Geometrical Diagnostics For Modified Gravitational Theory With The Different Formalisms

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Geometrical diagnostics for modified gravitational theory with the different formalisms

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Abstract:

Geometrical diagnostic methods were often applied to distinguish the gravitational models. But it is scarce to investigate the differences between the different formalisms of modified gravitational theories (e.g. the metric formalism and the Palatini formalism). In this paper, we discriminate the gravitational theory with the different formalisms by using the geometrical diagnostic methods. For considered modified theory of gravity (e.g. the $f(R)$ theory or GBD theory), we can see that the difference between two formalisms are remarkable according to the diagnostic results. And relative to the Λ CDM model, there are the more deviation in metric formalism than those in Palatini formalism, according to the $\{r, s\}$ diagnostic. Given that the GBD (generalized Brans-Dicke theory) is a time-variable Newton gravitational constant (VG) theory, the differences between the VG theory and the constant- G theory are studied. It indicates that the variation of Newton gravitational constant could induce notable effects on geometrical quantities (e.g. r , s and q) in both the metric formalism and the Palatini formalism.

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Keywords: Time-variable gravitational constant; Metric formalism; Palatini formalism; Geometrical diagnostic.

I. Introduction

For exploring properties of gravity or solving questions in the theory of general relativity (GR) [1–6], lots of modified theories of GR have been constructed, e.g. the $f(R)$ theory [7, 8], the $f(G)$ theory [9, 10], the Brans-Dicke (BD) theory [11], and so on [12–28]. There is a prior assumption that we have to take in studying modified theories of gravity (MG), i.e. which one or ones of the dynamical variables should be chosen to describe the gravitational interaction? Correspondingly, the modified gravitational theories could be divided into two classifications: the metric formalism [8, 29] and the Palatini formalism [30, 31]. In the metric

formalism the Levi-Civita connection is related to metric, while in the Palatini formalism the metric and the connection are regarded as the independent dynamical variables. Usually, the different field equations are gained in the metric-formalism theories and the Palatini-formalism theories [8], respectively.

To find the "final" theory of describing the gravitational interaction, it is significant to explore the differences between the different modified theories (or the different formalisms). Then one can test them according to the observational results. Lots of gravitational models have been differentiated via the geometrical diagnostic methods [32–36]. But studying the distinctions between the different formalisms are scarce by using the so-called geometrical diagnostic. Obviously, which formalism should be chosen preferentially is an important issue, since it can decrease the uncertainty on the theoretical researching of gravity. In this paper, we probe the discrepancy between the different formalisms of modified theory, by selecting the $f(R)$ and the generalized Brans-Dicke (GBD) theories as examples. Also, we try to given an answer, i.e. under the observational limits on the Newton gravitational constant G , whether the change of G could lead to the remarkable geometrical effect in the generalized Brans-Dicke theory (a theory with the time-variable Newton gravitational constant). Due to the dynamics of BD field and the coupling between the Brans-Dicke field and the gravitational geometry, we find that the difference of some geometrical quantity (e.g. r , s or q) between the variable- G theory and the constant- G theory could be obvious for both the metric formalism and the Palatini formalism.

The constructions of this paper are as follows. In section II, we introduce the basic equations for $f(R)$ modified gravitational theory with the different formalisms, and apply the geometrical diagnostics to distinguish the different formalisms. Section III investigate the geometrical diagnostics for the different formalisms of GBD theory. Especially, we study the influences of variable G on the geometrical quantities in this theory. Section IV is the conclusions.

II. The geometrical diagnostics for $f(R)$ theory with the different

formalisms

A. Basic equations for $f(R)$ modified theory in both the metric formalism and the Palatini formalism

In this section, we show the field equations of $f(R)$ theory in the metric formalism and the Palatini formalism, respectively. $f(R)$ modified gravitational theory is a simple and popular extension relative to GR. In the metric formalism, the action of $f(R)$ theory is denoted by

$$S_{met} = S_g(g_{\mu\nu}) + S_m(g_{\mu\nu}, \Psi) = \int d^4x \sqrt{-g} \left[\frac{1}{16\pi G} f(R) + L_m \right], \quad (1)$$

where g denotes the determinant of metric $g_{\mu\nu}$, R is the Ricci scalar, $f(R)$ is an arbitrary function of R , G denotes the Newton gravitational constant, L_m denotes the Lagrangian density of matter, respectively. Using the variation principle, one gets the modified field equation of gravity

$$f_R R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} - \left[\nabla_\mu \nabla_\nu - g_{\mu\nu} \square \right] f_R = 8\pi G T_{\mu\nu}, \quad (2)$$

Here $f_R = \frac{\partial f(R)}{\partial R}$, and $\square \equiv \nabla^\mu \nabla_\mu$. $R_{\mu\nu}$ and $T_{\mu\nu}$ denote the Ricci tensor and the energy-momentum tensor of matter, respectively. The trace of Eq.(2) is

$$f_R(R)R - 2f(R) + 3\square f_R = 8\pi G T. \quad (3)$$

In the Palatini formalism, the action of $f(R)$ theory read as,

$$S_{Pala} = S_g(g_{\mu\nu}, \tilde{\Gamma}_{\mu\nu}^\lambda) + S_m(g_{\mu\nu}, \psi) = \int d^4x \sqrt{-g} \left[\frac{1}{16\pi G} f(\tilde{R}) + L_m \right]. \quad (4)$$

Here the metric $g_{\mu\nu}$ and the connection $\tilde{\Gamma}_{\mu\nu}^\lambda$ are regarded as the independent dynamical variables. $\tilde{R} = g^{\mu\nu} \tilde{R}_{\mu\nu}$ and the Ricci tensor $\tilde{R}_{\mu\nu}$ is defined by the independent Palatini connection

$$\tilde{R}_{\mu\nu} = \partial_\lambda \Gamma_{\mu\nu}^\lambda - \partial_\nu \Gamma_{\mu\lambda}^\lambda + \Gamma_{\lambda\sigma}^\lambda \Gamma_{\mu\nu}^\sigma - \Gamma_{\mu\sigma}^\lambda \Gamma_{\lambda\nu}^\sigma \quad (5)$$

Varying the action (4) with respect to $g_{\mu\nu}$, we gain the gravitational field equation in the Palatini formalism

$$F(\tilde{R}) \tilde{R}_{\mu\nu} - \frac{1}{2} f(\tilde{R}) g_{\mu\nu} = \kappa T_{\mu\nu} \bar{\nabla}_\lambda, \quad (6)$$

where $F(\tilde{R}) = \frac{\partial f(\tilde{R})}{\partial \tilde{R}}$. The trace of Eq.(6) is

$$F(\tilde{R}) \tilde{R} - 2f(\tilde{R}) = 8\pi G T. \quad (7)$$

Varying the action with respect to $\tilde{\Gamma}_{\mu\nu}^{\lambda}$ gives

$$\tilde{\nabla}_{\lambda} \left(\sqrt{-g} F(\tilde{R}) g^{\mu\nu} \right) = 0, \quad (8)$$

Where $\tilde{\nabla}$ is the covariant derivative with respect to the Palatini connection. Eq.(8) implies that the connection can be represented as the Christoffel symbol associated with the metric $h_{\mu\nu}$ by defining $h_{\mu\nu} = F(R)g_{\mu\nu}$. Then we arrive at a relation:

$$\tilde{\Gamma}_{\mu\nu}^{\lambda} = \Gamma_{\mu\nu}^{\lambda} + \frac{1}{2F} [-g_{\mu\nu} \partial^{\lambda} F + \delta_{\nu}^{\lambda} \partial_{\mu} F + \delta_{\mu}^{\lambda} \partial_{\nu} F], \quad (9)$$

where $\Gamma_{\mu\nu}^{\lambda}$ is the Livi-Civita connection associated with the metric $g_{\mu\nu}$. Thus, by using Eq.(5) the Ricci tensor and the Ricci scalar in the Palatini formalism are rewritten as

$$\tilde{R}_{\mu\nu} = R_{\mu\nu}(g) + \frac{3}{2F^2} \nabla_{\mu} F \nabla_{\nu} F - \frac{1}{F} \nabla_{\mu} \nabla_{\nu} F - \frac{1}{2F} g_{\mu\nu} \nabla_{\sigma} \nabla^{\sigma} F, \quad (10)$$

where $R_{\mu\nu}$ denotes the Ricci tensor defining in the metric formalism, and all covariant derivatives are taken with respect to the metric $g_{\mu\nu}$. Combining above equations, the modified Einstein equation in the Palatini-formalism $f(\tilde{R})$ theory can be reexpressed as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{\kappa T_{\mu\nu}^{\text{eff}}}{F} + 8\pi T_{\mu\nu}^{\text{eff}} \quad (11)$$

with $\kappa = 8\pi G$, $8\pi T_{\mu\nu}^{\text{eff}} = -\frac{1}{2} g_{\mu\nu} \left(\tilde{R} - \frac{f}{F} \right) + \frac{1}{F} (\nabla_{\mu} \nabla_{\nu} F - g_{\mu\nu} \square F) - \frac{3}{2} \frac{1}{F^2} [\nabla_{\mu} F \nabla_{\nu} F - \frac{1}{2} g_{\mu\nu} (\nabla F)^2]$.

B. The geometrical diagnostics for $f(R)$ theory with the different formalisms

In this part, we utilize the $f(R)$ theory with the different formalisms to cosmology. A flat Friedmann-Lemaitre-Robertson-Walker (FLRW) metric has a form:

$$ds^2 = -dt^2 + a^2(t) d\vec{x}^2. \quad (12)$$

Here a is the cosmic scale factor, t is the cosmic time. Inserting the FLRW metric into field equations of $f(R)$ theory with metric formalism and assuming the energy momentum tensor: $T_{\mu\nu} = (\rho + p) U_{\mu} U_{\nu} + p g_{\mu\nu}$, we have the cosmological equations [8]

$$H^2 = \frac{\kappa}{3f_R} \left[\rho + \frac{Rf_R - f}{2} - 3H \tilde{R} f_{RR} \right], \quad (13)$$

$$2\tilde{H} + 3H^2 = -\frac{\kappa}{f_R} \left[p + (R)^2 f_{RRR} + 2H \tilde{R} f_{RR} + \tilde{R} f_{RR} + \frac{1}{2}(f - Rf_R) \right]. \quad (14)$$

Here $\tilde{H} \equiv \frac{\dot{a}}{a}$ is a geometrical quantity, called the Hubble parameter. ρ and p denote the density and pressure of universal matter, respectively. U_μ represents the four-velocity of an observer comoving with the fluid. For solving Eqs.(13) and (14), we define two dimensionless quantities: $y_H = H^2 / m^2 - (1+z)^3$ and $y_R = R / m^2 - 3(1+z)^3$, with the cosmic redshift $z = 1/a - 1$. Then we receive two differential equations:

$$y_H' = -\frac{1}{1+z} \left(\frac{1}{3} y_R - 4y_H \right), \quad (15)$$

$$y_R' = \frac{[y_H + (1+z)^3]f_R - (1+z)^3 - \frac{f_R}{6}[y_R + 3(1+z)^3] + \frac{f}{6m^2} - 9(1+z)^2}{[y_H + (1+z)^3](1+z)f_{RR}m^2}, \quad (16)$$

Where $'$ denotes the derivative with respect to z . The initial conditions are provided by: $y_H|_{z=0} = H_0^2 / m^2 - 1$ and $y_R|_{z=0} = 6H_0^2(1 - q_0) / m^2 - 3$, with $m^2 = (8315 \text{ Mpc})^{-2} \left(\frac{\Omega_{0m} h^2}{0.13} \right)$. According to the observational results, we

consider the current dimensionless energy density of matter $\Omega_{0m} = 0.27$ [37], the current value of deceleration parameter $q_0 = -0.63$ [38], the current value of dimensionless Hubble constant $h = 0.673 \pm 0.010$ [39], with $H_0 = 100 h \text{ km s}^{-1} \text{ Mpc}^{-1}$.

In the Palatini- $f(R)$ theory, the cosmological equation could be exhibited as [8]:

$$\left(H + \frac{\dot{f}_R}{2f_R} \right)^2 = \frac{\kappa(\rho + 3p) + f}{6f_R} \quad (17)$$

Combining Eq.(17) and the conservation equation, we have

$$\tilde{R} = -\frac{3H[\dot{f}_R \tilde{R} - 2f]}{\dot{f}_{RR} \tilde{R} - \dot{f}_R} \quad (18)$$

Also, Eq.(17) can be rewritten as

$$H^2 = \frac{1}{6f_R} \frac{3f - f_R \tilde{R}}{\left[1 - \frac{3f_{RR}(\tilde{R} - 2f)}{2f_R(f_{RR}\tilde{R} - f_R)} \right]^2} \quad (19)$$

Furthermore, we gain

$$\frac{d\tilde{R}}{dz} = -\frac{9H_0^2\Omega_{m0}(1+z)^2}{f_{RR}\tilde{R} - f_R} \quad (20)$$

$$\frac{H^2}{H_0^2} = \frac{1}{6f_R} \frac{3\Omega_{m0}(1+z)^3 + f/H_0^2}{\left[1 + \frac{9H_0^2\Omega_{m0}(1+z)^3 f_{RR}}{2f_R(f_{RR}R - f_R)} \right]^2} \quad (21)$$

Thus, combining with trace equation in Palatini formalism of $f(R)$ theory, we can solve Eq.(21).

The geometrical quantities—statefinder parameters $\{r, s\}$ are introduced in Ref.[32], which are defined as follows:

$$r \equiv \frac{a^{\square}}{aH^3} = \frac{H''(z+1)^2}{H} - \frac{2H'(z+1)}{H} + \frac{H'^2(z+1)^2}{H^2} + 1, \quad (22)$$

$$s \equiv \frac{r-1}{3(q-\frac{1}{2})} = \frac{\frac{H''(z+1)^2}{H} - \frac{2H'(z+1)}{H} + \frac{H'^2(z+1)^2}{H^2}}{3\frac{H'(z+1)}{H} - \frac{9}{2}}. \quad (23)$$

Obviously, both r and s are the third-order derivative (the highest) of a with respect to t . Here

$$q \equiv \frac{-\ddot{a}}{aH^2} \quad (24)$$

is another geometrical quantity, called the deceleration parameter, which is the second derivative of a with respect to t .

Lots of models are diagnosed by using the statefinder parameters [33–36]. But applying the statefinder diagnose to distinguish the modified gravitational theory with the different formalisms are lack. In this paper, we investigate the differences between the Palatini formalism and the metric formalism by using the diagnostic method. From above, we can find that the gravitational field equation is the fourth-order PDE (partial differential equation) in the metric formalism, while the field equation in the Palatini formalism is the second-order PDE which is easier to solve and interpret [40]. Then the different methods are utilized to solve the cosmological equations numerically for the different formalisms. For

plotting pictures, we take a viable model $f(R) = R - \beta R_s (1 - e^{-R/R_s})$, which is proposed and developed by Refs.[41–43]. Here β and R_s are two constants, and could be related by $\beta R_s \simeq 18H_0^2 \Omega_{0m}$. Some viable conditions on this model can be found in Refs.[42, 43]. This model has an important feature that it owns only one more parameter than the Λ CDM model.

FIG. 1: The $\{r,s\}$ geometrical diagnostic for $f(R)$ model with the metric formalism (left) and the Palatini formalism (right), respectively. The selected values of model parameter β are marked on the pictures.

The graphs of $\{r,s\}$ geometrical diagnostic are plotted in Fig.1 with the metric formalism (left) and the Palatini formalism (right), respectively. The values of model parameter β are taken as $[0.9, 1.1, 1.3]$, and marked by β_1 , β_2 and β_3 , respectively. Considering that for the most popular Λ CDM model, we have $\{r,s\} = \{1,0\}$. Hence we could find the deviation of $f(R)$ model in both formalisms from the Λ CDM model, which show that the values of $\{r,s\}$ in metric formalism are larger (or more deviation) than those in Palatini formalism according to Fig.1. Also, Fig.1 shows that for the same function of $f(R)$, the difference between the metric formalism and the Palatini formalism are notable according to the $\{r,s\}$ diagnostic. In addition, difference between these two formalisms can be reflected by the values of model parameter β . We can notice that the more larger value of β , the $\{r,s\}$ pictures are more close to Λ CDM in Palatini formalism, while the opposite results are given in metric formalism. The arrow describes the evolution of universe from the early stage to the late stage.

Using the same model-parameter values with those in Fig.1, Fig.2

depicts geometric diagnostic of $\{r, q\}$. We can read that the shapes of $\{r, q\}$ in metric formalism are different from those in Palatini formalism. For seeing the effect of q on the $\{r, q\}$ diagnostic, we illustrate the evolutions of deceleration parameter q for the considered $f(R)$ model. The metric formalism are drawn in Fig.3 (left), while the Palatini formalism are plotted in Fig. 3 (right). Given that we have $q(z) = 1/2$ for the matter dominated universe, then there must be $q(z) \leq 1/2$ for any cosmological model. For selected model-parameter values in this paper, we can see that the evolutionary curves of $q(z)$ in Palatini formalism are consistent with the requirement: $q(z) \leq 1/2$.

FIG. 2: The $\{r, q\}$ geometrical diagnostic for $f(R)$ model with the metric formalism (left) and the Palatini formalism (right).

FIG. 3: The evolutions of deceleration parameter q for $f(R)$ model with the metric formalism (left) and the Palatini formalism (right).

III. The geometrical diagnostics for GBD theory with the different formalisms

A. Basic equations for GBD modified theory in both the metric formalism and the Palatini formalism

Some problems on $f(R)$ theory have been found in some reference [8], such as the inconsistent problem of γ between the theoretical value and the observational value (here γ is the parametrized post-Newtonian parameter). The $f(R)$ theory can be extended by considering some methods, such as the $f(G)$ theory (adding the higher-order terms) [9, 10], the GBD theory [44–46], etc. In this part, we apply the geometric diagnostic methods to distinguish the different formalisms of GBD modified theory. In addition, given that the time-variable gravitational constant G have been investigated in some theoretical and observational issues [47–54], we explore the effects of time-variable G in the modified theory with the different formalisms.

The action of system in the metric-formalism GBD theory is written as

$$S = S_g(g_{\mu\nu}, \phi) + S_\phi(g_{\mu\nu}, \phi) + S_m(g_{\mu\nu}, \psi_m) = \frac{1}{2} \int L_T d^4x = \frac{1}{2} \int d^4x \sqrt{-g} \left[\phi f(R) - \frac{\omega}{2\phi} \partial_\mu \phi \partial^\mu \phi + \frac{16\pi}{c^4} L_m \right]. \quad (25)$$

Using the variational principle, in the metric-formalism GBD theory we obtain the gravitational field equation and the BD scalar field equation as follows

$$\phi [f_R R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu}] - (\nabla_\mu \nabla_\nu - g_{\mu\nu} \square)(\phi f_R) + \frac{1}{2} \frac{\omega}{\phi} g_{\mu\nu} \partial_\sigma \phi \partial^\sigma \phi - \frac{\omega}{\phi} \partial_\mu \phi \partial_\nu \phi = 8\pi T_{\mu\nu}. \quad (26)$$

$$f(R) + 2\omega \frac{\square \phi}{\phi} - \frac{\omega}{\phi^2} \partial_\mu \phi \partial^\mu \phi = 0. \quad (27)$$

where ϕ is the BD scalar field, ω is the coupling constant, ∇_μ is the covariant derivative associated with the Levi-Civita connection of the metric. The trace of Eq. (26) is

$$f_R R - 2f(R) + \frac{3\square(\phi f_R)}{\phi} + \frac{\omega}{\phi^2} \partial_\mu \phi \partial^\mu \phi = \frac{8\pi T}{\phi}. \quad (28)$$

The action of GBD theory in the Palatini formalism read as,

$$S_p = S_g \left(g_{\mu\nu}, \overset{\square}{\Gamma}{}^\lambda_{\mu\nu}, \phi \right) + S_\phi(g_{\mu\nu}, \phi) + S_m(g_{\mu\nu}, \psi) = \frac{1}{2} \int d^4x L_T, \quad (29)$$

with the total Lagrange quantity $L_T = \sqrt{-g} \left[\phi f(\tilde{R}) - \frac{\omega}{\phi} \partial_\mu \phi \partial^\mu \phi + \frac{16\pi}{c^4} L_M \right]$. In the Palatini formalism, varying the action (29) with respect to $g_{\mu\nu}$ and ϕ , we gain two field equations as follows

$$\phi F(\tilde{R}) \tilde{R}_{\mu\nu} - \frac{1}{2} \phi f(\tilde{R}) g_{\mu\nu} - \frac{\omega}{\phi} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} g_{\mu\nu} \frac{\omega}{\phi} \partial_\sigma \phi \partial^\sigma \phi = 8\pi T_{\mu\nu} \quad (30)$$

$$\frac{2\omega}{\phi} \Box \phi - \frac{\omega}{\phi^2} \partial_\mu \phi \partial^\mu \phi + f(\tilde{R}) = 0 \quad (31)$$

The trace of Eq.(30) is

$$F(\tilde{R}) \tilde{R} - 2f(\tilde{R}) + \frac{\omega}{\phi^2} \partial_\mu \phi \partial^\mu \phi = \frac{8\pi T}{\phi}. \quad (32)$$

Varying the action with respect to $\tilde{\Gamma}^{\lambda}_{\mu\nu}$ gives

$$\tilde{\nabla}_\lambda \left(\sqrt{-g} \phi F(\tilde{R}) g^{\mu\nu} \right) = 0, \quad (33)$$

Where $\tilde{\nabla}$ is the covariant derivative with respect to the Palatini connection.

B. The geometrical diagnostics for GBD theory with the different formalisms

For a flat FLRW universe, using Eqs. (26) and (27) we can derive the evolutionary equations of background universe in the metric-GBD theory as,

$$3f_R H^2 = \frac{8\pi\rho_m}{\phi} + \frac{f_R R - f(R)}{2} - 3H\dot{f}_R + \frac{1}{2} \omega \left(\frac{\dot{\phi}}{\phi} \right)^2 - 3Hf_R \frac{\dot{\phi}}{\phi}, \quad (34)$$

$$-2f_R \dot{H} = \frac{8\pi}{\phi} (\rho_m + p_m) + \ddot{f}_R - H\dot{f}_R + \omega \left(\frac{\dot{\phi}}{\phi} \right)^2 - Hf_R \frac{\dot{\phi}}{\phi} + f_R \frac{\ddot{\phi}}{\phi} + 2\frac{\dot{\phi}}{\phi} \dot{f}_R, \quad (35)$$

$$f(R) - \omega \left(\frac{\dot{\phi}}{\phi} \right)^2 + 2\omega \frac{\ddot{\phi}}{\phi} + 6\omega H \frac{\dot{\phi}}{\phi} = 0. \quad (36)$$

Here $R = 6(2H^2 + \dot{H})$, and "dot" denotes the derivative with respect to cosmic time t . For $\phi = \text{constant}$, Eqs.(34-36) are reduced to the cases of $f(R)$ theory.

Using Eqs. (30)-(33), we can get the evolutionary equations of

background universe in the Palatini-GBD theory,

$$3FH^2 = \frac{8\pi(\rho+3p)}{2\phi} + \frac{1}{2}f + \omega\left(\frac{\dot{\phi}}{\phi}\right)^2 - \frac{3}{4}\left(\frac{\dot{\phi}}{\phi}\right)^2 F - \frac{3}{2}\frac{\dot{\phi}}{\phi}F - \frac{3}{4}\frac{F^2}{F} - 3HF\frac{\dot{\phi}}{\phi} - 3H\frac{F}{F}, \quad (37)$$

$$-2F\dot{H} = \frac{8\pi(\rho+p)}{\phi} + \omega\left(\frac{\dot{\phi}}{\phi}\right)^2 - \frac{3}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2 F - 3\frac{\dot{\phi}}{\phi}F - \frac{3}{2}\frac{F^2}{F} + \frac{\ddot{\phi}}{\phi}F + 2F\frac{\dot{\phi}}{\phi} + \frac{F}{F} - HF\frac{\dot{\phi}}{\phi} - H\frac{F}{F}, \quad (38)$$

$$-f(\tilde{R}) - \frac{\omega}{\phi^2}\dot{\phi}^2 + 2\omega\frac{\ddot{\phi}}{\phi} + 6\omega H\frac{\dot{\phi}}{\phi} = 0 \quad (39)$$

In the following, we solve the cosmological equations of GBD theory with the different formalisms. For solving equations (34)-(36) in the framework of metric formalism, we define the dimensionless variables: $y_H = H^2 / m^2 - (1+z)^3$, $y_R = R / m^2 - 3(1+z)^3$, $y_\phi = \phi / \phi_0$, $y'_\phi = \dot{\phi} / \phi_0$.

Then Eqs. (34-36) provide the differential equations for $\{y_H, y_R, y_\phi, y'_\phi\}$ as follows

$$y'_H = -\frac{1}{(1+z)}\left(\frac{1}{3}y_R - 4y_H\right), \quad (40)$$

$$y'_R = \frac{1}{[y_H + (z+1)^3](z+1)m^2 f_{RR}} \{ [y_H + (z+1)^3]f_R - [y_R + 3(z+1)^3]\frac{f_R}{6} + \frac{f}{6m^2} \\ - \frac{\omega}{6}(z+1)^2[y_H + (z+1)^3]\left(\frac{y'_\phi}{y_\phi}\right)^2 - [y_H + (z+1)^3](z+1)f_R\frac{y'_\phi}{y_\phi} - \frac{(z+1)^3}{\phi} \} - 9(z+1)^2 \quad (41)$$

$$y'_\phi = \dot{\phi} / \phi_0, \quad (42)$$

$$y''_\phi = \frac{y_\phi}{2\omega[y_H + (z+1)^3](z+1)^2} \left\{ \frac{f}{m^2} + \omega[y_H + (z+1)^3](z+1)^2 \left(\frac{y'_\phi}{y_\phi}\right)^2 \right. \\ \left. + \omega(z+1)\frac{y'_\phi}{y_\phi} \left[\frac{1}{3}y_R - 4y_H - 3(z+1)^3 \right] + 4\omega(z+1)\frac{y'_\phi}{y_\phi} [y_H + (z+1)^3] \right\} \quad (43)$$

To solve above differential equations, the initial conditions are selected as respectively: $y_\phi|_{z=0} = 1$, $y'_\phi|_{z=0} = 0.01$. The initial conditions of y_H and y_R are taken the same values with those in f(R) theory. The value of initial condition $y'_\phi|_{z=0}$ can be indicated by the following observations. For

example, the limits on the variation of G can be exhibited by: $\left| \frac{\dot{G}}{G} \right| = \left| \frac{\dot{\phi}}{\phi} \right| \leq$

$4.1 \times 10 y^{-1}$ from Pulsating white dwarf G117-B15A [48], $-4 \times 10 y^{-1} \leq \frac{\dot{\phi}}{\phi}$
 $\leq 2.5 \times 10^{-10} y^{-1}$ from Nonradial pulsations of white dwarfs [49], $\left| \frac{\dot{\phi}}{\phi} \right|$
 $\leq 2.3 \times 10^{-11} y^{-1}$ from Millisecond pulsar PSR J0437-4715 [50], $\left| \frac{\dot{\phi}}{\phi} \right| \leq 10^{-11} y^{-1}$
 from Type-Ia supernovae [51], $\frac{\dot{\phi}}{\phi} = (0.6 \pm 0.42) \times 10^{-12} y^{-1}$ from Neutron star
 masses [52], $\left| \frac{\dot{\phi}}{\phi} \right| \leq 1.6 \times 10^{-12} y^{-1}$ from Helioseismology [53], and $\frac{\dot{\phi}}{\phi} = (4 \pm 9) \times 10^{-13} y^{-1}$ from Lunar laser ranging experiment [54], etc. Taking a stringent
 bound $\left| \frac{\dot{\phi}}{\phi} \right| \leq 10^{-12} y^{-1}$, we can calculate to limit $|y'_\phi(z=0)| \leq 0.015$ by using
 the center value $H_0 = 67.3 \text{ kms}^{-1} \text{ Mpc}^{-1} = 6.87 \times 10^{-11} y^{-1}$. Here we take
 $y'_\phi(z=0) = 0.01$ as an initial condition in Eq. (42). Then the pictures of
 geometrical diagnostic are illustrated in Fig.4 and Fig.5 for the metric
 formalism of GBD theory.

For the Palatini formalism of GBD theory, using the trace equation
 in GBD theory we have

$$\dot{\bar{R}} = \frac{-8\pi(\dot{\rho}_m \phi - \dot{\phi} \rho_m) \phi + \omega(2\ddot{\phi} \phi \phi - 2\dot{\phi}^3)}{\phi^3(f_{\bar{R}\bar{R}} R - f_{\bar{R}})} \quad (44)$$

Changing the variable from t to z , with $\frac{d}{dt} = -H(1+z) \frac{d}{dz}$, we receive
 $\dot{\phi} = \phi'(-(1+z)H)$, and $\ddot{\phi} = \phi'' H^2 (z+1)^2 + \phi' H^2 (z+1) + \phi' H' H (z+1)^2$. Using the
 definition of geometrical quantities, we can plot the diagnostic pictures of
 $\{r, s\}$ and $\{r, q\}$ in the Palatini formalism of GBD theory.

FIG. 4: The $\{r,s\}$ geometrical diagnostic for GBD theory with the metric formalism (left) and the Palatini formalism (right).

In this part, we distinguish the different formalisms of GBD modified gravitational theory. For comparison, we take the same $f(R)$ function as above. From Figs. 4 and 5, we can see that the difference between the metric formalism and the Palatini formalism are still conspicuous according to the $\{r,s\}$ and $\{r,q\}$ diagnostics, respectively. In the GBD theory, the evolutions of deceleration parameter q are plotted in Fig.6, which exhibit the different evolutions between two formalisms. According to Fig.6, for case of $\beta = 1.1$ in Palatini formalism, it is satisfied with the requirement of $q(z) \leq 1/2$. And for this case we have $z_T = 0.11$ (at where $q = 0$) for metric formalism, and $z_T = 1.09$ for Palatini formalism. z_T is called the transition redshift, which describes the universal expansion from deceleration to acceleration. Obviously, the significantly different values of z_T are indicated in these two formalisms.

In order to explore the effects of time-variable G , we also compare the $f(R)$ theory (G is a constant) with the GBD theory (G is a variable quantity) according to the results of geometrical diagnostics. Some results could be exhibited as follows. (1) According to Fig.1 (left) and Fig.4 (left), in the metric formalism the variation of G affects obviously on the values of r (and s). For example, for case of $\beta = 1.3$, the value of r vary from 0 to -38.55 for $f(R)$ theory, and from 0 to -16.54 for GBD theory, respectively; the largest value of s vary from 2.88 (corresponding to GBD) to 4.61 (corresponding to $f(R)$ theory). Obviously, for the influence of

variable G , the values of r (or s) in GBD are smaller than those in $f(R)$ theory, which indicates that the variation of Newton gravitational constant can induce remarkable effects on geometrical quantities for the existence of BD scalar field (including the terms of its dynamics and the coupling between the BD field and the $f(R)$ in the action). (2) In the Palatini formalism (see Fig.1 (right) and Fig.4 (right)), for $\beta = 1.1$ case the difference between the variable- G theory (GBD) and the constant- G theory ($f(R)$) are not obvious, while for other cases the shapes of r - s curves are different. (3) According to Fig.3 and Fig.6, in the metric formalism we find that the values of q decay more early (about $z \sim 2$) in $f(R)$ theory than that (about $z \sim 0.8$) in GBD theory; while in the Palatini formalism, the difference between the GBD theory and the $f(R)$ theory are more reflected at the earlier stage of universe (higher redshift).

FIG. 5: The $\{r, q\}$ geometrical diagnostic for GBD theory with the metric formalism (left) and the Palatini formalism (right).

FIG. 6: The evolutions of a geometrical quantity-deceleration parameter q for GBD theory with the metric formalism (left) and the Palatini formalism (right).

IV. Conclusions

Several observational and theoretical motivations require us to investigate the modified theories of GR. In modified gravitational theory, there exist the metric formalism and the Palatini formalism. In these two formalisms, the dynamical variables are considered to be different. Geometrical-diagnostic methods were often applied to distinguish the gravitational models. But it is scarce to investigate the differences for the different formalisms (e.g. the metric formalism and the Palatini formalism) in the modified theories of gravity. In this paper, we discriminate the different formalisms of the modified gravity by using the diagnostic methods. For considered modified theories of gravity (including the $f(R)$ theory and the GBD theory), we can see that the difference between the two formalisms are notable according to the geometrical diagnostics. And relative to the Λ CDM model, there are the more deviation in metric formalism than those in Palatini formalism, according to the $\{r,s\}$ diagnostic.

Given that the GBD is a time-variable Newton gravitational constant theory, the differences between the variable- G theory and the constant- G theory are studied by using the diagnostic methods. According to the observational limits on G , we plot some pictures on the geometrical quantities. For the influence of variable G , the values of r (or s) in metric-formalism GBD are smaller than those in $f(R)$ theory, which indicates that the variation of Newton gravitational constant can induce remarkable effects on geometrical quantities for the existence of BD scalar field. In the Palatini formalism, the shapes of r - s curves between the GBD theory and the $f(R)$ theory could be obviously different, depending on the values of model parameter β . In addition, in the metric formalism we can find that the values of q decay more early (about $z \sim 2$) in $f(R)$ theory than that (about $z \sim 0.8$) in GBD theory; while in the Palatini formalism, the difference between the $f(R)$ theory and the GBD theory are more reflected at the higher-redshift universe. In summary, according to our study the effects of variable G could be found in both the metric formalism and the Palatini formalism, by testing the geometrical

quantities (e.g. r , s and q).

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