PT-symmetric distinction of three states and its implementation on a cloud quantum processor

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$\mathcal{PT}$ -symmetric distinction of three states and its implementation on a cloud quantum processor

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Abstract

We present a novel $\mathcal{PT}$ -symmetric approach for discriminating three pure qubit states and its implementation by the dilation method. Our approach based on the dilation method manipulating the Hilbert space of the system by two-staged $\mathcal{PT}$ -symmetric evolution expedites the three-state discrimination process at the cost of introducing an inconclusive outcome. This modification of the Hilbert space of the qubit makes it possible to eliminate one of three states in the limit $\alpha \rightarrow \pm \frac{\pi}{2}$ in the vicinity of the exceptional point corresponding to coalescing eigenvectors and eigenvalues, effectively reducing the problem to two-state discrimination, or reducing the geometry to the mirror-symmetric states for which an efficient discrimination strategy is developed. To demonstrate the effectiveness of our approach, we implement it on a superconducting quantum processor provided by IBM Quantum Experience employing an ancilla qubit. We conclude our paper by identifying potential applications, including those in quantum communication and cryptography, to facilitate further research and development.

Keywords: $\mathcal{PT}$ symmetry; Quantum state discrimination and elimination; IBM Quantum Experience.

I. INTRODUCTION

The problem of identifying information stored in a quantum system is fundamental in quantum computer science, and the simplest option is to use two-dimensional systems or qubits to store quantum information. In classical physics, the state variables of the system are also observables and there is no fundamental limitation on the precision with which they can be determined. By contrast, the quantum observables are represented by operators acting on the vector space, which in the general case do not commute, and the outcome of the measurement has statistical rather than deterministic properties. Quantum systems in different non-orthogonal states are impossible to perfectly distinguish even when arbitrary large but finite number of samples for the measurements are available [1–6]. In other words, no test exists which allows to guess correctly all of the time and this fact is highlighted by the quantum Chernoff bound [7].

Quantum state discrimination involves two parties who agreed on a set of allowed states in which the system can be, and their prior probabilities of occurrence. A measurement can obtain only a finite amount of information, and thus this set must be finite. Alice prepares a state from this set and sends it to Bob, who must determine it using the appropriate measurement. Quantum state discrimination has a number of important applications. In particular, it is strongly linked to a dimension witness of quantum systems [8], and represents an operational interpretation of conditional mutual entropy [9]. The quantum key distribution security is based on the hardness of a quantum state discrimination and on the no-cloning theorem [10]. The search over an unstructured database can be mapped to the discrimination of the states exponentially close to each other [11].

Quantum state discrimination is difficult apart from the $N = 2$ case, and the existing strategies for quantum state discrimination can be classified into the minimum error discrimination [1], unambiguous discrimination [12, 13], and maximum confidence discrimination [14], each with its own advantages and drawbacks. The minimum error discrimination solution was obtained for the states possessing particular symmetries such as “geometrically uniform” states [15] and mirror symmetric states [16]. In the general case of $N = 3$ states, the minimum error discrimination solution for pure qubit states was obtained in [17, 18]. The analytic solution for mixed qubit states without necessary and sufficient conditions was obtained in [19], and the complete analysis was performed in [20]. Unambiguous state discrimination can be achieved only for linearly independent states [21], and therefore is not possible for $N = 3$ qubit states.

Meanwhile, recent developments in $\mathcal{PT}$ -symmetric quantum mechanics [22–24], where the condition of Hermiticity is replaced by the condition of $\mathcal{PT}$ -symmetry, provide new opportunities for the quantum information science, which are not available in the usual

Hermitian case. Such theories possess an additional degree of freedom represented by the α parameter, and in the limit $\alpha \rightarrow 0$ the intersection of $\mathcal{PT}$ -symmetric and Hermitian cases are real symmetric Hamiltonians [25]. At certain values of the α parameter, the degeneracies occur, known as exceptional points, which correspond to coalescing eigenvectors and eigenvalues [26, 27]. These points can be used in multiple applications. First, in the $\mathcal{PT}$ -symmetric system, the time of quantum evolution is *theoretically zero*, while in the Hermitian one a finite time is needed. This effect was demonstrated both theoretically [28] and experimentally [29]. Second, it was used for enhanced sensing [30–35]. It has recently been shown that $\mathcal{PT}$ -symmetric sensors are 8.856 times superior to Hermitian ones [36]. Moreover, $\mathcal{PT}$ -symmetric operations increase the quantum Fisher information needed to increase the accuracy of quantum parameter estimation [37, 38], which in turn was used for the Bayesian parameter estimation [39–41]. The existence of exceptional points was demonstrated in multiple classical systems [42–50], and they found applications in laser mode management [51–53], and topological mode transfer [54–57].

In contrast to the above-mentioned existing strategies for quantum state discrimination, in our work we propose a novel approach based on the on the ability of $\mathcal{PT}$ symmetry to manipulate the Hilbert space of the qubit for $N = 3$ state discrimination. For $N = 2$ states, the $\mathcal{PT}$ -symmetric discrimination was developed in [58], and the focus of our work is an extension of this method for $N = 3$ states. The manipulation of the α parameter allows to eliminate one of the states or to reduce them to the mirror-symmetric ones. To demonstrate the efficiency of our approach, we implement $\mathcal{PT}$ -symmetric dynamics by the dilation procedure [59], and implement it on the IBM Quantum Experience [60].

II. RESULTS

First, we provide a necessary background on $\mathcal{PT}$ symmetry in Section III. In Section IV, we develop a $\mathcal{PT}$ symmetric approach for the $N = 3$ pure quantum states discrimination, which consists of two stages of $\mathcal{PT}$ symmetric evolution. On the first one, the two of three states are made mutually orthogonal by the Hermitian scalar product. On the second stage, the $\mathcal{PT}$ -symmetric evolution in the limit $\alpha \rightarrow \pm\frac{\pi}{2}$ approaching the exceptional point is taken, and this allows to eliminate one of three states since two of them coalesce in this limit. Alternatively, we find the values of the α parameter which reduces three states into mirror-symmetric ones. In Section V, we provide a comparison between the theoretical model and the results of the runs on IBM Quantum Experience. In Section VI, we reach conclusions and discuss possible applications of our approach in quantum information science. The details of implementation are provided in Methods.

III. BACKGROUND ON $\mathcal{PT}$ SYMMETRY

In order to have a complete description of the physical system, the energy eigenvalues must be real valued. Complex energies are often used to describe dissipative phenomena, when the probability of finding a particle decreases over time. The decaying particle, however, does not vanish but transforms to other particles, and therefore such description is incomplete. The condition of reality of the spectra can be achieved by constraining the Hamiltonian to be Hermitian $H = H^\dagger$. Nevertheless, this condition is not necessary and can be replaced by a condition of an unbroken $\mathcal{PT}$ -symmetry [22–24] which guarantees that all eigenvalues of the Hamiltonian are real. Additionally, it provides an extra degree of freedom not available in the Hermitian case which we describe further.

The Hamiltonian is $\mathcal{PT}$ -symmetric if it satisfies the condition $\mathcal{H} = \mathcal{H}^{\mathcal{PT}}$. The signs of the quantum mechanical coordinate and momentum, $\hat{x}$ and $\hat{p}$, are changed by the parity operator $\mathcal{P}$ as $\mathcal{P}\hat{x}\mathcal{P} = -\hat{x}$, $\mathcal{P}\hat{p}\mathcal{P} = -\hat{p}$, and for the case of qubit, up to a unitary transformation, is given by [28]:

$$\mathcal{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (1)$$

Time-reversal operator $\mathcal{T}$ changes the signs of the imaginary unit and the momentum operator as $\mathcal{T}i\mathcal{T} = -i$ and $\mathcal{T}\hat{p}\mathcal{T} = -\hat{p}$. The $\mathcal{PT}$ operator is a combination of $\mathcal{P}$ and $\mathcal{T}$. For the case of qubit, the most general $\mathcal{PT}$ -symmetric depends on three real parameters r , s and β as [58]:

$$\mathcal{H} = \mathcal{H}^{\mathcal{PT}} = \begin{pmatrix} re^{i\beta} & s \\ s & re^{-i\beta} \end{pmatrix} \quad (2)$$

The $\mathcal{PT}$ -symmetric Hamiltonian is called to be possessing an unbroken $\mathcal{PT}$ symmetry if each of its eigenfunctions is also an eigenfunction of the $\mathcal{PT}$ operator. This condition guarantees that all energy eigenvalues are real [61, 62]. Additionally, this condition provides an extra $\mathcal{C}$ operator which is represented by the sum of all eigenfunctions of the $\mathcal{PT}$ -symmetric Hamiltonian, Eqn. 2:

$$\mathcal{H}\psi_n(x) = E_n\psi_n(x), \quad \mathcal{C}(x, y) = \sum_{n=1}^2 \psi_n(x) \psi_n(y) \quad (3)$$

For the qubit case, it takes the form:

$$\mathcal{C} = \frac{1}{\cos(\alpha)} \begin{pmatrix} i \sin(\alpha) & 1 \\ 1 & -i \sin(\alpha) \end{pmatrix} \quad (4)$$

with α being expressed as $\sin(\alpha) = \frac{r}{s} \sin(\beta)$. As the result, the set of commuting operators in the $\mathcal{PT}$ -symmetric theory is bigger in comparison with the Hermitian case, $[\mathcal{C}, \mathcal{H}] = 0$ and $[\mathcal{C}, \mathcal{PT}] = 0$. The ket vector both in the Hermitian and $\mathcal{PT}$ -symmetric cases has the same form:

$$|\psi\rangle = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi} \sin\left(\frac{\theta}{2}\right) \end{pmatrix} \quad (5)$$

with θ and ϕ being the meridian and parallel of the Bloch sphere of the qubit respectively. The difference is in the scalar product which is fixed in the Hermitian case but in the $\mathcal{PT}$ -symmetric is *defined* by the $\mathcal{C}$ operator given in Eqn. 4 as $(\langle\psi|)_{\mathcal{CPT}} = (\mathcal{CPT}|\psi\rangle)^T$, $(\langle\mu|\nu\rangle)_{\mathcal{CPT}} = (\mathcal{CPT}|\mu\rangle)^T \cdot |\nu\rangle$, here the over-script T is matrix transposition and the $\mathcal{CPT}$ operation is a combination of $\mathcal{C}$ and $\mathcal{PT}$ operators which we defined previously. The limit $\alpha \rightarrow 0$ recovers the regular Hermitian case since $\lim_{\alpha \rightarrow 0} (\mathcal{C}) = \mathcal{P}$.

This property was used for $N = 2$ state discrimination [46] to manipulate the angle between state vectors effectively converting them into orthogonal ones. The no-cloning theorem [10] still applies both for the Hermitian and $\mathcal{PT}$ -symmetric case since this conversion happens at the cost of introducing an inconclusive outcome meaning that the $\mathcal{PT}$ -symmetric part of the total wave-function of the system has a norm less than one, in the general case. For two non-orthogonal states on the $\phi = -\frac{\pi}{2}$ parallel:

$$|\psi_{1,2}\rangle = \begin{pmatrix} \cos\left(\frac{\pi \mp 2\sigma}{4}\right) \\ -i \sin\left(\frac{\pi \mp 2\sigma}{4}\right) \end{pmatrix} \quad (6)$$

this conversion may be obtained by two possible *Solutions*:

- *Solution 1: zeroing the $\mathcal{CPT}$ product $(\mathcal{CPT}|\psi_1\rangle)^T \cdot |\psi_2\rangle = 0$ by setting the Hamiltonian in Eqn. 2 to make $\sin(\alpha) = \frac{r}{s} \sin(\beta) = \cos(\sigma)$.*
- *Solution 2: performing $\mathcal{PT}$ -symmetric Hamiltonian evolution to $(\langle\psi_1|\psi_2\rangle)_{\text{Hermitian}} = 0$ for a time τ_{Perp} :*

$$\sin^2(\omega\tau_{\text{Perp}}) = \frac{\cos^2(\alpha) \cos(\sigma)}{2 \sin(\alpha) (1 - \sin(\alpha) \cos(\sigma))}, \quad \omega = \sqrt{s^2 - r^2 \sin^2(\beta)} \quad (7)$$

effectively modifying the metrics as:

$$\cos^2(\alpha) e^{+i\mathcal{H}^\dagger t} e^{-i\mathcal{H}t} = \begin{pmatrix} \cos^2(\omega t - \alpha) + \sin^2(\omega t) & -2i \sin^2(\omega t) \sin(\alpha) \\ 2i \sin^2(\omega t) \sin(\alpha) & \cos^2(\omega t + \alpha) + \sin^2(\omega t) \end{pmatrix} \quad (8)$$

In particular, in contrast to the Hermitian case, in the $\mathcal{PT}$ -symmetric dynamics, the states $|0\rangle$ and $|1\rangle$ have an angular separation $\pi - 2|\alpha|$, and when approaching the exceptional point in the limit $\alpha \rightarrow \pm\frac{\pi}{2}$ they merge. In the next Section IV, we develop a $\mathcal{PT}$ -symmetric approach for the case of three states $N = 3$, by applying a double $\mathcal{PT}$ -symmetric evolution, and exploiting the properties of exceptional points $\alpha \rightarrow \pm\frac{\pi}{2}$.

IV. OUR SCHEME FOR $\mathcal{PT}$ -SYMMETRIC DISTINCTION OF $N = 3$ STATES

Our approach consists of three steps and can be summarized as:

- *Step 1*: evolve two of the states, $|\psi_1\rangle$ and $|\psi_2\rangle$, into the orthogonal ones $(\langle\psi_1|\psi_2\rangle)_{\text{Hermitian}} = 0$ by applying the first $\mathcal{PT}$ -symmetric evolution.

- *Step 2*: By applying a unitary rotation, convert these states into $|\psi_{1,2}\rangle \rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm i \end{pmatrix}$.

In these positions, they remain orthogonal under the $\mathcal{PT}$ operations with an arbitrary value of α parameter, and we use it to adjust a relative angle with the third state.

- *Step 3*: Perform a second $\mathcal{PT}$ -symmetric evolution to adjust the angle between $|\psi_{1,2}\rangle$ and $|\psi_3\rangle$, and perform the projective measurement.

Without loss of generality, an arbitrary set of three states $|\psi_i\rangle = \begin{pmatrix} \cos(\frac{\theta_i}{2}) \\ e^{i\phi_i} \sin(\frac{\theta_i}{2}) \end{pmatrix}$, $i \in [1, 3]$ can be adjusted to the following positions by unitary rotations provided in Eqn. 33 in Methods:

$$|\psi_{1,2}\rangle \rightarrow \begin{pmatrix} \cos(\frac{\pi \mp 2\sigma}{4}) \\ -i \sin(\frac{\pi \mp 2\sigma}{4}) \end{pmatrix}, |\psi_3\rangle \rightarrow \begin{pmatrix} \cos(\frac{\mu}{2}) \\ e^{i\nu} \sin(\frac{\mu}{2}) \end{pmatrix} \quad (9)$$

and σ , μ and ν parameters in the following equations.

In *Step 1*, we use $\mathcal{PT}$ -symmetric evolution controlled by the Hamiltonian in Eqn. 2, and perform it for a time τ_{Perp} in Eqn. 7. As a result, the first pair of states take a form:

$$|\psi_1\rangle \rightarrow \begin{pmatrix} \cos(\frac{\delta}{2}) \\ -i \sin(\frac{\delta}{2}) \end{pmatrix}, |\psi_2\rangle \rightarrow \begin{pmatrix} \sin(\frac{\delta}{2}) \\ i \cos(\frac{\delta}{2}) \end{pmatrix} \quad (10)$$

with the δ parameter provided by the Eqn. 11 and 12:

$$\cos\left(\frac{\delta}{2}\right) = \frac{\cos(\omega\tau_{\text{Perp}} - \alpha) \cos\left(\frac{\pi - 2\sigma}{4}\right) - \sin(\omega\tau_{\text{Perp}}) \sin\left(\frac{\pi - 2\sigma}{4}\right)}{\sqrt{\mathcal{V}}} \quad (11)$$

Figure 1. Result of the *Steps 1-3* when $\kappa_{23} \sim (\frac{\pi}{2} + \alpha) \rightarrow 0$.

Figure 2. Result of the *Steps 1-3* when $\kappa_{13} \sim (\frac{\pi}{2} - \alpha) \rightarrow 0$.

Figure 3. Reduction to the mirror-symmetric states.

$$\begin{aligned} \mathcal{V} = & 1 - \cos(2\omega\tau_{Perp}) \sin^2(\alpha) \\ & + 2 \sin(\omega\tau_{Perp}) \sin(\alpha) (\cos(\omega\tau_{Perp}) \cos(\alpha) \sin(\sigma) - \sin(\omega\tau_{Perp}) \cos(\sigma)) \end{aligned} \quad (12)$$

In *Step 2*, we apply the following gate with the χ parameter given by Eqn. 41 in Methods:

$$\mathcal{W} = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos(\frac{\delta}{2}) & i \sin(\frac{\delta}{2}) \\ i \sin(\frac{\delta}{2}) & \cos(\frac{\delta}{2}) \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & ie^{-i\chi} \end{pmatrix} \cdot \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \quad (13)$$

$$|\chi_{(1,2,3)}\rangle = \mathcal{W} |\psi_{(1,2,3)}\rangle \quad (14)$$

The resulting states take the following form where $\rho = \xi + \frac{\pi}{2}$ and ξ is given in Eqn. 41:

$$|\chi_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, \quad |\chi_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}, \quad |\chi_3\rangle = \begin{pmatrix} \cos(\frac{\rho}{2}) \\ i \sin(\frac{\rho}{2}) \end{pmatrix} \quad (15)$$

We proceed to the second $\mathcal{PT}$ -symmetric evolution constituting *Step 3*. After *Step 2* the first two states are orthogonal, we can adjust the relative angle to the third state. For completeness, we consider both Hermitian and $\mathcal{CPT}$ measurements.

A. Step 3 by Hermitian measurement

We apply the second $\mathcal{PT}$ -symmetric evolution for a time τ^{II} , and the relative angles between the evolved states κ_{12} , κ_{13} and κ_{23} are given by:

$$\begin{cases} \cos^2(\kappa_{12}) = \frac{2 \tan^2(\alpha) \sin^2(2\omega\tau^{II})}{1 + \sec^2(\alpha) - \tan^2(\alpha) \cos(4\omega\tau^{II})} \\ \cos^2(\kappa_{13}, \kappa_{23}) = \\ = \frac{(\sqrt{2} \sin(\frac{\pi \pm 2\rho}{4}) [(1 \pm 2 \sin(\alpha)) \sin^2(\omega\tau^{II}) + \cos^2(\omega\tau^{II} + \alpha)] + \sin(2\alpha) \cos(\frac{\rho}{2}) \sin(2\omega\tau^{II})^2}{2((1 \pm \sin(\alpha))^2 \sin^2(\omega\tau^{II}) + \cos^2(\alpha) \cos^2(\omega\tau^{II})) (\sin^2(\omega\tau^{II}) (1 + 2 \sin(\alpha) \sin(\rho)) - \sin(2\alpha) \sin^2(\frac{\rho}{2}) \sin(2\omega\tau^{II}) + \cos^2(\alpha - \omega\tau^{II}))} \end{cases} \quad (16)$$

After the time $\tau^{II} = \frac{\pi}{2\omega}$, these expressions take the form:

$$\begin{cases} \cos^2(\kappa_{12}) = 0 \\ \cos^2(\kappa_{13}, \kappa_{23}) = \frac{(1 \pm \sin(\alpha))^2 (1 \pm \sin(\rho))}{4 \sin(\alpha) \sin(\rho) - \cos(2\alpha) + 3} \end{cases} \quad (17)$$

In the limit $(\alpha + \frac{\pi}{2}) \rightarrow 0$ we obtain:

$$\cos^2(\kappa_{13}) = \frac{(1 + \sin(\rho)) (\frac{\pi}{2} + \alpha)^4}{16 (1 - \sin(\rho))} + O\left(\left(\frac{\pi}{2} + \alpha\right)^5\right) \quad (18)$$

$$\cos^2(\kappa_{23}) = 1 - \frac{(1 + \sin(\rho)) (\frac{\pi}{2} + \alpha)^4}{16 (1 - \sin(\rho))} + O\left(\left(\frac{\pi}{2} + \alpha\right)^5\right) \quad (19)$$

and for $(\frac{\pi}{2} - \alpha) \rightarrow 0$ we have:

$$\cos^2(\kappa_{13}) = 1 - \frac{(1 - \sin(\rho)) (\frac{\pi}{2} - \alpha)^4}{16 (1 + \sin(\rho))} + O\left(\left(\frac{\pi}{2} - \alpha\right)^5\right) \quad (20)$$

$$\cos^2(\kappa_{23}) = \frac{(1 - \sin(\rho)) (\frac{\pi}{2} - \alpha)^4}{16 (1 + \sin(\rho))} + O\left(\left(\frac{\pi}{2} - \alpha\right)^5\right) \quad (21)$$

The corresponding geometry of the states is shown in Fig. 1 and Fig. 2. In the limiting case when two of the states coalesce, and the third one is perpendicular to them both, we are able to eliminate one of three states by applying projective measurement.

Alternatively, the α parameter can be adjusted to achieve $\cos^2(\kappa_{13}) = \cos^2(\kappa_{23}) = \frac{1}{2}$ in Eqn.17 along $\phi = -\frac{\pi}{2}$ parallel:

$$\sin(\alpha) = \left\{ -\cot\left(\frac{\rho}{2}\right), -\tan\left(\frac{\rho}{2}\right) \right\} \quad (22)$$

depending on the value of ρ to ensure that $|\sin(\alpha)| \leq 1$. This reduces the three states to the mirror-symmetric ones as illustrated in Fig. 3 for which the minimum-error strategy is developed [16].

B. Step 3 $\mathcal{CPT}$ measurement

The same result can be achieved using the $\mathcal{CPT}$ measurement, since for an arbitrary α the states $|\psi_1\rangle$ and $|\psi_2\rangle$ are mutually orthogonal:

$$(\langle\psi_1|\psi_2\rangle)_{\mathcal{CPT}} = 0; (\langle\psi_{1,2}|)_{\mathcal{CPT}} = \frac{(1 \pm \sin(\alpha))}{\sqrt{2} \cos(\alpha)} (1, \mp i) \quad (23)$$

This allows to use the value of α to adjust the relative angles with the third state κ_{13} and κ_{23} :

$$\begin{cases} \cos^2(\kappa_{12}) = 0 \\ \cos^2(\kappa_{13}, \kappa_{23}) = \frac{(1 \pm \sin(\alpha))(1 \pm \sin(\rho))}{2(1 + \sin(\alpha) \sin(\rho))} \end{cases} \quad (24)$$

In the limit $(\frac{\pi}{2} + \alpha) \rightarrow 0$ corresponding to Fig. 1:

$$\cos^2(\kappa_{13}) = \frac{(\frac{\pi}{2} + \alpha)^2 (1 + \sin(\rho))}{4(1 - \sin(\rho))} + O\left(\left(\frac{\pi}{2} + \alpha\right)^3\right) \quad (25)$$

$$\cos^2(\kappa_{23}) = 1 - \frac{(\frac{\pi}{2} + \alpha)^2 (1 + \sin(\rho))}{4(1 - \sin(\rho))} + O\left(\left(\frac{\pi}{2} + \alpha\right)^3\right) \quad (26)$$

and for $(\frac{\pi}{2} - \alpha) \rightarrow 0$ represented in Fig. 2:

$$\cos^2(\kappa_{13}) = 1 - \frac{(\frac{\pi}{2} - \alpha)^2 (1 - \sin(\rho))}{4(1 + \sin(\rho))} + O\left(\left(\frac{\pi}{2} - \alpha\right)^3\right) \quad (27)$$

$$\cos^2(\kappa_{23}) = \frac{(\alpha - \frac{\pi}{2})^2 (1 - \sin(\rho))}{4(1 + \sin(\rho))} + O\left(\left(\alpha - \frac{\pi}{2}\right)^3\right) \quad (28)$$

Analogously to the Hermitian case, the $\mathcal{CPT}$ projection operators can be introduced which are the $\mathcal{CPT}$ observables:

$$\hat{P}_{1,2} = \left(\frac{|\psi_{1,2}\rangle\langle\psi_{1,2}|}{\langle\psi_{1,2}|\psi_{1,2}\rangle} \right)_{\mathcal{CPT}} = \frac{1}{2} \begin{pmatrix} 1 & \mp i \\ \pm i & 1 \end{pmatrix}, [\mathcal{CPT}, \hat{P}_{1,2}] = 0 \quad (29)$$

Similarly to the Hermitian case, by applying the measurement $\hat{\mathcal{M}} = \hat{P}_1 - \hat{P}_2$ on a sample, it is possible to eliminate one of three states as:

$$\begin{cases} \mathcal{M} = 1 \Rightarrow |\psi_1\rangle \\ \mathcal{M} = -1 \Rightarrow |\psi_2\rangle \text{ or } |\psi_3\rangle \end{cases} \quad \text{in the limit } \left(\frac{\pi}{2} + \alpha\right) \rightarrow 0, \quad \text{corresponding to Fig. 1} \quad (30)$$

$$\begin{cases} \mathcal{M} = 1 \Rightarrow |\psi_2\rangle \\ \mathcal{M} = -1 \Rightarrow |\psi_1\rangle \text{ or } |\psi_3\rangle \end{cases} \quad \text{in the limit } \left(\frac{\pi}{2} - \alpha\right) \rightarrow 0, \quad \text{corresponding to Fig. 2} \quad (31)$$

As a result, the state of the system is determined, or one of the states is excluded which allows to reduce the problem of two-state discrimination. Additionally, for $\alpha = -\rho$ it is possible to reduce our three states to mirror-symmetric ones as illustrated in Fig. 3.

In the next Section V, we implement the Hermitian approach using the dilation method and perform measurements on the IBM Quantum Experience. The $\mathcal{PT}$ -symmetric part of the total wave function is in the postselected space of ancilla being in $|0\rangle$. The norms of the states evolved under the $\mathcal{PT}$ -symmetric operations, $|\zeta_1\rangle$ and $|\zeta_2\rangle$, are proportional to $\sim (1 \pm \sin(\alpha))$, as a result, in the limit $(\frac{\pi}{2} \pm \alpha) \rightarrow 0$ the applied measurement on one of them becomes almost conclusive while on the other one almost inconclusive corresponding to higher or lower occurrence of $|0\rangle$ in the measurement applied on ancilla, correspondingly.

V. IMPLEMENTATION AND EXPERIMENTAL RESULTS

IBM Quantum Experience is a quantum processor operating on superconducting qubits which has become a leading candidate for scalable quantum computing platform, see a recent review [63]. These devices already enabled proof-of-concept results such as quantum error correction [64], fault-tolerant gates [65], experimental evidence of the violation of Mermin and Leggett-Garg inequalities [66, 67], non-local parity measurements [68, 69], simulations of paradigmatic models in open quantum systems [70], creation of highly entangled graph states [71], determining the ground-state energies of the molecules [72] as well as implementation of quantum witnesses [73]. Finally, $\mathcal{PT}$ -symmetric quantum mechanics provides an opportunity to increase entanglement by local operations [74], which is forbidden in the Hermitian case, and this property was experimentally demonstrated [75]. One of the main drawbacks of such systems, however, is the difficulty of improving the coherence time [76], and we will observe this problem later in the implementation of $\mathcal{PT}$ -symmetric operations. We provide the details of implementation of the both $\mathcal{PT}$ -symmetric evolutions in Methods.

For our numerical study, we take $\sigma = \frac{6}{5}$, and consider the third state in one of two forms, $|\psi_3\rangle$ and $|\psi'_3\rangle$:

$$|\psi_3\rangle = \begin{pmatrix} \cos\left(\frac{\pi+2\delta}{4}\right) \\ -i \sin\left(\frac{\pi+2\delta}{4}\right) \end{pmatrix}, |\psi'_3\rangle = \begin{pmatrix} \cos\left(\frac{\pi+2\Delta}{4}\right) \\ e^{i\phi} \sin\left(\frac{\pi+2\Delta}{4}\right) \end{pmatrix} \quad (32)$$

Figure 4. The experimental results for third state $|\psi_3\rangle$ being on the same meridian with the first two.

Figure 5. The experimental results for third state $|\psi'_3\rangle$ having $\Delta = \frac{\sigma}{2}$ depending on the parallel ϕ . An additional rotation is involved, and there is a sharper mismatch between theory and experiment due to depolarization effects.

The comparisons between theory and experimental results run on IBM Quantum Experience for the first part of the $\mathcal{PT}$ -symmetric evolution are presented in Figs. 4, 5, and 6. For the third state $|\psi_3\rangle$ along the same parallel with the first two $\phi = -\frac{\pi}{2}$ the results are represented in Fig. 4. When an additional rotation along ϕ is involved, as shown in Fig. 5 and 6, one may observe that the discrepancy between the theoretical and experimental results are more pronounced. This may be attributed to the presence of an additional gate implementing this operation and the depolarization effects inherent for real devices as described in [76]. The implementation of the $\mathcal{PT}$ -symmetric evolution comes at a price of introducing an inconclusive outcome, and some parts of the parameter space have different rates of conclusive results of the measurements. It can be varied by the input value in the

Figure 6. The experimental results for third state $|\psi'_3\rangle$ having $\Delta = \frac{\sigma}{4}$ depending on the parallel ϕ . An additional rotation is involved, and there is a sharper mismatch between theory and experiment due to depolarization effects.

Figure 7. *Step 3*, $\mathcal{PT}$ -symmetric elimination of one of the states. The result corresponds to numerical solution for $(\frac{\pi}{2} + \alpha) = 0.1$

differential equation $\mathcal{N}(0)$. Simultaneously, even in the case of an inconclusive outcome, one can obtain some information about the system under study since a part of the parameter space has a higher rate of an inconclusive outcomes.

The results of implementation of the *Step 3* and the second part of the $\mathcal{PT}$ -symmetric evolution for the value $(\frac{\pi}{2} + \alpha) = 0.1$ is represented in Fig. 7. One can observe that the states $|\zeta_2\rangle$ and $|\zeta_3\rangle$ effectively coalesce leading to $\cos^2(|\zeta_2\rangle, |\zeta_3\rangle) \approx 1$. Second, one can observe that the probability of decisive outcome is drastically different between different sides of the graph due to the presence of the factor $\sim (1 \pm \sin(\alpha))$. For example, for $\rho = \frac{\pi}{2}$ the probability of obtaining a decisive outcome is $\approx 98\%$ while for $\rho = -\frac{\pi}{2}$ it is $\approx 2\%$, and around $\sim 50\%$ in between these values. In practice, for the case $\rho < 0$ we need to take the limit $(\frac{\pi}{2} + \alpha) \rightarrow 0$

and vice versa $(\frac{\pi}{2} - \alpha) \rightarrow 0$ for $\rho > 0$. This results in a strong sensitivity to the $|\zeta_1\rangle$ state, which allows to eliminate it with high probability by applying the projection measurement $\mathcal{M} = P_1 - P_2$.

The features demonstrated in Fig. 7 for $(\frac{\pi}{2} + \alpha) = 0.1$ are summarized as follows:

	Postselected result, $ 00\rangle$ and $ 01\rangle$	Probability of decisive outcome, $\frac{N_{shots} - N_{ 10\rangle} - N_{ 11\rangle}}{N_{shots}}$
$ \zeta_1\rangle$	$ 00\rangle$, 100% probability	$\approx 98\%$
$ \zeta_2\rangle$	$ 01\rangle$, 100% probability	$\approx 2\%$
$ \zeta_3\rangle$	$ 01\rangle$, $> 98\%$ probability	$\sim 50\%$

Here $N_{shots} = 8192$ is a total number of shots on the IBM Quantum experience, and $N_{|ij\rangle}$, $i \in [0, 1]$ is a number of outcomes corresponding to $|ij\rangle$, such that $\sum_{i,j=1,2} N_{|ij\rangle} = N_{shots}$. The postselected result corresponds to $\frac{|00\rangle}{|00\rangle + |01\rangle}$ and $\frac{|01\rangle}{|00\rangle + |01\rangle}$, and probability of decisive outcome is $\frac{N_{shots} - N_{|10\rangle} - N_{|11\rangle}}{N_{shots}}$.

One can observe a sharp contrast between the three states which enables an elimination of one of them. For $|\zeta_1\rangle$, with the probability $\approx 98\%$, the decisive outcome $|00\rangle = +1$ will be obtained. In case of $|\zeta_2\rangle$, the number of conclusive outcomes will be low and produce the $|01\rangle = -1$ result. Finally, for $|\zeta_3\rangle$ the number of conclusive results will be $\sim 50\%$, and with $\approx 98\%$ probability $|01\rangle = -1$ will be obtained. This contrast increases as $\alpha \rightarrow \pm \frac{\pi}{2}$. This procedure can be repeated by swapping the states $1 \leftrightarrow 2 \leftrightarrow 3$ thus enabling to eliminate other states $|\zeta_2\rangle$ and $|\zeta_3\rangle$ in the same way.

VI. CONCLUSION AND DISCUSSIONS

$\mathcal{PT}$ -symmetric quantum mechanics has proven itself to be a powerful tool in multiple applications. In particular, it enables an enhancement of sensitivity and Quantum Fisher Information. Unlike previous research, we apply $\mathcal{PT}$ -symmetric methods for triple state discrimination which is a difficult problem apart from the highly symmetric geometries. A key element of our approach is to increase sensitivity to one of three states, which allows us to eliminate it. Alternatively, it enables the reduction of the triple states to the mirror-symmetric ones for which an efficient measurement strategy is already developed. Our approach enabling an elimination of one of the state may find applications in quantum cryptography and communications since the quantum state elimination has attracted much attention recently [77], and its applications for quantum communication and cryptography were suggested [78–80]. Additionally, our approach can find applications in the context of the three state quantum key distribution scheme [81] which possesses a unique property: the information cannot be extracted by the best eavesdropping strategy [82].

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DATA AND CODE AVAILABILITY

The code and results of the runs on the IBM Quantum Experience will be made public on GitHub upon publication.

METHODS

Adjusting to convenient positions

A set of arbitrary three states $|\psi_i\rangle = \begin{pmatrix} \cos(\frac{\theta_i}{2}) \\ e^{i\phi_i} \sin(\frac{\theta_i}{2}) \end{pmatrix}$, $i \in [1, 3]$ can be adjusted to the starting positions in Eqn. 9 by the following unitary rotation:

$$\mathcal{R} = \begin{pmatrix} \cos(\frac{\theta_1}{2}) & \sin(\frac{\theta_1}{2}) e^{-i\phi_1} \\ -\sin(\frac{\theta_1}{2}) e^{i\phi_1} & \cos(\frac{\theta_1}{2}) \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -ie^{-i\lambda-i\phi_2} \end{pmatrix} \cdot \begin{pmatrix} \cos(\frac{\pi-2\sigma}{4}) & -i \sin(\frac{\pi-2\sigma}{4}) \\ -i \sin(\frac{\pi-2\sigma}{4}) & \cos(\frac{\pi-2\sigma}{4}) \end{pmatrix} \quad (33)$$

$$\begin{aligned} \cos\left(\frac{\mu}{2}\right) &= |\beta| = \sqrt{(Re(\beta))^2 + (Im(\beta))^2}; \quad \nu \\ &= \arctan\left(\frac{Im(\gamma)}{Re(\gamma)}\right) - \arctan\left(\frac{Im(\beta)}{Re(\beta)}\right) \end{aligned} \quad (34)$$

with σ , and λ parameters given by the Eqns. 35, 36, 37, and 38 as:

$$\cos(\sigma) = \sqrt{\frac{1 + \cos(\theta_1) \cos(\theta_2) + \sin(\theta_1) \sin(\theta_2) \cos(\phi_1 - \phi_2)}{2}} \quad (35)$$

$$\begin{aligned} \lambda &= \arctan\left(\frac{\sin(\frac{\theta_1}{2}) \cos(\frac{\theta_2}{2}) \sin(\phi_2 - \phi_1)}{\cos(\frac{\theta_1}{2}) \sin(\frac{\theta_2}{2}) - \sin(\frac{\theta_1}{2}) \cos(\frac{\theta_2}{2}) \cos(\phi_2 - \phi_1)}\right) - \\ &\quad - \arctan\left(\frac{\sin(\frac{\theta_1}{2}) \sin(\frac{\theta_2}{2}) \sin(\phi_2 - \phi_1)}{\cos(\frac{\theta_1}{2}) \cos(\frac{\theta_2}{2}) + \sin(\frac{\theta_1}{2}) \sin(\frac{\theta_2}{2}) \cos(\phi_2 - \phi_1)}\right) \end{aligned} \quad (36)$$

$$\begin{aligned} \beta &= \cos\left(\frac{\theta_1}{2}\right) \cos\left(\frac{\theta_3}{2}\right) \cos\left(\frac{\pi-2\sigma}{4}\right) \left(1 + \tan\left(\frac{\theta_1}{2}\right) \tan\left(\frac{\pi-2\sigma}{4}\right) e^{i\phi_1-i\phi_2-i\lambda}\right) + \\ &+ \sin\left(\frac{\theta_1}{2}\right) \sin\left(\frac{\theta_3}{2}\right) \cos\left(\frac{\pi-2\sigma}{4}\right) e^{i\phi_3-i\phi_1} \left(1 - \cot\left(\frac{\theta_1}{2}\right) \tan\left(\frac{\pi-2\sigma}{4}\right) e^{i\phi_1-i\phi_2-i\lambda}\right) \end{aligned} \quad (37)$$

$$\begin{aligned} \gamma = & i \cos\left(\frac{\theta_1}{2}\right) \cos\left(\frac{\theta_3}{2}\right) \sin\left(\frac{\pi-2\sigma}{4}\right) \left(\tan\left(\frac{\theta_1}{2}\right) \cot\left(\frac{\pi-2\sigma}{4}\right) e^{i\phi_1-i\phi_2-i\lambda} - 1 \right) - \\ & -i \sin\left(\frac{\theta_1}{2}\right) \sin\left(\frac{\theta_3}{2}\right) \sin\left(\frac{\pi-2\sigma}{4}\right) e^{i\phi_3-i\phi_1} \left(1 + \cot\left(\frac{\theta_1}{2}\right) \cot\left(\frac{\pi-2\sigma}{4}\right) e^{i\phi_1-i\phi_2-i\lambda} \right) \end{aligned} \quad (38)$$

Unitary rotation, *Step 2*

The unitary rotation adjusting the states into the starting positions for the *Step 3* and the second $\mathcal{PT}$ -symmetric evolution has the following parameters:

$$\begin{aligned} \kappa = & \cos\left(\frac{\mu}{2}\right) \left(\cos(\omega\tau - \alpha) \cos\left(\frac{\delta}{2}\right) + \sin(\omega\tau) \sin\left(\frac{\delta}{2}\right) \right) \\ & + ie^{i\nu} \sin\left(\frac{\mu}{2}\right) \left(\cos(\omega\tau + \alpha) \sin\left(\frac{\delta}{2}\right) - \sin(\omega\tau) \cos\left(\frac{\delta}{2}\right) \right) \end{aligned} \quad (39)$$

$$\begin{aligned} \zeta = & i \cos\left(\frac{\mu}{2}\right) \left(\cos(\omega\tau - \alpha) \sin\left(\frac{\delta}{2}\right) - \sin(\omega\tau) \cos\left(\frac{\delta}{2}\right) \right) \\ & + e^{i\nu} \sin\left(\frac{\mu}{2}\right) \left(\cos(\omega\tau + \alpha) \cos\left(\frac{\delta}{2}\right) + \sin(\omega\tau) \sin\left(\frac{\delta}{2}\right) \right) \end{aligned} \quad (40)$$

$$\cos\left(\frac{\xi}{2}\right) = \frac{|\kappa|}{\sqrt{|\kappa|^2 + |\zeta|^2}}, \quad \chi = \arctan\left(\frac{\text{Im}(\zeta)}{\text{Re}(\zeta)}\right) - \arctan\left(\frac{\text{Im}(\kappa)}{\text{Re}(\kappa)}\right) \quad (41)$$

Implementation of $\mathcal{PT}$ symmetry by the dilation method

We implement the $\mathcal{PT}$ -symmetric Hamiltonian evolution on the IBM Quantum Experience by extending the original qubit with ancilla and employing the Neumark's theorem [59]. The combined qubit-ancilla system is governed by the following *Hermitian* Hamiltonian:

$$H_{a,q}^{Total}(t) = \hat{1} \otimes \Sigma(t) + \sigma_y \otimes \Upsilon(t) \quad (42)$$

and its elements are given by:

$$\Sigma(t) = \left[\mathcal{H}_q(t) + i \frac{d\zeta(t)}{dt} \zeta(t) + \zeta(t) \mathcal{H}_q(t) \zeta(t) \right] \mathcal{N}^{-1}(t) \quad (43)$$

$$\Upsilon(t) = i \left[\mathcal{H}_q(t) \zeta(t) - \zeta(t) \mathcal{H}_q(t) - i \frac{d\zeta(t)}{dt} \right] \mathcal{N}^{-1}(t) \quad (44)$$

$$\mathcal{N}(t) = T \exp \left[-i \int_0^t d\tau \mathcal{H}_q^\dagger(\tau) \right] \mathcal{N}(0) \tilde{T} \exp \left[i \int_0^t d\tau \mathcal{H}(\tau) \right] \quad (45)$$

where T and $\tilde{T}$ are the time and anti-time-ordering operators, respectively. The operator $\zeta(t) = (\mathcal{N}(t) - \hat{1})^{\frac{1}{2}}$ must maintain all its eigenvalues to be real, and the initial value $\mathcal{N}(0)$ must to be correspondingly chosen in order to ensure it. The following system of equations:

$$\begin{cases} \Sigma(t) - i\Upsilon(t)\eta(t) = \mathcal{H}_q(t) \\ \Sigma(t)\zeta(t) + i\Upsilon(t) = i\frac{d\zeta(t)}{dt} + \zeta(t)\mathcal{H}_q(t) \end{cases} \quad (46)$$

ensures that the driven qubit is evolved by the $\mathcal{PT}$ -symmetric Hamiltonian in Eqn. 2. The ancilla qubit must be initialized as $|\psi(0)\rangle_a = \frac{1}{\sqrt{\zeta(0)^2+1}}(|0\rangle_a + \zeta(0)|1\rangle_a)$. For both stages of the $\mathcal{PT}$ -symmetric evolution, 4×4 evolution matrix was obtained by numerical solution of differential equations by Mathematica [83]. Finally, the evolution matrices $U_{Evolution}$ were decomposed into the elementary gates U3 as defined by IBM. For the first part of the $\mathcal{PT}$ -symmetric evolution, they are denoted as U_j^i , and V_j^i for the second part, where $i, j \in [1, 4]$. This was done employing the method defined in [84, 85]. First, the rotation to the “magic basis” defined as:

$$\begin{cases} |\phi_1\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle); & |\phi_2\rangle = \frac{-i}{\sqrt{2}}(|00\rangle - |11\rangle) \\ |\phi_3\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle); & |\phi_4\rangle = \frac{-i}{\sqrt{2}}(|01\rangle + |10\rangle) \end{cases} \quad (47)$$

was performed. As a result, the evolution matrix was factorized as:

$$U_{Evolution} = (U_A \otimes U_B) \cdot U_D \cdot (V_A \otimes V_B) \quad (48)$$

$$U_D = e^{i\theta_0} MatrixExp\left(i \sum_{k=1}^3 \theta_k \sigma_k \otimes \sigma_k\right) = \sum_{k=1}^4 e^{i\Phi_k} |\phi_k\rangle \langle \phi_k| \quad (49)$$

The final transformation is carried out by M and Λ matrices:

$$M = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i & 0 & 0 \\ 0 & 0 & i & 1 \\ 0 & 0 & i & -1 \\ 1 & -i & 0 & 0 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 \end{pmatrix} \quad (50)$$

$$\theta = (\theta_0, \theta_1, \theta_2, \theta_3)^T; \quad \Phi = (\Phi_0, \Phi_1, \Phi_2, \Phi_3)^T; \quad \theta = \Lambda \cdot \Phi \quad (51)$$

The numerical results are as follows.

Figure 8. *Step 1* and the first stage of the $\mathcal{PT}$ -symmetric evolution.

First $\mathcal{PT}$ -symmetric evolution

$$U^{FirstStage} = \begin{pmatrix} 0.49542 - 0.320657i & -0.28768 - 0.44447i & -0.502231 + 0.325065i & 0.063182 + 0.097617i \\ -0.28768 - 0.44447i & 0.660279 - 0.427361i & 0.063182 + 0.097617i & 0.248411 - 0.160782i \\ 0.502231 - 0.325065i & -0.063182 - 0.097617i & 0.49542 - 0.320657i & -0.28768 - 0.44447i \\ -0.063182 - 0.097617i & -0.248411 + 0.160782i & -0.28768 - 0.44447i & 0.660279 - 0.427361i \end{pmatrix} \quad (52)$$

$$(\Phi_0, \Phi_1, \Phi_2, \Phi_3) = (-0.098992, -0.098992, -1.04988, -1.04988) \quad (53)$$

$$U_A = \begin{pmatrix} 0.720292 + 0.637273i & -0.181544 + 0.205194i \\ -0.205194 - 0.181544i & -0.637273 + 0.720292i \end{pmatrix} \quad (54)$$

$$U_B = \begin{pmatrix} -0.680335 - 0.444739i & 0.582539 + 0.i \\ 0.578199 + 0.0709821i & 0.729456 - 0.358527i \end{pmatrix} \quad (55)$$

$$V_A = \begin{pmatrix} 0.84926 - 0.232754i & -0.457048 + 0.125262i \\ -0.125262 - 0.457048i & -0.232754 - 0.84926i \end{pmatrix} \quad (56)$$

$$V_B = \begin{pmatrix} -0.522805 + 0.256958i & 0.812802 + 0.i \\ 0.414398 - 0.69923i & 0.487599 - 0.318747i \end{pmatrix} \quad (57)$$

Figure 9. *Steps 2 and 3.* Unitary rotation $\mathcal{W}$ puts the evolved state into convent positions, and the second $\mathcal{PT}$ -symmetric evolution eliminates one of three states or reduces them to the mirror-symmetric ones.

Second $\mathcal{PT}$ -symmetric evolution

$$U^{Second\ stage} = \begin{pmatrix} 0.454036i & -0.544835 & 0.521687i & 0.474186 \\ -0.544812 & -0.454078i & 0.474177 & -0.52168i \\ -0.521687i & -0.474186 & 0.454036i & -0.544835 \\ -0.474185 & 0.521686i & -0.544871 & -0.453998i \end{pmatrix} \quad (58)$$

$$V_A = \begin{pmatrix} 0.607617 & -0.2486i & 0.698146 & -0.285642i \\ -0.698146 + 0.285642i & 0.607615 & -0.248605i & \end{pmatrix} \quad (59)$$

$$V_B = \begin{pmatrix} -0.378676 + 0.925529i & & 0 & \\ & 0 & 0.378673 & -0.925531i \end{pmatrix} \quad (60)$$

$$U_A = U_B = \hat{1}; (\Phi_0, \Phi_1, \Phi_2, \Phi_3) = (0.807102, -0.807101, 0.807102, -0.807104) \quad (61)$$

The first part of the $\mathcal{PT}$ -symmetric evolution is shown in Fig. 8, and the unitary rotation with the second part of the $\mathcal{PT}$ -symmetric evolution in Fig. 9 respectively.

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