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Research Article

Keywords: Chaotic system, scaling matrix, disturbance, uncertain parameter, finite-time synchronization

Posted Date: February 23rd, 2022

DOI: <https://doi.org/10.21203/rs.3.rs-1378505/v1>

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Finite-time synchronization of different dimensional chaotic systems with uncertain parameters and external disturbances

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Abstract: This paper proposes a new control scheme using two scaling matrices that realizes the finite-time synchronization of different-dimensional chaotic systems with parameter uncertainties and external disturbances. Firstly, based on Lyapunov stability theorem and finite-time stability theorem, the definition of finite-time synchronization of chaotic systems with different dimensions is introduced. Secondly, in the case of external disturbance and parameter uncertainty, an adaptive feedback hybrid controller and parameter adaptive laws are designed to synchronize different dimensional uncertain chaotic systems in finite-time. Then, according to the characteristics of the unknown parameters of the system, a transformation matrix is constructed to meet the needs of chaotic systems with different dimensions, and a simplified synchronization control scheme is designed. Finally, two numerical experiments are carried out to verify the effectiveness of the proposed methods.

Keywords: Chaotic system • scaling matrix • disturbance • uncertain parameter • finite-time synchronization

1 Introduction

Chaotic systems are sensitive to initial values, and small changes in initial conditions will lead to large differences in the final dynamic behavior of chaotic systems ^[1]. Since Pecora and Carroll put forward the principle of chaos synchronization for the first time in 1990 ^[2], which has aroused the interest of many scholars and began to be widely researched, especially in secure communication and statistical prediction ^[3-6].

Finite-time synchronization is the application of the finite-time stability theorem to the synchronization research of chaotic system ^[7-8], according to the obtained error dynamic systems, to design a suitable controller so that the error system converges to zero in a finite time. Based on the finite-time stability theorem, a class of nonlinear feedback controllers is proposed in reference [9], which realizes the finite-time function projection synchronization in complex networks with time delay, and the criterion of convergence time is given. In reference [10], the finite-time synchronization of a class of hyperchaotic systems is researched based on the principle of feedback passivation control.

However, in the actual design process of the synchronization scheme, some uncertain factors of a chaotic system may have an impact on the synchronization of the system, and how to reduce the impact of these factors is particularly important. References [11] designed a nonlinear adaptive control synchronization method for two identical Lorenz systems with unknown parameters. References [12-13] respectively researched the finite-time synchronization of integer order or fractional order of same

dimensional chaotic systems with random disturbances and unknown parameters using the adaptive control method and Lyapunov stability theorem. These results show that the finite-time theorem is valuable in the synchronization control of uncertain chaotic systems and practical engineering.

With the discovery of synchronization between cardiopulmonary systems with different dimensions in the biological world [14], the synchronization of chaotic or hyperchaotic systems with different dimensions began to attract people's interest [15-17]. Reference [18] researched the generalized synchronization of chaotic systems with different dimensions based on the finite-time stability theorem. In reference [19], by reducing the dimension of a high-dimensional chaotic system, an adaptive controller is designed to realize the synchronization of two chaotic systems with different dimensions. References [20-21] researched the generalized finite-time synchronization of different dimensional chaotic systems with uncertain parameters using the finite-time stability theorem. In reference [22], taking external disturbances and unknown parameters as the overall uncertainty factors, the finite-time synchronization of uncertain chaotic or hyperchaotic systems with different dimensions is researched by increasing or reducing dimensions.

At present, many researchers have done much research on the finite-time synchronization control of two identical or unidentical chaotic systems, the above literature gives us great research motivation.

1. Based on the existing literature, according to the characteristics of the chaotic system, we try to design a new control scheme with faster convergence speed.
2. External disturbance exists objectively. In many literatures, disturbance is considered a constant value and is directly cancelled out in the controller. In this paper, we try to control the disturbance as a bounded parameter.
3. The complex behavior of chaotic system is closely related to its dimension. Synchronizing chaotic systems of different dimensions in a limited time puts forward higher requirements for controller design, and also brings great challenges to our work.
4. In communication, the design of different dimensions of driving system and response system can increase the confidentiality of information. Therefore, the synchronization of chaotic systems with different dimensions began to attract interest in communication encryption, which also greatly motivates our work.

Therefore, based on the above literature, this paper further researches the finite-time synchronization of chaotic systems with the above factors. The main work of this paper is to research the application of scaling matrices finite-time synchronization in different dimensional chaotic or hyperchaotic systems with external disturbances and uncertain parameters. For a class of chaotic systems with different dimensions, in the case of external disturbance and parameter uncertainty, based on Lyapunov stability theorem and finite-time stability theorem, the corresponding controller and parameter adaptive laws are designed by using scaling matrices so that the chaotic systems with different dimensions can achieve synchronization under the same dimension in finite time, and the expression of the time when the system reaches stability is given.

This paper is arranged as follows: In Section 2, the finite-time synchronization theory of different dimensional chaotic systems is introduced in detail. In Section 3, the synchronization schemes of different dimensional chaotic systems with external disturbances and uncertain parameters are introduced. In Section 4, we will choose two groups of examples to verify the validity of the proposed methods. Finally, the conclusions and some prospects are given in Section 5.

2 Preliminaries

Consider the drive system that can be described by

$$\dot{x} = f(x)\alpha + F(x) + h(t) \quad (1)$$

where $x \in \mathbf{R}^n$ is the state variable, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{\tau_1})^T$ are parameters, τ_1 is a constant, $f(x): \mathbf{R}^n \rightarrow \mathbf{R}^{n \times \tau_1}$ is the function matrix, $F(x): \mathbf{R}^n \rightarrow \mathbf{R}^n$ is the nonlinear function vector, $h(t) = (h_1(t), \dots, h_n(t))^T \in \mathbf{R}^{n \times 1}$ is the external disturbance function vector.

The response system is given as

$$\dot{y} = g(y)\beta + G(y) + H(t) + u \quad (2)$$

where $y \in \mathbf{R}^m$ is the state variable, $\beta = (\beta_1, \beta_2, \dots, \beta_{\tau_2})^T$ are parameters, τ_2 is a constant, $g(y): \mathbf{R}^m \rightarrow \mathbf{R}^{m \times \tau_2}$ is the function matrix, $G(y): \mathbf{R}^m \rightarrow \mathbf{R}^m$ is the nonlinear function vector, $H(t) = (H_1(t), \dots, H_m(t))^T \in \mathbf{R}^{m \times 1}$ is the external disturbance function vector, $u = (u_1, \dots, u_m)^T \in \mathbf{R}^m$ is the controller to be designed.

Definition 1 For the chaotic or hyperchaotic systems (1) and (2), if there exists a controller $u \in \mathbf{R}^m$ and given matrices $\Theta \in \mathbf{R}^{d \times m}$ and $\Phi \in \mathbf{R}^{d \times n}$ such that the synchronization error $e = \Theta y - \Phi x$, satisfies that

$$\begin{aligned} \lim_{t \rightarrow T} \|e(t)\| &= 0 \\ \|e(t)\| &\equiv 0, t > T \end{aligned} \quad (3)$$

where d is a certain constant, $\|\bullet\|$ is the vector norm. Then, the drive system (1) and the response system (2) are said to be finite-time synchronization under dimension d , where T is the synchronization time of the two chaotic systems.

Remark 1. For our work, we give the following definitions in advance.

(i) For matrix $A \in \mathbf{R}^{m \times n}$:

$A(i) (i=1, 2, \dots, m)$ is defined as the vector composed of the elements of the i -th row of a matrix A .

$A(ij) (i=1, 2, \dots, m, j=1, 2, \dots, n)$ is defined as the element of the i -th row and j -th column of a matrix A .

(ii) Define vector function $S(\kappa) = (\text{sign}(\kappa_1), \dots, \text{sign}(\kappa_m))^T$, $|\kappa|^\lambda = (|\kappa_1|^\lambda, \dots, |\kappa_m|^\lambda)^T$, λ is any real number, where $\kappa = (\kappa_1, \dots, \kappa_m)^T$.

(iii) Define vector matrix $D(\kappa) = \text{diag}(\kappa_1, \dots, \kappa_m)$, where $\kappa = (\kappa_1, \dots, \kappa_m)^T$.

Assumption 1. The constant d satisfies $0 < d \leq \max\{m, n\}$. We choose $d = m$ in this paper for convenience.

Assumption 2. The matrix Θ is row full rank matrix. Defined Θ^{-1} as the right inverse of a matrix Θ , i.e. $\Theta\Theta^{-1} = E$.

Assumption 3. The external disturbance functions $h_i(t) (i = 1, 2, \dots, n)$ and $H_j(t) (j = 1, 2, \dots, m)$ are bounded.

Proposition 1. Consider the following system

$$\dot{z}(t) = \xi(z(t)), z(0) = z_0 \quad (4)$$

where $\xi: D \rightarrow R$ is continuous. Suppose there exist a Lyapunov function $V: D \rightarrow R$, positive real numbers p, q and $\mu \in (0, 1)$, and an open neighborhood $U \subseteq D$ of the origin such that

$$\begin{aligned} V(z) &> 0 \\ \dot{V}(z) &\leq -2^{\frac{1+\mu}{2}} p V(z)^{\frac{1+\mu}{2}} - 2qV(z), \forall z \in U \setminus \{0\} \end{aligned} \quad (5)$$

then, the origin of system (4) is finite-time stable. The settling time depending on the initial state $z(0)$, and satisfies

$$T(z(0)) \leq \frac{\ln\left(1 + 2^{\frac{1-\mu}{2}} \frac{q}{p} V(z(0))^{1-\mu}\right)}{q(1-\mu)} \quad (6)$$

and T is continuous on U . Furthermore, if $D = R^n$, and $V > 0$, $\dot{V} < 0$ on $R^n \setminus \{0\}$, then the origin of system (4) is globally finite-time stable.

Proof. First, we give a scalar system

$$\dot{w}(t) = -2^{\frac{1+\mu}{2}} p \cdot [w(t)]^{\frac{1+\mu}{2}} - 2q \cdot w(t), w(0) = w_0 \quad (7)$$

where $[w]^{\frac{1+\mu}{2}} = |w|^{\frac{1+\mu}{2}} \cdot \text{sign}(w)$ is continuous and locally Lipschitz everywhere except at the origin.

Construct $\exp^{2qt} w(t)$, and derivation with respect to t gives

$$\begin{aligned} \frac{d(\exp^{2qt} w(t))}{dt} &= 2q \exp^{2qt} w(t) + \exp^{2qt} \left(-2^{\frac{1+\mu}{2}} p \cdot |w(t)|^{\frac{1+\mu}{2}} \cdot \text{sign}(w(t)) - 2q \cdot w(t) \right) \\ &= -p 2^{\frac{1+\mu}{2}} \exp^{2qt} \cdot |w(t)|^{\frac{1+\mu}{2}} \cdot \text{sign}(w(t)) \\ &= -p 2^{\frac{1+\mu}{2}} \cdot |w(t) \exp^{2qt}|^{\frac{1+\mu}{2}} \exp^{2qt \left(1 - \frac{1+\mu}{2}\right)} \cdot \text{sign}(w(t)) \end{aligned} \quad (8)$$

According to reference [8], we can get the solution $\gamma(t, w_0)$ of system (7) under the initial value w_0 are described by

$$\gamma(t, w_0) = \begin{cases} \text{sign}(w_0) \exp^{-2qt} \left[|w_0|^{\frac{1-\mu}{2}} + 2^{\frac{\mu-1}{2}} \frac{p}{q} (1 - \exp^{(1-\mu)qt}) \right]^{\frac{2}{1-\mu}}, & t < \frac{\ln \left(1 + 2^{\frac{1-\mu}{2}} \frac{q}{p} |w_0|^{\frac{1-\mu}{2}} \right)}{q(1-\mu)}, w_0 \neq 0 \\ 0, & t \geq \frac{\ln \left(1 + 2^{\frac{1-\mu}{2}} \frac{q}{p} |w_0|^{\frac{1-\mu}{2}} \right)}{q(1-\mu)} \\ 0, & t \geq 0, w_0 = 0 \end{cases} \quad (9)$$

Obviously, the solution $\gamma(t, w_0)$ converges to 0 in finite time. Therefore, we choose an appropriate

Lyapunov function satisfies $V(z) > 0$ and $\dot{V}(z) \leq -2^{\frac{1+\mu}{2}} p V(z)^{\frac{1+\mu}{2}} - 2qV(z), \forall z \in U \setminus \{0\}$. Then, it can be obtained that the time for the system (4) to reach the origin is finite, and the settling time

$$T(z(0)) \leq \frac{\ln \left(1 + 2^{\frac{1-\mu}{2}} \frac{q}{p} V(z(0))^{1-\mu} \right)}{q(1-\mu)}.$$

3 Main results

In this section, we will propose two adaptive feedback hybrid control methods for finite-time synchronization of different dimensional chaotic systems with uncertain parameters and external disturbances.

Because the external disturbance functions $h_i(t) (i=1,2,\dots,n)$ and $H_j(t) (j=1,2,\dots,m)$ are bounded,

there are constants $d_i (i=1,2,\dots,n)$ and $D_j (j=1,2,\dots,m)$ such that

$$|h_i(t)| \leq d_i \text{ and } |H_j(t)| \leq D_j \quad (10)$$

For

$$\begin{aligned} (\Theta H(t) - \Phi h(t))(i) &= (\Theta)(i) H(t) - (\Phi)(i) h(t) \\ &= \sum_{j=1}^m (\Theta)(ij) H_j(t) - \sum_{j=1}^n (\Phi)(ij) h_j(t) \end{aligned} \quad (11)$$

therefore

$$\begin{aligned} |(\Theta H(t) - \Phi h(t))(i)| &= \left| \sum_{j=1}^m (\Theta)(ij) H_j(t) - \sum_{j=1}^n (\Phi)(ij) h_j(t) \right| \\ &\leq \sum_{j=1}^m |(\Theta)(ij)| D_j + \sum_{j=1}^n |(\Phi)(ij)| d_j \end{aligned} \quad (12)$$

Denote

$$d_i^* = \left| \sum_{j=1}^m (\Theta)(ij) \right| D_j + \left| \sum_{j=1}^n (\Phi)(ij) \right| d_j \quad (13)$$

for $\Theta(ij), \Phi(ij)$ are constant, $h_i(t)$ and $H_j(t)$ are bounded, so d_i^* is also bounded.

To sum up, we can draw a conclusion

$$\left| (\Theta H(t) - \Phi h(t))(i) \right| \leq d_i^* \quad (14)$$

where d_i, D_j ($i = 1, 2, \dots, n, j = 1, 2, \dots, m$) are constant values.

The error system is given as follows

$$\begin{aligned} \dot{e} &= \Theta \dot{y} - \Phi \dot{x} \\ &= \Theta g(y) \beta - \Phi f(x) \alpha + \Theta G(y) - \Phi F(x) + \Theta H(t) - \Phi h(t) + \Theta u \end{aligned} \quad (15)$$

Denote

$$R_1 = \Theta G(y) - \Phi F(x) + \Theta g(y) \hat{\beta} - \Phi f(x) \hat{\alpha} + S(e) \hat{d} + q_1 \cdot e + p_1 \cdot D(|e|^{\mu_1}) S(e) \quad (16)$$

where $\Theta \in \mathbf{R}^{m \times m}, \Phi \in \mathbf{R}^{m \times n}$ are scaling constant matrices, $\hat{\alpha}, \hat{\beta}, \hat{d}$ are the adaptive estimations of parameters

α, β, d^* , respectively, $d^* = (d_1^*, d_2^*, \dots, d_m^*)$ is a constant vector, and $S(e) = (\text{sign}(e_1), \dots, \text{sign}(e_m))^T$,

$D(e) = \text{diag}(e_1, \dots, e_m)$, $D(|e|^{\mu_1}) = \text{diag}(|e_1|^{\mu_1}, \dots, |e_m|^{\mu_1})$, $D(|e|^{\mu_2}) = \text{diag}(|e_1|^{\mu_2}, \dots, |e_m|^{\mu_2})$, $0 < \mu_1 < 1$, p_1 and q_1 are control gain constants, all positive real numbers.

Then, based on adaptive control method [23], the controller in equation (2) is designed as follows

$$u = -\Theta^{-1} R_1 \quad (17)$$

The parameter adaptive laws designed as follows

$$\begin{aligned} \Delta \dot{d} &= -D(e) S(e) - p_1 \cdot D(|\Delta d|^{\mu_1}) S(\Delta d) - q_1 (\Delta d) \\ \Delta \dot{\alpha} &= (\Phi f(x))^T e - p_1 \cdot D(|\Delta \alpha|^{\mu_1}) S(\Delta \alpha) - q_1 (\Delta \alpha) \\ \Delta \dot{\beta} &= -(\Theta g(y))^T e - p_1 \cdot D(|\Delta \beta|^{\mu_1}) S(\Delta \beta) - q_1 (\Delta \beta) \end{aligned} \quad (18)$$

where $\Delta \alpha = \alpha - \hat{\alpha}$, $\Delta \beta = \beta - \hat{\beta}$, $\Delta d = d^* - \hat{d}$, and $S(\Delta d) = (\text{sign}(\Delta d_1), \dots, \text{sign}(\Delta d_m))^T$,

$S(\Delta \alpha) = (\text{sign}(\Delta \alpha_1), \dots, \text{sign}(\Delta \alpha_{r_1}))^T$, $S(\Delta \beta) = (\text{sign}(\Delta \beta_1), \dots, \text{sign}(\Delta \beta_{r_2}))^T$;

$D(|\Delta d|^{\mu_1}) = \text{diag}(|\Delta d_1|^{\mu_1}, \dots, |\Delta d_m|^{\mu_1})$, $D(|\Delta \alpha|^{\mu_1}) = \text{diag}(|\Delta \alpha_1|^{\mu_1}, \dots, |\Delta \alpha_{r_1}|^{\mu_1})$ and

$D(|\Delta \beta|^{\mu_1}) = \text{diag}(|\Delta \beta_1|^{\mu_1}, \dots, |\Delta \beta_{r_2}|^{\mu_1})$.

Theorem 1. For chaotic or hyperchaotic systems, if their expressions can be expressed as (1) and (2)

$\mathbf{x}(0), \mathbf{y}(0)$ and given scaling matrices Θ

and Φ , the systems can achieve finite-time synchronization by the controller (17) and the parameter

adaptive laws (18) with the settling time, given by $T_1 = \frac{\ln\left(1 + 2^{\frac{1-\mu_1}{2}} \frac{q_1}{p_1} V(0)^{\frac{1-\mu_1}{2}}\right)}{q_1(1-\mu_1)}$.

Proof. Substituting (17) into (15), we have

$$\dot{\mathbf{e}} = \Theta \mathbf{g}(\mathbf{y}) \Delta \boldsymbol{\beta} - \Phi \mathbf{f}(\mathbf{x}) \Delta \boldsymbol{\alpha} + \Theta \mathbf{H}(t) - \Phi \mathbf{h}(t) - S(\mathbf{e}) \hat{\mathbf{d}} - p_1 \cdot D(|\mathbf{e}|^{\mu_1}) S(\mathbf{e}) - q_1 \cdot \mathbf{e} \quad (19)$$

We introduce the following Lyapunov function

$$V = \frac{1}{2} \left(\mathbf{e}^T \mathbf{e} + (\Delta \boldsymbol{\beta})^T \Delta \boldsymbol{\beta} + (\Delta \boldsymbol{\alpha})^T \Delta \boldsymbol{\alpha} + (\Delta \mathbf{d})^T (\Delta \mathbf{d}) \right) \quad (20)$$

Take the derivative of $V(t)$ along equations (18) and (19)

$$\begin{aligned} \dot{V} &= \mathbf{e}^T \dot{\mathbf{e}} + (\Delta \boldsymbol{\beta})^T \Delta \dot{\boldsymbol{\beta}} + (\Delta \boldsymbol{\alpha})^T \Delta \dot{\boldsymbol{\alpha}} + (\Delta \mathbf{d})^T \Delta \dot{\mathbf{d}} \\ &= \mathbf{e}^T (\Theta \mathbf{H}(t) - \Phi \mathbf{h}(t)) - \mathbf{e}^T S(\mathbf{e}) \hat{\mathbf{d}} - p_1 \cdot \mathbf{e}^T D(|\mathbf{e}|^{\mu_1}) S(\mathbf{e}) - q_1 \cdot \mathbf{e}^T \mathbf{e} \\ &\quad - p_1 \cdot (\Delta \boldsymbol{\beta})^T D(|\Delta \boldsymbol{\beta}|^{\mu_1}) S(\Delta \boldsymbol{\beta}) - q_1 (\Delta \boldsymbol{\beta})^T (\Delta \boldsymbol{\beta}) - p_1 \cdot (\Delta \boldsymbol{\alpha})^T D(|\Delta \boldsymbol{\alpha}|^{\mu_1}) S(\Delta \boldsymbol{\alpha}) - q_1 (\Delta \boldsymbol{\alpha})^T (\Delta \boldsymbol{\alpha}) \\ &\quad + (\mathbf{d}^* - \hat{\mathbf{d}})^T \left(-D(\mathbf{e}) S(\mathbf{e}) - p_1 \cdot D(|\Delta \mathbf{d}|^{\mu_1}) S(\Delta \mathbf{d}) - q_1 (\Delta \mathbf{d}) \right) \\ &\leq \mathbf{e}^T \mathbf{d}^* - \mathbf{e}^T S(\mathbf{e}) \hat{\mathbf{d}} - p_1 \cdot \mathbf{e}^T D(|\mathbf{e}|^{\mu_1}) S(\mathbf{e}) - q_1 \cdot \mathbf{e}^T \mathbf{e} - p_1 \cdot (\Delta \boldsymbol{\beta})^T D(|\Delta \boldsymbol{\beta}|^{\mu_1}) S(\Delta \boldsymbol{\beta}) - q_1 (\Delta \boldsymbol{\beta})^T (\Delta \boldsymbol{\beta}) \\ &\quad - p_1 \cdot (\Delta \boldsymbol{\alpha})^T D(|\Delta \boldsymbol{\alpha}|^{\mu_1}) S(\Delta \boldsymbol{\alpha}) - q_1 (\Delta \boldsymbol{\alpha})^T (\Delta \boldsymbol{\alpha}) \\ &\quad + (\mathbf{d}^* - \hat{\mathbf{d}})^T \left(-D(\mathbf{e}) S(\mathbf{e}) - p_1 \cdot D(|\Delta \mathbf{d}|^{\mu_1}) S(\Delta \mathbf{d}) - q_1 (\Delta \mathbf{d}) \right) \\ &\leq -q_1 \cdot \mathbf{e}^T \mathbf{e} - q_1 (\Delta \boldsymbol{\beta})^T (\Delta \boldsymbol{\beta}) - q_1 (\Delta \boldsymbol{\alpha})^T (\Delta \boldsymbol{\alpha}) - q_1 (\Delta \mathbf{d})^T (\Delta \mathbf{d}) - p_1 \cdot \mathbf{e}^T D(|\mathbf{e}|^{\mu_1}) S(\mathbf{e}) \\ &\quad - p_1 \cdot (\Delta \boldsymbol{\beta})^T D(|\Delta \boldsymbol{\beta}|^{\mu_1}) S(\Delta \boldsymbol{\beta}) - p_1 \cdot (\Delta \boldsymbol{\alpha})^T D(|\Delta \boldsymbol{\alpha}|^{\mu_1}) S(\Delta \boldsymbol{\alpha}) - p_1 \cdot (\Delta \mathbf{d})^T D(|\Delta \mathbf{d}|^{\mu_1}) S(\Delta \mathbf{d}) \quad (21) \\ &< 0 \end{aligned}$$

Because $0 < \mu_1 < 1$, $\frac{1}{2} < \frac{1+\mu_1}{2} < 1$, there are

$$\begin{aligned} \dot{V} &= \mathbf{e}^T \dot{\mathbf{e}} + (\Delta \boldsymbol{\beta})^T \Delta \dot{\boldsymbol{\beta}} + (\Delta \boldsymbol{\alpha})^T \Delta \dot{\boldsymbol{\alpha}} + (\Delta \mathbf{d})^T \Delta \dot{\mathbf{d}} \\ &\leq -q_1 \cdot \mathbf{e}^T \mathbf{e} - q_1 (\Delta \boldsymbol{\beta})^T (\Delta \boldsymbol{\beta}) - q_1 (\Delta \boldsymbol{\alpha})^T (\Delta \boldsymbol{\alpha}) - q_1 (\Delta \mathbf{d})^T (\Delta \mathbf{d}) - p_1 \cdot \mathbf{e}^T D(|\mathbf{e}|^{\mu_1}) S(\mathbf{e}) \\ &\quad - p_1 \cdot (\Delta \boldsymbol{\beta})^T D(|\Delta \boldsymbol{\beta}|^{\mu_1}) S(\Delta \boldsymbol{\beta}) - p_1 \cdot (\Delta \boldsymbol{\alpha})^T D(|\Delta \boldsymbol{\alpha}|^{\mu_1}) S(\Delta \boldsymbol{\alpha}) - p_1 \cdot (\Delta \mathbf{d})^T D(|\Delta \mathbf{d}|^{\mu_1}) S(\Delta \mathbf{d}) \\ &= -p_1 \cdot \left(\mathbf{e}^T D(|\mathbf{e}|^{\mu_1}) S(\mathbf{e}) + (\Delta \boldsymbol{\beta})^T D(|\Delta \boldsymbol{\beta}|^{\mu_1}) S(\Delta \boldsymbol{\beta}) + (\Delta \boldsymbol{\alpha})^T D(|\Delta \boldsymbol{\alpha}|^{\mu_1}) S(\Delta \boldsymbol{\alpha}) + (\Delta \mathbf{d})^T D(|\Delta \mathbf{d}|^{\mu_1}) S(\Delta \mathbf{d}) \right) \\ &\quad - q_1 \left(\mathbf{e}^T \mathbf{e} + (\Delta \boldsymbol{\beta})^T (\Delta \boldsymbol{\beta}) + (\Delta \boldsymbol{\alpha})^T (\Delta \boldsymbol{\alpha}) + (\Delta \mathbf{d})^T (\Delta \mathbf{d}) \right) \end{aligned}$$

$$\begin{aligned}
&\leq -p_1 \cdot \left((e^T e)^{\frac{1+\mu_1}{2}} + ((\Delta\beta)^T (\Delta\beta))^{\frac{1+\mu_1}{2}} + ((\Delta\alpha)^T (\Delta\alpha))^{\frac{1+\mu_1}{2}} + ((\Delta d)^T (\Delta d))^{\frac{1+\mu_1}{2}} \right) \\
&\quad - q_1 \cdot \left(e^T e + (\Delta\beta)^T (\Delta\beta) + (\Delta\alpha)^T (\Delta\alpha) + (\Delta d)^T (\Delta d) \right) \\
&\leq -p_1 \cdot \left(e^T e + (\Delta\beta)^T (\Delta\beta) + (\Delta\alpha)^T (\Delta\alpha) + (\Delta d)^T (\Delta d) \right)^{\frac{1+\mu_1}{2}} \\
&\quad - q_1 \cdot \left(e^T e + (\Delta\beta)^T (\Delta\beta) + (\Delta\alpha)^T (\Delta\alpha) + (\Delta d)^T (\Delta d) \right) \\
&= -2^{\frac{1+\mu_1}{2}} p_1 \cdot V^{\frac{1+\mu_1}{2}} - 2q_1 \cdot V
\end{aligned} \tag{22}$$

From equations (20) and (21), it can be seen that $V > 0$ and $\dot{V} < 0$ near the origin, according to Lyapunov stability theorem [24] and Definition 1, the synchronization error system is asymptotically stable at the origin. Furthermore, from equation (22) and Proposition 1, it can be obtained that the error

$$\text{system (19) is stable for a finite time at the origin, and the stability time } T_1 = \frac{\ln \left(1 + 2^{\frac{1-\mu_1}{2}} \frac{q_1}{p_1} V(0)^{\frac{1-\mu_1}{2}} \right)}{q_1 (1 - \mu_1)},$$

that is, the driving system (1) and the response system (2) can achieve finite-time synchronization.

The proof is completed.

Remark 2. Let denote α_i or β_i be i -th unknown parameter of i -th equation in the system.

Assumption 4. System (1) and (2) are linear at unknown parameter α_i or β_i .

When systems (1) and (2) satisfy Assumption 4, then systems can be written as

$$\begin{aligned}
\dot{x} &= A_\alpha x + F(x) \\
\dot{y} &= B_\beta y + G(y) + u
\end{aligned} \tag{23}$$

where $A_\alpha = \text{diag}(\alpha_1, \dots, \alpha_n)$, $B_\beta = \text{diag}(\beta_1, \dots, \beta_m)$.

Denote

$$\tilde{e} = \Omega e \tag{24}$$

where $\Omega = (\Omega_{ij}) \in \mathbf{R}^{n \times m}$ is the transformation matrix, $\tilde{e} = (e_1, \dots, e_n)^T$, $e = (e_1, \dots, e_m)^T$. If $n \leq m$, then

$$\Omega_{ij} = \begin{cases} 1 & i = j \\ 0 & \text{others} \end{cases}; \text{ if } n > m, \text{ then } \Omega_{ij} = \begin{cases} 1 & i = j, i = 1, \dots, m \\ 1 & j = m, i = m + 1, \dots, n \\ 0 & \text{others} \end{cases}$$

The error system is obtained as

$$\begin{aligned}
\dot{e} &= \Theta \dot{y} - \Phi \dot{x} \\
&= \Theta B_\beta y - \Phi A_\alpha x + \Theta G(y) - \Phi F(x) + \Theta H(t) - \Phi h(t) + \Theta u
\end{aligned} \tag{25}$$

Denote

$$R_2 = \Theta (B_\beta y + G(y) - B_{\Delta\beta} e) + \Phi A_{\Delta\alpha} \tilde{e} - \Phi A_\alpha x - \Phi F(x) + S(e) \hat{d} + q_2 e + p_2 \cdot D(|e|^{\mu_2}) S(e) \tag{26}$$

where $A_{\Delta\alpha} = A_{\alpha} - A_{\dot{\alpha}} = \text{diag}(\Delta\alpha_1, \dots, \Delta\alpha_n)$, $B_{\Delta\beta} = B_{\beta} - B_{\dot{\beta}} = \text{diag}(\Delta\beta_1, \dots, \Delta\beta_m)$.

Then, to achieve systems synchronization, based on adaptive control principle, the controller in equation (23) is designed as follows

$$\mathbf{u} = -\Theta^{-1}R_2 \quad (27)$$

The parameter adaptive laws designed as follows

$$\begin{aligned} \Delta\dot{\mathbf{d}} &= -D(\mathbf{e})S(\mathbf{e}) - p_2 \cdot D(|\Delta\mathbf{d}|^{\mu_2})S(\Delta\mathbf{d}) - q_2(\Delta\mathbf{d}) \\ \Delta\dot{\boldsymbol{\alpha}} &= D(\tilde{\mathbf{e}})\Phi^T\tilde{\mathbf{e}} - p_2 \cdot D(|\Delta\boldsymbol{\alpha}|^{\mu_2})S(\Delta\boldsymbol{\alpha}) - q_2(\Delta\boldsymbol{\alpha}) \\ \Delta\dot{\boldsymbol{\beta}} &= -D(\mathbf{e})\Theta^T\mathbf{e} - p_2 \cdot D(|\Delta\boldsymbol{\beta}|^{\mu_2})S(\Delta\boldsymbol{\beta}) - q_2(\Delta\boldsymbol{\beta}) \end{aligned} \quad (28)$$

Theorem 2. For chaotic or hyperchaotic systems, if their expressions can be expressed as (23) and satisfy the assumptions (1) to (4), for any initial value $\mathbf{x}(0), \mathbf{y}(0)$ and given scaling matrices Θ and Φ , the systems can achieve finite-time synchronization by the controller (27) and the parameter adaptive

laws (28) with the settling time, given by $T_2 = \frac{\ln\left(1 + 2^{\frac{1-\mu_2}{2}} \frac{q_2 V(0)^{\frac{1-\mu_2}{2}}}{p_2}\right)}{q_2(1-\mu_2)}$.

Proof. Substituting (27) into (25), we have

$$\dot{\mathbf{e}} = \Theta B_{\Delta\beta}\mathbf{e} - \Phi A_{\Delta\alpha}\tilde{\mathbf{e}} + \Theta\mathbf{H}(t) - \Phi\mathbf{h}(t) - S(\mathbf{e})\hat{\mathbf{d}} - p_2 \cdot D(|\mathbf{e}|^{\mu_2})S(\mathbf{e}) - q_2\mathbf{e} \quad (29)$$

We introduce the following Lyapunov function

$$V = \frac{1}{2}(\mathbf{e}^T\mathbf{e} + (\Delta\boldsymbol{\beta})^T\Delta\boldsymbol{\beta} + (\Delta\boldsymbol{\alpha})^T\Delta\boldsymbol{\alpha} + (\Delta\mathbf{d})^T(\Delta\mathbf{d})) \quad (30)$$

Take the derivative of $V(t)$ along equations (28) and (29)

$$\begin{aligned} \dot{V} &= \mathbf{e}^T\dot{\mathbf{e}} + (\Delta\boldsymbol{\beta})^T\Delta\dot{\boldsymbol{\beta}} + (\Delta\boldsymbol{\alpha})^T\Delta\dot{\boldsymbol{\alpha}} + (\Delta\mathbf{d})^T\Delta\dot{\mathbf{d}} \\ &= \mathbf{e}^T(\mathbf{H}(t) - \mathbf{h}(t)) - S(\mathbf{e})\hat{\mathbf{d}} - p_2 \cdot \mathbf{e}^T D(|\mathbf{e}|^{\mu_2})S(\mathbf{e}) - q_2 \cdot \mathbf{e}^T\mathbf{e} \\ &\quad - p_2 \cdot (\Delta\boldsymbol{\beta})^T D(|\Delta\boldsymbol{\beta}|^{\mu_2})S(\Delta\boldsymbol{\beta}) - q_2(\Delta\boldsymbol{\beta})^T(\Delta\boldsymbol{\beta}) \\ &\quad - p_2 \cdot (\Delta\boldsymbol{\alpha})^T D(|\Delta\boldsymbol{\alpha}|^{\mu_2})S(\Delta\boldsymbol{\alpha}) - q_2(\Delta\boldsymbol{\alpha})^T(\Delta\boldsymbol{\alpha}) \\ &\quad + (\mathbf{d}^* - \hat{\mathbf{d}})^T \left(-D(\mathbf{e})S(\mathbf{e}) - p_2 \cdot D(|\Delta\mathbf{d}|^{\mu_2})S(\Delta\mathbf{d}) - q_2(\Delta\mathbf{d}) \right) \\ &\leq \mathbf{e}^T\mathbf{d}^* - \mathbf{e}^T S(\mathbf{e})\hat{\mathbf{d}} - p_2 \cdot \mathbf{e}^T D(|\mathbf{e}|^{\mu_2})S(\mathbf{e}) - q_2 \cdot \mathbf{e}^T\mathbf{e} \\ &\quad - p_2 \cdot (\Delta\boldsymbol{\beta})^T D(|\Delta\boldsymbol{\beta}|^{\mu_2})S(\Delta\boldsymbol{\beta}) - q_2(\Delta\boldsymbol{\beta})^T(\Delta\boldsymbol{\beta}) \\ &\quad - p_2 \cdot (\Delta\boldsymbol{\alpha})^T D(|\Delta\boldsymbol{\alpha}|^{\mu_2})S(\Delta\boldsymbol{\alpha}) - q_2(\Delta\boldsymbol{\alpha})^T(\Delta\boldsymbol{\alpha}) \\ &\quad + (\mathbf{d}^* - \hat{\mathbf{d}})^T \left(-D(\mathbf{e})S(\mathbf{e}) - p_2 \cdot D(|\Delta\mathbf{d}|^{\mu_2})S(\Delta\mathbf{d}) - q_2(\Delta\mathbf{d}) \right) \end{aligned}$$

$$\begin{aligned}
&\leq -q_2 \cdot e^T e - q_2 (\Delta \beta)^T (\Delta \beta) - q_2 (\Delta \alpha)^T (\Delta \alpha) - q_2 (\Delta d)^T (\Delta d) \\
&\quad - p_2 \cdot e^T D(|e|^{\mu_2}) S(e) - p_2 \cdot (\Delta \beta)^T D(|\Delta \beta|^{\mu_2}) S(\Delta \beta) \\
&\quad - p_2 \cdot (\Delta \alpha)^T D(|\Delta \alpha|^{\mu_2}) S(\Delta \alpha) - p_2 \cdot (\Delta d)^T D(|\Delta d|^{\mu_2}) S(\Delta d) \\
&< 0
\end{aligned} \tag{31}$$

Because $0 < \mu_2 < 1$, $\frac{1}{2} < \frac{1+\mu_2}{2} < 1$, there are

$$\begin{aligned}
\dot{V} &= e^T \dot{e} + (\Delta \beta)^T \Delta \dot{\beta} + (\Delta \alpha)^T \Delta \dot{\alpha} + (\Delta d)^T \Delta \dot{d} \\
&\leq -q_2 \cdot e^T e - q_2 (\Delta \beta)^T (\Delta \beta) - q_2 (\Delta \alpha)^T (\Delta \alpha) - q_2 (\Delta d)^T (\Delta d) - p_2 \cdot e^T D(|e|^{\mu_2}) S(e) \\
&\quad - p_2 \cdot (\Delta \beta)^T D(|\Delta \beta|^{\mu_2}) S(\Delta \beta) - p_2 \cdot (\Delta \alpha)^T D(|\Delta \alpha|^{\mu_2}) S(\Delta \alpha) \\
&\quad - p_2 \cdot (\Delta d)^T D(|\Delta d|^{\mu_2}) S(\Delta d) \\
&= -p_2 \cdot \left(e^T D(|e|^{\mu_2}) S(e) + (\Delta \beta)^T D(|\Delta \beta|^{\mu_2}) S(\Delta \beta) + (\Delta \alpha)^T D(|\Delta \alpha|^{\mu_2}) S(\Delta \alpha) + (\Delta d)^T D(|\Delta d|^{\mu_2}) S(\Delta d) \right) \\
&\quad - q_2 \cdot \left(e^T e + (\Delta \beta)^T (\Delta \beta) + (\Delta \alpha)^T (\Delta \alpha) + (\Delta d)^T (\Delta d) \right) \\
&\leq -p_2 \cdot \left((e^T e)^{\frac{1+\mu_2}{2}} + ((\Delta \beta)^T (\Delta \beta))^{\frac{1+\mu_2}{2}} + ((\Delta \alpha)^T (\Delta \alpha))^{\frac{1+\mu_2}{2}} + ((\Delta d)^T (\Delta d))^{\frac{1+\mu_2}{2}} \right) \\
&\quad - q_2 \cdot \left(e^T e + (\Delta \beta)^T (\Delta \beta) + (\Delta \alpha)^T (\Delta \alpha) + (\Delta d)^T (\Delta d) \right) \\
&\leq -p_2 \cdot \left(e^T e + (\Delta \beta)^T (\Delta \beta) + (\Delta \alpha)^T (\Delta \alpha) + (\Delta d)^T (\Delta d) \right)^{\frac{1+\mu_2}{2}} \\
&\quad - q_2 \cdot \left(e^T e + (\Delta \beta)^T (\Delta \beta) + (\Delta \alpha)^T (\Delta \alpha) + (\Delta d)^T (\Delta d) \right) \\
&= -2^{\frac{1+\mu_2}{2}} p_2 \cdot V^{\frac{1+\mu_2}{2}} - 2q_2 \cdot V
\end{aligned} \tag{32}$$

From equations (30) and (31), it can be seen that $V > 0$ and $\dot{V} < 0$ near the origin, according to Definition 1, the synchronization error system is asymptotically stable at the origin. Furthermore, from equation (32) and Proposition 1, it can be obtained that the error system (29) is stable for a finite

time at the origin, and the stability time $T_2 = \frac{\ln \left(1 + 2^{\frac{1-\mu_2}{2}} \frac{q_2}{p_2} V(0)^{\frac{1-\mu_2}{2}} \right)}{q_2(1-\mu_2)}$, that is, the driving system and the

response system (23) can achieve finite-time synchronization.

The proof is completed.

4 Numerical simulations

Example 1. The Cai system is described as follows:

$$\begin{cases} \dot{x}_1 = a_1(x_2 - x_1) \\ \dot{x}_2 = b_1 x_1 + c_1 x_2 - x_1 x_3 \\ \dot{x}_3 = x_1^2 - l_1 x_3 \end{cases} \tag{33}$$

and the Rossler system is described as follows:

$$\begin{cases} \dot{y}_1 = -y_2 - y_3 + u_1 \\ \dot{y}_2 = y_1 + a_2 y_2 + y_4 + u_2 \\ \dot{y}_3 = b_2 + y_1 y_3 + u_3 \\ \dot{y}_4 = -c_2 y_3 + l_2 y_4 + u_4 \end{cases} \quad (34)$$

where the parameters are selected as $a_1 = 20, b_1 = 14, c_1 = 10.6, l_1 = 2.8$ and $a_2 = 0.25, b_2 = 3, c_2 = 0.5, l_2 = 0.05$, the system (33) and (34) are chaotic and hyperchaotic, respectively [25,26]. Rewrite the system equations (33) and (34) with external disturbances and uncertain parameters into the following forms to synchronize them in four-dimensions.

$$\dot{\mathbf{x}} = f(\mathbf{x})\boldsymbol{\alpha} + F(\mathbf{x}) + \mathbf{h}(t) \quad (35)$$

$$\text{where } f(\mathbf{x}) = \begin{bmatrix} x_2 - x_1 & 0 & 0 & 0 \\ 0 & x_1 & x_2 & 0 \\ 0 & 0 & 0 & -x_3 \end{bmatrix}, \quad \boldsymbol{\alpha} = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ l_1 \end{bmatrix}, \quad F(\mathbf{x}) = \begin{bmatrix} 0 \\ -x_1 x_3 \\ x_1^2 \end{bmatrix}.$$

$$\dot{\mathbf{y}} = g(\mathbf{y})\boldsymbol{\beta} + G(\mathbf{y}) + \mathbf{H}(t) + \mathbf{u} \quad (36)$$

$$\text{where } g(\mathbf{y}) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ y_2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -y_3 & y_4 \end{bmatrix}, \quad \boldsymbol{\beta} = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \\ l_2 \end{bmatrix}, \quad G(\mathbf{y}) = \begin{bmatrix} -y_2 - y_3 \\ y_1 + y_4 \\ y_1 y_3 \\ 0 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}.$$

$$\text{We choose } \Theta = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \Phi = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}. \text{ From equation (18), the adaptive laws of parameters}$$

can be designed as follows

$$\begin{cases} \Delta \dot{a}_1 = (x_2 - x_1)(e_1 + e_4) - p_1 \cdot \text{sign}(\Delta a_1) |\Delta a_1|^{\mu_1} - q_1 \cdot (\Delta a_1) \\ \Delta \dot{b}_1 = x_1(e_1 + e_2 + e_3) - p_1 \cdot \text{sign}(\Delta b_1) |\Delta b_1|^{\mu_1} - q_1 \cdot (\Delta b_1) \\ \Delta \dot{c}_1 = x_2(e_1 + e_2 + e_3) - p_1 \cdot \text{sign}(\Delta c_1) |\Delta c_1|^{\mu_1} - q_1 \cdot (\Delta c_1) \\ \Delta \dot{l}_1 = -x_3(e_3 + e_4) - p_1 \cdot \text{sign}(\Delta l_1) |\Delta l_1|^{\mu_1} - q_1 \cdot (\Delta l_1) \\ \Delta \dot{a}_2 = -y_2(e_1 + e_2) - p_1 \cdot \text{sign}(\Delta a_2) |\Delta a_2|^{\mu_1} - q_1 \cdot (\Delta a_2) \\ \Delta \dot{b}_2 = -e_3 - p_1 \cdot \text{sign}(\Delta b_2) |\Delta b_2|^{\mu_1} - q_1 \cdot (\Delta b_2) \\ \Delta \dot{c}_2 = y_3 e_4 - p_1 \cdot \text{sign}(\Delta c_2) |\Delta c_2|^{\mu_1} - q_1 \cdot (\Delta c_2) \\ \Delta \dot{l}_2 = -y_4 e_4 - p_1 \cdot \text{sign}(\Delta l_2) |\Delta l_2|^{\mu_1} - q_1 \cdot (\Delta l_2) \\ \Delta \dot{d}_1 = -e_1 \text{sign}(e_1) - p_1 \cdot \text{sign}(\Delta d_1) |\Delta d_1|^{\mu_1} - q_1 \cdot (\Delta d_1) \\ \Delta \dot{d}_2 = -e_2 \text{sign}(e_2) - p_1 \cdot \text{sign}(\Delta d_2) |\Delta d_2|^{\mu_1} - q_1 \cdot (\Delta d_2) \\ \Delta \dot{d}_3 = -e_3 \text{sign}(e_3) - p_1 \cdot \text{sign}(\Delta d_3) |\Delta d_3|^{\mu_1} - q_1 \cdot (\Delta d_3) \\ \Delta \dot{d}_4 = -e_4 \text{sign}(e_4) - p_1 \cdot \text{sign}(\Delta d_4) |\Delta d_4|^{\mu_1} - q_1 \cdot (\Delta d_4) \end{cases} \quad (37)$$

From equation (17), the controller can be designed as follows

$$\begin{cases} u_1 = y_2 + y_3 + \hat{a}_1(x_2 - x_1) - \hat{d}_1 \text{sign}(e_1) - p_1 \cdot \text{sign}(e_1) |e_1|^{\mu_1} - q_1 e_1 + \hat{d}_2 \text{sign}(e_2) + p_1 \cdot \text{sign}(e_2) |e_2|^{\mu_1} + q_1 e_2 \\ u_2 = -y_1 - y_4 - \hat{a}_2 y_2 + \hat{b}_1 x_1 + \hat{c}_1 x_2 - x_1 x_3 - \hat{d}_2 \text{sign}(e_2) - p_1 \cdot \text{sign}(e_2) |e_2|^{\mu_1} - q_1 e_2 \\ u_3 = -y_1 y_3 - \hat{b}_2 + \hat{b}_1 x_1 + \hat{c}_1 x_2 - x_1 x_3 + x_1^2 - \hat{l}_1 x_3 - \hat{d}_3 \text{sign}(e_3) - p_1 \cdot \text{sign}(e_3) |e_3|^{\mu_1} - q_1 e_3 \\ u_4 = \hat{c}_2 y_3 - \hat{l}_2 y_4 + \hat{a}_1(x_2 - x_1) + x_1^2 - \hat{l}_1 x_3 - \hat{d}_4 \text{sign}(e_4) - p_1 \cdot \text{sign}(e_4) |e_4|^{\mu_1} - q_1 e_4 \end{cases} \quad (38)$$

In the controller (38), the obtained error system is as follows

$$\begin{cases} \dot{e}_1 = -\Delta a_1(x_2 - x_1) + \Delta a_2 y_2 - \Delta b_1 x_1 - \Delta c_1 x_2 - \tilde{d}_1 \text{sign}(e_1) - p_1 \cdot \text{sign}(e_1) |e_1|^{\mu_1} - q_1 e_1 \\ \dot{e}_2 = \Delta a_2 y_2 - \Delta b_1 x_1 - \Delta c_1 x_2 - \tilde{d}_2 \text{sign}(e_2) - p_1 \cdot \text{sign}(e_2) |e_2|^{\mu_1} - q_1 e_2 \\ \dot{e}_3 = \Delta b_2 - \Delta b_1 x_1 - \Delta c_1 x_2 + \Delta l_1 x_3 - \tilde{d}_3 \text{sign}(e_3) - p_1 \cdot \text{sign}(e_3) |e_3|^{\mu_1} - q_1 e_3 \\ \dot{e}_4 = -\Delta c_2 y_3 + \Delta l_2 y_4 - \Delta a_1(x_2 - x_1) + \Delta l_1 x_3 - \tilde{d}_4 \text{sign}(e_4) - p_1 \cdot \text{sign}(e_4) |e_4|^{\mu_1} - q_1 e_4 \end{cases} \quad (39)$$

The initial values are choose $(x_1(0), x_2(0), x_3(0)) = (4, -3, 4)$, $(y_1(0), y_2(0), y_3(0), y_4(0)) = (5, -6, 3, 3)$,

and the initial value of parameter estimations are $(\hat{a}_1(0), \hat{b}_1(0), \hat{c}_1(0), \hat{l}_1(0)) = (0.1, 0.1, 0.1)$,

$(\hat{a}_2(0), \hat{b}_2(0), \hat{c}_2(0), \hat{l}_2(0)) = (0.1, 0.1, 0.1, 0.1)$, $(\hat{d}_1(0), \hat{d}_2(0), \hat{d}_3(0), \hat{d}_4(0)) = (0.1, 0.1, 0.1, 0.1)$, two sets of

gain constants are selected $p_1 = 2, q_1 = 2$ and $p_1 = 8, q_1 = 8$, μ_1 is a constant of 0.5. The external

disturbance function are selected $\mathbf{h}(t) = (0.01 \sin(10t), -0.02 \cos(10t), 0.05 \sin(10t))^T$,

$\mathbf{H}(t) = (0.01 \sin(10t), -0.02 \cos(10t), 0.01 \cos(10t), 0.02 \sin(10t))^T$. Fig. 1(a) shows that the synchronization

error of the error system with the controller gradually tends to zero in $T_1 = 1.2809s$, and Fig. 2(a) shows

that the synchronization error of the error system with the controller gradually tends to zero in

$T_1 = 0.3202s$. The variation of parameter estimates of driving system and response system with time

is shown in Fig. 1(b) (c) and Fig. 2(b) (c), it can be seen that the parameter estimates also converge

to the value in a finite time.

As shown from Fig. 1 and Fig. 2, if the control gain constant is large, the convergence speed of synchronization will be faster, that is, the time for the system to realize synchronization is shorter. The value of the control gain constant only affects the synchronization speed. In consideration of space, the following example can take any set of experimental values.

Fig.1 Trajectories: (a) Errors; (b) The unknown parameter estimates of Cai system; (c) The unknown parameter estimates of Rossler system. ($p_1 = q_1 = 2$)

Fig.2 Trajectories: (a) Errors; (b) The unknown parameter estimates of Cai system; (c) The unknown parameter estimates of Rossler system. ($p_1 = q_1 = 8$)

Example 2. The financial system is described as follows:

$$\begin{cases} \dot{x}_1 = x_3 + (x_2 - r_1)x_1 \\ \dot{x}_2 = 1 - r_2x_2 - x_1^2 \\ \dot{x}_3 = -x_1 - r_3x_3 \end{cases} \quad (40)$$

and the Chen-Lee system is described as follows:

$$\begin{cases} \dot{y}_1 = \omega_1 y_1 - y_2 y_3 + u_1 \\ \dot{y}_2 = \omega_2 y_2 + y_1 y_3 + u_2 \\ \dot{y}_3 = \omega_3 y_3 + 0.2 y_4 + \frac{1}{3} y_1 y_2 + u_3 \\ \dot{y}_4 = 2.2 y_1 + 0.05 y_4 + 0.5 y_2 y_3 + u_4 \end{cases} \quad (41)$$

where the parameters are selected as $r_1 = 0.8, r_2 = 0.2, r_3 = 1.9$ and $\omega_1 = 5, \omega_2 = -10, \omega_3 = -3.8$, the system (40) and (41) are chaotic and hyperchaotic, respectively [27,28]. Rewrite the system equations (40) and (41) with external disturbances and uncertain parameters into the following forms to synchronize them in four-dimensions.

$$\dot{\mathbf{x}} = A_{\alpha} \mathbf{x} + F(\mathbf{x}) + h(t) \quad (42)$$

$$\text{where } A_{\alpha} = \begin{bmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{bmatrix}, \quad F(\mathbf{x}) = \begin{bmatrix} x_3 + x_1 x_2 \\ 1 - x_1^2 \\ -x_1 \end{bmatrix}.$$

$$\dot{\mathbf{y}} = B_{\beta} \mathbf{y} + G(\mathbf{y}) + H(t) + \mathbf{u} \quad (43)$$

$$\text{where } B_{\beta} = \begin{bmatrix} \omega_1 & 0 & 0 & 0 \\ 0 & \omega_2 & 0 & 0 \\ 0 & 0 & \omega_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad G(\mathbf{y}) = \begin{bmatrix} -y_2 y_3 \\ y_1 y_3 \\ 0.2 y_4 + \frac{1}{3} y_1 y_2 \\ 2.2 y_1 + 0.05 y_4 + 0.5 y_2 y_3 \end{bmatrix}.$$

We choose $\Theta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$, $\Phi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. From equation (28), the adaptive laws of parameters

can be designed as follows

$$\begin{cases} \Delta \dot{r}_1 = e_1^2 + e_1 e_4 - p_2 \cdot \text{sign}(\Delta r_1) |\Delta r_1|^{\mu_2} - q_2 \cdot (\Delta r_1) \\ \Delta \dot{r}_2 = e_2^2 + e_2 e_4 - p_2 \cdot \text{sign}(\Delta r_2) |\Delta r_2|^{\mu_2} - q_2 \cdot (\Delta r_2) \\ \Delta \dot{r}_3 = e_3^2 - p_2 \cdot \text{sign}(\Delta r_3) |\Delta r_3|^{\mu_2} - q_2 \cdot (\Delta r_3) \\ \Delta \dot{\omega}_1 = -e_1^2 - e_1 e_4 - p_2 \cdot \text{sign}(\Delta \omega_1) |\Delta \omega_1|^{\mu_2} - q_2 \cdot (\Delta \omega_1) \\ \Delta \dot{\omega}_2 = -e_2^2 - p_2 \cdot \text{sign}(\Delta \omega_2) |\Delta \omega_2|^{\mu_2} - q_2 \cdot (\Delta \omega_2) \\ \Delta \dot{\omega}_3 = -e_3^2 - p_2 \cdot \text{sign}(\Delta \omega_3) |\Delta \omega_3|^{\mu_2} - q_2 \cdot (\Delta \omega_3) \\ \Delta \dot{d}_1 = -e_1 \text{sign}(e_1) - p_2 \cdot \text{sign}(\Delta d_1) |\Delta d_1|^{\mu_2} - q_2 \cdot (\Delta d_1) \\ \Delta \dot{d}_2 = -e_2 \text{sign}(e_2) - p_2 \cdot \text{sign}(\Delta d_2) |\Delta d_2|^{\mu_2} - q_2 \cdot (\Delta d_2) \\ \Delta \dot{d}_3 = -e_3 \text{sign}(e_3) - p_2 \cdot \text{sign}(\Delta d_3) |\Delta d_3|^{\mu_2} - q_2 \cdot (\Delta d_3) \\ \Delta \dot{d}_4 = -e_4 \text{sign}(e_4) - p_2 \cdot \text{sign}(\Delta d_4) |\Delta d_4|^{\mu_2} - q_2 \cdot (\Delta d_4) \end{cases} \quad (44)$$

From equation (27), the controller can be designed as follows

$$\begin{cases} u_1 = y_2 y_3 + x_2 x_1 + x_3 - \omega_1 x_1 - r_1 y_1 - \hat{\omega}_1 e_1 + \hat{r}_1 e_1 - \hat{d}_1 \text{sign}(e_1) - p_2 \cdot \text{sign}(e_1) |e_1|^{\mu_2} - q_2 \cdot (e_1) \\ u_2 = -y_1 y_3 + 1 - x_1^2 - \omega_2 x_2 - r_2 y_2 - \hat{\omega}_2 e_2 + \hat{r}_2 e_2 - \hat{d}_2 \text{sign}(e_2) - p_2 \cdot \text{sign}(e_2) |e_2|^{\mu_2} - q_2 \cdot (e_2) \\ u_3 = -0.2 y_4 - \frac{1}{3} y_1 y_2 - x_1 - \omega_3 x_3 - r_3 y_3 - \hat{\omega}_3 e_3 + \hat{r}_3 e_3 - \hat{d}_3 \text{sign}(e_3) - p_2 \cdot \text{sign}(e_3) |e_3|^{\mu_2} - q_2 \cdot (e_3) \\ u_4 = -2.2 y_1 - 0.05 y_4 - 0.5 y_2 y_3 + 1 - x_1^2 - r_2 y_2 + \hat{r}_2 e_2 - \hat{d}_4 \text{sign}(e_4) - p_2 \cdot \text{sign}(e_4) |e_4|^{\mu_2} - q_2 \cdot (e_4) \end{cases} \quad (45)$$

In the controller (45), the obtained error system is as follows

$$\begin{cases} \dot{e}_1 = (\Delta \omega_1 - \Delta r_1) e_1 - \hat{d}_1 \text{sign}(e_1) - p_2 \cdot \text{sign}(e_1) |e_1|^{\mu_2} - q_2 \cdot (e_1) \\ \dot{e}_2 = (\Delta \omega_2 - \Delta r_2) e_2 - \hat{d}_2 \text{sign}(e_2) - p_2 \cdot \text{sign}(e_2) |e_2|^{\mu_2} - q_2 \cdot (e_2) \\ \dot{e}_3 = (\Delta \omega_3 - \Delta r_3) e_3 - \hat{d}_3 \text{sign}(e_3) - p_2 \cdot \text{sign}(e_3) |e_3|^{\mu_2} - q_2 \cdot (e_3) \\ \dot{e}_4 = (\Delta \omega_1 - \Delta r_1) e_1 - \Delta r_2 e_2 - \hat{d}_4 \text{sign}(e_4) - p_2 \cdot \text{sign}(e_4) |e_4|^{\mu_2} - q_2 \cdot (e_4) \end{cases} \quad (46)$$

The initial values are choose $(x_1(0), x_2(0), x_3(0)) = (2, 6, 4)$ $(y_1(0), y_2(0), y_3(0), y_4(0)) = (1, 2, 3, 2)$,

the initial value of parameter estimations are $(\hat{r}_1(0), \hat{r}_2(0), \hat{r}_3(0)) = (0.1, 0.1, 0.1)$,

$(\hat{\omega}_1(0), \hat{\omega}_2(0), \hat{\omega}_3(0)) = (0.1, 0.1, 0.1)$ $(\hat{d}_1(0), \hat{d}_2(0), \hat{d}_3(0), \hat{d}_4(0)) = (0.1, 0.1, 0.1, 0.1)$, gain constants are

selected $p_2 = 6, q_2 = 6$, μ_2 is a constant of 0.5. The external disturbance function are selected as

$\mathbf{h}(t) = (0.05 \sin(5t), -0.01 \sin(5t), 0.02 \cos(5t))^T$ $\mathbf{H}(t) = (0.01 \cos(5t), -0.02 \cos(5t), -0.01 \cos(5t), 0.02 \sin(5t))^T$.

Fig. 3(a) shows that the synchronization error of the error system with the controller gradually tends

to zero in $T_2 = 0.3988s$. The variation of parameter estimates of driving system and response system with time is shown in Fig. 3(b) (c), it can be seen that the parameter estimates also converge to the value in a finite time.

Fig. 3 Trajectories: (a) Errors; (b) The unknown parameter estimates of the financial system; (c) The unknown parameter estimates of Chen-Lee system. ($p_2 = q_2 = 6$)

5 Conclusions

This paper realized finite-time synchronization of different dimensional chaotic systems with external disturbances and uncertain parameters. Based on the characteristics of chaotic systems, several synchronization schemes are given using the scaling matrices. In all, numerical experiments have been employed to validate the proposed methods. Although some conclusions of finite-time synchronization of chaotic systems are obtained in this paper, the obtained control scheme is designed based on the adaptive control method. Our future work will continue to design more novel and convenient synchronization schemes according to the characteristics of chaotic systems and the method of sliding mode control.

Data Availability

The main results of our work are proved in detail, which can be seen in the context. No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was partly supported by the National Natural Science Foundation of China (Grant Nos.11901073).

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