Collective and individual mathematical progress: Layering explanations in the case of Sierpiński triangle

Tommy Dreyfus (tommyd@tauex.tau.ac.il)
Tel Aviv University https://orcid.org/0000-0002-5656-8987
Naneh Apkarian
Chris Rasmussen
Michal Tabach

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Abstract

Qualitative research on classroom-based mathematics learning in inquiry-oriented classrooms is scarce. This paper presents a methodology aimed at developing a rich understanding of the interplay of mathematical progress in the different settings in which learning in such classrooms occurs - individuals, small groups, and the whole class. For this purpose, we enhance a theoretical-methodological approach of coordinating Documenting Collective Activity and the RBC-model of Abstraction in Context that has been developed in earlier studies. We do this using an intact lesson on the area and perimeter of the Sierpiński triangle in a mathematics education master's level course on Chaos and Fractals. The enhancement of the methodology allowed integrating the Collective and Individual Mathematical Progress (CIMP) by Layering the Explanations (LE) provided by the two approaches and thus exhibiting the complexity of learning processes in inquiry-oriented classrooms.

Introduction

Learning in inquiry-oriented classrooms (Laursen & Rasmussen, 2019) is becoming more and more common at all levels of schooling. There is evidence from meta-analyses that student-centered approaches have advantages for learning outcomes (Freeman et al., 2014; Theobald et al., 2020), in particular at the tertiary level. It is important to understand why and how these advantages come about. Yet, qualitative research on classroom-based mathematics learning in inquiry-oriented classrooms is scarce, possibly because it presents methodological difficulties. One of these difficulties stems from the different scales of relevant social settings in inquiry-oriented classrooms: individuals, small groups, and the class as a whole. These require different research questions and different methodological frameworks. In earlier work (Hershkowitz et al., 2014; Author et al., 2014), we used and coordinated two frameworks in order to achieve a detailed qualitative analysis of learning in classrooms in which knowledge construction occurred in small groups followed by a teacher-led whole class discussion in which the knowledge constructed earlier was institutionalized. However, there are other ways mathematical progress may be made in inquiry-oriented classrooms; for example, new knowledge construction may also occur in whole class discussion that starts with review of small group work, which presents yet another layer of complexity for qualitative analyses. A central aim of this paper is to offer a methodological approach that effectively deals with this complexity.

This paper furthers the operationalization of the theoretical-methodological approach we developed in earlier studies so as to allow analysis and rich understanding of the interplay of mathematical progress in the different settings in which learning in inquiry-oriented classrooms occurs - individuals, small groups, and the whole class. With this in mind the research presented in this paper has two intertwined aims which develop our CIMPLE approach:

- Document and explain collective and individual mathematical progress (CIMP), in different classroom settings

- Layer these explanations (LE) to develop a rich story of the interplay of knowledge emerging in different settings and how such interplay contributes to our understanding of learning in inquiry-oriented classrooms .

Background And Theoretical Framing

A more complete accounting of student mathematical progress as it occurs in an inquiry-based classroom requires attending to the mathematical reasoning of individuals, of small groups, and of the whole class. Different research groups have taken up this challenge in slightly different, albeit compatible ways. For example, Towers and colleagues (Martin et al., 2006; Towers & Martin, 2015) analyze the collective from the perspective of enactivism and improvisational theory and layer this with an analysis of individuals informed by Pirie-Kieren theory of dynamic growth. For their part, Saxe and colleagues (Saxe, 1999; Saxe & Esmonde, 2005; Saxe & Farid, 2021) use a cultural development framework to investigate the interplay between individual and collective activity. Individual cognitive development is traced through an individual's speech forms, which are taken to serve both communicative and individual cognitive functions. Collective activity is documented through an analysis of a common ground of talk and action that is produced and reproduced through activity. We take another approach than both these research teams.

To theoretically situate our investigation of the complexity between individuals, small groups, and the whole class, we draw on the emergent perspective (Cobb & Yackel, 1996), which frames mathematical progress as both a process of active individual development and a process of collective progress. The term "emergent" refers to the stance that, while for experts mathematical ideas and ways of reasoning may have a life of their own, for learners in inquiry-oriented classrooms, these ideas and ways of reasoning come to life via interaction (Blumer, 1969). That is, they emerge through a complex construction process that cuts across individual and group settings.

This broad theoretical framing that views mathematical progress as both an individual and collective accomplishment necessitates further theoretical framing that focuses on (1) individual knowledge construction and (2) on collective mathematical progress. In the paragraphs that follow we explicate our specific theoretical stances on individual and collective mathematical progress. In the methodology section, we articulate how we coordinate these different levels.

Our analysis of individual mathematical progress draws on the Abstraction in Context framework and the Recognizing-Building-Constructing (RBC) model of epistemic actions (Hershkowitz et al., 2001), which for analyzing learners' construction and consolidation of knowledge. An RBC analysis starts from and is based on an a priori analysis of the tasks the learners are asked to deal with. In the a priori analysis, the researchers ask what new (to the students) elements of knowledge are helpful or necessary to deal with the task at hand. The aim of the a priori analysis is to identify these elements of knowledge, typically concepts, procedures, or strategies. For each knowledge element, we give a definition in terms of the mathematical meaning of the element in the context in which it is being used and an operational definition. The operational definition is methodologically important: It fixes under what circumstances the researcher will say that a student is using or expressing a construct that represents this knowledge element. Hence, the operational definition will take into account the definition of constructing, namely that constructing refers to the first time a new construct is expressed or used by the learner.

The main use of the a priori analysis is to focus, at least initially, the researchers' attention on the identified knowledge elements when carrying out the RBC micro-analysis of students' learning processes. The a priori analysis may fail to predict some of the students' ways of going about a task. In such cases, the a posteriori analysis of the data according to the RBC-model will provide modifications to the a priori analysis. Hence, the a priori analysis serves as a working hypothesis that may later be confirmed or modified by the micro-analysis of the data. Unexpected knowledge elements have been identified in a number of previous studies (Author et al., 2015a).

An RBC analysis, however, tells only half of a good story. The other half of the story is the collective mathematical progress, both at the small group level and at the whole class level. Our analysis of collective mathematical progress draws on the Documenting Collective Activity (DCA) approach, developedthrough the emergent perspective (Author et al., 2008; Author et al., 2015b). From this point of view, collective activity is a social phenomenon in which mathematical ideas come to function-as-if shared through interaction. The term functions-as-if shared means that claims made about the group are not meant to be deterministic of every individual within the group and while individual variation is expected, the talk patterns across individuals suggest that the group acts "as if" others in the group hold similar or compatible ways of reasoning. Such normative ways of reasoning are rigorously documented by tracing the evolving *function* of ideas within the group's patterns of interaction.

It is the change in function of ideas during group discourse that allows researchers to make claims about when a particular way of reasoning becomes normative. In particular, the DCA approach makes use of Toulmin's model of argumentation (Toulmin, 1958), which consists of Data, Claim, Warrant, Backing, Rebuttals, and Qualifiers, to trace the changing function of ideas. Data consists of facts or procedures that lead to the claim that is made. To further improve the strength of the argument, speakers often provide more clarification that connects the data to the claim, which serves as a warrant. It is not uncommon, however, for rebuttals or qualifiers to arise once a claim, data, and warrant have been presented. Backing provides further support for the core of the argument.

The following three criteria are used to determine when a way of reasoning becomes normative: 1) When the backing and/or warrants for particular claim are initially present but then drop off, 2) When certain parts of an argument shift position within subsequent arguments in a way that represents progress (e.g., a claim shifts to data or warrant), or 3) When a particular idea is repeatedly used as either data or warrant for different claims across multiple days. When the discourse including a particular idea satisfies any one of these criteria, it is considered to function-as-if-shared.

In our prior work, our approach was to combine DCA analyses of WCDs with RBC analyses of SGW. This was based on the idea that a DCA analysis is designed to analyze the production of knowledge by groups and RBC analysis is designed to analyze knowledge construction by individuals. The act of networking (Bikner-Ahsbahs & Prediger, 2014) by coordinating between the two types of analyses was a main aim of the research and allowed us to follow the evolution of ideas from their construction in small groups to their becoming a normative way of reasoning during whole class discussion, or vice versa by linking between the knowledge elements that were constructed and the ideas that functioned-as-if shared (Author et al., 2014; see also Hershkowitz et al., 2014).

In further work (Author et al., 2020), we expanded the theoretical basis for the above networking studies by identifying and illuminating environmental and internal-theoretical commonalities across the two approaches, commonalities that contribute to the productivity of networking; we went on to show how and why these commonalities provide the rationale and logic of combining the two theories and hence lend coherence to the process and results of the networking. In other words, we provided an argumentative grammar (Kelly, 2004; Bakker, 2017) for networking DCA and RBC at the levels of method and content.

Methodology

In the present research study, we attempt to achieve the aims set out in the introduction by analyzing the transcripts from both, small group work (SGW) and whole class discussions (WCD) by means of DCA and RBC. That is, we carried out a complete DCA analysis of all episodes of the lesson, as well as a complete RBC analysis of all episodes of the lesson. The rationale for this methodological innovation is to use all data along an entire lesson of alternating SGW and WCD episodes in order to portray a picture about processes of construction of knowledge and about knowledge functioning-as-if-shared that is as complete as possible, given the data we have, and to gain insight into the interplay between these different levels by coordinating the analyses obtained by the two approaches.

Following these analyses, we used the results of the DCA analysis on the WCDs to identify ideas that function-as-if-shared in the whole class, and then worked backward from these ideas; the aim of working backward was to investigate the ways in which mathematical progress (in the sense of both, DCA and RBC)was made previously with the aim of explaining the emergence of these ideas on the basis of prior constructing actions, as identified by the RBC analysis, and ideas functioning-as-if-shared in groups, as identified by the DCA analysis on the SGW.

This enhanced methodology was chosen in view of our theoretical considerations and our research aims. It allows us to follow the mathematical progress on the level of the individuals in the groups, and the groups, as well as the interplay between them, up to the emergence of an idea as functioning-as-if-shared in the class as a whole.

Setting

The context for this study was a semester-long intact graduate level mathematics course on chaos and fractals. The course was taught by a member of the research team (henceforth, the professor) with another research team member (the instructor) at times contributing to instruction, and a third research team member attending each session, assisting with thematic and technological support. The instructional approach stressed SGW on tasks followed by WCD, with sporadic periods of lecture and presentation. During SGW, students were invited to use huddle boards - one table sized white board per group - to share their thinking, promote group communication, and facilitate subsequent presentation of their work during WCD. The professor and the instructor went from group to group, trying to understand student thinking and attempting to focus students' activity on what they saw as the main issues; they did this mainly by asking questions but did not otherwise intervene in the SGW.

There were 11 students in the class, 10 of whom had a bachelor's degree in mathematics and were pursuing a master's degree in mathematics education. The remaining student was an undergraduate pursuing her bachelor's degree in mathematics. All students were or intended to be secondary school teachers or community college instructors. The chaos and fractals course qualified as part of the substantial mathematics component of their program. Throughout the course, students worked in four stable groups: A (Carmen, Jen and Joy); B (Kevin, Elise and Mia); C (Soo, Kay and Shani); and D (Curtis and Sam). All names are pseudonyms. Groups A and B were video-recorded during SGW; the class was video-recorded during WCDs, one camera focusing on the professor and another one on (part of) the class.

The course included 23 lessons of 75 minutes. After an introductory overview related to determinism and chaos, the main topic in lessons 1–8, were dynamic processes modelled by sequences of real numbers generated by repeatedly applying a function such as $f(x) = 3x(1-x)$ to an initial number x_0. The class dealt with notions such as fixpoints and orbits, converging, periodic, diverging and chaotic ones. This paper focuses on Lesson 9, the first lesson dealing with fractals.

In Lesson 9, six WCDs (numbered 1, 3, 5, 7, 9 and 11) alternated with five SGWs (numbered 2, 4, 6, 8, and 10). The lesson started with watching excerpts from a video about fractals (Peitgen et al., 1990/2003) with examples including a cauliflower, mountains, a magnetic pendulum, the coast of Britain, and Julia sets (WCD1); the students discussed to what extent they could see parallels between these different examples (SGW2), and reported back to the plenum, mentioning, among others, that in all of them, the same or similar patterns were recurring at different scales (WCD3). From the professor's point of view, this was meant to provide the background for a worksheet about the Sierpiński triangle, with tasks about its construction, its area and perimeter, and self-similarity. The first two of these tasks are presented in Fig. 1.

The students spent almost 20 minutes in the small groups (SGW4), carrying out iterations of the Sierpiński triangle construction on their huddle boards according to Task 1 of the activity; they colored, at each stage, the middle triangles, imagining them to be removed from the figure. The professor and the instructor moved among the groups, asking them to reflect on part (d), which includes the self-referential command to “repeat (b), (c) and (d)”; during the ensuing WCD5 the professor explicitly asked about the meaning of “repeating the repeat” in part (d); the students responded using terms such as “infinite loop” and “zooming in”, and connected this to the dynamic processes they had encountered in earlier lessons. The professor then led students’ attention to Task 2 by drawing part of the figure that remained white (uncolored) at each stage; he invited the students to imagine the eventual shape, and to develop conjectures about its area and its perimeter. In SGW6, Groups A and B both focused on the area. While Group B came up rather quickly with “three fourth to the *n* of our A_1” (turn B241), Group A spent time on a formula for the area of an equilateral triangle, and eventually got to “that’s one-fourth of it, so each term maybe three-fourths of it” (turn A520). After brief reports by the students in WCD7, the professor sent them back to the groups, asking them to produce conjectures rather than computations. In SGW8, Group A soon conjectured that the area tends to 0, and spent most of the time discussing the nature of the perimeter, and how the perimeter at stage *n* can be found from the one at stage *n*-1. Group B answered this same question (“and it's increasing by a scale of three over two”, turn B336), but did not make a conjecture about convergence of the perimeter.

Toward the end of SGW8, the instructor joined Group A, asking them about the area and perimeter of the Sierpiński triangle. While the students agreed about the area tending to zero, the instructor’s question raised a controversy with respect to the perimeter, which led the instructor to gather the class and initiate WCD9. This is the starting point of our data analysis in the Findings section.

A Priori Analysis

A necessary methodological aspect of any RBC analysis is an a priori analysis of the task, in this case the activity presented in Fig. 1, in terms of knowledge elements intended to be constructed by the students during their work. In a complete a priori analysis, each knowledge element is presented, if appropriate together with its component elements, and each component element is given by a general as well as an operational definition. The aim of the operational definition is to give the researcher a well-defined tool to decide whether a specific component has been constructed by a learner. Since in this paper, we have no space to enter the details of the RBC analysis, we present only a summary of the a priori analysis, including those knowledge elements (*italicized* in the following list) and those operational definitions we will refer to.

- *Repeating Process*: A process that includes an instruction to repeat itself is necessarily an infinite process. In this paper, the relevant processes are geometric.
- *Area Features*: The shape at any stage of the process of generating the Sierpiński triangle according to the activity consists of more and smaller triangles than the shape at the previous stage. The triangles form a repeating pattern. Either of the statements (or similar ones) will be considered as evidence of having constructed *Area Features*.
- *Area Sequence*: The area of the shape generated by the *Repeating Process*, at any stage, can be obtained from the area at the previous stage by multiplying the area of the previous stage by ¾ or by

subtracting ¼ of the area at the previous stage, because each shape is obtained from the previous one by cutting all triangles into 4 congruent triangles and keeping three of the four. Note: We do not distinguish between a student's additive view of the area as a sequence or their multiplicative view of it as a series.

- *Area Limit 0*: The sequence of areas is monotonically decreasing and converges to 0.
- Comment: The students did not relate to the fact that the sequence is geometric.
- *Perimeter Features*: The perimeter at any stage of the process goes all around all white triangles at this stage; it is obtained from the perimeter at the previous stage by adding the midlines of the triangles belonging to the shape at the previous stage. Thus, the perimeter at each stage has more segments than the perimeter at the previous stage. The perimeter is cumulative. The added segments are smaller than the previous ones. The perimeter will thus look more and more intricate.

Findings

Following the methodology outlined above, we selected the first part of WCD9 as the starting point for this paper because the DCA analysis showed that no ideas that function-as-if-shared emerged earlier, and three ideas that function-as-if-shared emerged in the first part of WCD9. We asked ourselves what was underlying the emergence of each of these ideas that function-as-if-shared, and this question led us to three rather complex stories of mathematical progress, in terms of the emergence of knowledge constructs as well as of the ideas that function-as-if-shared, including ideas that function-as-if-shared already in discussions of the small groups. In this section, we present these three stories.

The story underlying Area Goes to Zero

As mentioned above, the instructor initiated WCD9 to bring out the controversy that had arisen in Group A to the entire class.

W78	**Instructor**	**Guys, so… Let me ask, let me ask the class a question. This group here has been talking about area and perimeter, can you do recount… Wait, first of all, you said area… you did some computations, and you just conceptually thought the area was going to…**
W79	Joy	Zero
W80	Instructor	Zero, right? Okay. And then tell me about… about the perimeter. Tell us about what… Because you guys had different ideas.
W81	Jen	Yeah, mhm
W82	Instructor	So tell us about… Carmen, tell us about your idea, and then Joy, tell us about your idea
W83	Joy	Okay
W84	Carmen	I was thinking if we keep zooming… Okay, for our area thing, we were going to keep zooming in, keep coloring in, so eventually we're gonna color all in. It's going to be black, so there’s no area, so there's nothing to… No area there’s nothing to put a fence around it. So, there’d be no perimeter... and then
The DCA analysis of this part of WCD9 yielded three arguments:		

Argument 0

Claim 0

Area goes to zero (Joy W79)

Comment: What we call here Argument 0 is not a complete argument but only a claim; this is the reason we assigned it the number 0 (rather than 1). Our detailed analysis in this section will justify the placement of this claim here, as if it were part of a complete argument.

Argument 1

Data 1 Keep zooming in, keep coloring in (Carmen W84)

Claim 1

There is … no area (Carmen W84)

Warrant 1 Eventually it’s going to be all black (Carmen W84)

Argument 2

Data 2 No area (Carmen W84)

Claim 2

There is no perimeter (Carmen W84)

Warrant 2 Nothing to put a fence around (Carmen W84)

Using the criteria for an idea to function-as-if-shared, these arguments led us to determine that *Area Goes to Zero* functions-as-if-shared according to Criterion 2, as it has been the claim of Argument 0 by Joy (W79) as well as of Argument 1 by Carmen (W84), and then the data of Argument 2 by Carmen (W84). We note already here that although the formal justification is based on Criterion 2, our detailed analysis in this section will show the importance of Criterion 1 in *Area Goes to Zero* functioning-as-if-shared.

Relevant Knowledge Elements

Following our methodology for coordinating DCA and RBC, we identify the knowledge elements that are or may be relevant for a given idea to function-as-if-shared. *Area Goes to Zero* is mathematically identical to the *Area Limit 0* knowledge element. From the a priori analysis, we conclude that *Area Limit 0* may be constructed on the basis of the following previous knowledge elements: In order to even consider a limit, one needs an infinite process, in this case *Repeating Process*, and a sequence of numbers, in this case the areas at consecutive stages of the process (*Area Sequence*). *Aera Features* enriches the imagery underlying the *Area Sequence*, but is not strictly necessary to construct the *Area Sequence* and the fact that the sequence of areas converges to zero. We next analyze whether this knowledge was constructed by the students in Groups A and B. We note that, of course, we only report constructing actions for which we have evidence; students may have constructed more without giving evidence for it.

Group A

We first outline our RBC analysis of Group A; this is followed by the DCA analysis within Group A. Of the three group members, Jen was less verbose than Joy and Carmen, thus we have no evidence of her construction of *Area Limit 0*. However, we can provide evidence that both Joy and Carmen constructed and consolidated this knowledge element, as well as some of the predicted constituent knowledge elements. We begin with Joy, using evidence from small group work episodes prior to WCD9 (see Table 1).

Table 1
Excerpts of RBC analysis of Joy during SGW

Constructing *Repeating Process*	**“this would be you keep repeating… Because d is the one to repeat, so then… we infinitely repeat” (A257, 259)**
Consolidating *Repeating Process* (building with *Repeating Process* to construct *Area Features*)	“That we start an infinite loop of continuing to draw more and more triangles… The instruction never ends” (A410, 414)
Constructing *Area Features*	“Pattern. It's the cauliflower, or the…” (A441) “Mhm. And then you keep iterating, you keep getting the same and same thing. So if you zoomed in, the picture looks the same. … And looks the same, and then looks the same.” (A444, 446)
Consolidating *Area Features*	“Yeah, we change the… key, change measurement key” (A452)
Constructing *Area Sequence*	“So each time you would subtract a fourth” (A499) “And then three-fourths of this… And three-fourths of that… So is it approaching zero then?” (A596, 598, 600)
Constructing *Area Limit 0*	“And then three-fourths of this… And three-fourths of that… So is it approaching zero then?” (A596, 598, 600) “Here it says it keeps multiplying by… Okay, so you are right, it approaches zero” (A60, 610)
Consolidating *Area Limit 0*	“We said that the area went to zero, because if you looked at the white space, and you kept drawing it in, eventually it would all look black. And if you did the calculation, you're… Every time you multiply by three-fourths, you get closer and closer to zero.” (A688, 691)

The evidence for Carmen is similar to the evidence for Joy, though of course not identical. Throughout the group work, there is a lot of reinforcement and interaction between Joy and Carmen; they frequently co-construct. Carmen also integrated Joy’s statements into her own thinking. For example, while constructing *Area Sequence*, Carmen established that “The area at each iteration is three-fourths of the previous” (A520). A little later Joy’s “Because each time we're taking away a quarter of it” (A546) was immediately followed by Carmen’s “Which gives us three-fourths of it” (A548) “and then three-fourths of our three-fourths” (A550), and a little later again “Maybe three… Maybe three-fourths to the N” (A590).

For brevity, we omit a full description of Carmen’s construction and consolidation of *Repeating Process* and *Area Sequence*, but include evidence of construction and consolidation of the focal idea, *Area Limit 0* (Table 2).

Table 2
Excerpts of RBC analysis of Carmen's during SGW

Constructing *Area Limit 0*	**"Is it approach… zero? I think it does, because otherwise, like, you know, we can have like an asymptote that's like three-fourths and minus five. And maybe it approaches negative five, or like…" (A605)**
Consolidating *Area Limit 0*	"If you keep filling it in, there's not going to be any white area. Okay. It keeps getting small." (A619)

Given the close collaboration, it is not surprising that a DCA analysis of Group A yields a number of ideas that function-as-if-shared within the group. We identified, among others, the following three arguments:

Argument A1

Data A1 And then three-fourths of this… and three-fourths of that (Joy 596, 598)

Claim A1 Okay, so you are right, it approaches zero (Joy 610)

Warrant A1 If it's an infinite loop and you just keep doing it (Jen 602)

Argument A2

Data A2 Because there'll be nothing (Joy A614)

Claim A2 It'll all be black (Joy A614)

Warrant A2 If you keep filling it in (Joy A614)

Argument A3

Data A3 If you keep filling it in, there's not going to be any white area (Carmen A619)

Claim A3 Okay. It keeps getting small (Carmen A619)

On the basis of these arguments, we identify "Area Approaches Zero" as an idea which, in Group A, functions-as-if-shared using Criterion 2. This idea is the claim of A1, made by Joy, and then used, in the form "there'll be nothing" as Data in A2, also by Joy. Similarly, the closely connected claim of A2 "It'll all be black" made by Joy is then used in the form "there's not going to be any white area" as Data in A3 by Carmen. Thus, by the time WCD9 occurred, Joy and Carmen had jointly constructed and consolidated *Area Limit 0*, and *Area Goes to Zero* (unchallenged by Jen) functioned-as-if-shared in Group A.

Group B

Group B, consisting of Elise, Kevin, and Mia, worked in parallel to Group A. As with the first group, we begin with an overview of the RBC analysis and then the DCA of mathematical progress within group work. For this group we did *not* find evidence for constructing *Area Limit 0*, but they did, in some cases partially, co-construct the constituent knowledge elements *Repeating Process*, *Area Features* and *Area*

Sequence. As we view these as co-constructed ideas, we present work from all three students in the same Table 3.

Table 3

Excerpts of RBC analysis of Group B during SGW

Constructing *Repeating Process*	**[see Task 1e in the activity, Fig. 1]** **...repeats the whole process again (Kevin B100)** **So it goes b, c, d, b, c, d, because d is the one telling you to repeat the process (Kevin B102)** **Oh, right. So the process consists of b and c… and d tells us to repeat… b and c (Mia 103)** **So not just ... repeat the whole, but that you start repeating it more and more, right? (Elise B109)**
Partially constructing *Area Features*	Basically we're taking out a certain … of the entire [triangle], and it's probably some… umm… scale factor (Mia B52) And then these pieces get smaller and smaller, right? (Elise B141)
Constructing *Area Sequence*	It's always been cut in… cut by a fourth (Kevin B39) So you're left with three fourths (Kevin B169) So, like, A0 would be three fourths... (Kevin B198) ...to the n (Mia B199) Oh, wait, no. Yeah, three fourths to the n (Kevin B200) so we always lose one fourth (Elise, B231) so it's three fourths to the n of our A1 (Elise, B241)

The students constructed *Repeating Process* rather early in their group work; Kevin and Elise provide evidence that they have completed this construction. They occasionally paid some attention to *Area Features*, thus partially co-constructing it but this construction was not completed. They discussed *Area Sequence* at some length, both as a series and a sequence but did not discuss convergence explicitly. It was mentioned briefly early on, with Mia asking "Is it tending… oh yeah you're right" (B40), but did not appear again until the instructor explicitly asked "any guess what will come out at the end, of the area?" (B308) To this, Kevin answered "it's going to be zero" (B309), but no additional justification was provided. We note that the justification (i.e., as $n \to \infty$, $(3/4)^n \to 0$) may appear obvious to the reader, but we have no evidence as to how Kevin came up with his answer and in fact many students in the class argued, at various times, that monotonic decrease was enough to justify a sequence converging to zero. Thus, we say that Group B co-constructed the knowledge elements which might lead to constructing *Area Limit 0*, but did not explicitly construct *Area Limit 0* itself during this session.

As with Group A, given the close collaboration, it is not surprising that a DCA analysis of Group B yields a number of ideas that function-as-if-shared within the group. We identified, among others, the following arguments:

Argument B1

Data B1 The second area is three-fourths of the first, and then, and then that continues (Elise B235, B237)

Claim B1 So, in other words, three fourths squared of the one before (Kevin B238)

Warrant B1 The third area is three-fourths of the second (Elise B237)

Argument B2

Data B2 The area at any stage is ¾ of the area at the previous stage (This was not said in these words but is implied by Argument B1)

Claim B2 So it's three-fourths to the N of our A1 (Elise B241)

These arguments are part of the students' construction of *Area Sequence* as noted in Table 3. In argument B1, the students convince themselves that $A_2 = (¾)^2 \cdot A_0$, as they develop a formulation for a sequence describing the area of the iteratively developed figure. In this argument, being able to calculate the area two stages along by multiplying the starting area by ¾ · ¾ functions as a claim, but in argument B2 it is used as data to support the more general formulation of the area of the figure at the *n*th step. Thus, by Criterion 2, the idea that the area is ¾ of the area of the previous stage in Group B functions-as-if-shared in Group B. We note that, had Kevin only hinted at $(¾)^n$ justifying *Area Limit 0*, that would have made $(¾)^n$ into another idea functioning-as-if-shared in the group. But as mentioned, we have no evidence for this linkage.

Whole Class

As noted, we found evidence that *Area Goes to Zero* functions-as-if-shared for the class. It was also noted that its status as functioning-as-if-shared is tentative, based on W78-W84.

In turn 84, Carmen presents two arguments. Argument 1 provides additional justification for Joy's unsubstantiated claim that the area is going to zero; Argument 2 uses that fact as data to justify her claim that there is "no perimeter" of the figure, since there is nothing left to outline. In doing so, *Area Goes to Zero* shifts from a claim needing justification to data which can support a new claim (Criterion 2 for classifying ideas as functioning-as-if-shared). In light of our analyses of the group work, we see that both Carmen and Joy have constructed and consolidated *Area Limit 0* prior to this public exchange, and this idea was already functioning-as-if-shared in Group A. We then must ask if this is simply a public presentation of what already functioned-as-if-shared for Group A or if it is something more. We argue that it *does* constitute an idea that functions-as-if-shared, using Criterion 1 and the analyses of the group work.

In line with Criterion 1, we note that this claim is never challenged in the whole class discussion, despite the presence of social and sociomathematical norms which would make that an expected course of action *if* someone objected to the statement or had some confusion about the mathematics. That is, in this class, the students would ask for justification if they felt it was needed (this is a methodological prerequisite for using DCA). In fact, Elise (turn 88, see Appendix A) does ask questions about what she doesn't agree with in Carmen's argument - but this is not about the area going to zero. Instead, she questions Carmen's claim that there will be "no perimeter," given that she understands the perimeter to increase after every iteration. We consider what we know about members of each group in turn.

In Group A, *Area Goes to Zero* functioned-as-if-shared, which is presumably why Joy did not support her declaration that area goes to zero (W79); we note that this is similar to Criterion 1, wherein justification drops away from claims that have been satisfactorily established. Thus Joy's declaration that the area goes to zero is perhaps further evidence that in Group A, this idea is accepted and functions-as-if-shared - thus we would not expect Joy, Carmen, or Jen to issue a challenge.

In Group B, the idea did *not* function-as-if-shared, but we have shown that Mia, Elise, and Kevin co-constructed many of the constituent knowledge elements of *Area Limit 0*, and Kevin said of the area "it's going to be zero" (B309). Although the discussion does not provide us with evidence that this idea had been constructed by anyone in Group B, let alone agreed upon by all three. It is our conjecture that the members of Group B recognize *Area Limit 0* as conclusion from their prior work and hence accept Carmen's statement, though they did not explicitly complete that construction during the group work segment. As noted previously, *Area Limit 0* does build directly upon ideas which Group B *did* construct and which function-as-if-shared among Elise, Mia, and Kevin, although they did not explicitly finish the construction of *Area Limit 0* nor provide evidence to say that *Area Goes to Zero* might function-as-if-shared among them.

The remaining five students in the class were in Group C and Group D, and we do not have access to their group discussions. However, during WCD9 two of these students make reference to the area going to zero in further conversation about what happens to the perimeter; these comments provide some idea of what may have happened - and bolster our claim that *Area Limit 0* functions-as-if-shared. When asked by the instructor about perimeter, Sam says "theoretically [...] we reach infinity in the end, it's going to go to zero. Then we don't have an area. So I'm, I'm not sure" (W130); in the same conversation Soo mentions that her group "tried writing the formulas for the area [...] eventually it's going to go to zero" (W126, 128). These comments were in the service of discussing perimeter, and while they are not part of complete arguments, they mark additional moments when area going to zero is mentioned and goes unquestioned. Furthermore, they suggest that Group C and Group D members engaged in the construction or co-construction of some of the knowledge elements related to *Area Limit 0*, if not *Area Limit 0* itself.

Concluding comment

The DCA analysis on WCD9 tentatively identified *Area Goes to Zero* as an idea which functions-as-if-shared in the classroom, based on a discussion which satisfies Criterion 2; that it is tentative is because the given arguments are brief and one of them is even incomplete; it is also a conversation between only two members of the class who were in the same group. We elected to identify this idea as functioning-as-if-shared in part due to the fact that no members of the class issue a rebuttal or question this result, which we could interpret as a variant of Criterion 1, supporting the decision that *Area Goes to Zero* functions-as-if-shared. Moreover, when combined with the RBC and DCA analysis of the group work of Group A and Group B, we become even more confident in this assessment. In particular, we note that Group A had reached this idea before sharing it with the whole class, and that Group B was close to constructing this idea. Thus, it may be that Carmen and Joy “finished” the partial arguments formed by Group B, rather than introduce new ideas. As Group B were most of the way there already, this did not present a perturbation nor a new line of thinking.

The story underlying **Keep Zooming In**

The excerpt of WCD9 relevant for this story follows immediately after the excerpt relevant for the previous story.

W84	**Carmen**	**I was thinking if we keep zooming… Okay, for our area thing, we were going to keep zooming in, keep coloring in, so eventually we're gonna color all in. It's going to be black, so there’s no area, so there's nothing to… No area there’s nothing to put a fence around it. So, there’d be no perimeter… and then**
W85	Instructor	So Carmen is thinking that the perimeter then would be zero, because there's no… There's nothing left to put a fence around. And Joy, you were thinking what?
W86	Joy	I kind of thought it's toward the opposite end - like, if you zoom in there's more to fence, and if you zoom in there's more to fence, and you just keep putting in more fencing material, because as you zoom in there's more and more to fence. Until... Except that you'd fill in the triangle, to some extent.
...		
W94	Soo	Umm… I think… Umm… Joy is saying, like, you keep zooming in you're going to get more triangles forming, so you have more areas, so you keep adding the numbers, right? ...
The DCA analysis of this part of WCD9 yielded three arguments (the first two were already presented in the previous story):		

Argument 1

Data 1 Keep zooming in, keep coloring in (Carmen W84)

Claim 1

There is … no area (Carmen W84)

Warrant 1 Eventually it’s going to be all black (Carmen W84)

Argument 3

Data 3 Keep zooming in (Joy W86)

Claim 3

Just the opposite [to C2] (Joy W86)

Warrant 3 The more you zoom in, the more there is to fence (Joy W86)

Argument 6

Data 6 You keep zooming in you're going to get more triangles forming (Soo W94)

Claim 6

You have more areas (Soo W94)

Warrant 6 You keep adding the numbers (Soo W94)

These arguments lead to *Keep Zooming In* to be identified as functioning-as-if-shared according to Criterion 3 since it has been used as data by three different students in three arguments with different claims: Carmen in Argument 1 (W84), Joy in Argument 3 (W86), and Soo in Argument 6 (W94).

Tracing the zooming in metaphor to earlier parts of the lesson leads all the way back to the movie the students watched at the beginning; the movie made extensive use of zooming, in order to show the coast of Britain as well as other natural fractals at various enlargements, in order to visually demonstrate how the images at different scales look alike.

When discussing the movie in Group A, Joy's statement "Especially when you, like, when you zoom in you get more of the same" (turn A7), was refined by her peers: "Maybe more of the same, but in an unexpected manner? … If you zoomed in, you'd expect to see kind of, like, just this. But instead, you're actually seeing…" (Carmen, turn A37); "Well kind of… Yeah, like what's the, the obvious example is not what you get when you zoom in on it" (Jen, turn A45). In parallel, in Group B, Elise said "Like, even as he zoomed to smaller and smaller pieces it looked like the big thing" (turn B3); and in the following WCD, Curtis (from Group D) noted: "We were thinking about how looking at the entire country, you sort of see the same amount of jaggedness or smoothness as you would if you zoomed in any closer. So, that sort of, we related that to the cauliflower" (turn W13). Curtis connects between zooming in on different fractals.

Next, the worksheets with the activity (Fig. 1) were shared out. The practice of zooming in influenced some of the students' work during the activity. Interestingly, no member of Group B used zooming in during the continuation of the lesson. We therefore focus on Group A.

After drawing the first few iterations of the Sierpiński triangle on their huddle board, the students were asked to think about the continuation of the process (Task 2a of the activity). In response, Joy used zooming in three times. In turn A435: “So… infinite loop? We said more… smaller triangles as we zoom in. I mean…”, Joy explicitly connects zooming in to the ‘infinite loop’, and in turn A444 to ‘keep iterating’: “And then you keep iterating, you keep getting the same and same thing. So if you zoomed in, the picture looks the same”. According to the RBC analysis, Joy uses zooming in as a tool to implement and build-with the *Repeating Process* knowledge element while constructing the *Area Features* knowledge element. In fact, Joy completes the construction of *Area Features* in A444.

Joy’s use of zooming in connection with *Repeating Process* is closely connected to Story 1. There, we concluded that *Repeating Process* had been constructed by the majority of students in class; moreover, the term ‘infinite loop’ had been used repeatedly, for example (but not only) as Warrant A2.

According to the DCA analysis of the group work, in turn A435 Joy used “as we zoom in” as Data 2 for Claim 2, which is “more smaller triangles”. Hence, not only is zooming in a tool according to both analyses, RBC and DCA, but it is a tool for related issues since the smaller triangles are a central component of *Area Features*.

After the students discussed the sequence of area measures without any further reference to zooming in, and came to the conclusion that the area tends to zero (see Story 1), the instructor joined the group and focused their attention on the perimeter.

A694	Instructor	And perimeter?
A695	Joy	I would say it would go the opposite
A699	Carmen	But if you have nowhere to fence off, you… you couldn’t build a fence.
A707	Carmen	So I was like, like the same sort of thing, if we keep zooming in, there's no area, there can be no fence.
A710	Joy	I thought it went infinitely, because if you zoom in, there's more fencing to put in. And if you zoom in there's more fence to put in.

Here Joy and Carmen both strongly rely on zooming in as a thinking tool and as data for their claims. The DCA analysis produced two arguments:

Argument A3

Data A3 If we keep zooming in there is no area (Carmen, A707)

Claim A3 There is no fence (Carmen, A707)

We remind the reader that Carmen uses fence as a metaphor for perimeter.

Argument A4

Data A4 If you zoom in there's more fence to put in (Joy, A710)

Claim A4 It (the fence) increases infinitely (Joy, A710)

In Arguments A3 and A4, Carmen and Joy use the same data, namely zooming in, to make opposite claims. Hence, by Criterion 3, *Zooming In* functions-as-if-shared for Group A. Specifically, Joy used zooming in as the basis of her justification. Joy later made a closely related argument in W86 (Argument 3). Carmen linked zooming in to her earlier statement (A619) that “If you keep filling it in, there's not going to be any white area”. For Carmen, zooming in connects to removing area, and since this eventually left no white triangles (the area is zero), she concluded that there was nothing to fence off, hence no fence and no perimeter. Here Carmen linked between the area and the perimeter of the Sierpiński triangle, the eventual shape the students were asked to imagine (Fig. 1: Activity, Task 2a). She later makes a closely related argument in W84 (Argument 1).

The different conclusions of Joy and Carmen can be explained by the RBC analysis: Joy is thinking in terms of the perimeter limit (PL) knowledge element and completes its construction in A710. Carmen, on the other hand, uses the *Area Limit 0* knowledge element, and considers the limiting process as completed, that is the area being equal to zero. In other words, she constructed (A699-A707) the actual infinity of the area of the Sierpiński triangle equaling zero, and from there concluded on the perimeter.

Based on this Group A discussion just before the instructor initiated WCD9, it is not surprising that Joy and Carmen used zooming in as data in their arguments at the beginning of WCD9.

The story underlying the **Perimeter of the White is also the Perimeter of the Black**

The excerpt of WCD9 relevant for this story follows.

W88	**Elise**	**So, what I feel, like, what Carmen's saying is when you zoom in… Or she says you color it all in so it's all black, but… What you're coloring in, is perimeter, to some extent. Not totally, because it's also area. But, like, every time you build a little triangle, you have more perimeter in there, right? So, then, all those… I don't know… Does all the black become all the tiny little pieces of all the tiny triangles?**
W89	Kevin	So what… So the, the perimeter is… also can be considered the perimeter of the black. Part of the perimeter is the perimeter of the black. 'Cause see, when you… When you shade it in, you're adding the perimeter of the black.
W90	Carmen	Oh, I see what you're saying - so it's actually, like, it's a… the fence is guarding both properties.
W91	Kevin	Yeah
W92	Carmen	Not just yours, but it's doing the other one too. Ok that makes sense haha.
...		
W95	Instructor	Curtis, you were nodding your head when Kevin was talking. Can you say a little about what you interpreted Kevin to say?
...		
W98	Curtis	There was no area to the… Yeah. But then, umm, Kevin was saying that the perimeter of the… the white is also the same as the perimeter of the… perimeter of the black part. So, since there's area… There is some area of the black… But we didn't talk about that. There could be a…
The DCA analysis of this part of WCD9 yielded two arguments:		

Argument 5

Data 5 When you shade it in, you're adding the perimeter of the black (Kevin W89)

Claim 5

the perimeter is… also can be considered the perimeter of the black. Part of the perimeter is the perimeter of the black (Kevin W89)

Warrant 5 The fence is guarding both properties; not just yours, but it's doing the other one too (Carmen W90, W92)

Argument 7

Data 7 since there's area… there is some area of the black (W98)

Claim 7

[unfinished idea]

Warrant 7 The perimeter of the… the white is also the same as the perimeter of the… perimeter of the black part (Curtis W98)

Using the criteria for an idea to function-as-if-shared, these arguments lead to *The Perimeter of the White is also the Perimeter of the Black* (abbreviated *PW = PB*) to be identified as functioning-as-if-shared by Criterion 2.

The story of *PW = PB* functioning-as-if-shared evolved in turns W88-W98 of WCD9, in a short time span. It was an unexpected idea for us as designers of the activity and as researchers. *PW = PB* functions-as-if-shared according to Criterion 2, as it is the claim of Argument 5 by Kevin (W89), and then the warrant of Argument 7 by Curtis (W98). We note that the use this warrant might be questioned because Argument 7 is missing a claim and is therefore not properly an argument; however, we feel confident that this idea functions-as-if-shared since, in addition to Kevin and Curtis, Carmen in Warrant 5 (W90) contributed her own independent interpretation of it as the fence guarding both properties: She interpreted the statement in terms of the fence metaphor as "the fence is guarding both properties," where "both" refers to the black (already removed) triangles on one hand, and to the white triangles that form part of the figure under consideration at this stage.

Following our methodology for coordinating DCA and RBC, we now consider the knowledge elements that may be relevant for *PW = PB*: *Repeating Process* and *Perimeter Features*. *Repeating Process* is so basic that it is necessary for "everything else"; also, it has been constructed by almost all students, as shown in Story 1 about *Area Goes to Zero*. On the other hand, *Repeating Process* has no direct bearing on *PW = PB*. We thus focus on *Perimeter Features*.

The analysis of the work of the groups on perimeter shows that Group A raised the question "So what counts as the perimeter?" (Joy, A648). This led to a brief discussion about what exactly they were supposed to find, but according to our RBC analysis, it did not lead to a constructing action. Group B, on the other hand, while spending quite some effort on computations relating to the length of the perimeter, up until the beginning of WCD9 never even asked themselves about the features of this perimeter.

This raises the question whether *PW = PB* could emerge and function-as-if-shared without any apparent basis of knowledge construction. The RBC analysis of WCD9, a component of our enhanced methodology, allowed us to answer this question.

The design of the activity placed the Area Task 2b right before the Perimeter Task 2c (see Fig. 1), but apart from that gave no indication of a link between area and perimeter; such a link was neither intended nor expected by the designers. And indeed, up to this point in time, we have not found any evidence for students linking the process of the area decreasing with the process of the perimeter growing either in the group work or in the whole class discussions. The presentation of the two different points of view by Carmen and Joy at the beginning of WCD9, however, led to an immediate linkage of area and perimeter by Elise (WCD9, 88). Elise's reaction is an attempt to build-with the knowledge she had constructed about the area (*Area Sequence*, and partially *Area Features*) and about the perimeter in order to connect area and perimeter during the process: “What you're coloring in, is perimeter, to some extent. Not totally, because it's also area. But, like, every time you build a little triangle, you have more perimeter in there, right?” (W88). We observe that Elise’s is a novel way of looking at the process. She connected two knowledge elements that were separate up to this point, and merged them into a new structure, which on the one hand is more complex but on the other hand is more of a unity because of the connection established - such structuring is strongly indicative of knowledge construction according to RBC.

Kevin (W89) picks up right where Elise left off: He notes that while “perimeter” at each stage of iteration refers to the perimeter of the region that has not been removed (or blackened in), i.e. the white region, this same perimeter, according to Elise’s insight, is surrounding the just removed black triangles. Both Elise’s and Kevin’s thinking constitute vertical reorganization of knowledge elements that had been previously constructed into new, deeper insights by establishing a new connection – the very essence of a constructing action. This was not lost on the rest of the class: Carmen expressed Kevin’s insight using the fence metaphor that had been prominent in her own thinking: “the fence is guarding both properties” (W90, W92); Curtis (W98) showed that he was thinking along with Elise, Kevin and Carmen; and just a bit later Mia provided similar evidence from still a slightly different point of view: “…if you imagine actually having a piece of paper triangle, shading in the middle triangle, taking it out - you're going to have all these shaded triangles, with perimeters. And that's how I see it” (W100).

Hence, Elise and Kevin co-constructed an unexpected knowledge element, *Combining Area and Perimeter*, which we defined (a posteriori) as follows:

- *Combining Area and Perimeter*: At any stage of the process, it is the very act of removing area that causes the addition of perimeter; therefore, the perimeter of all the triangles remaining in the shape at any stage of the iteration is at the same time the perimeter of all the triangles that have been removed from the shape (apart from the perimeter of the original triangle at stage 0).

We observe that Elise constructed the first part of *Combining Area and Perimeter*, and Kevin the second. Carmen’s interpretation stressed the second one while Mia’s stressed the first one.

This knowledge constructing process identified by RBC not only provided the ideas appearing in the DCA analysis but also explains how *PW = PB* could function-as-if-shared; indeed, according to the DCA analysis presented earlier, the same turns and contributions by Elise, Kevin, Carmen and Curtis are what demonstrates that *PW = PB* does indeed function-as-if-shared.

Discussion And Conclusion

In this paper weframed SGW and WCD in terms of three stories from an inquiry-based lesson with two aims: to gain insight into the complexity of the interplay between collective and individual mathematical progress (CIMP) in inquiry-oriented classrooms, and to demonstrate a methodological approach of layering explanations (LE) to the analysis of mathematical progress in such classrooms. The two aims are, of course, closely linked into CIMPLE: It is the layered explanations provided by the methodological approach that allow us to illustrate the complexity of CIMP. We begin the discussion by summarizing the three stories and then turn to issues related to methodology and complexity.

Students' Mathematical Progress

We begin with Story 2 which deals with *Zooming In*, a way of reasoning that functions-as-if-shared in the class. Story 2 highlights that ideas which function-as-if-shared in a class can be underlying imagery that, for example, imagery that supports intuitive thinking about difficult issues (in this case, infinite processes). Story 2 originates at the beginning of Lesson 9 with a video; zooming in continues to be used throughout much of the students' work, providing language and a tool to grapple with the infinite nature of the process under consideration. This practice is a tool connected to everyday experiences (e.g., using a mobile phone or other touch screen) and is used by the students to engage with the mathematical issue of infinity.

The background of Story 1 is formed by a rather classical process of knowledge construction in Group A, as revealed by the RBC analysis. Students in Group A progress from constructing *Repeating Process* via *Area Features* and *Area Sequence* to *Area Limit 0*. The progress toward *Area Limit 0* was not immediate for the students. They used numerical examples and recursive relations, but without concluding that the sequence is geometric and therefore tends to 0. The parallel analysis of Group B shows that all three students constructed detailed preparatory knowledge for *Area Limit 0*, mainly *Repeating Process* and *Area Sequence*; however, we have no evidence for any of them having constructed *Area Limit 0*.

Part of the enhancement of the methodology implemented in the present research is that we carried out not only an RBC analysis but also a DCA analysis of the group work. This analysis revealed that *Area Goes to Zero* functioned-as-if-shared in Group A. At the very beginning of WCD9, two of the Group A students were asked by the instructor to present their view of the development of the perimeter while constructing the Sierpiński Triangle. During this presentation, both students relied on *Zooming In*; also, both relied on *Area Goes to Zero*. Their justification received no rebuttal, nor even comment, from other members of the class, and such began to function-as-if-shared among the whole class - this is supported by the preparatory work done by Group B, and likely the other groups had made some of this progress as well.

We conclude from Story 1 that ideas may be constructed, consolidated, and function-as-if-shared within small groups before they come to function-as-if-shared among the whole class. We also conclude that ideas may function-as-if-shared in the whole class, even if the associated constructing process has occurred only partially.

In contrast to Story 1, we learn from Story 3 that the *PW = PB* came to function-as-if-shared in the class without any preparatory constructing process of this relationship in the SGW leading up to WCD9. This is where our analysis was enriched by another part of the enhancement of the methodology implemented in the present research, namely to carry out an RBC analysis not only on the SGW but also on the WCD. Specifically, the RBC analysis of WCD9 reveals that the relevant constructing process took place entirely in WCD9, essentially in parallel with the relationship coming to function-as-if-shared. No link between the processes of the growing perimeter and of the decreasing area was made by any of the students before Elise brought it up in WCD9 (turn W88). Without the RBC analysis of WCD9, we would likely have missed the construction of CAP by Kevin, Elise, Curtis and Carmen, a link which was somewhat unexpected and not part of the initial design.

Figure 2 integrates the three stories graphically. It shows results of the RBC analysis in solid (blue) lines and the DCA analysis in dashed (red) ones.

Enhanced Methodology

The collective and individual mathematical progress (CIMP) which occurred during Lesson 9, and was documented in the three stories in particular, is of mathematical interest because of its close connection to the concept of self-similarity, which was discussed later in the course (Hershkowitz et al., 2022) and to the apparent paradox inherent in some fractals such as the Sierpiński triangle. Our focus in the present paper is less on the mathematical progress itself than (1) on the enhancement of our methodology for investigating mathematical progress in inquiry-based classrooms and (2) on the insights into the complexity of mathematical progress in such classrooms that the methodological enhancement reveals.

Our methodology of coordinating the RBC and DCA theoretical frameworks has been enhanced in the present paper as follows: not only did we analyze small group work using RBC and whole class discussions using DCA (e.g., Hershkowitz et al., 2014), but we also carried out DCA analyses of the small group work and RBC analyses of the whole class discussions. This enhancement allowed Layering the Explanations (LE) for the CIMP in a manner that makes it transparent and fully integrates the collective with the individual mathematical progress – CIMPLE.

As apparent from the summary in the previous subsection, the combination of analysis methods was crucial for identifying and explaining the students' mathematical progress in these three stories from one single day in class; stories which interact and support each other. Our enhanced methodology of systematically applying analyses stemming from two theoretical frameworks thus provides the layers of explanation that allowed us to make sense of the lesson as a whole. We conclude that to make sense of learning in classrooms, using and coordinating two or more methodologies with somewhat different foci and grain sizes is insightful, in line with earlier work on networking theories in general (Bikner & Prediger, 2014) and in situations like the current one (Author et al., 2020), in which we provided an argumentative grammar for the coordination of our two approaches. Here we further our understanding as of the benefits and insights stemming from coordinating the results of the analyses.

Complexity

Our enhanced methodology of analyzing all episodes of the lesson by means of both, RBC and DCA, thus enabled us to exhibit the complexity of mathematical progress in the specific lesson analyzed here. Our analysis shows, by means of the three stories, a multiplicity of ways in which knowledge has developed and mathematical progress has been achieved, by individual contributions to small group work and whole class discussions. We showed that knowledge may be constructed by two or three students in a small group, such as, for example, the area elements, *Area Features*, *Area Sequence*, and *Area Goes to Zero* in Group A (Story 1); knowledge may also be constructed by a large group of individuals collaborating and arguing together in a whole class setting such as happened in constructing the connection between area and perimeter (Story 3). Ideas may begin to function-as-if-shared without a preceding constructing process (Story 2), or in proximity to a relevant constructing process (Stories 1 and 3); and if in proximity, the constructing process can be prior (Story 1) or almost simultaneous (Story 3).

Knowledge that is new to the students is constructed not only in SGWs but also in WCDs. Hence, the role of WCDs, at least in an inquiry-oriented classroom, goes far beyond summarizing group work and institutionalizing constructs from the group work. This conclusion should be seen in tandem with the conclusion that ideas may begin to function-as-if-shared not only in WCDs but also in SGW. This is relevant for the complexity of what happens in the classroom as well as for the social to individual connection in the interpretative framework / emergent perspective.

Concluding remarks

We believe that this complexity is not exceptional but that similarly complex interactions between processes of knowledge construction and processes of ideas becoming normative are typical for inquiry-oriented classrooms. However, such generalization should be accompanied by a few caveats. While we consider that constructing knowledge in WCDs should be taken into account as a possibility, the success of the RBC analysis of the WCD in the current research may have been due to the small size of the class, since such analysis depends on having rather dense information about the participating students. Similarly, the closeness in time of constructing and beginning to function-as-if-shared by the whole class observed in Story 3 may not be generalizable.

The research presented here raises a number of questions that merit further research. The most obvious one is the question of generalizability just addressed. Research on other, possibly larger inquiry-based classrooms, other student age groups, and other mathematical content areas using the enhanced methodology proposed here is needed. Another area for research concerns the issue of teacher moves: In view of the instructor's move initiation WCD9, the question arises what teacher moves might lead to whole class discussion with a high potential for new ideas to emerge, be constructed and become normative?

To conclude, the emergence of knowledge in inquiry-oriented classrooms, whether its construction by individuals, its co-construction by several learners, or its becoming normative for a group or for the entire class, occurs in settings from the individual via the small group to the whole class, and in a large variety of processes. In the present paper, we demonstrate that our enhanced methodology of coordinating the RBC-model of Abstraction in Context and Documenting Collective Activity at the levels of individuals, small group work and whole class discussions, holds promise for enabling researchers to understand learning in inquiry-oriented classrooms in depth.

Statements And Declarations

All authors contributed to the study conception, design, data analysis and writing. All authors read and approved the final manuscript. There are no competing interests.

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Statement

On behalf of all authors, the corresponding author states that there is no conflict of interest

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Figures

Activity Sierpiński's triangle (Waclaw Sierpiński, 1915)

1. Triangles
   (a) Sketch an equilateral triangle of side a = 16 cm (16 has been chosen for convenience).
   (b) Connect the midpoints of the triangle's sides, so as to generate four congruent triangles of side $a/2$.
   (c) "Take away" the triangle in the middle (you may cut it out or simply color it in a dark color).
   (d) You are now left with three equilateral triangles. For each one of them, repeat (b), (c), and (d).
   (e) Discuss the instruction, given in (d), to repeat not only (b) and (c) but also (d). Write your considerations and conclusions, and present them to the class.
2. Stop after carrying out (b) and (c) six times.
   (a) Imagine the shape that results from repeating (b) and (c) "forever". (Remember that the shape under consideration is the part that is not colored.)
   (b) Can you assign an area to this shape? Write your considerations and conclusions, and present them to the class.
   (c) Can you assign a perimeter to this shape? Write your considerations and conclusions, and present them to the class.

Figure 1

Sierpiński triangle activity – Tasks 1 and 2

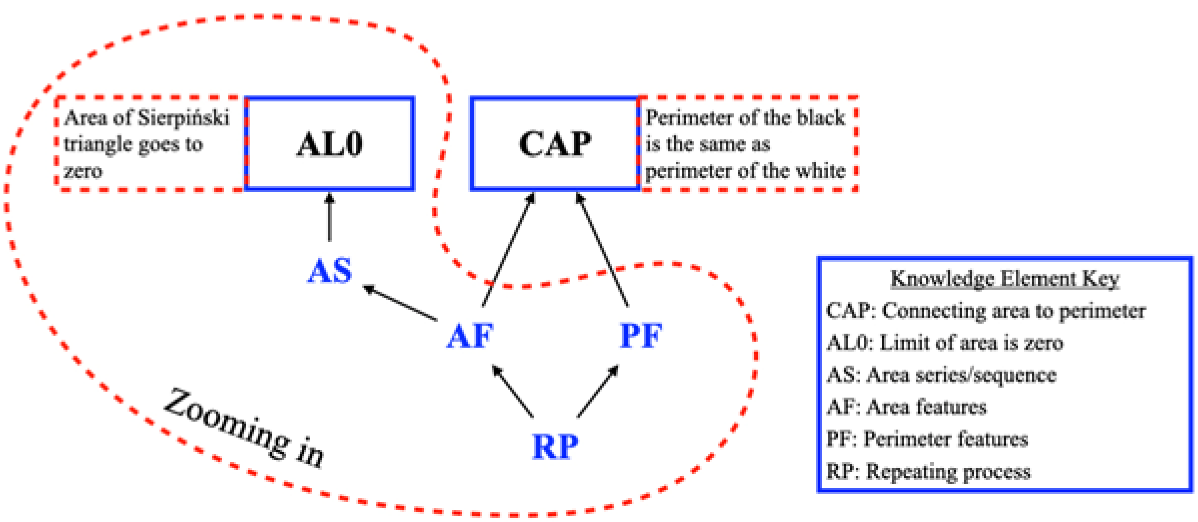

Figure 2

Diagrammatic summary of the class' mathematical progress

Supplementary Files

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- AppendixA.docx