A study of libration points and analysis of their stability behavior in-ring body problem under albedo effect

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A study of libration points and analysis of their stability behavior in-ring body problem under albedo effect

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Abstract Our intention in this article is to analyze the fundamental analysis of the zones and their existing libration points under the solar radiation pressure and albedo effects. We can feel about albedos at any Nature's property wherever the Sun rays exist. As we know that important phase of the dynamical system of any celestial body is its stability and the factors which change its behavior of motion. This research article presents an explicit analysis of the albedo effect for ring body problems for a specific value of $v = 6$, which explains the behavior of existing zone's stability, shifting of their libration points with monotonicity and the characteristic roots of δ confirms the behavior of stability condition for existing zones as well as their existing libration points in a synodic frame of references. We observed that in this problem from an experimental point of view, a little change in v, changes the complete behavior of the problem and leaves a nice platform of the proof. In this problem, β plays a significant role, which affects firstly the whole dynamics of peripheral primaries, the mass parameters $\beta = 0$ and $\beta \neq 0$ generally confirm that this synodic system wears, an absence of a center of mass and the presence of the center of mass, respectively. We have observed in this article is the libration point L_0, which exists in the center and has peripheral primary P_0, remains unchanged throughout applying the albedo effect. The stability analysis confirms the location of the libration points, whether it is stable, unstable, etc., but here we have seen the more interesting things *i.e.*, most of the locations of the libration points except exceptionals are found unstable. This article also shows the momentum of the position of all the libration points with their

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existing zone as well as peripheral primaries, by the effect of the radiation pressure and albedo. It is also analyzed here that this complete problem works in a certain short and very sensitive range set $(-1.75, 1.75)$ because this problem is filled with trigonometric periodic functions. In this problem, we have considered only the real roots of the axis x, y, whereas imaginary roots are not considered. For stability analysis, we have taken both real and complex roots of δ. Results of this article for $v = 6$ may help us to analyze an imaginary behavior of our solar system's planets in a blank paper, provided by certain rules and regulations by equations of motion because real behavior may be far away from us. By the present study, it is found that the whole behavior of the system for various v whenever change, which can be extended later in another way.

Keywords Celestial Bodies Albedo Effect Libration Points Stability Ring Body.

1 Introduction

The study of momentum of the celestial bodies in the dynamical system and their stability has great importance, as we know that albedo is dimensionless quantity as well as non-gravitational force existing valuable effects on the movement of masses as well as their stability. Generally, the albedo effect is measured on a scale of mass 0 to 1. As we know that almost complete source of natural radiation in the solar system is Sun. By the study of article [5]; a celestial body or surface having zero albedos implies that the body is 'black-body' and that suck all the occurred radiations, whereas the unit albedo of a celestial body depicts a 'white-body' which is a surpassing reflector that reflects all the occurred radiations absolutely and homogeneously in all the directions. For example, the albedo of the snow is 0.95 which reflects about 95% of the total incoming radiation, whereas the reflection of the incoming radiation of the water is approximately 10%, so in the resulting sense, the albedo of the water is 0.1.

In fact, there doesn't exist any planet in our solar system whose albedos are negligible or zero, *i.e*, no planets have completely 'black-body' in the solar system. A high albedo of a celestial body has a lower temperature because it reflects the majority of the radiation that hits it and sucks the rest of the radiation. The Earth's albedo is approximate 0.30,*i.e.*, The reflection of the total incident radiation of the entire Earth is approximately 30%. An albedo is a fraction of an expression of the ability of reflected radiation to incident radiation:

$$Albedo = \frac{Radiation\ reflected\ back\ to\ the\ space.}{incident\ radiation}.$$

[7] has analyzed the Albedo Effects on Libration Points in the Elliptic Restricted Three-Body Problem. Table 1: represents some facts about our planet's albedos. [6], have discussed, Non-collinear libration points in(Elliptic restricted

three-body problem) ER3BP with albedo effect and oblateness. This confirms that the planet Venus has high albedo whereas Mercury has very low albedo under this table. The study of the $(N+1)$ ring body problem has great impor-

Table 1 This table represents some facts about albedos of our planets.

Planet	Earth	Jupiter	Mars	Mercury	Neptune	Uranus	Saturn	Venus
Albedo	0.30	0.34	0.16	0.12	0.29	0.30	0.34	0.75

tance in celestial mechanics. One beautiful thing we studied in this problem is that a little change in values of v in equations of motion changes everything with a nice proof. It is also observed that the little change of values in v wears RTBP, R4BP, R5BP etc., the valuable thing, its stability which defines the problem's behavior in celestial mechanics. Further β plays a significant role in this problem, $\beta = 0$ and $\beta \neq 0$ express the system is working with, without a center of mass, and with a center of mass in the synodic frame of reference, respectively. In this problem, the existence of the zones depends on the change of values of β. This article's brief and important symbols, in which this article completely works are $v = 6, \Omega, \beta = 0, \beta \neq 0, Q_{A_i}, \sigma_i$, which are explained in further important sections.

For a few decades ago, many scientists, astronomers, and researchers have inquired about the motion of the infinitesimal mass under the combined effect of $(N+1)$ ring body problem. [11] and [8] have analyzed the dynamics of the $(N+1)$ ring body problem in planar case, Symmetric periodic motion in a ring problem discussed by[18], Periodic solutions in the ring problem studied by [9],[10]. [12] have expressed the fundamentals of the Ring Problem of (N+ 1) Bodies: An Overview. [16,15] has studied the analysis of the distribution of times of escape in the N-body ring problem.[1] has discussed about the Equilibrium dynamics of a circular restricted three-body problem with Kerr-like primaries The dynamical properties of the restricted four-body problem with radiation pressure.[20], have studied the study of Newton-Raphson basins of convergence in the three-dipole problem.[3], has studied the Periodic solutions activities in the planar (n+ 1) ring problem with oblateness and on a property of the zero-velocity curves in the regular polygon problem of N+ 1 bodies with a quasi-homogeneous potential analyzed by [4], An asymptotic behavior of orbits in the (N+ 1)-body ring problem analyzed by [17].A study of restricted three-body analysed in some important circumstances by [2,19,24]

The brief concernment of this article is governed by onwards, the present article is distributed into seven sections. Section-2 explains the significant relation between radiation factor σ_0 and albedo effect σ_i. Section-3 wears fundamental information about equations of the motion of ring body problem with the creation of radiation factor as well as albedo effect at the peripheral primaries, *i.e.* problems invention section. Where the information about equilibrium points and governing equation for their stability has been defined in Section-4. Where Section-5 explains the treatments about libration points

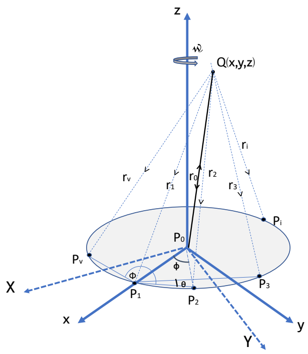

Fig. 1 A general arrangement for the ring body problem is, peripheral primaries $P_i, i = 1, 2.........v$ which have equal masses m are vested at the vertices of a regular polygon with v sides, here P_0 represents the central primaries with mass $\beta = \frac{m_0}{m}$ is vested at the center of the regular polygon.

and the existence of their zones for some specific value of β. The analysis of the numerical application of the existing libration points and their possible range set under the albedo effect is discussed in Section-6. The last Section-7 becomes the conclusion section.

2 Relation between σ_i and σ_0

Nevertheless, in some circumstances, the gravitational pattern is not enough to find the behavior of the dynamics of the problem. So, some extra hypotheses must be introduced whose effect is taken into account. We then primarily consider that the primaries are radiating sources that do not affect each other, but their influence occurs only with the motion of the particle. Let the distance between the primaries are unit, F_0 and F_i are the gravitational forces acting on infinitesimal mass due to m_0 and m_i, $i = 1, 2, 3, ...6$ mass respectively, F_p be the force due to solar radiation pressure by m_0 on the infinitesimal body and F_A is the Albedo force due to solar radiation reflected by an infinitesimal body, where $\sigma_0 = \frac{F_p}{F_0}$ and $\sigma_i = \frac{F_p}{F_i}$. Let the line joining m_0 and m_i is taken as X-axis and P_0 represent their center of mass as m_0.

By the study of article [23], the following important relations become,

$$(1 - \sigma_i)$$

and

$$\sigma_0 \quad i = 1, 2, 3, ...6$$

$$Q_{A_0} F_0 = F_0(1 - \sigma_0); \quad \text{where} \quad \frac{F_p}{F_0} << 1$$

$$Q_{A_i} F_i = F_i(1 - \sigma_i); \quad \text{where} \quad \frac{F_A}{F_i} << 1$$

$$\sigma_0 = \frac{L_0}{2\pi G m_0 ck}; \quad \sigma_i = \frac{L_i}{2\pi G m_i ck};$$

$$\frac{\sigma_i}{\sigma_0} = \frac{Li}{L_0}\frac{m_0}{m_i} \Rightarrow \frac{L_i}{L_0}\beta \Rightarrow \frac{L_i}{L_0}\left(\frac{1-\mu}{\mu}\right)$$

$$\Rightarrow \sigma_i = \sigma_0(\frac{1-\mu}{\mu})k \quad \text{or} \quad \sigma_0 \beta k; \quad \beta = \frac{1-\mu}{\mu}; \quad \frac{L_i}{L_0} = k = constant.$$

$$F_{A_0} \in (0, 1) \iff (1 - \sigma_0) \in (0, 1), \text{whenever} \;\; \sigma_0 \in (0, 1).$$

$$F_{A_i} \in (0, 1) \iff (1 - \sigma_i) \in (0, 1), \text{whenever} \;\; \sigma_i \in (0, 1)\forall, i = 1, 2, ...6.$$

The following condition is that zero albedos of the celestial body imply 'black body' in which the celestial body absorbs all incident radiations,

$$F_{A_0} \in (0, 1] \iff (1 - \sigma_0) \in (0, 1], \text{whenever} \;\; \sigma_0 \in [0, 1).$$

$$F_{A_i} \in (0, 1] \iff (1 - \sigma_i) \in (0, 1], \text{whenever} \;\; \sigma_i \in [0, 1) \;\; \forall, i = 1, 2, ...6.$$

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The following condition implies unit albedo of the celestial body represents 'white body' which reflects all the coming radiation absolutely and homogeneously in all directions. This condition for

$$\sigma_i = 0, \forall i = 0, 1, 2, 3, ...6.$$

return to general ring body problem.

$$F_{A_0} \in [0, 1) \iff (1 - \sigma_0) \in [0, 1), \quad \text{whenever} \quad \sigma_0 \quad \in (0, 1].$$

$$F_{A_i} \in [0, 1) \iff (1 - \sigma_i) \in [0, 1), \quad \text{whenever} \quad \sigma_i \quad \in (0, 1] \;\; \forall, i = 1, 2, ...6.$$

It is confirmed here by relation, Q_{A_i} strictly increases $\iff$ σ_i strictly decreases $\forall, i = 0, 1, 2, ...6$.

By research article [5], where L_0 is the luminosity of the center of mass primary m_0, L_i is the luminosity of the primary of mass $m_i = m, i = 1, 2, ...6$, G is the universal gravitational constant, c is the speed of light and k is mass per unit area and $\sigma_i = \sigma$ for all, $i = 1, 2, ...6$; the luminosity $L_i = L$,$\forall$, $i = 1, 2, ...6$. Whenever $\sigma_i \equiv 0, \forall, i = 1, 2, ...6; \sigma_0$ is radiation factor and $\sigma_i, i = 1, 2, ...6$ is the albedo factor of the peripheral primaries. The system or equations of motion said to be non-luminous and peripheral primaries P_i with masses $m_i = m$ for all, $i = 1, 2, ...6$ return to the general $(N + 1)$ ring body problem.

3 Fundamental equations of the motion of celestial bodies

The fundamental of consists $(N+1)$ body ring configuration with $(v = N-1)$ homogeneously spaced, spherical and analogous bodies P_i with multiple masses $m_i = m$, *i.e.*, $m_1 = m_2 = ... = m_v = m$, which are spherical in shape and vested on a periphery of radius d.

A different primary P_0 of mass $m_0 = \beta m$ is located at the center of the mass of the given system [8]. The peripheral primaries P_i, $i = 1, 2, ...v$ moving on their plane around their common center of mass in circular orbits under their mutual gravitational attraction with, ω, as a constant angular velocity. The particle $Q(x, y, z)$ is orbiting in the connected gravitational field of the complete primaries system. For comparison purpose, by the article [21], the mass parameter of RTBP, we define $\mu = \frac{1}{(1+\beta)}$ or $\beta = \frac{(1-\mu)}{\mu}$, as a mass parameter, consequently, the range of, $\mu \in (0, 1]$ whenever $\beta \in [0, \infty)$.

In the ring body problem of the primaries, by the article by [8], the general relations between the angle formed by a regular polygon is as follows (Figure.1);

$$\theta = \frac{\phi}{2} = \frac{\pi}{v}, \quad \Phi = \frac{(v-2)\pi}{2}, \quad \phi = \frac{2\pi}{v}, \tag{1}$$

Here, we denote the side of the polygon as,

$$P_i P_{i+1} = P_v P_1 = \alpha, i = 1, 2, 3, ..., (v-1). \tag{2}$$

Thus, the equations of motion written by using [14], and [22], the test particle in the rotating(synodic) coordinate system, in dimensionless quantities, are:

$$\ddot{x} - 2\dot{y} = \Omega_x, \tag{3}$$

$$\ddot{y} + 2\dot{x} = \Omega_y, \tag{4}$$

$$\Omega(x, y) = \frac{1}{\Delta}\left(\frac{\beta Q_{A_0}}{r_0} + \sum_{i=1}^{v} \frac{Q_{A_i}}{r_i}\right) + \frac{1}{2}(x^2 + y^2), \tag{5}$$

Supportive ingredients of equation (5) are the following, which are filled with trigonometric periodic functions.

$$\Delta = M(\Lambda + \beta M^2),$$

$$M = \sqrt{2(1 - \cos\phi)},$$

$$\Lambda = \sum_{i=2}^{v} \frac{\sin^2\theta \cos(\frac{v}{2} - i + 1)\theta}{\sin^2(v + 1 - i)},$$

$$r_0 = \sqrt{x^2 + y^2},$$

$$r_j = \sqrt{(x - x_j)^2 + (y - y_j)^2},$$
$$x_j = \Phi \cos(j-1)\phi,$$
$$y_j = \Phi \sin(j-1)\phi, j = 1, 2, ...v,$$
$$\Phi = \alpha \frac{1}{\sqrt{2(1-\cos\phi)}},$$

Where x_j, y_j, Λ, M, and Φ are periodic functions, so the whole system wears trigonometric periodic functions. Thus we can say this whole system is periodic in a certain range set. Here r_j presents the distances of infinitesimal mass from the j^{th} peripheral primaries,x_j and y_j are the periodic coordinates of the peripheral primaries. The zero-velocity curve of the dimensionless distances of the particle is calculated from the equation, by [13].

$$2\frac{d\Omega(x,y)}{dt} = \frac{d}{dt}(\dot{x}^2 + \dot{y}^2),$$

By integrating,

$$2\Omega(x,y) - C = \dot{x}^2 + \dot{y}^2,$$

where $\dot{x}^2 + \dot{y}^2 = v^2$ and C is the constant of integration which constitutes the integral of the equations of motion.

$$2\Omega(x,y) - C = v^2, \tag{6}$$

4 Zones of the stationary solutions

The treatment of the stationary solutions for the equilibrium points obtained by using conditions by solving [14],

$$\dot{x} = \dot{y} = \ddot{x} = \ddot{y} \equiv 0;$$
$$\Omega_x = 0 \quad \text{and} \quad \Omega_y = 0,$$

$$\Omega_x = x - \frac{1}{\Delta}\left(\frac{\beta x Q_{A_0}}{{r_0}^3} + \sum_{i=1}^{6}\frac{Q_{A_i}(x - x_i)}{r_i^3}\right) = f(x,y) = 0, \tag{7}$$

$$\Omega_y = y - \frac{1}{\Delta}\left(\frac{\beta y Q_{A_0}}{{r_0}^3} + \sum_{i=1}^{6}\frac{Q_{A_i}(y - y_i)}{r_i^3}\right) = g(x,y) = 0, \tag{8}$$

The stability of every solution governed by the following equation's characteristic roots by solving[22];

$$\delta^4 + \delta^2(4 - \Omega_{xx} - \Omega_{yy}) + (\Omega_{xx}\Omega_{yy} - \Omega_{xy}^2) = 0, \tag{9}$$

which is 4^{th} degree polynomial in δ.

5 The treatment of libration points and their zones

With the help of research article [8], the equilibria are established numerically and the invention of their partition for the various v has shown that these points are systematized in an equal arc on the concentric circles centered at the origin, which mentioned five zones are $C1, C2, A1, A2$, and B. The existence of these important zones completely depends on β. An exclusive zone that wears all the existing equilibrium points is obtained by the same Jacobian constant, so the Jacobi energy curve is defined as $C_{C1}, C_{C2}, C_{A1}, C_{A2}, C_B$.

In this problem, β plays a significant role, which affects firstly the whole dynamics of peripheral primaries, the mass parameter $\beta = 0$ and $\beta \neq 0$ generally confirms that this synodic system wears, an absence of a center of mass and presence of center of mass, respectively. In this problem, we are interested only in real roots, for the existence of libration points, while for stability analysis we are interested in both real and complex roots of δ. One thing, we have seen in this article is the libration point L_0, which exists in the center and has peripheral primary P_0, remains unchanged throughout applying the albedo effect. This article somewhere analyses the whole system of behavior for various v whenever change. Therefore masses of peripheral primaries are alike, there are many axes of symmetry, such as the bisector of the angle $P_0P_iP_{i+1}$. Because of symmetry, the configuration will repeat the rotation about the central body with the angle $\frac{\phi}{2} = \frac{\pi}{v}$. Here for v is positive even integer *i.e.* $v = 6$, then there are $\frac{v}{2}$ axes of symmetry that exists affixing with two adverse primaries and $\frac{v}{2}$ axes that parallel with the bisectrix of two successive primaries.

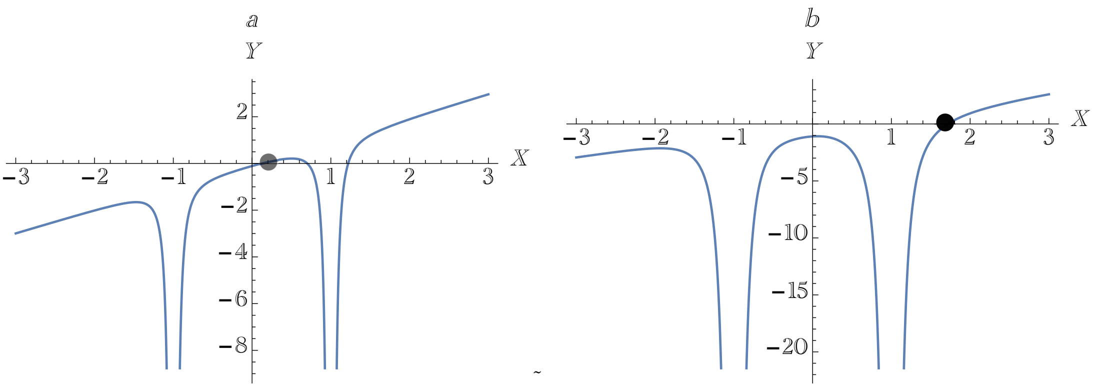

Fig. 2 (a) Display of $\beta = 0, Q_{A_i} = 0.1 \iff \sigma_i = 0.9, \forall, i = 0, 1, 2, ...6.$, absence of center of mass and at high albedo, we get radius of $C1 = 0.1215$; (b) $\beta = 0, Q_{A_i} = 1 \iff \sigma_i = 0.0, \forall, i = 0, 1, 2, ...6.$, absence of center of mass as well as albedo which provides radius of $C1 = 1.7172$.

Table 2 Shows the variation of the albedo effect, $\sigma_i, \forall i = 0, 1, 2, ...6$ at zone $C1$ for the absence of center of mass ($\beta = 0$), and C_{C1} is the energy curve of outer collinear points of the peripheral primaries. The range set of the $\sigma_i \in [0, 1) \Rightarrow C1 \in (0.10, 1.73)$ and the range set of energy curve $C_{C1} \in (5.46, 10.58)$ for possible albedo effect. It is observed that the radius of zone $C1$ is strictly monotonic increasing as the radiation factor and albedo effect, whenever σ_i, $\forall i = 0, 1, 2, ...6$ decreases strictly.

Q_{A_i}	x	y	σ_i	$\delta_{1,2}$	$\delta_{3,4}$	C_{C1}	Stability
0.1	0.121510255	0.0	0.9	$1.2784717 \pm 0.9999699i$	$1.2784717 \pm 0.9999699i$	10.5731	Unstable
0.2	1.310568219	0.0	0.8	± 5.9420169	$\pm 4.2753789i$	5.8256	Unstable
0.3	1.381268311	0.0	0.7	± 4.3038833	$\pm 3.8514983i$	5.6050	Unstable
0.4	1.442067851	0.0	0.6	± 3.3810819	$\pm 2.5475704i$	5.5085	Unstable
0.5	1.496560830	0.0	0.5	± 2.7709988	$\pm 2.1575982i$	5.4715	Unstable
0.6	1.546532202	0.0	0.4	± 2.3267488	$\pm 1.8847971i$	5.4685	Unstable
0.7	1. 593018436	0.0	0.3	± 1.9804592	$\pm 1.6813441i$	5.4873	Unstable
0.8	1.636685790	0.0	0.2	± 1.6955614	$\pm 1.5218174i$	5.6272	Unstable
0.9	1.677994833	0.0	0.1	± 1.4495713	$\pm 1.3911351i$	5.4715	Unstable
1.0	1.717281746	0.0	0.0	± 1.2264780	$\pm 1.2792374i$	7.2625	Unstable

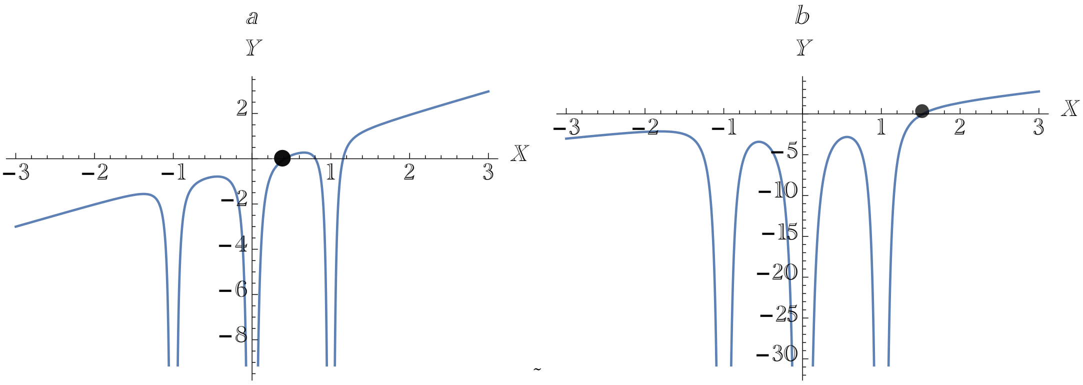

Fig. 3 (a) Plot of $\beta = 2, Q_{A_i} = 0.1 \iff \sigma_i = 0.9, \forall, i = 0, 1, 2, ...6.$, presence of center of mass and at high albedo, we get radius of $C1 = 0.4127$; (b) $\beta = 2, Q_{A_i} = 1 \iff \sigma_i = 0.0, \forall, i = 0, 1, 2, ...6.$, presence of center of mass and absence of albedos which provides radius of zone $C1 = 1.5268$.

Table 3 Shows the albedo effect, $\sigma_i, \forall i = 0, 1, 2, ...6$ at the zone $C1$ for the presence of center of mass $\beta = 2$, and C_{C1} is the energy curve of outer collinear points of the peripheral primaries. The range set of the $\sigma_i \in [0, 1) \Rightarrow C1 \in (0.40, 1.54)$ and the range set of energy curve $C_{C1} \in (4.20, 6.42)$ for possible albedo effect. It is observed that the radius of zone $C1$ is strictly monotonic increasing, whenever $\sigma_i, \forall i = 0, 1, 2, ...6$ decreases strictly.

Q_{A_i}	x	y	σ_i	$\delta_{1,2}$	$\delta_{3,4}$	C_{C1}	Stability
0.1	0.412688583	0.0	0.9	± 3.88506296	$\pm 2.70242746i$	4.2143	Unstable
0.2	1.221686705	0.0	0.8	± 6.85898405	$\pm 4.93507469.i$	6.4097	Unstable
0.3	1.273479139	0.0	0.7	± 4.95684644	$\pm 3.62720645i$	6.0126	Unstable
0.4	1.318378491	0.0	0.6	± 3.89384587	$\pm 2.91203291i$	5.7936	Unstable
0.5	1.358936814	0.0	0.5	± 3.19713432	$\pm 2.45549748i$	5.6596	Unstable
0.6	1.396418933	0.0	0.4	± 2.69514697	$\pm 2.136449671i$	5.5745	Unstable
0.7	1. 431553417	0.0	0.3	± 2.30914101	$\pm 1.899482091i$	5.5206	Unstable
0.8	1.464802529	0.0	0.2	± 1.99731697	$\pm 1.71527697i$	5.4822	Unstable
0.9	1.496480578	0.0	0.1	± 1.73479933	$\pm 1.56659782i$	5.4715	Unstable
1.0	1.526812890	0.0	0.0	± 1.50528238	$\pm 1.44248918i$	5.4666	Unstable

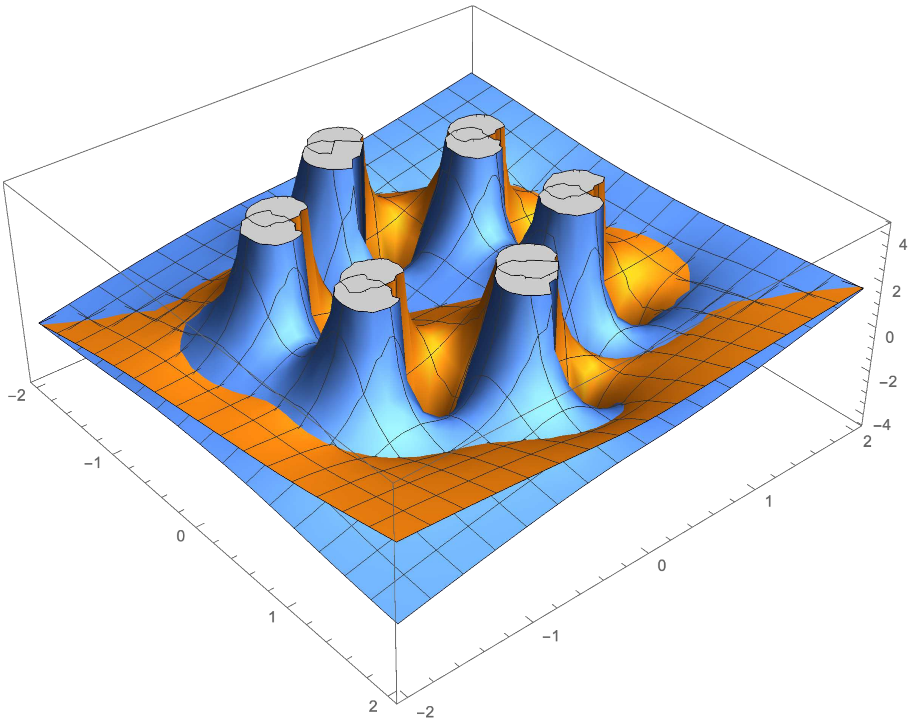

Fig. 4 $\beta = 0$, the plot of $f(x,y), g(x,y)$simultaneous at the albedo effect, the absence of center.

Table 4 The libration point L_0 doesn't have an albedo effect and stability remains the same.

$\beta = 0$	$Q_{A_i} = 1$	Or	$\sigma_i = 0.0$		$Q_{A_i} = 0.9$	Or	$\sigma_i = 0.1$	
Libration Points	x	y	C	Stability	x	y	C	Stability
L_0	0.0	0.0	6.55668	Unstable	0.0	0.0	6.5566	Unstable

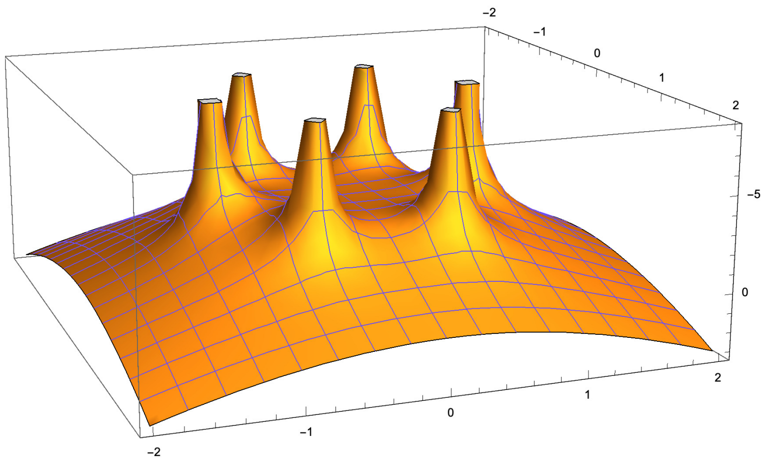

Fig. 5 Display of $\beta = 0$, the zero-velocity curve(ZVC) represents all peripheral primaries for the absence of a center of mass for $v = 6$, at the effect of albedo.

Table 5 Shows the albedo behavior from at the zone B for the absence of a center of mass ($\beta = 0$) and C_B is the energy curve of equilibria between two successive peripheral primaries, for the absence of a center of mass and absence of albedo the range libration points are $L_{1,2,....,6} \in [\pm 0.766, \pm 0.884]$, presence of albedo the range libration points are $L_{1,2,....,6} \in [\pm 0.449, \pm 0.894]$.

$\beta = 0$	$Q_{A_i} = 1$		Or	$\sigma_i = 0.0$	$Q_{A_i} = 0.9$		Or	$\sigma_i = 0.1$
Libration Points	x	y	C_B	Stability	x	y	C_B	Stability
L_1	-0.765856	0.442167	7.99927	Unstable	-0.773646	0.449347	7.9986	Unstable
L_2	0.765856	-0.442167	7.99927	Unstable	0.773646	-0.449347	7.9986	Unstable
L_3	-0.765856	-0.442167	7.99927	Unstable	-0.773646	-0.449347	7.9986	Unstable
L_4	0.765856	0.442167	7.99927	Unstable	0.773646	0.449347	7.9986	Unstable
L_5	0	0.8843211	7.99827	Unstable	0.0	-0.894334	7.9985	Unstable
L_6	0	-0.8843211	7.99827	Unstable	0.0	0.894334	7.9985	Unstable

Table 6 We analyze the albedo effect at zone $C2$ for the center of mass ($\beta = 0$) and C_{C2} is the energy curve of outer triangular points of the peripheral primaries. The range of libration points $L_{7,8,9,10,11,12} \in [\pm 0.796, \pm 1.592]$ to $[\pm 0.843, \pm 1.602]$.

$\beta = 0$	$Q_{A_i} = 1$		Or	$\sigma_i = 0.0$	$Q_{A_i} = 0.9$		Or	$\sigma_i = 0.1$
Libration Points	x	y	C_{C2}	Stability	x	y	C_{C2}	Stability
L_7	-1.37887	0.796089	7.04546	Unstable	-1.3078	0.842567	7.0647	Unstable
L_8	1.37887	-0.796089	7.04546	Unstable	1.3078	-0.842567	7.0647	Unstable
L_9	-1.37887	-0.796089	7.04546	Unstable	-1.3078	-0.842567	7.0647	Stable
L_{10}	1.37887	0.796089	7.04546	Unstable	1.3078	0.842567	7.0647	Unstable
L_{11}	0.0	1.59218	7.03636	Unstable	0.0	-1.60223	7.0550	Stable
L_{12}	0.0	-1.59218	7.03636	Unstable	0.0	1.60223	7.0550	Unstable

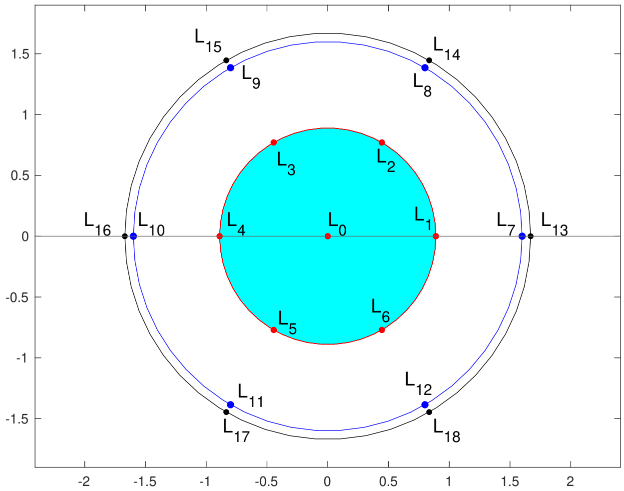

Fig. 6 Represents $\beta = 0$ with albedo effect , In this case there exists 19 libration points exists including L_0 central equilibrium point at the primary P_0, $L_{1,2,3,4,5,6}$ is located at the zone $A1$, zone $C2$ wears libration points $L_{7,8,9,10,11,12}$ and the libration point $L_{13,14,15,16,17,18}$ exist zone $C1$.

Table 7 Represents the analysis of the albedo effect at zone $C1$ for the absence of a center of mass ($\beta = 0$) and C_{C1} is the energy curve of outer collinear points of the peripheral primaries. The range of libration points $L_{13,14,15,16,17,18} \in [\pm 0.859, \pm 1.717]$ to $[\pm 0.975, \pm 1.678]$.

$\beta = 0$	$Q_{A_i} = 1$		Or	$\sigma_i = 0.0$	$Q_{A_i} = 0.9$		Or	$\sigma_i = 0.1$
Libration Points	x	y	C_{C1}	Stability	x	y	C_{C1}	Stability
L_{13}	-0.85862	1.48718	7.25543	Unsable	-0.974978	1.38842	7.2507	Unstable
L_{14}	0.85862	-1.48718	7.25543	Unstable	0.974978	-1.38842	7.2507	Unstable
L_{15}	-0.85862	-1.48718	7.25543	Unstable	-0.974978	-1.38842	7.2507	Unstable
L_{16}	0.85862	1.48718	7.25543	Unstable	0.974978	1.38842	7.2507	Unstable
L_{17}	-1.7172	0.0	7.24463	Unstable	-1.67796	0.0	6.3637	Stable
L_{18}	1.7172	0.0	7.24463	Unstable	1.67796	0.0	6.3637	Unstable

Table 8 Analyze the albedo effect at zone $A1$ for the presence of a center of mass ($\beta = 2$) and C_{A1} is the energy curve of inner collinear points of the peripheral primaries. The range of libration points $L_{1,2,....,6} \in [\pm 0.285, \pm 0.570]$ to $[\pm 0.291, \pm 0.565]$.

$\beta = 2$	$Q_{A_i} = 1$		Or	$\sigma_i = 0.0$	$Q_{A_i} = 0.9$		Or	$\sigma_i = 0.1$
Libration Points	x	y	C_{A1}	Stability	x	y	C_{A1}	Stability
L_1	- 0.285054	0.4937288	5.66859	Unstable	-0.291273	0.488694	5.6684	Unstable
L_2	0.285054	-0.493728	5.66859	Unstable	0.291273	-0.488694	5.6684	Unstable
L_3	-0.285054	-0.493728	5.66859	Unstable	-0.291273	-0.488694	5.6684	Unstable
L_4	0.285054	0.493728	5.66859	Unstable	0.291273	0.488694	5.6684	Unstable
L_5	-0.570108	0.0	5.66749	Unstable	-0.565162	0.0	5.6679	Unstable
L_6	0.570108	0.0	5.66749	Unstable	0.565162	0.0	5.6679	Unstable

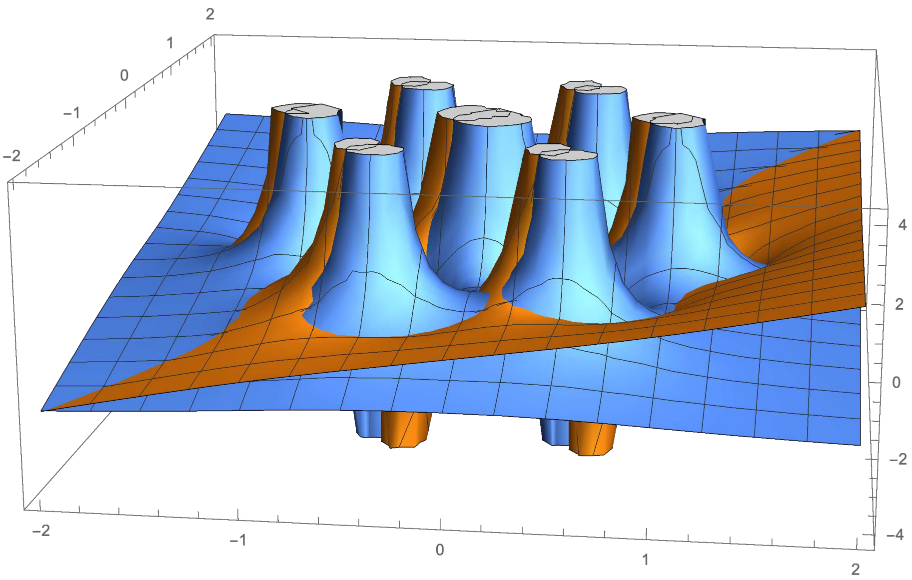

Fig. 7 $\beta = 2$, representing the plot of $f(x, y), g(x, y)$ simultaneous at the albedo effect, the presence of center.

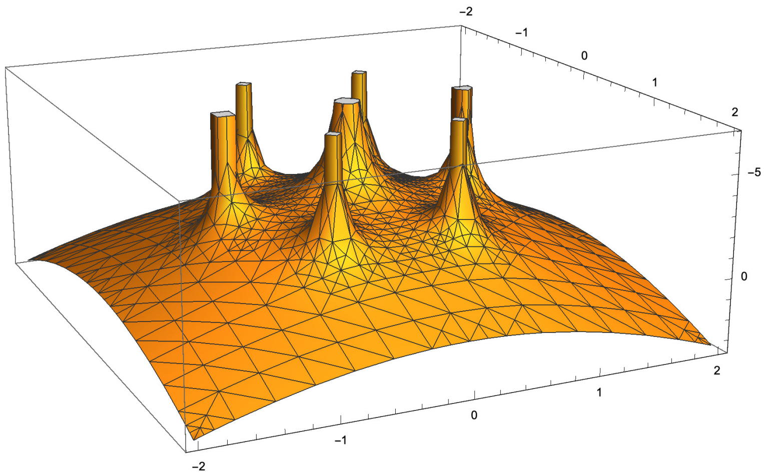

Fig. 8 The plot of $\beta = 2$, the zero-velocity curve(ZVC) represents the peripheral primaries for the presence of a center of mass for $v = 6$, at the effect of the albedo and the radiation pressure.

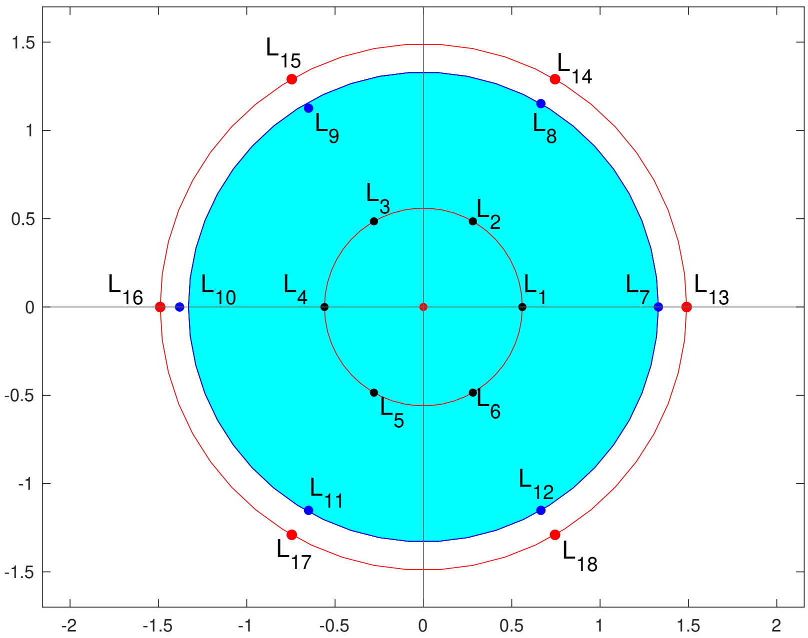

Fig. 9 $\beta = 2$ at the albedo effect, in this case every zone there exists six equilibrium, total 18 libration point exists.The locations of the libration point $L_{1,2,3,4,5,6}$ lies in zone B, $L_{7,8,9,10,11,12}$ exists in zone $C2$, $L_{13,14,15,16,17,18}$ lies in zone $C1$.

Table 9 Represents the albedo effect at zone $C2$ for the presence of a center of mass ($\beta = 2$) and C_{C2} is the energy curve outer triangular points of the peripheral primaries. The range of libration points $L_{7,8,....,12.} \in [\pm 0.661, \pm 1.414]$ to $[\pm 0.66, \pm 1.332]$.

$\beta = 2$	$Q_{A_i} = 1$		Or	$\sigma_i = 0.0$	$Q_{A_i} = 0.9$		Or	$\sigma_i = 0.1$
Libration Points	x	y	C_{C2}	Stability	x	y	C_{C2}	Stability
L_7	-1.14438	0.660707	5.1766	Unstable	-1.09084	0.660684	5.1852	Unstable
L_8	1.14438	-0.660707	5.1766	Unstable	1.09084	-0.660684	5.1852	Unstable
L_9	-1.14438	-0.660707	5.1766	Unstable	-1.09084	-0.660684	5.1852	Unstable
L_{10}	1.14438	0.660707	5.1766	Unstable	1.09084	0.660684	5.1852	Unstable
L_{11}	0.0	1.32141	5.1788	Unstable	0.0	-1.33224	5.1791	Unstable
L_{12}	0.0	-1.32141	5.1788	Unstable	0.0	1.33224	5.1791	Unstable

Table 10 Represents the albedo effect at zone $C1$ for the presence of a center of mass ($\beta = 2$) and C_{C1} is the energy curve of outer collinear points of the peripheral primaries. The range of libration points $L_{13,14,....,18} \in [\pm 0.763, \pm 1.526]$ to $[\pm 0.823, \pm 1.496]$.

$\beta = 2$	$Q_{A_i} = 1$		Or	$\sigma_i = 0.0$	$Q_{A_i} = 0.9$		Or	$\sigma_i = 0.1$
Libration Points	x	y	C_{C1}	Stability	x	y	C_{C1}	Stability
L_{13}	-0.763399	1.32224	5.4670	Unstable	-0.823354	1.2704	5.2626	Unstable
L_{14}	0.763399	-1.32224	5.4670	Unstable	0.823354	-1.2704	5.2626	Unstable
L_{15}	-0.763399	-1.32224	5.4670	Unstable	-0.823354	-1.2704	5.2626	Unstable
L_{16}	0.763399	1.32224	5.4670	Unstable	0.823354	1.2704	5.2626	Unstable
L_{17}	-1.5268	0.0	5.4666	Unstable	-1.49647	0.0	5.4715	Unstable
L_{18}	1.5268	0.0	5.4666	Unstable	1.49647	0.0	5.4715	Stable

6 Numerical application of libration points

Table 2. Analysis of zone $C1$ for $\beta = 0$ and $v = 6$, which explains the complete behavior of zone $C1$, throughout the effect of albedo and radiation pressure. The radius of zone $C1 \in [0.122, 1.717]$ and the range set of Jacobi constant *i.e.* energy curve $C_{C1} \in [5.469, 10.5734]$, all are found unstable.

Table 3. Analysis of zone $C1$ for $\beta \neq 0$ *i.e.* $\beta = 2$ and $v = 6$, which explains the complete behavior of zone $C1$, throughout the effect of albedo and radiation pressure. The radius of zone $C1 \in [0.413, 1.527]$ and the range set of Jacobi constant *i.e.* energy curve $C_{C1} \in [4.214, 6.409]$, all are found unstable.

With the help of tables (2,3), our purpose is to analyze the complete behavior of at least one zone *i.e*, $C1$. This zone can help to visualize the other existing zones as well as their libration points approximation of behavior under applied effects.

Table: 5,6,7. Explain the calculation for $\beta = 0$, zones $B, C2, C1$, respectively, where each zone wear there exist 6 libration points, range set of libration points for zone B *i.e.* $L_{1,2,3,4,5,6} \in [\pm 0.442, \pm 0.894]$, in zone $C2$ *i.e.* $L_{7,8,9,10,11,12} \in [\pm 0.796, \pm 1.602]$, for zone $C1$ *i.e.* $L_{13,14,15,16,17,18} \in [\pm 0.859, \pm 1.717]$ and a libration point L_0, exist at the center of mass, which remains unchanged throughout albedo effect, so under this condition total number of existing libration points are 19. The albedo effect is also guaranteed here, the stability may or may not change but the location of libration points as well as their zone's radius is affected by radiation pressure and the albedo effect.

The table: 8,9,10. Explain the calculation for $\beta \neq 0$ *i.e.* $\beta = 2$, now the possible existing zones are $A1, C2, C1$, here also each zone wear there exist 6 libration, possible range set of libration points for zone $A1$ *i.e.* $L_{1,2,3,4,5,6} \in [\pm 0.285, \pm 0.570]$, in zone $C2$ *i.e.* $L_{7,8,9,10,11,12} \in [\pm 0.660, \pm 1.332]$, for zone $C1$ *i.e.* $L_{13,14,15,16,17,18} \in [\pm 0.763, \pm 1.526]$, so under this circumstances total number of existing libration points are 18.

For $\beta = 0$, after albedo effect except for libration points $L_{9,11,17}$, all are found unstable,*i.e.* the libration point $L_{9,11,17}$, found stable and for $\beta = 2$,

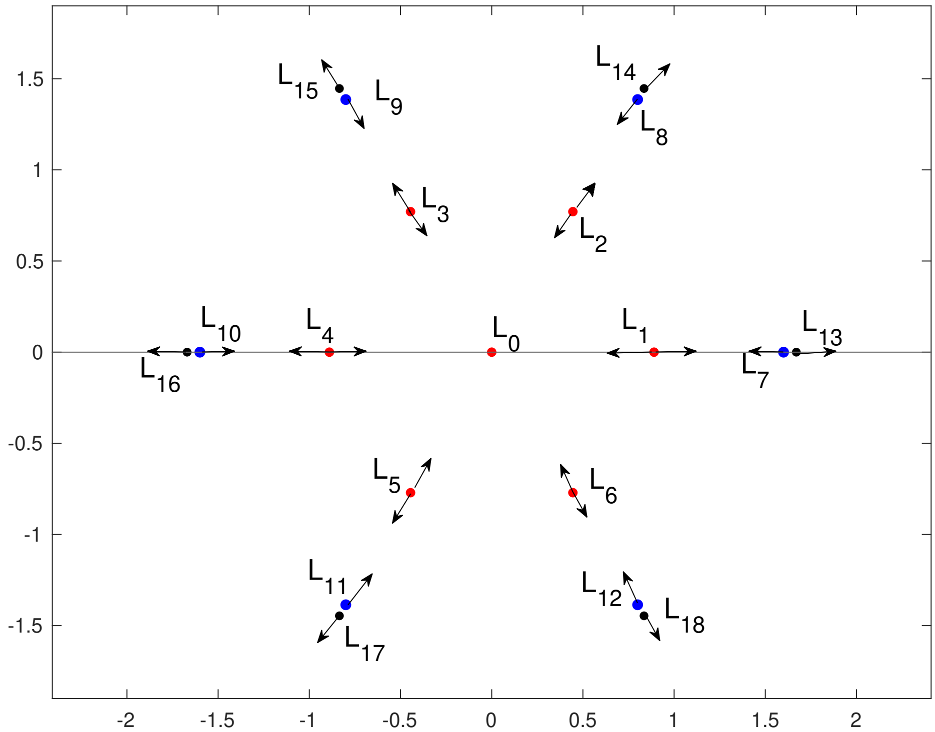

Fig. 10 For $\beta = 0$, $\sigma_i \in (0, 1)$, *i.e.* σ_i belongs to 0.9 to 0.1 and Q_{A_i}, $\forall i = 0, 1, 2, ...6$ lies 0.1 to 0.9, so the position of the libration points as well as their existing zones shown by arrow $(\uparrow, \downarrow)$, here it is found that all the libration points somewhere increasing mode *i.e.* $(\uparrow)$, decreasing mode *i.e.* $(\downarrow)$, and somewhere fluctuate$(\uparrow, \downarrow)$, provided in a certain range set $(-1.72, 1.72)$. Also here, in every zone, there exists six equilibrium, a total of 19 libration points exists, including L_0 at the center of mass, the position having no change throughout effects.The position of the libration points $L_{1,2,3,4,5,6}$ by red dots , blue dots for libration points $L_{7,8,9,10,11,12}$ and black dots for libration points $L_{13,14,15,16,17,18}$.

after albedo effect except the libration point L_{18}, all are found unstable *i.e.* the only libration point L_{18}, found stable condition.

7 Conclusion

Our purpose in this research article is to analyze the fundamental analysis of, change of, lots of libration points location whenever albedo force applies to the equations of the motion. Here we recorded the change of the radius of possible

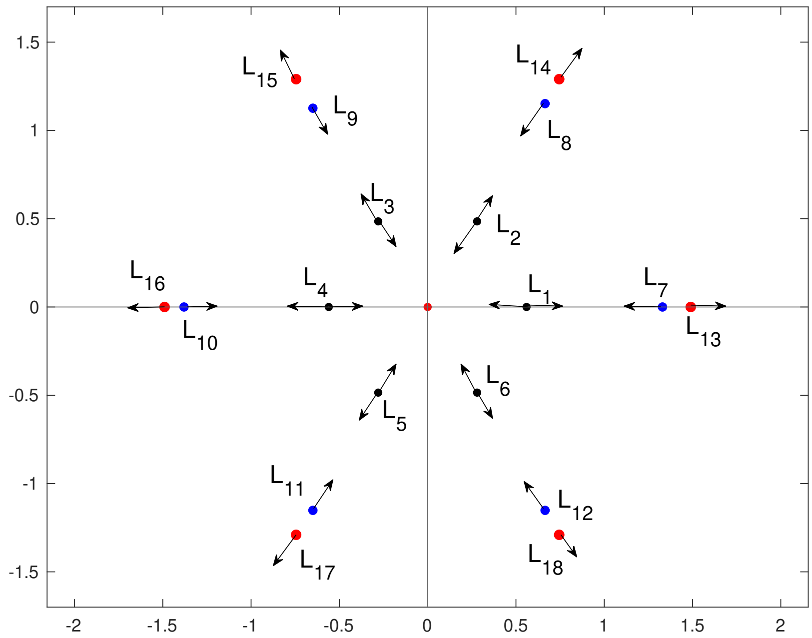

Fig. 11 For $\beta = 2$ or $(\mu = 1/3)$, $\sigma_i \in (0, 1)$, *i.e.* σ_i belongs to 0.9 to 0.1 and Q_{A_i}, lies 0.1 to 0.9, $\forall i = 0, 1, 2, ...6$, thus so the position of the libration points as well as their existing zones shown by arrow $(\uparrow, \downarrow)$, here it is found that all the libration points somewhere increasing mode *i.e.*$(\uparrow)$, decreasing mode *i.e.* $(\downarrow)$, and somewhere fluctuate$(\uparrow, \downarrow)$, provided in a certain range set $(-1.53, 1.53)$. Also here, in every zone there exists six equilibrium, a total of 18 libration points exists, with a center of mass. The position of the libration points $L_{1,2,3,4,5,6}$ by black dots, blue dots for libration points $L_{7,8,9,10,11,12}$ and red dots for libration points $L_{13,14,15,16,17,18}$.

existing zone and monotonicity of all libration points for specific zones. The libration points are found symmetric, along the axis of rotations.

The author makes a possible effort in his study to visualize the complete structure of this problem for specific value $v = 6$ under radiation pressure albedo effect, the restriction of $\beta = 0$ and $\beta \neq 0$ *i.e.* $\beta = 2$, explained nicely in every important figure and table. This article somewhere makes an effort to create a difference in general ring body problem and ring body problem's behavior after effect of the albedo and radiation pressure.

It is also found that the general ring body problem is somewhere in the extended form of radiation pressure, somewhere not under the albedo effect, it's confirmed that the radius of all the possible zones after the albedo effect is somewhere less or equal to and somewhere greater or equal to as compared to the general ring body problem. In this problem, it is also observed that the existence of the zones also depends on the change of values of β. This article also shows the movement of the position of all libration points with their existing zone as well as peripheral primaries, by the effect of radiation pressure and albedo. One beautiful thing also we studied in this problem, a little change in v, changes complete behavior of problem and leave a nice platform of proof. It is also observed that this complete problem works in a certain range set $(-1.75, 1.75)$. The author has observed the whole behavior of the system for various v whenever change. For future study, this $(N+1)$ ring body problem's expansion in various directions is possible, which we can think about closely.

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9 Conflict of interest

The authors declare that they have no conflict of interest.

10 Data availability

The analyzed data has been taken from the study of the article [8]. All computational analysis have been made with the help of MATLAB19a and Mathematica software, which are present in this article.

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