Theorem 1.1 except that we have worked with a truncated problem, as explained in Remark 1.2. The full statement of Theorem 1.1 will be established once we prove uniform bounds in L^∞ of the solutions constructed so far. So, in this section, let us suppose that $p, q > 2$ are such that $\frac{1}{p} + \frac{1}{q} > \frac{N-2s}{N}$ with say, $2 < p < 2_s^*$ and $p < q$.

Given $n \in \mathbb{N}$, we can define the truncated functions,

$$g_n(t) = \begin{cases} g(t), & t \leq n, \\ A_n t^{p-1} + B_n, & t > n, \end{cases}$$

where the coefficients are chosen in such a way that g_n is C^1 . Thus, in view of (H_2) , we see that $A_n = (\frac{l_2}{p-1} + o(1)) \cdot n^{q-p}$, $B_n = (\frac{l_2(p-q)}{(p-1)(q-1)} + o(1)) \cdot n^{q-1}$. The energy functionals associated to the modified problem of (2.2) are given by

$$\begin{aligned} J_n(U, V) &= k_s \int_{\mathbb{R}_+^{N+1}} y^{1-2s} \langle \nabla U, \nabla V \rangle dx dy + \lambda \int_{\mathbb{R}^N \times \{0\}} UV dx \\ &\quad - \int_{\mathbb{R}^N \times \{0\}} F(U) dx - \int_{\mathbb{R}^N \times \{0\}} G_n(V) dx, \end{aligned}$$

where G_n is the primitive of g_n . They are C^2 functionals defined over the Hilbert space E . The critical points of J_n correspond to weak solutions of the modified problem

$$\begin{cases} (-\Delta)^s u + \lambda u = g_n(v) & \text{in } \mathbb{R}^N, \\ (-\Delta)^s v + \lambda v = f(u) & \text{in } \mathbb{R}^N. \end{cases} \quad (4.1)$$

For a fixed n , thanks to the Section 3, there are positive solutions (u_n, v_n) of the modified problem (4.1) satisfying the conclusion of Theorem 1.1.

Remark 4.1. According to the Section 3, we find the solutions (u_n, v_n) of the modified problem (4.1) having relative Morse index ≤ 1 .

Now we state the main result of this section.

Theorem 4.2. Assume $(H_1) - (H_4)$. For any given $n \in \mathbb{N}$, let (U_n, V_n) be solutions of the problem (4.1). If there exists $k \in \mathbb{N}$ such that $m(u_n, v_n) \leq k$ for every n , then there exists $M > 0$ such that

$$\|u_n\|_\infty + \|v_n\|_\infty \leq M, \quad \forall n.$$

In particular, u_n and v_n are solutions of the problem (1.3), for large values of n .

The proof of Theorem 4.2 is based on the following simple fact whose proof is the same as Lemma 1.2 in [24].

Lemma 4.3. Assume $(H_1) - (H_4)$ and let (U_n, V_n) be any solutions of the problem (4.1). If there exist $\beta > 0$ and $k + 1$ functions $\Psi_1, \Psi_2, \dots, \Psi_{k+1} \in X$ having disjoint supports, such that

$$J_n''(U_n, V_n)(\Psi_i, \beta \Psi_i)(\Psi_i, \beta \Psi_i) < 0, \quad \forall i = 1, \dots, k + 1,$$

then $m(U_n, V_n) \geq k + 1$.

Next, we will prove a Liouville-type theorem which is crucial for the proof of Theorem 4.2.

Proposition 4.4. Let $f_\infty, g_\infty \in C^1(\mathbb{R})$ and (u, v) satisfy

$$\begin{cases} (-\Delta)^s u = g_\infty(v), & \text{in } \mathbb{R}^N, \\ (-\Delta)^s v = f_\infty(u), & \text{in } \mathbb{R}^N. \end{cases} \quad (4.2)$$

and $m(u, v) < +\infty$ in the sense of Definition 2.2 and 2.3. Let $f_\infty(t) = c|t|^{p-1}t$ with $c > 0$ and $2 < p < 2_s^*$.

(i) If $g_\infty = 0$, then $u = 0$;

(ii) If g_∞ satisfies following conditions, for $p \leq q$, $\frac{1}{p} + \frac{1}{q} > \frac{N-2s}{N}$ and some $C_1, C_2 > 0$,

(a) $C_1|t|^q \leq g_\infty(t)t \leq C_2|t|^q$;

(b) $g_\infty(t)t \leq qG_\infty(t)$;

(c) $(p-1)g_\infty(t)t \leq g'_\infty(t)t^2$;

then $u = 0 = v$.

Proof. (i) It is obvious.

(ii) We may assume $c = 1$, suppose $(-\Delta)^s u = g_\infty(v)$, $(-\Delta)^s v = |u|^{p-1}u$, with g_∞ satisfying the conditions (a) – (c). The associated energy functionals $J_\infty : X^s(\mathbb{R}_+^{N+1}) \times X^s(\mathbb{R}_+^{N+1}) \rightarrow \mathbb{R}^1$ to the extension problem of (4.2) is given by

$$J_\infty(U, V) = k_s \int_{\mathbb{R}_+^{N+1}} y^{1-2s} \langle \nabla U, \nabla V \rangle dx dy - \frac{1}{p} \int_{\mathbb{R}^N} |u|^p dx - \int_{\mathbb{R}^N} G_\infty(v) dx.$$

Fix any smooth function $\Psi \in C_0^\infty(\mathbb{R}_+^{N+1})$ such that $\Psi = 0$ in $B_{N+1}^+(0, 0.5R_0)$, $\Psi = 1$ in $B_{N+1}^+(0, 2R_0) \setminus B_{N+1}^+(0, R_0)$ and $\text{supp}\Psi \subset B_{N+1}^+(0, 3.5R_0) \setminus B_{N+1}^+(0, 0.5R_0)$. For any large R , let $\Phi \in C_0^\infty(\mathbb{R}_+^{N+1}, [0, 1])$ supported on $B_{N+1}^+(0, 2R) \subset \mathbb{R}_+^{N+1}$ satisfying $\Phi = 1$ on $B_{N+1}^+(0, R)$ and $|\nabla\Phi|^2 \leq |\Phi|$.

In view of $m(u, v) < +\infty$, replace ϕ with $U\Phi^m\Psi$ in (2.3), then the assumption (2.3) reads as

$$\begin{aligned} & (p-1) \int_{\mathbb{R}^N \times \{0\}} U^p \Phi^{2m} \Psi^2 dx + \int_{\mathbb{R}^N \times \{0\}} g'_\infty(V) U^2 \Phi^{2m} \Psi^2 dx \\ \leq & C k_s \int_{\mathbb{R}_+^{N+1}} y^{1-2s} (m^2 U^2 \Phi^{2(m-1)} \Psi^2 |\nabla\Phi|^2 + U^2 \Phi^{2m} |\nabla\Psi|^2 \\ & + \Phi^{2m} \Psi^2 |\nabla U|^2) dx dy + \int_{\mathbb{R}^N \times \{0\}} g_\infty(V) U \Phi^{2m} \Psi^2 dx, \end{aligned}$$

which implies that

$$\begin{aligned} & (p-1) \int_{\mathbb{R}^N \times \{0\}} U^p \Phi^{2m} \Psi^2 dx \\ \leq & C(R_0) k_s \int_{\text{supp}\Psi} y^{1-2s} (U^2 \Phi^{2(m-1)} |\nabla\Phi|^2 + U^2 \Phi^{2m} + |\nabla U|^2 \Phi^{2m}) dx dy \\ & + C(R_0) \int_{\text{supp}\Psi(x,0) \times \{0\}} g_\infty(V) U \Phi^{2m} dx. \end{aligned} \quad (4.3)$$

Now we estimate the right terms of the above inequality, it follows from Proposition 2.1 and Hölder inequality with m large that

$$k_s \int_{\text{supp}\Psi} y^{1-2s} (U^2 \Phi^{2(m-1)} |\nabla\Phi|^2 + U^2 \Phi^{2m} + |\nabla U|^2 \Phi^{2m}) dx dy < C. \quad (4.4)$$

On the other hand, in view of (4.2) and the similar arguments as (4.4), we arrive at

$$\begin{aligned} & \int_{\text{supp}\Psi(x,0) \times \{0\}} g_\infty(V) U \Phi^{2m} dx \\ = & k_s \int_{\text{supp}\Psi} y^{1-2s} \langle \nabla U, \nabla(U \Phi^{2m}) \rangle dx dy \\ \leq & k_s \int_{\text{supp}\Psi} y^{1-2s} (\Phi^{2m} |\nabla U|^2 + m^2 \Phi^{4m-2} U^2 + |\nabla U|^2 |\nabla\Phi|^2) dx dy < C. \end{aligned} \quad (4.5)$$

We conclude by combing (4.3)-(4.5), which together lead to $u \in L^p(\mathbb{R}^N)$. Thanks to $\int_{\mathbb{R}^N} (-\Delta)^s u v dx = \int_{\mathbb{R}^N} g_\infty(v) v dx = \int_{\mathbb{R}^N} u^p dx < +\infty$ and the condition (a), $v \in L^q(\mathbb{R}^N)$ also.

Making use of well-known Pohožăve-Rellich type identity,

$$\int_{\mathbb{R}^N} (-\Delta)^s u v dx = \frac{N}{N-2s} \int_{\mathbb{R}^N} (F_\infty(u) + G_\infty(v)) dx.$$

By the condition (c), we deduce that

$$\begin{aligned} & \frac{N-2s}{N} \int_{\mathbb{R}^N} u^p dx = \frac{1}{p} \int_{\mathbb{R}^N} u^p dx + \int_{\mathbb{R}^N} G_\infty(v) dx \\ \geq & \frac{1}{p} \int_{\mathbb{R}^N} u^p dx + \frac{1}{q} \int_{\mathbb{R}^N} g_\infty(v) v dx = \left(\frac{1}{p} + \frac{1}{q}\right) \int_{\mathbb{R}^N} u^p dx \end{aligned}$$

Since $\frac{1}{p} + \frac{1}{q} > \frac{N-2s}{N}$, this implies that $u = v = 0$ and concludes the proof of Proposition 4.4. $\square$

Once Proposition 4.4 is settled, we may use the classical blow-up argument to give the proof of Theorem 4.2 that is the similar with [25], we omit it.

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Competing interests

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S.H. Li and L.M. Wu carried out the proof of the theorems. S.Q. He and L.M. Zhang carried out the check of the manuscript. All authors read and approved the final manuscript.

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