Some results concerning block Toeplitz matrices

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SOME RESULTS CONCERNING BLOCK TOEPLITZ MATRICES

MUHAMMAD AHSAN KHAN

ABSTRACT. Given A, B, C , and D , block Toeplitz matrices, we will prove the necessary and sufficient condition for $AB - CD = 0$, and $AB - CD$ to be a block Toeplitz matrix. In addition, with respect to change of basis, the characterization of normal block Toeplitz matrices with entries from the algebra of diagonal matrices is also obtained.

1. INTRODUCTION

A scalar Toeplitz matrix is an $n \times n$ matrix with the following structure:

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_{-1} & a_0 & a_1 & \cdots & a_{n-2} \\ a_{-2} & a_{-1} & a_0 & \cdots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \\ a_{1-n} & a_{2-n} & a_{3-n} & \cdots & a_0 \end{pmatrix}.$$

The entries depend upon the difference $i - j$ and hence they are constant down all the diagonals. These matrices are ubiquitous and play an important role in different fields of mathematics, as well as applied areas as signal processing or time series analysis. The subject is many decades old; among monograph dedicated to the subject are [12, 16] and [10].

The related area of block Toeplitz matrices is less studied, one of the reason being the new difficulties that appear with respect to the scalar case. Besides its theoretical interests, the subject is also important to multivariate control theory. As references for block Toeplitz matrices one can use [1, 11, 13, 14, 15]. In [2], the authors have proved the variety of algebraic results about scalar Toeplitz matrices. They have formulated those algebraic results in terms of products of Toeplitz matrices. Given Toeplitz matrices A, B, C , and D , their main result

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determines if the matrix $AB - CD$ is Toeplitz. The necessary and sufficient condition is a rank two matrix equation involving tensor products of the vectors defining A, B, C , and D . They have also proved the necessary and sufficient condition for $AB - CD = 0$. In addition to that they have also completely characterized normal Toeplitz matrices. The characterization of normal Toeplitz matrices has been discussed in [3, 17, 4, 5, 6, 7, 18].

The purpose of the present paper is to generalize some of the results of [2] concerning the product of block Toeplitz matrices by way of introducing the special structure of the displacement matrix of a block Toeplitz matrix. Our main result concerning normal block Toeplitz matrices namely Theorem 4.3 is far from being exhaustive, but it will serve as a basis for further research.

The paper is organized as follows : Notations and some basic facts about displacement matrices are presented in section 2. Then as we will be interested in block Toeplitz matrices, some basic properties concerning the product of these matrices are derived in section 3. The last section is devoted to the study of normal Block Toeplitz matrices with entries from the algebra of diagonal matrices.

2. PRELIMINARIES

Let $\mathbb{C}$ denote the set of complex numbers. We designate by $\mathcal{M}_n$ the algebra of $n \times n$ matrices and by $\mathcal{D}_d$ the algebra of $d \times d$ diagonal matrices with entries from $\mathbb{C}$. Throughout in this paper, we prefer to label the indices from 0 to $n - 1$; so $A \in \mathcal{M}_n$ is written $A = (a_{i,j})_{i,j=0}^{n-1}$ with $a_{i,j} \in \mathbb{C}$. Then $\mathcal{T}_n \subset \mathcal{M}_n$ is the space of Toeplitz matrices $A = (a_{i-j})_{i,j=0}^{n-1}$. We will mostly be interested in block matrices, that is, matrices whose elements are not necessarily scalars, but elements in $\mathcal{M}_d$. Thus a block Toeplitz matrix is actually an $nd \times nd$ matrix, but which has been decomposed in n^2 blocks of dimension d , and these blocks are constant parallel to the main diagonal. we will use the following notations:

- $\mathcal{M}_n \otimes \mathcal{M}_d$ is the collection of $n \times n$ block matrices whose entries all belong to $\mathcal{M}_d$;
- $\mathcal{T}_n \otimes \mathcal{M}_d$ is the collection of $n \times n$ block Toeplitz matrices whose entries all belong to $\mathcal{M}_d$;
- $\mathcal{D}_n \otimes \mathcal{M}_d$ is the collection of $n \times n$ diagonal block Toeplitz matrices whose entries all belong to $\mathcal{M}_d$.
- $\mathcal{C}_1 \otimes \mathcal{M}_d$ is the collection of all $n \times 1$ block matrices whose entries all belong to $\mathcal{M}_d$;

- $\mathcal{R}_1 \otimes \mathcal{M}_d$ is the collection of all $1 \times n$ block matrices whose entries all belong to $\mathcal{M}_d$.

Obviously $\mathcal{D}_n \otimes \mathcal{M}_d \subset \mathcal{T}_n \otimes \mathcal{M}_d \subset \mathcal{M}_n \otimes \mathcal{M}_d$. For block diagonal matrices we will use the notation

$$\text{diag}(A_1 \ A_2 \ \cdots \ A_n) = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n \end{pmatrix}.$$

In many cases it will suffice to consider block Toeplitz matrices with zero diagonals. In other cases $\tilde{A} + \tilde{A}_0$ will describe the most general block Toeplitz matrix, where $\tilde{A}$ is a block Toeplitz matrix with 0 on the main diagonal and $\tilde{A}_0$ is the diagonal block Toeplitz matrix.

If $a = \begin{pmatrix} 0 \\ a_{-1} \\ \vdots \\ a_{1-n} \end{pmatrix}$, then we define $\hat{a} = \begin{pmatrix} 0 \\ a_{1-n} \\ \vdots \\ a_{-1} \end{pmatrix}$ and if $b = \begin{pmatrix} 0 \\ a_1 \\ \vdots \\ a_{n-1} \end{pmatrix}$, then

$$\bar{\hat{b}} = \begin{pmatrix} 0 \\ \overline{a_{n-1}} \\ \vdots \\ \overline{a_1} \end{pmatrix}.$$

Let $I \in \mathcal{M}_d$ be the identity matrix and $S \in \mathcal{M}_n \otimes \mathcal{M}_d$ consisting of zero matrices, except for identity matrices along the subdiagonal, i.e.,

$$S = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ I & 0 & 0 & \cdots & 0 & 0 \\ 0 & I & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I & 0 \end{pmatrix}.$$

For any $M \in \mathcal{M}_n \otimes \mathcal{M}_d$ the displacement matrix is defined as

$$\Delta(M) = M - SMS^*.$$

We will use this matrix to determine whether the difference of the matrix products is Toeplitz. See [9] and [19] for other kinds of displacement matrices. We denote the vector

$\begin{pmatrix} I \\ 0 \\ \vdots \\ 0 \end{pmatrix}^T \in \mathcal{R}_1 \otimes \mathcal{M}_d$ by P_+ ,

then its adjoint is the vector $P_+^* \in \mathcal{C}_1 \otimes \mathcal{M}_d$.

Lemma 2.1. *If $M \in \mathcal{M}_n \otimes \mathcal{M}_d$, then $M = \sum_{k=0}^{n-1} S^k(\Delta(M))S^{k*}$.*

Proof.

$$\begin{aligned} \sum_{k=0}^{n-1} S^k(\Delta(M))S^{k*} &= \sum_{k=0}^{n-1} S^k(M - SMS^*)S^{k*} \\ &= \sum_{k=0}^{n-1} (S^k MS^{k*} - S^{k+1} MS^{k+1*}) \\ &= M - S^n MS^{n*} = M \end{aligned}$$

□

Thus if want to show that $M = 0$, then it will sufficient to show that $\Delta(M) = 0$.

3. BLOCK TOEPLITZ PRODUCT

In this section, we will generalized some important results of [2]. The following Lemma describes the necessary and a sufficient condition for a block matrix A to be a block Toeplitz matrix.

Lemma 3.1. *$A \in \mathcal{M}_n \otimes \mathcal{M}_d$ is Toeplitz if and only if there exist $X, X' \in \mathcal{C}_1 \otimes \mathcal{M}_d$ such that, $\Delta(A) = XP_+ + P_+^* X'^*$.*

Proof. Suppose that $A = (A_{i-j})_{i,j=0}^{n-1} \in \mathcal{T}_n \otimes \mathcal{M}_d$. Since, the displacement matrix for A is defined as $\Delta(A) = A - SAS^*$. Then simple computation yields that

$$A = \begin{pmatrix} A_0 & A_1 & A_2 & \dots & A_{n-1} \\ A_{-1} & 0 & 0 & \dots & 0 \\ A_{-2} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ A_{-(n-1)} & 0 & 0 & \dots & 0 \end{pmatrix}$$

If we take $X = \begin{pmatrix} A_0 \\ A_{-1} \\ \vdots \\ A_{1-n} \end{pmatrix}$ and $X' = \begin{pmatrix} 0 \\ A_1 \\ \vdots \\ A_{n-1} \end{pmatrix}$, then one can easily check

that

$$\Delta(A) = XP_+ + P_+^* X'^*.$$

For the converse, let $A = (A_{i,j})_{i,j=0}^{n-1} \in \mathcal{M}_n \otimes \mathcal{M}_d$. Suppose then that

$X = \begin{pmatrix} X_0 \\ X_1 \\ \vdots \\ X_{n-1} \end{pmatrix}$ and $X' = \begin{pmatrix} X'_0 \\ X'_1 \\ \vdots \\ X'_{n-1} \end{pmatrix}$ be vectors in $\mathcal{C}_1 \otimes \mathcal{M}_d$, since we have

$$\Delta(A) = XP_+ + P_+^*X'^*$$

$\Rightarrow$

$$A = SAS^* + XP_+ + P_+^*X'^*$$

$\Rightarrow$

$$A = \begin{pmatrix} X_0 + X_0'^* & X_1'^* & X_2'^* & \cdots & X_{n-1}'^* \\ X_1 & A_{0,0} & A_{0,1} & \cdots & A_{0,n-2} \\ \vdots & \vdots & \vdots & \ddots & \\ X_{n-1} & A_{n-2,0} & A_{n-1,1} & \cdots & A_{n-2,n-2} \end{pmatrix}$$

Comparing corresponding entries yields $A_{i_1,j_1} = A_{i_2,j_2}$, whenever $i_1 - j_1 = i_2 - j_2$, where $0 \leq i_1, i_2, j_1, j_2 \leq n-1$, i.e., A is a block Toeplitz matrix. $\square$

In the rest of this section, if $A = (A_{i-j})_{i,j=0}^{n-1}$ is any block Toeplitz matrix then for simplification we write $A = \tilde{A} + \tilde{A}_0$, where $\tilde{A} \in \mathcal{T}_n \otimes \mathcal{M}_d$, with 0 on the main diagonal and $\tilde{A}_0 \in \mathcal{D}_n \otimes \mathcal{M}_d$.

Lemma 3.2. Let $\tilde{A}_- = \begin{pmatrix} 0 \\ A_{-1} \\ \vdots \\ A_{-(n-1)} \end{pmatrix}$, $\tilde{B}_+ = \begin{pmatrix} 0 \\ B_1 \\ \vdots \\ B_{n-1} \end{pmatrix}^T$, $\tilde{A}_+ = \begin{pmatrix} 0 \\ A_{n-1} \\ \vdots \\ A_1 \end{pmatrix}$

and $\tilde{B}_- = \begin{pmatrix} 0 \\ B_{1-n} \\ \vdots \\ B_{-1} \end{pmatrix}^T$ with entries from $\mathcal{M}_d$. If $A = \tilde{A} + \tilde{A}_0$ and

$B = \tilde{B} + \tilde{B}_0$, then there exist $Y, Y' \in \mathcal{C}_1 \otimes \mathcal{M}_d$, such that $\Delta(AB) = YP_+ + P_+^*Y'^* + \tilde{A}_-\tilde{B}_+ - \tilde{A}_+\tilde{B}_-$.

Proof. We have

$$\begin{aligned} \Delta(AB) &= \Delta[(\tilde{A} + \tilde{A}_0)(\tilde{B} + \tilde{B}_0)] \\ &= \Delta[\tilde{A}\tilde{B} + \tilde{A}\tilde{B}_0 + \tilde{A}_0\tilde{B} + \tilde{A}_0\tilde{B}_0] \\ &= \Delta(\tilde{A}\tilde{B}) + \Delta(\tilde{A}\tilde{B}_0) + \Delta(\tilde{A}_0\tilde{B}) + \Delta(\tilde{A}_0\tilde{B}_0) \end{aligned}$$

Since $\tilde{B}_0 \in \mathcal{D}_n \otimes \mathcal{M}_d$, then $\tilde{A}\tilde{B}_0 \in \mathcal{T}_n \otimes \mathcal{M}_d$, therefore by Lemma 3.1, there exist $U, U' \in \mathcal{C}_1 \otimes \mathcal{M}_d$ such that $\Delta(\tilde{A}\tilde{B}_0) = UP_+ + P_+^*U'^*$.

Similarly $\Delta(\tilde{A}_0\tilde{B}) = VP_+ + P_+^*V'^*$ and $\Delta(\tilde{A}_0\tilde{B}_0) = WP_+ + P_+^*W'^*$, with $V, V', W, W' \in \mathcal{C}_1 \otimes \mathcal{M}_d$.

$$(3.1) \implies \Delta(AB) = \Delta(\tilde{A}\tilde{B}) + (U + V + W)P_+ + (U'^* + V'^* + W'^*)P_+^*.$$

Now we have entry at the position (i, j) of $\tilde{A}\tilde{B}$ is

$$(3.2) \quad (\tilde{A}\tilde{B})_{i,j} = \sum_{k=0}^{n-1} A_{k-i}B_{j-k}, \quad 0 \leq i, j \leq n-1,$$

Using the formula (3.2) and definition of S , one obtains

$$(3.3) \quad (\tilde{A}\tilde{B} - S\tilde{A}\tilde{B}S^*)_{i,j} = \begin{cases} A_{-i}B_j - A_{n-i}B_{j-n} & \text{for } 1 \leq i, j \leq n-1, \\ (\tilde{A}\tilde{B})_{0,j} & \text{for } 1 \leq j \leq n-1, \\ (\tilde{A}\tilde{B})_{i,0} & \text{for } 0 \leq i \leq n-1. \end{cases}$$

Therefore

$$(3.4) \quad \Delta(\tilde{A}\tilde{B}) = XP_0 + P_0^*X'^* + \tilde{A}_-\tilde{B}_+ - \tilde{A}_+\tilde{B}_-,$$

$$\text{where } X = \begin{pmatrix} (\tilde{A}\tilde{B})_{0,0} \\ (\tilde{A}\tilde{B})_{1,0} \\ \vdots \\ (\tilde{A}\tilde{B})_{n-1,0} \end{pmatrix}, \quad X' = \begin{pmatrix} 0 \\ (\tilde{A}\tilde{B})_{0,0} \\ \vdots \\ (\tilde{A}\tilde{B})_{0,n-1} \end{pmatrix}.$$

(3.4) yields

$$\begin{aligned} \Delta(AB) &= (U + V + W + X)P_+ + (U'^* + V'^* + W'^* + X'^*)P_+^* + \tilde{A}_-\tilde{B}_+ - \tilde{A}_+\tilde{B}_- \\ &= YP_+ + P_+^*Y'^* + \tilde{A}_-\tilde{B}_+ - \tilde{A}_+\tilde{B}_-, \end{aligned}$$

where $Y = U + V + W + X_0$ and $Y'^* = U'^* + V'^* + W'^* + X_0'^*$. $\square$

The next result is the most important result of this paper.

Theorem 3.3. *Let $\tilde{A}_-, \tilde{A}_+, \tilde{C}_-, \tilde{C}_+ \in \mathcal{C}_1 \otimes \mathcal{M}_d$ and $\tilde{B}_+, \tilde{B}_-, \tilde{D}_+, \tilde{D}_- \in \mathcal{R}_1 \otimes \mathcal{M}_d$ with 0 in the zeroth component. If $A = \tilde{A} + \tilde{A}_0$, $B = \tilde{B} + \tilde{B}_0$, $C = \tilde{C} + \tilde{C}_0$ and $D = \tilde{D} + \tilde{D}_0$ then: $AB - CD \in \mathcal{T}_n \otimes \mathcal{M}_d$ if and only if $\tilde{A}_-\tilde{B}_+ - \tilde{A}_+\tilde{B}_- = \tilde{C}_-\tilde{D}_+ - \tilde{C}_+\tilde{D}_-$.*

Proof. By Lemma 3.2

$$\begin{aligned} \Delta(AB - CD) &= \Delta(AB) - \Delta(CD) \\ &= (Y - Z)P_+ + P_+^*(Y'^* - Z'^*) + \tilde{A}_-\tilde{B}_+ - \tilde{A}_+\tilde{B}_- - \tilde{C}_-\tilde{D}_+ + \tilde{C}_+\tilde{D}_- \end{aligned}$$

One can check that the last four terms on the right side of the above equation involve vectors with 0 in the zeroth component. By Lemma

3.1, $AB - CD$ is Toeplitz if and only if $\tilde{A}_- \tilde{B}_+ - \tilde{A}_+ \tilde{B}_- = \tilde{C}_- \tilde{D}_+ - \tilde{C}_+ \tilde{D}_-$.

□

Corollary 3.4. *Let $\tilde{A}_-, \tilde{A}_+ \in \mathcal{C}_1 \otimes \mathcal{M}_d$, and $\tilde{B}_+, \tilde{B}_- \in \mathcal{R}_1 \otimes \mathcal{M}_d$ with 0 in the zeroth component. If $A = \tilde{A} + \tilde{A}_0$ and $B = \tilde{B} + \tilde{B}_0$, then $AB \in \mathcal{T}_n \otimes \mathcal{M}_d$ if and only if $\tilde{A}_- \tilde{B}_+ - \tilde{A}_+ \tilde{B}_- = 0$.*

Proof. The proof follows immediately from Theorem 3.3 by taking $C = D = 0$. □

Theorem 3.5. *Let Y, Y', Z and Z' be vectors in $\mathcal{C}_1 \otimes \mathcal{M}_d$ with 0 on the zero component of X' and Z' . If $AB - CD \in \mathcal{T}_n \otimes \mathcal{M}_d$, then $AB = CD$ if and only if $Y = Z$ and $Y' = Z'$.*

Proof. If $AB - CD \in \mathcal{T}_n \otimes \mathcal{M}_d$, then by Lemma 3.1, there exist Y, Y', Z and Z' in $\mathcal{C}_1 \otimes \mathcal{M}_d$, such that

$$\Delta(AB - CD) = (Y - Z)P_+ + P_+(Y'^* - Z'^*).$$

By Lemma 2.1, $AB = CD$ if and only if $\Delta(AB - CD) = 0$. The latter equation holds if and only if the vectors which form the product with P_+ are 0. That is, $AB = CD$ if and only if $Y = Z$ and $Y' = Z'$. □

4. NORMAL BLOCK TOEPLITZ MATRICES

The general problem of characterizing normal block Toeplitz matrices with entries from $\mathcal{M}_d$ is a very harder problem and no work has been done hitherto. As a first step, we take entries of block Toeplitz matrices from $\mathcal{D}_d$, then we will obtain the characterization of normal block Toeplitz matrices with entries from $\mathcal{D}_d$. The next result from [2] characterized normal Toeplitz matrices among all Toeplitz matrices.

Theorem 4.1. *Let $A = (a_{i-j})_{i,j=0}^{n-1} \in \mathcal{T}_n$, then A is normal if and only if either $a = \lambda \hat{b}$ for some $|\lambda| = 1$ or $a = \lambda \tilde{b}$ for some $|\lambda| = 1$.*

Lemma 4.2. *Suppose $A = (A_{i-j})_{i,j=0}^{n-1} \in \mathcal{T}_n \otimes \mathcal{D}_d$. Then there is a change of basis that brings A into the following form*

$$A' = \text{diag} (A'_1 \quad A'_2 \quad \cdots \quad A'_d),$$

where for every $k = 1, 2, \dots, d$, $A'_k \in \mathcal{T}_d$.

The following result is the main result of this section. Although, it is a trivial implication of Lemma 4.2 but it will provide a path for future research concerning the characterization of normal block Toeplitz matrices.

Theorem 4.3. *If $A \in \mathcal{T}_n \otimes \mathcal{D}_d$, then A is normal if and only if there exist scalars $\lambda_1, \lambda_2, \dots, \lambda_d$ with $|\lambda_k| = 1$, such that for every $k = 1, 2, \dots, d$, either $a_k = \lambda_k \hat{b}$ or $a_k = \lambda_k \bar{\hat{b}}_k$.*

Proof. If $A \in \mathcal{T}_n \otimes \mathcal{D}_d$, then by Lemma 4.2, A has the form

$$A' = \text{diag} \begin{pmatrix} A'_1 & A'_2 & \cdots & A'_d \end{pmatrix},$$

where for every $k = 1, 2, \dots, d$, $A'_k = (a_{r-s,k})_{r,s=0}^d$. Now $AA^* = A^*A$ if and only if $A'_k A'^*_k = A'^*_k A'_k$. By Theorem 4.1, each A'_k is normal if and only if there exist scalars $\lambda_1, \lambda_2, \dots, \lambda_d$, with $|\lambda_k| = 1$, such that

either $a_k = \lambda_k \hat{b}$ or $a_k = \lambda_k \bar{\hat{b}}_k$, where $a_k = \begin{pmatrix} a_{-1,k} \\ a_{-2,k} \\ \vdots \\ a_{-d,k} \end{pmatrix}$, and $b_k = \begin{pmatrix} a_{1,k} \\ a_{2,k} \\ \vdots \\ a_{d,k} \end{pmatrix}$,
 $k = 1, 2, \dots, d$. □

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REFERENCES

- [1] A. Böttcher and B. Silbermann, Analysis of Toeplitz operators, Springer-Verlag, 1990.
- [2] C. Gu, L. Patton, Commutation relation for Toeplitz and Hankel matrices, SIAM J. Matrix Anal. Appl. 24 (2003), 728–746.
- [3] D. Farenick, M. Krupnik, N. Krupnik, W. Lee, Normal Toeplitz matrices, SIAM J. Matrix Anal. Appl., 17 (1996), 1037–1043.
- [4] D. Ikramov, On a description of normal Toeplitz matrices, Comput. Math. Math. Phys., 34 (1994), 399–404.
- [5] D. Ikramov, Classification of normal Toeplitz matrices with real elements, Math. Notes, 57 (1995), 463–469.
- [6] D. Ikramov, N. Chugunov, A criterion for the normality of a complex Toeplitz matrix, Comput. Math. Math. Phys., 36 (1996), pp. 131–137.
- [7] D. Ikramov, N. Chugunov, On the skew-symmetric part of Toeplitz matrices, Math. Notes, 63 (1998), 124–127.
- [8] D.L. Zimmerman: Block Toeplitz products of block Toeplitz matrices, *Linear and Multilinear Algebra* 25 (1989), 185–190.
- [9] G. Heinig, K. Rost, Algebraic Methods for Toeplitz-Like Matrices and Operators, Birkhauser Verlag, Basel, 1984.
- [10] H. Widom, “Toeplitz Matrices,” in Studies in Real and Complex Analysis, (J. I.I. Hirschmann, ed.), MAA Studies in Mathematics, Prentice-Hall, Englewood Cliffs, NJ, 1965.
- [11] I. Gohberg, S. Goldberg and M.A. Kaashoek, Classes of Linear Operators, Vol. 2, Birkhauser Verlag Basel, 1993.

- [12] I. S. Iohvidov, *Hankel and Toeplitz Matrices and Forms*, Birkhuser Boston, Cambridge, MA, 1982.
- [13] M. A. Khan, A Family of Maximal Algebras of Block Toeplitz Matrices, *An. St. Univ. Ovidius, Constanta Seria Matematica* 26 (3) (2018), 127-142.
- [14] M. A. Khan, D. Timotin, Algebras of block Toeplitz Matrices with commuting entries, *Linear and Multilinear Algebra* DOI10.1080/03081087.2019.1693955.
- [15] T. Shalom, On Algebras of Toeplitz matrices, *Linear Algebra and its applications* 96 (1987), 211-226.
- [16] U. Grenander and G. Szego, *Toeplitz Forms and Their Applications*. University of Calif. Press, Berkeley and Los Angeles, 1958.
- [17] V. Gel'fgat, A normality criterion for Toeplitz matrices, *Comput. Math. Math. Phys.*, 35 (1995), 1147-1150.
- [18] K. Ito, Every normal Toeplitz matrix is either of type I or of type II, *SIAM J. Matrix Anal. Appl.*, 17 (1996), 998-1006.
- [19] T. Kailath, H. Sayed, *Displacement structure: Theory and applications*, *SIAM Rev.*, 37 (1995), 297-386.

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