Evolution Behaviour of Breathers and Lump-N-Solitons ($N \rightarrow \infty$) for the (2+1)-Dimensional Benjamin-Bona-Mahony-Burgers Equation

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Research Article

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Evolution behaviour of breathers and lump- N -solitons ($N \rightarrow \infty$) for the (2+1)-dimensional Benjamin-Bona-Mahony-Burgers equation

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Abstract Several types of exact solution of the (2+1)-dimensional Benjamin-Bona-Mahony-Burgers (BBMB) equation will be reported in this research. On the basis of bilinear neural network method (BNNM), some specific activation functions are precisely taken into the single hidden layer neural network model, different forms of breather wave solutions, lump solutions and novel hybrid solutions are proposed. By means of symbolic computation and making use of Mathematica software, exact breather wave solutions, lump solutions and lump- N -soliton solutions are constructed completely. The obtained results enrich the knowledge of BBMB equation in the current studies. Various figures are plotted to exhibit the interesting evolution characteristic of theses waves vividly. The proposed methods effectively enable us to understand and investigate the nonlinear evolution equations in plasma physics, oceanography, optical fibers and fluid mechanics.

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1 Introduction

Nonlinear partial differential equations (NPDEs) involved both space and time variables, are fundamental models in nonlinear science, especially in nonlinear dynamics. The wave phenomenon of NLEEs are playing an essential role in studying various areas of physics and mathematics, such as plasmas physics, nonlinear optics, fluid mechanics, and Bose-Einstein condensates [1-6]. As typical wave models in nonlinear science, Solitons, breathers, lump and rogue waves have been investigated in [7,8,9]. When the energy concentration and local properties in all directions, and a specific expression of the wave is a rational function, this wave is called lump wave [10-15]. In the train of the study, a series of methods has been quickly developed to obtain lump wave including the first mathematical description of lump wave [16], an effective method named the long wave limit method was proposed to obtain the M -lump solutions of the integrable systems [17,18], many researchers have devoted to this positive method for

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research. the direct method [19-25], the parameter limit method [26-30], complexification method [31,32], Zhang et al proposed bilinear neural network method in [33-35]. Through these methods, so many valuable and interesting results have sprung up. Inspired by the previous studies, we mainly committed to the (2+1)-dimensional BBMB equation in order to study the evolution and degeneration behaviour of breathers and interaction solutions with N -solitons in this work. By applying the Horita bilinear method [36,37] and other systematic method, the structures of solutions for this equation are analyzed.

The (2+1)-dimensional BBMB equation is expressed on the following form

$$\begin{cases} u_t + u_{xxx} - 3(uv)_x = 0, \\ u_x = v_y. \end{cases} \quad (1)$$

where $u(x, y, t)$ and $v(x, y, t)$ indicate the fluid velocity component, uv is a C^2 -smooth nonlinear function. This model frequently occurs in the phenomena for propagation of bore and small-amplitude long waves in nonlinear dispersive media. Actually, how to define the propagation of small-amplitude long waves in a nonlinear dispersive media, the mechanisms leading to the degradation of the wave are quite complex and not understood well. So it is very necessary to study dissipative mechanisms to completely reflect real situations. BBMB equation has gained some currency when need to append dissipation to nonlinearity and dispersion arises in modeling unidirectional propagation of planar waves [38]. Many different forms of exact solutions of Eq.(1) are studied by different approaches. Numerical solutions of BBMB equation has been studied in [39-43]. Solitary and periodic solution obtained via exp-function method [44,45]. Collision phenomena consisting of lump, periodic and stripe soliton solutions were investigated in [46]. Through appropriate analysis, we assume

$$\begin{cases} u(x, y, t) = 2(\ln f)_{xy}, \\ v(x, y, t) = v_0 + 2(\ln f)_{xx}. \end{cases} \quad (2)$$

the real free parameter v_0 is a seed solution of Eq.(1), and f is an undetermined real-valued function associated with x, y, t . Eq.(1) is converted to the bilinear operator D as follows

$$(D_y D_t + D_x^3 D_y - 3v_0 D_x D_y) f \cdot f = 0 \quad (3)$$

The definition of Hirota bilinear operator $D_x^m D_t^n$ in [21] ($n, m \geq 0$) is

$$D_x^m D_t^n f(x, t) \cdot g(x, t) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^n f(x, t) g(x', t') \Big|_{x'=x, t'=t}.$$

However, as far as we know, there are many unknown properties and exact solutions of Eq.(1) deserve further study and discussion, such as evolution behavior of breather wave solutions, degradation process of breather wave solutions, and interaction phenomena between lump and N -soliton ($N \rightarrow \infty$) with different kinds. In this study, breather wave solutions are constructed firstly, as degeneration of breather

waves, lump waves emerge. Then the interaction solutions of Eq.(1) are expressed as lump- N -soliton solutions are given. The structure with following sections: In section 2, we will give the details of BNNM and the tensor formula corresponds to neural network model. In section 3, based on the single layer network model, breather waves with different forms and lump waves of Eq.(1) derived from breather waves by the limit method will be given. In section 4, Interaction solutions expressed by lump- N -soliton solutions will be obtained via the corresponding model. The dynamical characteristics of relevant waves will be shown by depicting different kinds of plots. In section 5. we will give some conclusions about this paper.

2 Bilinear neural network method

In order to find out the exact solutions of the BBMB equation, inspired by neural network algorithm, Fig.1 is displayed as a nonlinear neural network model and a neural network frame of solutions will be designed, taking the tensor formula as [33]:

$$f = W_{l_n, f} F_{l_n}(\xi_{l_n}), \xi_{l_j} = W_{l_{j-1}, l_j} F_{l_{j-1}}(\xi_{l_{j-1}}) + d_{l_j}, j = 1, 2, \dots, n. \quad (4)$$

where $W_{\lambda, \eta}$ is the weight coefficient of neuron λ to η . F is a generalized arbitrarily activation function, especially the function F must satisfy $F_{l_n}(\xi) \geq 0$ in the last layer. $l_0 = \{x, y, \dots, t\}$, $l_1 = \{1, 2, \dots, m_1\}$, $\dots$, $l_j = \{m_{j-1} + 1, m_{j-1} + 2, \dots, m_j\}$ ($j = 2, 3, \dots, n-1$), the constant d means a threshold. The main process of BNNM can be developed as:

Step 1: Taking the effective transformation in Eq.(2), Eq.(1) is converted to the bilinear equation Eq.(3).

Step 2: Inserting Eq.(4) into the bilinear equation Eq.(3), a useful equation can be obtained.

Step 3: Making the coefficients for each item equal to 0, such as $x, y, t, \dots, F^{(n)}(\xi), \dots$, we obtain a undetermined algebraic equations.

Step 4: Under help of Mathematica or Maple, according to solving the previous algebraic equations, solutions combining the coefficients of W with d could be obtained perfectly.

Step 5: Inserting the obtained solutions into the following equation

$$\begin{cases} u(x, y, t) = 2(\ln f)_{xy}, \\ v(x, y, t) = v_0 + 2(\ln f)_{xx}. \end{cases}$$

we will construct the exact solution of NPDEs.

Some classical methods which is proposed to construct solutions of NPDEs and BNNM could cover them. Classical methods can be considered as the case when there is only one hidden layer in neural network models. Specifically, the single-layer network model will cover the test functions commonly used in classical bilinear equations, when taking the proper functions into single-layer.

Fig. 1: Nonlinear neural network of (4): $l_0 = \{x, y, \dots, t\}$, $l_1 = \{1, 2, \dots, m_1\}$, $\dots$, $l_j = \{m_{j-1} + 1, m_{j-1} + 2, \dots, m_j\} (j = 3, \dots, n - 1)$.

3 Evolution and degeneration of breather wave in neural network model

To obtain the breather wave solutions of the bilinear Eq.(3), a diversity of testing functions are covered in the tensor formula (4) of neural network model (Fig.1), especially when making multiple hidden layers reduce to a single layer (see Fig.2). On the basis of [9,21] and the network model in Fig.1, when assigning $F_1(\xi_1) = \cosh \xi_1$, $F_2(\xi_2) = \cos \xi_2$, $F_3(\xi_3) = \cosh \xi_3$, $W_{1,f} = b_1, W_{2,f} = b_2, W_{3,f} = b_3$ in the single layer, Eq.4 is reduced to the three-wave breather-type soliton:

$$f(x, y, t) = b_1 \cosh \xi_1 + b_2 \cos \xi_2 + b_3 \cosh \xi_3 \quad (5)$$

where $\xi_i = k_i(a_i x + p_i y + q_i t + \gamma_i)$, undetermined parameters $k_i, a_i, b_i, p_i, q_i, \gamma_i (i = 1, 2, 3)$ are free and real. According to Fig.2, we can understand the structure of this test function intuitively.

Fig. 2: Three-wave method applied to neural network model.

Substituting Eq.(5) into Eq.(3) with computer algebra software Mathematic 12, the parameter relations are given as:

Case I:

$$\begin{cases} a_2 = \frac{a_3 p_3 k_3^2 (b_1^2 - b_3^2)}{p_2 b_2^2 k_2^2}, k_1 = \frac{a_3 k_3}{a_1}, p_1 = -\frac{a_1 p_3}{a_3}, q_1 = -a_1 (k_3^2 a_3^2 - 3v_0 - \frac{3a_3^2 p_3^2 k_3^2 (b_1^2 - b_3^2)^2}{p_2^2 b_2^4 k_2^2}), \\ q_2 = \frac{a_3 p_3 k_3^2 (b_1^2 - b_3^2)}{p_2 b_2^2 k_2^2} (3v_0 - 3k_3^2 a_3^2 + \frac{a_3^2 p_3^2 k_3^2 (b_1^2 - b_3^2)^2}{p_2^2 b_2^4 k_2^2}), q_3 = a_3 (3v_0 - k_3^2 a_3^2 + 3 \frac{a_3^2 p_3^2 k_3^2 (b_1^2 - b_3^2)^2}{p_2^2 b_2^4 k_2^2}). \end{cases} \quad (6)$$

A class of particular breather wave solution is constructed by putting Eq.(5) and Eq.(6) into Eq.(2) as follows

$$\begin{cases} u(x, y, t) = 2 \left(\frac{a_1 b_1 p_1 k_1^2 \cosh \xi_1 - a_2 b_2 p_2 k_2^2 \cos \xi_2 + a_3 b_3 p_3 k_3^2 \cosh \xi_3}{b_1 \cosh \xi_1 + b_2 \cos \xi_2 + b_3 \cosh \xi_3} - \frac{(a_1 b_1 k_1 \sinh \xi_1 - a_2 b_2 k_2 \sin \xi_2 + a_3 b_3 k_3 \sinh \xi_3)(b_1 p_1 k_1 \sinh \xi_1 - b_2 p_2 k_2 \sin \xi_2 + b_3 p_3 k_3 \sinh \xi_3)}{(b_1 \cosh \xi_1 + b_2 \cos \xi_2 + b_3 \cosh \xi_3)^2} \right) \\ v(x, y, t) = v_0 + 2 \left(\frac{a_1 b_1 p_1 k_1^2 \cosh \xi_1 - a_2 b_2 p_2 k_2^2 \cos \xi_2 + a_3 b_3 p_3 k_3^2 \cosh \xi_3}{b_1 \cosh \xi_1 + b_2 \cos \xi_2 + b_3 \cosh \xi_3} - \frac{(a_1 b_1 k_1 \sinh \xi_1 - a_2 b_2 k_2 \sin \xi_2 + a_3 b_3 k_3 \sinh \xi_3)^2}{(b_1 \cosh \xi_1 + b_2 \cos \xi_2 + b_3 \cosh \xi_3)^2} \right) \end{cases} \quad (7)$$

where

$$\begin{cases} \xi_1 = \frac{a_3 k_3}{a_1} (a_1 x - \frac{a_3 k_3}{a_1} y - a_1 (k_3^2 a_3^2 - 3v_0 - \frac{3a_3^2 p_3^2 k_3^2 (b_1^2 - b_3^2)^2}{p_2^2 b_2^4 k_2^2}) t + \gamma_1), \\ \xi_2 = k_2 \left(\frac{a_3 p_3 k_3^2 (b_1^2 - b_3^2)}{p_2 b_2^2 k_2^2} x + p_2 y - \frac{a_3 p_3 k_3^2 (b_1^2 - b_3^2)}{p_2 b_2^2 k_2^2} (3k_3^2 a_3^2 - 3v_0 - \frac{3a_3^2 p_3^2 k_3^2 (b_1^2 - b_3^2)^2}{p_2^2 b_2^4 k_2^2}) t + \gamma_2 \right), \\ \xi_3 = k_3 (a_3 x - p_3 y + a_3 (3v_0 - k_3^2 a_3^2 + \frac{3a_3^2 p_3^2 k_3^2 (b_1^2 - b_3^2)^2}{p_2^2 b_2^4 k_2^2}) t + \gamma_3). \end{cases} \quad (8)$$

So a pair of breather wave solution (u, v) is displayed in Eq.(7), the property of double solitary waves is shown with the two expressions ξ_1 and ξ_3 . At the same time, the property of periodic solitary waves is given with the expression ξ_2 . In addition, under the action of the function $\cos \xi_2$, (u, v) also has the outstanding feature of local oscillation. Based on these characteristics, the obtained breather wave solutions are named as double breather waves [47,48]. The deformation behaviour of double breathers are presented in Fig.3(a,b,c), when we choose the parameters as $a_1 = a_3 = 1, b_1 = b_3 = 1, b_2 = \frac{1}{2}, k_2 = k_3 = \frac{1}{4}, p_3 = -2, v_0 = -1, \gamma_1 = \gamma_2 = \gamma_3 = 0$.

Fig. 3: Spatial oscillation of double breathers in Eq.(7) as the time goes by: (a) $t = -5$, (b) $t = 0$, (c) $t = 5$. (d) Spatial oscillation of lump solution in Eq.(9) as $v_0 = \alpha = r = l = n = p_2 = 1, \beta = a_1 = a_3 = p_3 = m = 2, \gamma_i = 0 (i = 1, 2, 3), t = 0$.

Here, we will construct an extended lump solution from Eq.(7) by parameter limit method, the neural network model as

Fig. 4: From double breather wave to lump wave in neural network model.

so the parameters $b_i (i = 1, 2, 3)$ must satisfy:

$$\lim_{k_i \rightarrow 0} (b_1 + b_2 + b_3) = 0 \quad (9)$$

and we take $k_2 = rk_3$, $b_1 = \alpha \cosh(lk_3)$, $b_2 = -(\alpha + \beta) \cosh(mk_3)$, $b_3 = \beta \cosh(nk_3)$, where α, β are real numbers, m, n, l, r are real number or pure imaginary numbers. Letting $k_3 \rightarrow 0$, lump solution are given by

$$\begin{cases} u(x, y, t) = 4 \left(\frac{a_3 p_3 (\beta - \alpha) + (\alpha + \beta) p_2 r^2 \Delta}{\alpha \theta_1^2 + (\alpha + \beta) r^2 \theta_2^2 + \beta \theta_3^2 + (\alpha + \beta) m^2 - (\alpha l^2 + \beta n^2)} + \frac{2(p_3(\beta \theta_3 - \alpha \theta_1) + (\alpha + \beta) p_2 r^2 \theta_2)(a_3(\alpha \theta_1 + \beta \theta_3) + (\alpha + \beta) r^2 \Delta \theta_2)}{(\alpha \theta_1^2 + (\alpha + \beta) r^2 \theta_2^2 + \beta \theta_3^2 + (\alpha + \beta) m^2 - (\alpha l^2 + \beta n^2))^2} \right), \\ v(x, y, t) = v_0 + \frac{8(\alpha + \beta)(a_3^2 + r^2 \Delta^2)(\alpha(\theta_1^2 - l^2) + (\alpha + \beta)(m^2 + r^2 \theta_2^2) + \beta(\theta_3^2 - n^2)) - 8(\alpha a_3 \theta_1 + (\alpha + \beta) r^2 \Delta \theta_2 + \beta a_3 \theta_3)^2}{(\alpha \theta_1^2 + (\alpha + \beta) r^2 \theta_2^2 + \beta \theta_3^2 + (\alpha + \beta) m^2 - (\alpha l^2 + \beta n^2))^2} \end{cases} \quad (10)$$

where

$$\lim_{k_3 \rightarrow 0} a_2 = \frac{a_3 p_3 (\alpha - \beta)}{p_2 r^2 (\alpha + \beta)} \stackrel{\text{def}}{=} \Delta, \quad \lim_{k_i \rightarrow 0} \frac{\xi_i}{k_i} = \theta_i (i = 1, 2, 3).$$

$\theta_1 = a_3 x - p_3 y + 3v_0 t + \frac{a_3}{a_1} \gamma_1$, $\theta_2 = \Delta x - p_2 y + 3v_0 \Delta t + \gamma_2$, $\theta_3 = a_3 x + p_3 y + 3v_0 t + \gamma_3$. When $k_3 \rightarrow 0$, the periodic property disappears and the breather wave in Eq.(7) are degenerated as rational function form in Eq.(10). Besides, the solution (u, v) in Eq.(10) goes to 0 when taking $|x|, |y|$ tend to ∞ in any direction, and the feature of polynomial attenuation appear obviously on all spatio-temporal variables. We also notice that the solution is non-singular due to the parameters satisfy $(\alpha + \beta)m^2 - (\alpha l^2 + \beta n^2) > 0$ from the expression of Eq.(10). From Fig.3(d), the lump solution u includes two upward peaks and one downward valley, and v is constructed by one upward peak and two downward valleys, the lump solutions with these structures is called bright lump [14]. Therefore, the solution presented by Eq.(10) can be called lump solution [14,26] or rogue wave solution [7].

Case II:

$$b_3 = 0, k_1 = k_2, q_1 = 3a_1 v_0 + (3a_1 a_2^2 - a_1^3) k_2^2, q_2 = 3a_1 v_0 + (a_1^3 - 3a_1^2 a_2) k_2^2, p_1 = -\frac{a_2 p_2 b_2^2}{a_1 b_1^2} \quad (11)$$

so we get

$$f(x, y, t) = b_1 \cosh(k_1(a_1 x + p_1 y + q_1 t + \gamma_i)) + b_2 \cos(k_2(a_2 x + p_2 y + q_2 t + \gamma_2)) \quad (12)$$

According to Eq.(2) and Eq.(12), another specific breather wave solution for Eq.(1) is obtained. Fig.3(a) depicts the spatial structure of the solution. Then the neural network model in Fig.2 reduced to following two equivalent neural network model, as following

Fig. 5: From breather to lump in neural network model.

Similarly, we take $k_2 = rk_1$, $b_1 = \cos(lk_1)$, $b_2 = -\cos(mk_1)$, and let $k_1 \rightarrow 0$. New type lump solution is obtained (Fig.6(b)).

$$\begin{cases} u(x, y, t) = \frac{4(a_1 p_2 r^2 - \frac{m a_2^2 p_2}{l})(\theta_1^2 + \theta_2^2 + m^2 - l^2) - 8(a_1 \theta_1 + a_1 r \theta_2)(r p_2 \theta_2 - \frac{m a_2^2 p_2}{l} \theta_1)}{(\theta_1^2 + \theta_2^2 + m^2 - l^2)^2}, \\ v(x, y, t) = v_0 + \frac{4a_1^2(r^2 + 1)(\theta_1^2 + \theta_2^2 + m^2 - l^2) - 8a_1^2(\theta_1^2 + r \theta_2)^2}{(\theta_1^2 + \theta_2^2 + m^2 - l^2)^2}. \end{cases} \quad (13)$$

where $\theta_1 = a_1 x - \frac{m p_2}{l} y + 3a_1 v_0 t + \gamma_1$, $\theta_2 = r(a_1 x + p_2 y + 3a_1 v_0 t + \gamma_2)$, $l \neq 0$, and $m^2 - l^2 > 0$. When $l = 0$, lump solution as follows:

$$\begin{cases} u(x, y, t) = \frac{4a_1 p_2 r^2(\theta_1^2 + \theta_2^2 + m^2) - 8a_1 p_2 r(\theta_1 + r \theta_2)\theta_2}{(\theta_1^2 + \theta_2^2 + m^2)^2}, \\ v(x, y, t) = v_0 + \frac{4a_1 p_2 r^2(\theta_1^2 + \theta_2^2 + m^2) - 8a_1^2(\theta_1 + r \theta_2)^2}{(\theta_1^2 + \theta_2^2 + m^2)^2}. \end{cases}$$

where $\theta_1 = a_1 x + 3a_1 v_0 t + \gamma_1$, $\theta_2 = r(a_1 x + p_2 y + 3a_1 v_0 t + \gamma_2)$.

Fig. 6: Spatial degradation process: Periodic breather wave deforms to lump wave with the parameters (a) $a_1 = \frac{1}{2}$, $a_1 = 2$, $b_1 = 2$, $b_2 = 1$, $p_2 = 1$, $\gamma_1 = \gamma_2 = 0$; (b) $a_1 = 2$, $l = -1$, $m = 3$, $r = 1$, $p_2 = 1$, $\gamma_1 = \gamma_2 = 0$ at $t = 0$.

We extract a single layer from Fig.1, and take $F_1(\xi_1) = \xi_1^2$, $F_2(\xi_2) = \xi_2^2$, $F_3(\xi_3) = C$, $W_{j,f} = 1$ ($j = 1, 2, 3$), classical lump solutions model (Fig.7) will be derived from Eq.(4):

$$f = \xi_1^2 + \xi_2^2 + C$$

Fig. 7: Classical lump model.

Usually, setting some regular quadratic polynomial functions as the test functions in Fig.7, this way is widely applied to obtain lump solutions, for instance

$$f(x, y, t) = a_0 + (a_1x + b_1y + c_1t + \gamma_1)^2 + (a_2x + b_2y + c_2t + \gamma_2)^2 \quad (14)$$

where $a_0, a_i, b_i, c_i, \gamma_i$ ($i = 1, 2$) are undetermined real parameters. By inserting Eq.(14) into Eq.(3), the parameters satisfy the following relations:

Case I:

$$b_1 = 0, c_1 = 3v_0a_1, a_2 = 0, c_2 = 0.$$

Case II:

$$a_1 = -\frac{a_2b_2}{b_1}, c_1 = -3v_0\frac{a_2b_2}{b_1}, c_2 = 3v_0a_2.$$

Then, the specific expression of lump solution for Eq.(1) is given as

$$\begin{cases} u(x, y, t) = \frac{4(a_1b_1+a_2b_2)(a_0+\theta_1^2+\theta_2^2)-8(a_1\theta_1+a_2\theta_2)(b_1\theta_1+b_2\theta_2)}{(a_0+\theta_1^2+\theta_2^2)^2}, \\ v(x, y, t) = v_0 + \frac{4(a_1^2+a_2^2)(a_0+\theta_1^2+\theta_2^2)-8(a_1\theta_1+a_2\theta_2)^2}{(a_0+\theta_1^2+\theta_2^2)^2}. \end{cases}$$

where

$$\theta_1 = a_1x + b_1y + c_1t + \gamma_1, \theta_2 = a_2x + b_2y + c_2t + \gamma_2,$$

and $a_1, b_1, c_1, \gamma_1, a_2, b_2, c_2, \gamma_2$ satisfy the above relations Case I or Case II.

4 Evolution behavior of interaction solution in single hidden layer

We can get some interaction solutions such as lump- N -exponential soliton (Fig.9), lump- N -cosh soliton (Fig.10), and other types of solitons (Fig.11). We obtain interaction solutions by combining the specific activation function with several neurons in the single hidden layer neural network model (Fig.(8)). Various of 3-dimensional plots are drawn in order to present the local properties and dynamic behaviors of theses interaction phenomena.

Fig. 8: Interaction neural network model.

The expression ψ is an arbitrary function, different forms of interaction solutions will be displayed by taking specific activation function.

Fig. 9: Different interaction between lump and exponential-soliton.

Fig. 10: Different interaction between lump and cosh-soliton.

Fig. 11: Different interaction phenomenon including: lump and cos-soliton, lump and cos-exponential function soliton, lump and cosh-cos-cosh soliton.

4.1. Theoretical proof of superposition behaviour

In this subsection, the interaction phenomenon between lump solutions and N -solitons will be inves-

tigated, and a theorem is given and prove it completely. Now, we assume a test function with sum type as follows:

$$f(x, y, t) = a_0 + \sum_{i=1}^M (a_i x + b_i y + c_i t + \gamma_i)^2 + \sum_{j=1}^N \delta_j e^{(p_j x + q_j y + r_j t + s_j)} \stackrel{\text{def}}{=} \Phi(x, y, t) + \Psi(x, y, t). \quad (15)$$

where $a_0, a_i, b_i, c_i (i = 1, 2, 3, \dots, M)$ and $\delta_j, p_j, q_j, r_j, s_j (j = 1, 2, 3, \dots, N)$ are some free real parameters, $\Phi(x, y, t)$ represents a quadratic polynomial function, and $\Psi(x, y, t)$ is a multi-soliton function. Substituting Eq.(15) into Eq.(2), some new interaction solutions of (2+1)-dimensional BBMB equation can be obtained.

Theorem: Assume the function Φ is a solution of Eq.(3), and the parameters satisfy the relations $\sum_{i=1}^M a_i b_i = 0$, $\sum_{j=1}^N b_j c_j = 0$, $q_j = 0$ and $r_j = -(p_j^3 - 3v_0 p_j)$, then $f(x, y, t)$ is also a solution of Eq.(3).

Proof: By the Hirota bilinear operator D_- , we can get

$$P(D_x, D_y, D_t) f \cdot f = ((D_t + D_3^3 + 3v_0 D_x) D_y) (\Phi \cdot \Phi + \Phi \cdot \Psi + \Psi \cdot \Phi + \Psi \cdot \Psi). \quad (16)$$

As we know that $\Phi(x, y, t)$ is a solution of Eq.(3) and $\Psi_y = 0$, we have

$$P(D_x, D_y, D_t) (\Phi \cdot \Phi) = 0, P(D_x, D_y, D_t) (\Psi \cdot \Psi) = 0, \quad (17)$$

so that

$$\begin{aligned} P(D_x, D_y, D_t) f \cdot f &= 2((D_t + D_3^3 + 3v_0 D_x) D_y) (\Phi \cdot \Psi) = 2(D_t + D_3^3 + 3v_0 D_x) (\Phi_y \Psi - \Phi \Psi_y) \\ &= 4((\Phi_{yt} \Psi - \Phi_y \Psi_t) + (\Phi_{yxx} \Psi - 3\Phi_{yxx} \Psi_x + 3\Phi_{yx} \Psi_{xx} - \Phi_y \Psi_{xxx}) - 3v_0 (\Phi_{yx} \Psi - \Phi_y \Psi_x)) \\ &= 4(\Phi_{yt} \Psi + 3\Phi_{yx} (\Psi_{xx} + v_0 \Psi) - \Phi_y (\Psi_{xxx} + \Psi_t - 3v_0 \Psi_x)). \end{aligned} \quad (18)$$

According to the form of Φ and Ψ , some relationship of parameters as follows:

$$\Phi_{yx} = 2 \sum_{i=1}^M a_i b_i = 0, \Phi_{yt} = 2 \sum_{j=1}^M b_j c_j = 0, \quad (19)$$

and

$$\Psi_{xxx} + \Psi_t - 3v_0 \Phi_x = \sum_{j=1}^N \delta_j p_j^3 e^{\eta_j} + \sum_{j=1}^N \delta_j (-p_j^3 + 3v_0 p_j) e^{\eta_j} - 3v_0 \sum_{j=1}^N \delta_j p_j e^{\eta_j} = 0, \quad (20)$$

where $\eta_j = p_j x - (p_j^3 - 3v_0 p_j) t + s_j$. Substituting Eq.(19) and (20) into Eq.(18), the theorem is proved completely.

Corollary: Assume

$$g(x, y, t) = a_0 + \sum_{i=1}^M (a_i x + b_i y + c_i t + \gamma_i)^2 + \Phi(x, y, t). \quad (21)$$

The function $\Phi(x, y, t)$ satisfies $\Phi_y = 0$ and $\Phi_{xx} + \Phi_t - 3v_0 \Phi_x = 0$, and the parameters satisfy the relations $\sum_{i=1}^M a_i b_i = 0$, $\sum_{j=1}^N b_j c_j = 0$, that is to say, the function $g(x, y, t)$ satisfies Eq.(3).

4.2. Specific hybrid solutions

Some examples will be given to explain the effectiveness of the above theorem and corollary. Because of the change of soliton number N and time t , we will study the diversity of evolution behaviour displayed by spatio-temporal structure of lump- N -solitons.

Taking $M = 2, N = 0$ in Eq.(15), substituting Eq.(15) into Eq.(3), we obtain the following two solutions with the aid of Mathematica:

$$b_1 = 0, c_1 = 3v_0a_1, a_2 = 0, c_2 = 0. \quad (22)$$

and

$$a_1 = -\frac{a_2b_2}{b_1}, c_1 = -3v_0\frac{a_2b_2}{b_1}, c_2 = 3v_0a_2. \quad (23)$$

Firstly, we give the application of the theorem, and draw three-dimensional plots (Fig.12) to exhibit the evolution behaviour of lump- N -solitons with different soliton number N . Combining Eq.(2), (15) and Eq.(22), the interaction solutions are given by

$$\begin{cases} u(x, y, t) = -\frac{4b_2y(2a_1^2(x+3v_0a_1t)+\sum_{j=1}^N\delta_jp_je^{\eta_j})}{(a_0+a_1^2(x+3v_0t)^2+b_2^2y^2+\sum_{j=1}^N\delta_je^{\eta_j})^2} \\ v(x, y, t) = v_0 + 2\left(\frac{2a_1^2+\sum_{j=1}^N\delta_jp_j^2e^{\eta_j}}{a_0+a_1^2(x+3v_0t)^2+b_2^2y^2+\sum_{j=1}^N\delta_je^{\eta_j}} - \frac{(2a_1^2(x+3v_0a_1t)+\sum_{j=1}^N\delta_jp_je^{\eta_j})^2}{(a_0+a_1^2(x+3v_0t)^2+b_2^2y^2+\sum_{j=1}^N\delta_je^{\eta_j})^2}\right) \end{cases} \quad (24)$$

Fig.12 are plots which demonstrate the evolutionary behavior of interaction solutions with the parameter selections

$$\begin{cases} (a) : N = 0, a_0 = 6, a_1 = b_2 = \frac{1}{4}, p_1 = \frac{1}{8}, v_0 = 1, \gamma_1 = \gamma_2 = t = 0, \\ (b) : N = 1, a_0 = 4, a_1 = \frac{1}{10}, b_2 = \frac{1}{8}, p_1 = -\frac{1}{8}, v_0 = -1, \delta_1 = 1, \gamma_1 = s_1 = t = 0, \\ (c) : N = 2, a_0 = 4, a_1 = \frac{1}{10}, b_2 = \frac{1}{8}, p_1 = -\frac{1}{8}, p_2 = \frac{1}{8}, v_0 = -1, \delta_1 = 1, \delta_2 = 1, \gamma_1 = \gamma_2 = s_1 = s_2 = t = 0, \\ (d) : N = 3, a_0 = 4, a_1 = \frac{1}{10}, b_2 = \frac{1}{8}, p_1 = -\frac{1}{8}, p_2 = \frac{1}{8}, p_3 = -\frac{1}{4}, v_0 = -1, \delta_1 = 1, \delta_2 = 1, \delta_3 = 1, \gamma_1 = \gamma_2 = \gamma_3 = 0, \\ s_1 = s_2 = s_3 = t = 0. \end{cases}$$

From Fig.12, the oscillation behavior of interaction solutions change with the increasing of the number N , or as the increase of the t (Fig.13), we can observe that the lump solutions are swallowed up by soliton solutions as the increase of N . It means that, the degeneration phenomena of rational solutions occurs in the process of oscillation between lump and N -solitons.

Fig. 12: Spatial superposition behavior of interaction Eq.(19) as (a) $N = 0$, (b) $N = 1$, (c) $N = 2$, (d) $N = 3$.

Fig. 13: Fusion evolution behavior of Lump- N -soliton as $p_1 = -2$, (a) $t = -20$, (b) $t = -10$, (c) $t = 0$, (d) $t = 10$.

Remark 1: In Fig.12, we mainly discuss the evolution behaviour of lump and 1-soliton. In addition, substituting Eq.(24) into Eq.(15), the oscillation behavior is similar to Fig.12. Actually when $N > 1$, the evolution structures of later interaction solutions are similar to the previous one.

Subsequently, we investigate the application of the corollary. Actually, there is an infinite number of functions $\Phi(x, y, t)$, which satisfy $\Phi_y = 0$ and $\Phi_{xxx} + \Phi_t - 3v_0\Phi_x = 0$, we will give two examples as follows:

Example 1: Taking a new interaction phenomenon into consideration. On the basis of Eq.(23) and the corollary, we write a test function as

$$f(x, y, t) = (a_1x + c_1t + \gamma_1)^2 + (b_2y + c_2t + \gamma_2)^2 + a_0 + \sum_{l=1}^L \sigma_l \cosh(r_lx - (r_l^3 - 3v_0r_l)t + s_l). \quad (25)$$

Substituting Eq.(25) into Eq.(2), lump- L -cosh type hybrid solutions are obtained. Fig.14 are drawn to better demonstrate the spatial structure, when the number of cosh type L increases and the parameters are taken as $v_0 = -1$, $a_0 = 4$, $a_1 = 2$, $b_2 = \frac{1}{2}$, $\sigma_l = 1$, $s_1 = s_2 = s_3 = 0$, $r_1 = \frac{1}{8}$, $r_2 = \frac{1}{2}$, $r_3 = 1$, at $t = 0$.

Fig. 14: Spatial evolution behavior of Lump- L -solitons : (a)lump-cosh type, (b)lump-cosh-cosh type, (c)lump-cosh-cosh-cosh type.

Example 2: We also take different types of lump- N -solitons into consideration. Combining Eq.(24) with the corollary, another test function is given as follows

$$f(x, y, t) = a_0 + \sum_{i=1}^M (a_i x + b_i y + c_i t + \gamma_i)^2 + \sum_{j=1}^N \delta_j e^{(p_j x - (p_j^3 - 3v_0 p_j)t + s_j)} + \sum_{k=1}^K \varrho_k \cos(q_j x - (q_j^3 - 3v_0 q_j)t + s_k) + \sum_{l=1}^L \sigma_l \cosh(r_j x - (r_j^3 - 3v_0 r_j)t + s_l). \quad (26)$$

Several novel interaction solutions are obtained, the spatial structures of the lump and N -solitons with different types are given in Fig.15.

Fig. 15: Spatial evolution behavior of lump and N -solitons with different types.

Remark 2: In Fig.15, different types of interaction as follows:

(a) Interaction of lump-cos type as $v_0 = 1, a_0 = 4, a_2 = \frac{1}{10}, b_1 = \frac{1}{16}, b_2 = \frac{1}{8}, q_1 = -1, \varrho_1 = 1$ at time $t = 0$.

(b) Interaction of lump-cos-exponential type as $v_0 = 1, a_0 = 4, a_2 = \frac{1}{10}, b_1 = \frac{1}{16}, b_2 = \frac{1}{8}, p_1 = -\frac{1}{2}, q_1 = -2, \delta_1 = 1, \varrho_1 = 1$ at time $t = 0$.

(c) Interaction of lump-cosh-cos-cosh type as $v_0 = 1, a_0 = 4, a_2 = \frac{1}{10}, b_1 = \frac{1}{16}, b_2 = \frac{1}{8}, p_1 = -\frac{1}{2}, q_1 = -2, \delta_1 = 1, \sigma_l = 1, \sigma_l = \frac{1}{100}, \varrho_1 = 10$ at time $t = 0$.

5 Conclusion

On the basis on the bilinear form and neural network model, the researchers proposed a new approach called BNNM to find out the exact solution of NPDEs, which can construct a variety of test function flexibly and lay the foundation for obtaining various solutions, such as periodical breather, lump soliton, muliti-soliton solutions, etc. In this study, by applying BNNM, different forms of breather wave solutions and their the evolution and degradation behaviours are investigated, novel lump solutions are derived from breathers via limit method applied to the parameters. Besides, inspired by a lot of researches about interaction phenomenon, the neural network solutions of different models and the interaction solutions are obtained by choosing appropriate values and activation functions, we have discussed the different types of interaction solutions for (2+1)-dimensional BBMB equation, and demonstrated the theoretical proof firstly in this paper. The structures of the solutions can be intuitively seen through the relevant neural network model, and neural network model can fit the NPDEs well due to its unique nonlinearity. According to various plots, local mathematical and physical properties of breather waves, lump solutions and hybrid solutions are observed. It could be extended to study some meaningful nonlinear issues in the fields of optics, fluid dynamics, plasmas, oceanography, Bose-Einstein condensates, gas, mechanics and so on.

Declarations of Competing Interest

The authors declare that there are no conflicts of interests with publication of this paper.

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Data availability statement

The authors declare that data supporting the findings of this study are available within the article, the figures are concrete expression.

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