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Pattern dynamics of a vegetation model with nonlocal interactions

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Abstract

In order to understand the mechanism of water uptake by vegetation, we propose a vegetation-water model which represents nonlocal effect via nonlocal delay in this paper. By mathematical analysis, the condition of producing steady pattern is obtained. Furthermore, the amplitude equation which determines the type of Turing pattern is obtained by nonlinear analysis method. The corresponding vegetation pattern and evolution process under different intensity of nonlocal effect in roots of vegetation are given by numerical simulations. There is a positive correlation between the intensity of nonlocal effect and vegetation density, that is, the vegetation density increases with the increase of the intensity of nonlocal effect. Numerical simulation shows that as intensity of nonlocal effect increased, the isolation degree of vegetation pattern increases which indicates that the robustness of the ecosystem decreases. Besides, the results reveal that with the water diffusion coefficient increase, the vegetation pattern transitions: stripe pattern→mix pattern→spot pattern emerge which predicts transitions from stripe vegetation to bare soil. Our results show the effects of intensity of nonlocal effect and diffusion coefficient on vegetation distribution, which provides theoretical basis for the study of vegetation.

Keywords: Vegetation; Pattern; Nonlocal delay; Multi-scale theory

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1 Introduction

Vegetation plays an extremely important role in ecological protection. It is beneficial to changing the abiotic environment and redistribution of resources [1]. In particular, in the process of photosynthesis, plants absorb a large amount of carbon dioxide and release oxygen, which can achieve the purpose of purifying air and improving air quality. In addition, vegetation has a strong role in soil and water conservation. The roots of the vegetation protect the slopes and retain water to form fertile patterns [2] and the leaves of vegetation can be used to make the land less prone to drought. Therefore, the destruction of vegetation will inevitably lead to the imbalance of ecological balance.

Lefever et al. gave a univariate vegetation model and described the dynamics of vegetation growth [3]. In arid and semi-arid areas, water is the decisive factor restricting the growth of vegetation. In 1999, Klausmeier first proposed the reaction-diffusion equation of two-variable vegetation [4] and water. Then, a three-dimensional model was proposed by Rietkerk et al., which was divided into vegetation, surface water and ground water [5]. So far, there has been much research on vegetation models [6–9]. Vegetation pattern is the direct reflection of vegetation spatial distribution. The spatial density distribution of vegetation and the stability of spatial structure can be seen. There are also many studies on vegetation pattern [10–18]. Sherratt used mathematical model to study the banding pattern and obtained the parameter region generated by the pattern [19]. Vegetation pattern can provide early warning of desertification and theoretical basis for vegetation protection [20–24].

The studies mentioned above are all about the local effects of vegetation and water. That is to say, vegetation absorb water at the current position and time. However, there are two processes for vegetation to absorb water. First, precipitation which is the main source of water resources permeates the soil. Second, the roots of vegetation absorb water, and then convert it into their own biomass [25]. Consequently, in arid or semi-arid areas, the roots of the vegetation absorb water in a certain area around it at the same time through a certain time, this phenomenon is called nonlocal delay mathematically.

At present, there are many researches on nonlocal actions and nonlocal delay interaction. Some work studied the traveling wave solutions of PDE model [26–32]. Nonlocal actions are generally expressed as integrals in mathematical models [33]. Zaytseva et al. constructed a grass sedimension dynamic model with nonlocal term, the integral term was represented by a Mexican kernel function which could reflect the scale dependence mechanism. They deduced the conditions for the generation of pattern and the factors affecting the structure of pattern [34]. Thompson et al. used FK model to predict the dynamic pattern of hens and chicks, which is produced by nonlocal activation and local inhibition. The results showed that the deviation between the prediction of FK model and random dynamic model was reduced on a long time scale [35]. Guo et al. constructed a reaction-diffusion equation of infectious diseases with nonlocal delay and studied the effect of delay parameters on the pattern structure [36].

We mainly study the nonlocal process of water uptake by plants in this paper, a integra is introduced as follows [37, 38] :

$$\tilde{V} = \int_{\Phi} \int_{-\infty}^t G(x_1, x_2, t-s) f(t-s) W(x_2, s) ds dx_2,$$

where, $\Phi \in [\tilde{a}, \tilde{b}] \times [\tilde{a}, \tilde{b}]$ and $x_1, x_2 \in \Phi$, $G(x_1, x_2, t)f(t)$ represents the weight of the water

from other position to current position before time t . The integral is the average value of water absorption by the roots of vegetation at position x_1 .

$G(x_1, x_2, t)$ is the solution of the following equation

$$\frac{\partial G}{\partial t} = D\left(\frac{\partial^2 G}{\partial x_1^2} + \frac{\partial^2 G}{\partial x_2^2}\right)$$

and

$$\frac{\partial G}{\partial n} = 0, G(x_1, x_2, 0) = \delta(x_1 - x_2).$$

Time distribution function is divided into the following forms [38]:

$$f_0(t) = \frac{1}{\tau}e^{-\frac{t}{\tau}}, f(t) = \frac{t}{\tau^2}e^{-\frac{t}{\tau}},$$

which represent weak and strong kernel, respectively. Figure 1 shows the functions of the weak kernel and the strong kernel with respect to time t . As can be seen from Figure 1(A), the weak kernel function $f_0(t)$ decreases with t , which biologically reflects that the water absorption strength of the roots decreases over time. For strong kernel $f(t)$, its graph increases firstly and then decreases over time, which reflects in arid and semi-arid areas vegetation is severely short of water, the strength of water absorption of the root is strong at first, then water resource near the root of vegetation is reduced and the water absorption intensity is weakened. Obviously, strong kernel are more practical. Besides, there are few studies on nonlocal delay with strong kernel in vegetation system. Hence, this paper mainly considers the effect of strong kernel on vegetation pattern.

Figure 1: Image of the weak kernel (A) and strong kernel (B) over time.

The two-variable mathematical model which was composed by Klausmeier in 1999 is the following form [4]:

$$\begin{cases} \frac{\partial N}{\partial T} = FJWN^2 - BN + D\Delta N, \\ \frac{\partial W}{\partial T} = R - IW - FWN^2 + \bar{V}\frac{\partial W}{\partial X}. \end{cases} \quad (1.1)$$

N and W represent plant biomass (N) and water (W), respectively. R represents the rate of rainfall which is the supply of water and IW denotes lost rate due to evaporation. Plant absorbs water at a rate of FWN^2 and the rate of conversion to biomass is J . BN is the natural mortality of plant. $\bar{V}\frac{\partial W}{\partial X}$ is the run-off and the water flows downhill. D is the diffusion rate of plant and $\Delta = \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}$.

In order to be more consistent with the actual situation, we consider the diffusion of water and substitute nonlocal delay integral $\tilde{V}$ into (1.1), the equation as follows is obtained:

$$\begin{cases} \frac{\partial N}{\partial T} = FJN^2 \int_{\Phi} \int_{-\infty}^t G(x_1, x_2, t-s)f(t-s)W(x_2, s)dsdx_2 - BN + D\Delta N, \\ \frac{\partial W}{\partial T} = R - IW - FN^2 \int_{\Phi} \int_{-\infty}^t G(x_1, x_2, t-s)f(t-s)W(x_2, s)dsdx_2 + \bar{V}\frac{\partial W}{\partial X} + \bar{D}\Delta W. \end{cases} \quad (1.2)$$

Suppose the boundary condition of system (1.2) is Neuman boundary, and the initial value is nonnegative.

In this paper, a two-variable mathematical model with nonlocal delay is studied. Through stability analysis, we get the conditions for the generation of stationary pattern. Next, amplitude equation is obtained by multiscale analysis. Finally, the influence of intensity of nonlocal effect τ and diffusion coefficient β on the pattern structure are analyzed by numerical simulations and some important conclusions are given.

2 Stability analysis

We mainly consider the growth of vegetation on flat ground. Consequently, system (1.2) can be rewritten as follows:

$$\begin{cases} \frac{\partial N}{\partial T} = FJN^2 \int_{\Phi} \int_{-\infty}^t G(x_1, x_2, t-s)f(t-s)W(x_2, s)dsdx_2 - BN + D\Delta N, \\ \frac{\partial W}{\partial T} = R - IW - FN^2 \int_{\Phi} \int_{-\infty}^t G(x_1, x_2, t-s)f(t-s)W(x_2, s)dsdx_2 + \bar{D}\Delta W. \end{cases} \quad (2.1)$$

Let $\tilde{V} = \int_{\Phi} \int_{-\infty}^t G(x_1, x_2, t-s)\frac{t-s}{\tau^2}e^{-\frac{t-s}{\tau}}W(x_2, s)dsdx_2$, system (2.1) becomes the following form [38]:

$$\begin{cases} \frac{\partial N}{\partial T} = FJN^2\tilde{V} - BN + D\Delta N, \\ \frac{\partial W}{\partial T} = R - IW - FN^2\tilde{V} + \bar{D}\Delta W, \\ \frac{\partial \tilde{V}}{\partial T} = \frac{1}{\tau}(P - \tilde{V}) + D\Delta \tilde{V}, \\ \frac{\partial P}{\partial T} = \frac{1}{\tau}(W - P) + D\Delta P. \end{cases} \quad (2.2)$$

We perform nondimensionalization:

$$n = \frac{\sqrt{F}}{\sqrt{I}}N, w = \frac{\sqrt{F}J}{\sqrt{I}}W, t = IT, \gamma = \frac{B}{I}, \beta = \frac{\bar{D}}{D}, \eta = \frac{\sqrt{F}J}{I\sqrt{I}}R,$$

$$v = \frac{\sqrt{F}J}{\sqrt{I}}\tilde{V}, p = \frac{\sqrt{F}J}{\sqrt{I}}P, x = \frac{\sqrt{I}}{\sqrt{D}}X, y = \frac{\sqrt{I}}{\sqrt{D}}Y.$$

Then the resulted nondimensionalized system is:

$$\begin{cases} \frac{\partial n}{\partial t} = n^2v - \gamma n + \Delta n, \\ \frac{\partial w}{\partial t} = \eta - w - n^2v + \beta \Delta w, \\ \frac{\partial v}{\partial t} = \frac{1}{\tau}(p - v) + \Delta v, \\ \frac{\partial p}{\partial t} = \frac{1}{\tau}(w - p) + \Delta p. \end{cases} \quad (2.3)$$

System (2.3) has three equilibria:

$$E_0 = (n_0, w_0, v_0, p_0) = (0, \eta, \eta, \eta),$$

$$E_1 = (n_1, w_1, v_1, p_1) = \left(\frac{2\gamma}{\eta + \sqrt{\eta^2 - 4\gamma^2}}, \frac{\eta + \sqrt{\eta^2 - 4\gamma^2}}{2}, \frac{\eta + \sqrt{\eta^2 - 4\gamma^2}}{2}, \frac{\eta + \sqrt{\eta^2 - 4\gamma^2}}{2} \right),$$

$$E_2 = (n_2, w_2, v_2, p_2) = \left(\frac{2\gamma}{\eta - \sqrt{\eta^2 - 4\gamma^2}}, \frac{\eta - \sqrt{\eta^2 - 4\gamma^2}}{2}, \frac{\eta - \sqrt{\eta^2 - 4\gamma^2}}{2}, \frac{\eta - \sqrt{\eta^2 - 4\gamma^2}}{2} \right).$$

The equilibrium E_0 is also called as the bare-soil and the two nonnegative equilibria should satisfy $\eta > 2\gamma$. Next, we mainly analyze the stability of the positive equilibria E_1 and E_2 .

2.1 Stability of the equilibrium E_1

Next we linearize system (2.3) at the equilibrium E_1 when there is no diffusion, and obtain the following system:

$$\begin{cases} \frac{dn}{dt} = a_{11}n + a_{12}w + a_{13}v + a_{14}p, \\ \frac{dw}{dt} = a_{21}n + a_{22}w + a_{23}v + a_{24}p, \\ \frac{dv}{dt} = a_{31}n + a_{32}w + a_{33}v + a_{34}p, \\ \frac{dp}{dt} = a_{41}n + a_{42}w + a_{43}v + a_{44}p, \end{cases} \quad (2.4)$$

where,

$$\begin{aligned} a_{11} &= \gamma, a_{12} = 0, a_{13} = \frac{\eta - \sqrt{\eta^2 - 4\gamma^2}}{\eta + \sqrt{\eta^2 - 4\gamma^2}}, a_{14} = 0, \\ a_{21} &= -2\gamma, a_{22} = -1, a_{23} = -\frac{\eta - \sqrt{\eta^2 - 4\gamma^2}}{\eta + \sqrt{\eta^2 - 4\gamma^2}}, a_{24} = 0, \\ a_{31} &= 0, a_{32} = 0, a_{33} = -\frac{1}{\tau}, a_{34} = \frac{1}{\tau}, \end{aligned}$$

$$a_{41} = 0, a_{42} = \frac{1}{\tau}, a_{43} = 0, a_{44} = -\frac{1}{\tau}.$$

The characteristic equation of system (2.4) is as follows:

$$\lambda^4 + \bar{\Upsilon}_1(0)\lambda^3 + \bar{\Upsilon}_2(0)\lambda^2 + \bar{\Upsilon}_3(0)\lambda + \bar{\Upsilon}_4(0) = 0,$$

where,

$$\begin{aligned}\bar{\Upsilon}_1(0) &= 1 - \gamma + \frac{2}{\tau}, \bar{\Upsilon}_2(0) = \frac{2}{\tau^2} + \frac{2(1-\gamma)}{\tau} - \gamma, \\ \bar{\Upsilon}_3(0) &= \frac{a_{13} + 1 - 2\gamma}{\tau^2} - \frac{2\gamma}{\tau}, \bar{\Upsilon}_4(0) = \frac{\gamma}{\tau^2}(a_{13} - 1).\end{aligned}$$

According to Routh-Hurwite criterion, E_2 is stable if and only if $Re\lambda_i < 0$, then the following conditions hold:

$$\begin{cases} \bar{\Upsilon}_1(0) > 0, \bar{\Upsilon}_2(0) > 0, \bar{\Upsilon}_4(0) > 0, \\ \bar{\Upsilon}_1(0)\bar{\Upsilon}_2(0)\bar{\Upsilon}_3(0) > \bar{\Upsilon}_3^2(0) + \bar{\Upsilon}_1^2(0)\bar{\Upsilon}_4(0), \\ \bar{\Upsilon}_1(0)\bar{\Upsilon}_2(0) > \bar{\Upsilon}_3(0). \end{cases} \quad (2.5)$$

Since $a_{13} < 1$, it is easy to see that $\bar{\Upsilon}_4(0) < 0$. This is a contradiction with (2.5). Therefore, E_1 is unstable.

2.2 Stability of the equilibrium E_2

Linearizing system (2.3) near $E_2(n_2, w_2, v_2, p_2)$, we obtain the linear system as follows:

$$\begin{cases} \frac{\partial n}{\partial t} = b_{11}n + b_{12}w + b_{13}v + b_{14}p + \Delta n, \\ \frac{\partial w}{\partial t} = b_{21}n + b_{22}w + b_{23}v + b_{24}p + \beta\Delta w, \\ \frac{\partial v}{\partial t} = b_{31}n + b_{32}w + b_{33}v + b_{34}p + \Delta v, \\ \frac{\partial p}{\partial t} = b_{41}n + b_{42}w + b_{43}v + b_{44}p + \Delta p, \end{cases} \quad (2.6)$$

where,

$$\begin{aligned}b_{11} &= \gamma, b_{12} = 0, b_{13} = \frac{\eta + \sqrt{\eta^2 - 4\gamma^2}}{\eta - \sqrt{\eta^2 - 4\gamma^2}}, b_{14} = 0, \\ b_{21} &= -2\gamma, b_{22} = -1, b_{23} = -\frac{\eta + \sqrt{\eta^2 - 4\gamma^2}}{\eta - \sqrt{\eta^2 - 4\gamma^2}}, b_{24} = 0, \\ b_{31} &= 0, b_{32} = 0, b_{33} = -\frac{1}{\tau}, b_{34} = \frac{1}{\tau}, \\ b_{41} &= 0, b_{42} = \frac{1}{\tau}, b_{43} = 0, b_{44} = -\frac{1}{\tau}.\end{aligned}$$

Let

$$\begin{pmatrix} n \\ w \\ v \\ p \end{pmatrix} = \begin{pmatrix} n^* \\ w^* \\ v^* \\ p^* \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} e^{\lambda t + i\vec{k}\vec{r}} + c.c + O(\varepsilon^2),$$

where $\vec{k} = (k_x, k_y)$, $\vec{r} = (X, Y)$ and $i^2 = -1$. We substitute the above equation into (2.6) to get the characteristic equation.

$$\det A = \begin{vmatrix} b_{11} - \kappa^2 - \lambda & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} - \beta k^2 - \lambda & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} - k^2 - \lambda & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} - \kappa^2 - \lambda \end{vmatrix} = 0.$$

which is equivalent to:

$$\lambda^4 + \bar{\Upsilon}_1(k)\lambda^3 + \bar{\Upsilon}_2(k)\lambda^2 + \bar{\Upsilon}_3(k)\lambda + \bar{\Upsilon}_4(k) = 0,$$

where,

$$\bar{\Upsilon}_1(k) = (\beta + 3)k^2 + \frac{-\gamma\tau + \tau + 2}{\tau},$$

$$\bar{\Upsilon}_2(k) = (3\beta + 3)k^4 - \frac{\beta\gamma\tau + 2\gamma\tau - 2\beta - 3\tau - 4}{\tau}k^2 - \frac{\gamma\tau^2 + 2\gamma\tau - 2\tau - 1}{\tau^2},$$

$$\begin{aligned} \bar{\Upsilon}_{3k} = & (3\beta + 1)k^6 - \frac{2\beta\gamma\tau + \gamma\tau - 4\beta - 3\tau - 2}{\tau}k^4 - \frac{2\beta\gamma\tau + 2\gamma\tau^2 + 2\gamma\tau - \beta - 4\tau - 1}{\tau^2}k^2 \\ & - \frac{1}{\tau^2(\eta\rho - \eta^2 + 2\gamma^2)}(2\rho\eta\gamma\tau - 2\eta^2\gamma\tau + 4\gamma^3\tau + \rho\eta\gamma - \eta^2\gamma + 2\gamma^3 - \eta\rho + \eta^2), \end{aligned}$$

$$\begin{aligned} \bar{\Upsilon}_{4k} = & \beta k^8 - \frac{\beta\gamma\tau - 2\beta - \tau}{\tau}k^6 - \frac{2\beta\gamma\tau + \gamma\tau^2 - \beta - 2\tau}{\tau^2}k^4 - \frac{1}{\tau^2(\eta\rho - \eta^2 + 2\gamma^2)} \\ & (\rho\eta\beta\gamma + 2\rho\eta\gamma\tau - \eta^2\beta\gamma - 2\eta^2\gamma\tau + 2\beta\gamma^3 + 4\gamma^3\tau - \eta\rho + \eta^2)k^2 - \frac{\gamma(\eta\rho - \eta^2 + 4\gamma^2)}{\tau^2(\eta\rho - \eta^2 + 2\gamma^2)}, \end{aligned}$$

where, $\rho = \sqrt{(\eta + 2\gamma)(\eta - 2\gamma)}$.

The characteristic equation of the system (2.3) without the diffusion term is:

$$\mu^4 + \bar{\Upsilon}_1(0)\mu^3 + \bar{\Upsilon}_2(0)\mu^2 + \bar{\Upsilon}_3(0)\mu + \bar{\Upsilon}_4(0) = 0,$$

where,

$$\bar{\Upsilon}_1(0) = \frac{-\gamma\tau + \tau + 2}{\tau}, \bar{\Upsilon}_2(0) = -\frac{\gamma\tau^2 + 2\gamma\tau - 2\tau - 1}{\tau^2},$$

$$\begin{aligned}\bar{\Upsilon}_3(0) &= -\frac{1}{\tau^2(\eta\rho - \eta^2 + 2\gamma^2)}(2\rho\eta\gamma\tau - 2\eta^2\gamma\tau + 4\gamma^3\tau + \rho\eta\gamma - \eta^2\gamma + 2\gamma^3 - \eta\rho + \eta^2), \\ \bar{\Upsilon}_4(0) &= -\frac{\gamma(\gamma\rho - \eta^2 + 4\gamma^2)}{\tau^2(\eta\rho - \eta^2 + 2\gamma^2)}.\end{aligned}$$

We have that E_2 is stable if and only if (2.5) holds. In the case when there is diffusion in system (2.3), the conditions for stability of the equilibrium are as follows:

$$\begin{cases} (1) \bar{\Upsilon}_1(k) > 0, \bar{\Upsilon}_2(k) > 0, \bar{\Upsilon}_4(k) > 0, \\ (2) \bar{\Upsilon}_1(k)\bar{\Upsilon}_2(k)\bar{\Upsilon}_3(k) > \bar{\Upsilon}_3^2(k) + \bar{\Upsilon}_1^2(k)\bar{\Upsilon}_4(k), \\ (3) \bar{\Upsilon}_1(k)\bar{\Upsilon}_2(k) > \bar{\Upsilon}_3(k). \end{cases} \quad (2.7)$$

The steady state E_2 of the model (2.3) without diffusion is stable and if it is unstable in presence of diffusion, then Turing instability occurs. Hence, the characteristic equation must have at least one positive eigenvalue or complex eigenvalues with positive real part. When (2.5) is established and one of (1)-(3) of (2.7) is not satisfied, the equilibrium becomes unstable and the Turing pattern is induced. Next, we will consider two cases in which patterns are induced.

Case 1.

$$(a) \bar{\Upsilon}_1(k) = (\beta + 3)k^2 + \frac{-\gamma\tau + \tau + 2}{\tau}.$$

Since all the parameters are nonnegative, $\bar{\Upsilon}_1(k) > 0$ is always established under the condition $\bar{\Upsilon}_1(0) > 0$.

(b) Let $\bar{\Upsilon}_2(k) = f_{22}z^2 + f_{21}z + f_{20}$, $z = k^2$, where,

$$\begin{aligned}f_{22} &= 3\beta + 3, \\ f_{21} &= -\frac{\beta\gamma\tau + 2\gamma\tau - 2\beta - 3\tau - 4}{\tau}, \\ f_{20} &= -\frac{\gamma\tau^2 + 2\gamma\tau - 2\tau - 1}{\tau^2}.\end{aligned}$$

Since $f_{22} > 0$, then $\bar{\Upsilon}_2(k)$ takes the minimum value when $z = -\frac{f_{21}}{2f_{22}}$,

$$\bar{\Upsilon}_2(k)_{min} = \frac{4f_{22}f_{20} - f_{21}^2}{4f_{22}}.$$

In summary, Turing patterns are induced by combining (2.5) and the following conditions:

$$\begin{cases} 4f_{22}f_{20} - f_{21}^2 < 0, \\ f_{21} < 0. \end{cases}$$

(c) Let $\bar{\Upsilon}_4(k) = R(k^2) = f_{44}z^4 + f_{43}z^3 + f_{42}z^2 + f_{41}z + f_{40}$ and $z = k^2$, where,

$$\begin{aligned}f_{44} &= \beta, f_{43} = -\frac{\beta\gamma\tau - 2\beta - \tau}{\tau}, f_{42} = -\frac{2\beta\gamma\tau + \gamma\tau^2 - \beta - 2\tau}{\tau^2}, \\ f_{41} &= -\frac{1}{\tau^2(\eta\rho - \eta^2 + 2\gamma^2)}(\rho\eta\beta\tau + 2\rho\eta\gamma\tau - \eta^2\beta\gamma - 2\eta^2\gamma\tau + 2\beta\gamma^3 + 4\gamma^3\tau - \eta\rho + \eta^2),\end{aligned}$$

$$f_{40} = -\frac{\gamma(\eta\rho - \eta^2 + 4\gamma^2)}{\tau^2(\eta\rho - \eta^2 + 2\gamma^2)}.$$

It is noted that $f_{44} > 0$ and $f_{40} > 0$. The first derivative of $R(z)$ is:

$$\frac{dR(z)}{dz} = 4f_{44}z^3 + 3f_{43}z^2 + 2f_{42}z + f_{41},$$

let $e = 4f_{44}$, $f = 3f_{43}$, $g = 2f_{42}$ and $h = f_{41}$, then we have:

$$\frac{dR(z)}{dz} = ez^3 + fz^2 + gz + h.$$

Suppose that

$$\varphi = f^2 - 3eg, \chi = fg - 9eh,$$

$$\psi = g^2 - 3fh, \Delta = \chi^2 - 4\varphi\psi,$$

we can obtain some properties about polynomials $R(z)$:

(1) For any $z = k^2$, $f_{40} > 0$.

(2) For $z = k^2$, $R(z) \rightarrow +\infty$ as $z \rightarrow +\infty$.

(3) According to Shengjin's criterion, we have:

(i) When $\varphi = \chi = 0$, $\frac{dR(z)}{dz} = 0$ has three equal real roots:

$$z_1 = z_2 = z_3 = -\frac{f}{3e} = -\frac{g}{f} = -\frac{3h}{g}.$$

(ii) When $\Delta = 0$, $\frac{dR(z)}{dz} = 0$ has the following roots:

$$z_1 = -\frac{f}{e} + K, z_2 = z_3 = -\frac{K}{2},$$

where, $K = \frac{\chi}{\psi} (\psi \neq 0)$.

(iii) When $\Delta > 0$, $\frac{dR(z)}{dz} = 0$ has the following roots:

$$z_1 = -\frac{f + \sqrt[3]{y_1} + \sqrt[3]{y_2}}{3e},$$

$$z_{2,3} = \frac{-f + \frac{1}{2}(\sqrt[3]{y_1} + \sqrt[3]{y_2}) \pm \frac{\sqrt{3}}{2}(\sqrt[3]{y_1} - \sqrt[3]{y_2})i}{3e},$$

where, $y_{1,2} = \varphi f + 3e(\frac{-\chi \pm \sqrt{\chi^2 - 4\varphi\psi}}{2})$.

(iv) When $\Delta < 0$, $\frac{dR(z)}{dz} = 0$ has the following roots :

$$z_1 = -\frac{f + 2\sqrt{\varphi}\cos(\frac{\phi}{3})}{3e},$$

$$z_{2,3} = \frac{-f + \sqrt{\varphi}(\cos\frac{\phi}{3} \pm \sqrt{3}\sin\frac{\phi}{3})}{3e},$$

where, $\phi = \arccos T$, $T = \frac{2\varphi f - 3e\chi}{2\varphi\sqrt{\varphi}} (\varphi > 0, -1 < T < 1)$.

When $\bar{\Upsilon}_4(k) < 0$ and (2.5) hold, Turing pattern occurs. As a consequence, we have the following conclusion:

(1) If $\varphi = \chi = 0$, $R(z)$ has an extreme point $z_1 = -\frac{f}{3e} = -\frac{g}{f} = -\frac{3h}{g}$ and it is a minimum point and z_2 is the maximum point. The sufficient conditions for the occurrence of Turing pattern are as follows:

$$\begin{cases} z_1 > 0, \\ R(z_1) < 0. \end{cases}$$

(2) If $\Delta = 0$, $R(z)$ has two extreme points z_1, z_2 and the minimum is to the right of the maximum. The sufficient conditions for the occurrence of Turing pattern are as follows:

$$\begin{cases} \max(z_1, z_2) > 0, \\ R(\max(z_1, z_2)) < 0. \end{cases}$$

(3) If $\Delta > 0$, $R(z)$ has a real extreme point z_1 . The following inequalities are the conditions for producing Turing pattern:

$$\begin{cases} z_1 > 0, \\ R(z_1) < 0. \end{cases}$$

(4) If $\Delta < 0$, $R(z)$ has three extreme points z_1, z_2, z_3 . Suppose $z_1 < z_2 < z_3$, then z_1 and z_3 are the minimum points. We consider the conditions for the occurrence of Turing pattern in two cases:

(a) When $z_1 > 0$,

$$\min\{R(z_1), R(z_3)\} < 0.$$

(b) When $z_1 < 0$,

$$\begin{cases} z_3 > 0, \\ R(z_3) < 0. \end{cases}$$

Case 2. $\bar{\Upsilon}_1(k)\bar{\Upsilon}_2(k) < \bar{\Upsilon}_3(k)$.

Let $R_1(k^2) = \bar{\Upsilon}_1(k)\bar{\Upsilon}_2(k) - \bar{\Upsilon}_3(k)$ and $z = k^2$, then

$$R_1(z) = r_{13}z^3 + r_{12}z^2 + r_{11}z + r_{10},$$

where,

$$r_{13} = 3\beta^2 + 9\beta + 8,$$

$$r_{12} = -\frac{1}{\tau}(\beta^2\gamma\tau + 10\beta\gamma\tau - 2\beta^2 - 6\beta\tau + 10\gamma\tau - 20\beta - 15\tau - 20),$$

$$r_{11} = \frac{1}{\tau^2}(\beta\gamma^2\tau^2 - 2\beta\gamma\tau^2 + 2\gamma^2\tau^2 - 8\beta\gamma\tau - 10\gamma\tau^2 + 4\beta\tau - 16\gamma\tau + 3\tau^2 + 6\beta + 20\tau + 12),$$

$$r_{10} = \frac{1}{\tau^3(\eta\rho - \eta^2 + 2\gamma^2)}(\rho\eta\gamma^2\tau^3 - \eta^2\gamma^2\tau^3 + 2\gamma^4\tau^3 + 2\rho\eta\gamma^2\tau^2 - \rho\eta\gamma\tau^3 - 2\eta^2\gamma^2\tau^2 + \eta^2\gamma\tau^3 + 4\gamma^2\tau^2 - 2\gamma^3\tau^3 - 8\rho\eta\gamma\tau^2 + 8\eta^2\gamma\tau^2 - 16\gamma^3\tau^2 - 6\rho\eta\gamma\tau + 2\rho\eta\tau^2 + 6\eta^2\gamma\tau - 2\eta^2\tau^2 - 12\gamma^3\tau + 4\gamma^2\tau^2 + 6\rho\eta\tau - 6\eta^2\tau + 10\gamma^2\tau + 2\eta\rho - 2\eta^2 + 4\gamma^2).$$

We get two properties of $R_1(z)$:

- (1) For any $z = k^2$, we have $R_1(z) \rightarrow +\infty$ when $z \rightarrow +\infty$.
- (2) The first derivative of $R_1(z)$ is as follows:

$$\frac{dR_1(z)}{dz} = 3r_{13}z^2 + 2r_{12}z + r_{11} = 0.$$

This equation has two roots, which implies $R_1(z)$ has two extreme points:

$$z_1 = \frac{-r_{12} + \sqrt{r_{12}^2 - 3r_{13}r_{11}}}{3r_{13}}$$

and

$$z_2 = \frac{-r_{12} - \sqrt{r_{12}^2 - 3r_{13}r_{11}}}{3r_{13}}.$$

Applying the above properties, it yields

$$z_{max} = z_2 < z_{min} = z_1.$$

When $R_1(z)_{min} = R_1(z_1) < 0$, the pattern is induced. The minimum point z_1 is the square of the wave number k , and it should be guaranteed to be positive. Based the above analysis, the system (2.3) produces Turing bifurcation when the conditions

$$\begin{cases} r_{12}^2 - 3r_{13}r_{11} > 0, \\ z_1 > 0, \\ R_1(z_1) < 0. \end{cases}$$

are certified.

Case 3. $\bar{\Upsilon}_1(k)\bar{\Upsilon}_2(k)\bar{\Upsilon}_3(k) > \bar{\Upsilon}_3^2(k) + \bar{\Upsilon}_1^2(k)\bar{\Upsilon}_4(k)$.

This case is complicated to analyze and we do not consider it in this paper.

We show the dispersion relation in Fig. 2 with fixed parameters: $\eta = 2.6, \gamma = 1.2, \beta = 30$, and τ has different values. Turing patterns are induced under the conditions $\text{Re}(\lambda) = 0$ when $k = 0$ and $\text{Re}(\lambda) > 0$ when $k > 0$.

3 Multiple Scale Analysis

In this part, we derive the amplitude equation to reveal the spatiotemporal behavior of the attachment of Turing bifurcation points [25, 27]. Only when the wave number disturbance approaches the critical value k_T , the steady state solution will become unstable. This paper mainly studies the vegetation-water model, and the process of changing vegetation pattern structure by

Figure 2: Dispersion relation of system (2.3). In the case of fixed other parameters, τ takes four different values. When τ takes three different values, with the increase of k , the real part of the characteristic root is greater than zero, the spatial pattern of system (2.3) is generated.

controlling the parameters of amplitude equation is studied. In order to obtain the control parameter τ , the critical wave number $k = k_T$ is calculated at first. Substituting k_T into $\Upsilon_4(k) = 0$, bifurcation threshold τ_T is obtained.

We rewrite the systems (2.3) at the equilibrium $E_2 = (N_2, W_2, V_2, P_2)$:

$$\begin{cases} \frac{\partial n}{\partial t} = b_{11}n + b_{12}w + b_{13}v + b_{14}p + N_1(n, w, v, p) + \Delta n, \\ \frac{\partial w}{\partial t} = b_{21}n + b_{22}w + b_{23}v + b_{24}p + N_2(n, w, v, p) + \beta\Delta w, \\ \frac{\partial v}{\partial t} = b_{31}n + b_{32}w + b_{33}v + b_{34}p + N_3(n, w, v, p) + \Delta v, \\ \frac{\partial p}{\partial t} = b_{41}n + b_{42}w + b_{43}v + b_{44}p + N_4(n, w, v, p) + \Delta p, \end{cases} \quad (3.1)$$

where,

$$N_1(n, w, v, p) = n^2w,$$

$$N_2(n, w, v, p) = -n^2w,$$

$$N_3(n, w, v, p) = 0,$$

$$N_4(n, w, v, p) = 0.$$

Since near the onset $\tau = \tau_T$, the following expression is a form of the solution of system (2.3):

$$U = U_u + \sum_{j=1}^3 U_0 [A_j e^{i\vec{k}_j \cdot \vec{r}} + \bar{A}_j e^{-i\vec{k}_j \cdot \vec{r}}],$$

where $\vec{k}_j$ and $-\vec{k}_j$ are a pair of oscillatory wave vectors and $|k_j| = k_T$, and the direction is different.

Take advantage of the above formula, we can obtain the solution of (2.3) as the following form:

$$U^0 = \sum_{j=1}^3 U_0 [A_j e^{i\vec{k}_j \cdot \vec{r}} + \bar{A}_j e^{-i\vec{k}_j \cdot \vec{r}}].$$

Let $U = (n, w, v, p)^T$ and $N = (N_1, N_2, N_3, N_4)^T$, then system (2.3) can be rewritten as:

$$\frac{\partial U}{\partial t} = LU + N, \quad (3.2)$$

where,

$$L = \begin{pmatrix} b_{11} + \Delta & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} + \delta\Delta & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} + \Delta & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} + \Delta \end{pmatrix}.$$

Let

$$L = L_T + (\tau_T - \tau)M, \quad (3.3)$$

where

$$L_T = \begin{pmatrix} a_{11}^* + \Delta & a_{12}^* & a_{13}^* & a_{14}^* \\ a_{21}^* & a_{22}^* + \delta\Delta & a_{23}^* & a_{24}^* \\ a_{31}^* & a_{32}^* & a_{33}^* + \Delta & a_{34}^* \\ a_{41}^* & a_{42}^* & a_{43}^* & a_{44}^* + \Delta \end{pmatrix},$$

and

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix}.$$

$$\begin{aligned} a_{11}^* &= b_{11}, & a_{12}^* &= 0, & a_{13}^* &= b_{13}, & a_{14}^* &= 0, \\ a_{21}^* &= b_{21}, & a_{22}^* &= b_{22}, & a_{23}^* &= b_{23}, & a_{24}^* &= 0, \\ a_{31}^* &= 0, & a_{32}^* &= 0, & a_{33}^* &= -\frac{1}{\tau}, & a_{34}^* &= \frac{1}{\tau}, \\ a_{41}^* &= 0, & a_{42}^* &= \frac{1}{\tau}, & a_{43}^* &= 0, & a_{44}^* &= -\frac{1}{\tau}, \end{aligned}$$

$$\begin{aligned} m_{11} &= \frac{b_{11} - a_{11}^*}{\tau_T - \tau}, & m_{12} &= \frac{b_{12} - a_{12}^*}{\tau_T - \tau}, & m_{13} &= \frac{b_{13} - a_{13}^*}{\tau_T - \tau}, & m_{14} &= \frac{b_{14} - a_{14}^*}{\tau_T - \tau}, \\ m_{21} &= \frac{b_{21} - a_{21}^*}{\tau_T - \tau}, & m_{22} &= \frac{b_{22} - a_{22}^*}{\tau_T - \tau}, & m_{23} &= \frac{b_{23} - a_{23}^*}{\tau_T - \tau}, & m_{24} &= \frac{b_{24} - a_{24}^*}{\tau_T - \tau}, \\ m_{31} &= \frac{b_{31} - a_{31}^*}{\tau_T - \tau}, & m_{32} &= \frac{b_{32} - a_{32}^*}{\tau_T - \tau}, & m_{33} &= \frac{b_{33} - a_{33}^*}{\tau_T - \tau}, & m_{34} &= \frac{b_{34} - a_{34}^*}{\tau_T - \tau}, \\ m_{41} &= \frac{b_{41} - a_{41}^*}{\tau_T - \tau}, & m_{42} &= \frac{b_{42} - a_{42}^*}{\tau_T - \tau}, & m_{43} &= \frac{b_{43} - a_{43}^*}{\tau_T - \tau}, & m_{44} &= \frac{b_{44} - a_{44}^*}{\tau_T - \tau}. \end{aligned}$$

Next we will use multiscale analysis, let

$$\tau_T - \tau = \varepsilon\tau_1 + \varepsilon^2\tau_2 + \varepsilon^3\tau_3 + o(\varepsilon^4), \quad (3.4)$$

$$U = \begin{pmatrix} n \\ w \\ v \\ p \end{pmatrix} = \varepsilon \begin{pmatrix} n_1 \\ w_1 \\ v_1 \\ p_1 \end{pmatrix} + \varepsilon^2 \begin{pmatrix} n_2 \\ w_2 \\ v_2 \\ p_2 \end{pmatrix} + \varepsilon^3 \begin{pmatrix} n_3 \\ w_3 \\ v_3 \\ p_3 \end{pmatrix} + o(\varepsilon^4), \quad (3.5)$$

$$N = \varepsilon^2 h_2 + \varepsilon^3 h_3 + o(\varepsilon^4), \quad (3.6)$$

where

$$h_2 = \begin{pmatrix} h_{21} \\ h_{22} \\ 0 \\ 0 \end{pmatrix}, \quad h_3 = \begin{pmatrix} n_1^2 w_1 \\ -n_1^2 w_1 \\ 0 \\ 0 \end{pmatrix},$$

$$h_{21} = \frac{\eta - \sqrt{\eta^2 - 4\gamma^2}}{2} n_1^2 + \frac{\gamma}{\eta - \sqrt{\eta^2 - 4\gamma^2}} n_1 w_1,$$

$$h_{22} = -\frac{\eta - \sqrt{\eta^2 - 4\gamma^2}}{2} n_1^2 - \frac{\gamma}{\eta - \sqrt{\eta^2 - 4\gamma^2}} n_1 w_1.$$

Let

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial T_0} + \varepsilon \frac{\partial}{\partial T_1} + \varepsilon^2 \frac{\partial}{\partial T_2} + o(\varepsilon^3). \quad (3.7)$$

$\frac{\partial}{\partial T_0}$ corresponds to the fast variable. Then we can obtain the following derivative:

$$\frac{\partial A}{\partial t} = \varepsilon \frac{\partial A}{\partial T_1} + \varepsilon^2 \frac{\partial A}{\partial T_2} + o(\varepsilon^3), \quad (3.8)$$

where $T_0 = t, T_1 = \varepsilon t, T_2 = \varepsilon^2 t$.

Noting (3.2) and (3.3), it can be verified the following equation:

$$\frac{\partial U}{\partial t} = (L_T + (\tau_T - \tau)M)U + N = L_T U + (\tau_T - \tau)MU + N. \quad (3.9)$$

We substitute (3.4)-(3.7) into (3.9) and obtain the following equations:

$$L_T \begin{pmatrix} n_1 \\ w_1 \\ v_1 \\ p_1 \end{pmatrix} = 0, \quad (3.10)$$

$$L_T \begin{pmatrix} n_2 \\ w_2 \\ v_2 \\ p_2 \end{pmatrix} = \frac{\partial}{\partial T_1} \begin{pmatrix} n_1 \\ w_1 \\ v_1 \\ p_1 \end{pmatrix} - \tau_1 M \begin{pmatrix} n_1 \\ w_1 \\ v_1 \\ p_1 \end{pmatrix} - h_2, \quad (3.11)$$

$$L_T \begin{pmatrix} n_3 \\ w_3 \\ v_3 \\ p_3 \end{pmatrix} = \frac{\partial}{\partial T_1} \begin{pmatrix} n_2 \\ w_2 \\ v_2 \\ p_2 \end{pmatrix} + \frac{\partial}{\partial T_2} \begin{pmatrix} n_1 \\ w_1 \\ v_1 \\ p_1 \end{pmatrix} - \tau_1 M \begin{pmatrix} n_2 \\ w_2 \\ v_2 \\ p_2 \end{pmatrix} - \tau_2 M \begin{pmatrix} n_1 \\ w_1 \\ v_1 \\ p_1 \end{pmatrix} - h_3. \quad (3.12)$$

Clearly, L_T can be thought of as a linear operator at a critical point. $(n_1, w_1, v_1, p_1)^T$ is a linear combination of eigenvectors which is corresponding to eigenvalue of zero. It then follows from (3.10) that:

$$\begin{pmatrix} n_1 \\ w_1 \\ v_1 \\ p_1 \end{pmatrix} = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \\ 1 \end{pmatrix} (\Theta_1 e^{i\vec{k}_1 \cdot \vec{r}} + \Theta_2 e^{i\vec{k}_2 \cdot \vec{r}} + \Theta_3 e^{i\vec{k}_3 \cdot \vec{r}}) + c.c., \quad (3.13)$$

where

$$l_1 = \frac{a_{13}^*}{(k_T^2 - a_{11}^*)(1 + \tau k_T^2)}, l_2 = 1 + \tau k_T^2, l_3 = \frac{1}{1 + \tau k_T^2}.$$

and $K_j = K_T, j = 1, 2, 3$. Θ_j is the amplitude of $e^{i\vec{k}_j \cdot \vec{r}}$ under the first order perturbation.

The following expression can be deduced by Eq. (3.11) directly:

$$\begin{aligned}
L_T \begin{pmatrix} n_2 \\ w_2 \\ v_2 \\ p_2 \end{pmatrix} &= \frac{\partial}{\partial T_1} \begin{pmatrix} n_1 \\ w_1 \\ v_1 \\ p_1 \end{pmatrix} - \tau_1 M \begin{pmatrix} n_1 \\ w_1 \\ v_1 \\ p_1 \end{pmatrix} - h_2 \\
&= \frac{\partial}{\partial T_1} \begin{pmatrix} n_1 \\ w_1 \\ v_1 \\ p_1 \end{pmatrix} - \tau_1 \begin{pmatrix} m_{11}n_1 + m_{12}w_1 + m_{13}v_1 + m_{14}p_1 \\ m_{21}n_1 + m_{22}w_1 + m_{23}v_1 + m_{24}p_1 \\ m_{31}n_1 + m_{32}w_1 + m_{33}v_1 + m_{34}p_1 \\ m_{41}n_1 + m_{42}w_1 + m_{43}v_1 + m_{44}p_1 \end{pmatrix} - \begin{pmatrix} h_{21} \\ h_{22} \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} F_n \\ F_w \\ F_v \\ F_p \end{pmatrix}.
\end{aligned} \tag{3.14}$$

Applying the Fredholm solvability condition, we conclude that a sufficient condition for (3.14) to have a nontrivial solution is that the function of the right of (3.14) is orthogonal to zero eigenvector of L_T^+ and the expression is as follows:

$$\begin{pmatrix} 1 \\ l_2^+ \\ l_3^+ \\ l_4^+ \end{pmatrix} e^{-i\vec{k}_j \cdot \vec{r}}, j = 1, 2, 3,$$

where

$$l_2^+ = \frac{d_1 k_T^2 - a_{11}^*}{a_{21}^*}, l_3^+ = \frac{\tau_T (d_1 k_T^2 - a_{11}^*) (d_2 k_T^2 - a_{22}^*) - \tau_T a_{12}^* a_{21}^*}{a_{21}^*}, l_4^+ = \frac{a_{11}^*}{a_{21}^*} l_2^+ l_3^+.$$

According to the orthogonal condition, we can obtain that

$$(1, l_2^+, l_3^+, l_4^+) \begin{pmatrix} F_n^j \\ F_w^j \\ F_v^j \\ F_p^j \end{pmatrix} = 0,$$

where $F_n^j, F_w^j, F_v^j, F_p^j$ represent the coefficients corresponding to $e^{i\vec{k}_j \cdot \vec{r}}$ in F_n, F_w, F_v, F_p , then we have:

$$\begin{pmatrix} F_n \\ F_w \\ F_v \\ F_p \end{pmatrix} = \begin{pmatrix} F_n \\ F_w \\ F_v \\ F_p \end{pmatrix} e^{i\vec{k}_1 \cdot \vec{r}} + \begin{pmatrix} F_n \\ F_w \\ F_v \\ F_p \end{pmatrix} e^{i\vec{k}_2 \cdot \vec{r}} + \begin{pmatrix} F_n \\ F_w \\ F_v \\ F_p \end{pmatrix} e^{i\vec{k}_3 \cdot \vec{r}}.$$

Combining (3.13) with (3.14), one has:

$$\begin{pmatrix} F_n^1 \\ F_w^1 \\ F_v^1 \\ F_p^1 \end{pmatrix} = \begin{pmatrix} l_1 \frac{\partial \Theta_1}{\partial T_1} \\ l_2 \frac{\partial \Theta_1}{\partial T_1} \\ l_3 \frac{\partial \Theta_1}{\partial T_1} \\ \frac{\partial \Theta_1}{\partial T_1} \end{pmatrix} - \tau_1 \begin{pmatrix} m_{11}l_1 + m_{12}l_2 + m_{13}l_3 + m_{14} \\ m_{21}l_1 + m_{22}l_2 + m_{23}l_3 + m_{24} \\ m_{31}l_1 + m_{32}l_2 + m_{33}l_3 + m_{34} \\ m_{41}l_1 + m_{42}l_2 + m_{43}l_3 + m_{44} \end{pmatrix} \Theta_1 + \begin{pmatrix} h_{21} \\ h_{22} \\ 0 \\ 0 \end{pmatrix} \bar{\Theta}_2 \bar{\Theta}_3, \quad (3.15)$$

$$\begin{pmatrix} F_n^2 \\ F_w^2 \\ F_v^2 \\ F_p^2 \end{pmatrix} = \begin{pmatrix} l_1 \frac{\partial \Theta_2}{\partial T_1} \\ l_2 \frac{\partial \Theta_2}{\partial T_1} \\ l_3 \frac{\partial \Theta_2}{\partial T_1} \\ \frac{\partial \Theta_2}{\partial T_1} \end{pmatrix} - \tau_1 \begin{pmatrix} m_{11}l_1 + m_{12}l_2 + m_{13}l_3 + m_{14} \\ m_{21}l_1 + m_{22}l_2 + m_{23}l_3 + m_{24} \\ m_{31}l_1 + m_{32}l_2 + m_{33}l_3 + m_{34} \\ m_{41}l_1 + m_{42}l_2 + m_{43}l_3 + m_{44} \end{pmatrix} \Theta_2 + \begin{pmatrix} h_{21} \\ h_{22} \\ 0 \\ 0 \end{pmatrix} \bar{\Theta}_1 \bar{\Theta}_3, \quad (3.16)$$

$$\begin{pmatrix} F_n^3 \\ F_w^3 \\ F_v^3 \\ F_p^3 \end{pmatrix} = \begin{pmatrix} l_1 \frac{\partial \Theta_3}{\partial T_1} \\ l_2 \frac{\partial \Theta_3}{\partial T_1} \\ l_3 \frac{\partial \Theta_3}{\partial T_1} \\ \frac{\partial \Theta_3}{\partial T_1} \end{pmatrix} - \tau_1 \begin{pmatrix} m_{11}l_1 + m_{12}l_2 + m_{13}l_3 + m_{14} \\ m_{21}l_1 + m_{22}l_2 + m_{23}l_3 + m_{24} \\ m_{31}l_1 + m_{32}l_2 + m_{33}l_3 + m_{34} \\ m_{41}l_1 + m_{42}l_2 + m_{43}l_3 + m_{44} \end{pmatrix} \Theta_3 + \begin{pmatrix} h_{21} \\ h_{22} \\ 0 \\ 0 \end{pmatrix} \bar{\Theta}_1 \bar{\Theta}_2. \quad (3.17)$$

Applying the Fredholm solvability condition, we get that:

$$\left\{ \begin{aligned} (l_1 + l_2 l_2^+ + l_3 l_3^+ + l_4^+) \frac{\partial \Theta_1}{\partial T_1} &= \tau_1 [(m_{11}l_1 + m_{12}l_2 + m_{13}l_3 + m_{14}) + l_2^+ (m_{21}l_1 + m_{22}l_2 + m_{23}l_3 \\ &\quad + m_{24}) + l_3^+ (m_{31}l_1 + m_{32}l_2 + m_{33}l_3 + m_{34}) + (m_{41}l_1 + m_{42}l_2 \\ &\quad + m_{43}l_3 + m_{44})] \Theta_1 - (h_{21} + l_2^+ h_{22}) \bar{\Theta}_2 \bar{\Theta}_3, \\ (l_1 + l_2 l_2^+ + l_3 l_3^+ + l_4^+) \frac{\partial \Theta_2}{\partial T_1} &= \tau_1 [(m_{11}l_1 + m_{12}l_2 + m_{13}l_3 + m_{14}) + l_2^+ (m_{21}l_1 + m_{22}l_2 + m_{23}l_3 \\ &\quad + m_{24}) + l_3^+ (m_{31}l_1 + m_{32}l_2 + m_{33}l_3 + m_{34}) + (m_{41}l_1 + m_{42}l_2 \\ &\quad + m_{43}l_3 + m_{44})] \Theta_2 - (h_{21} + l_2^+ h_{22}) \bar{\Theta}_1 \bar{\Theta}_3, \\ (l_1 + l_2 l_2^+ + l_3 l_3^+ + l_4^+) \frac{\partial \Theta_3}{\partial T_1} &= \tau_1 [(m_{11}l_1 + m_{12}l_2 + m_{13}l_3 + m_{14}) + l_2^+ (m_{21}l_1 + m_{22}l_2 + m_{23}l_3 \\ &\quad + m_{24}) + l_3^+ (m_{31}l_1 + m_{32}l_2 + m_{33}l_3 + m_{34}) + (m_{41}l_1 + m_{42}l_2 \\ &\quad + m_{43}l_3 + m_{44})] \Theta_3 - (h_{21} + l_2^+ h_{22}) \bar{\Theta}_1 \bar{\Theta}_2. \end{aligned} \right. \quad (3.18)$$

The second formula of equation (3.11) is solved as follows:

$$\begin{aligned}
\begin{pmatrix} n_2 \\ w_2 \\ v_2 \\ p_2 \end{pmatrix} &= \begin{pmatrix} N_0 \\ W_0 \\ V_0 \\ P_0 \end{pmatrix} + \sum_{i=1}^3 \begin{pmatrix} N_i \\ W_i \\ V_i \\ P_i \end{pmatrix} e^{i\vec{k}_i \cdot \vec{r}} + \sum_{i=1}^3 \begin{pmatrix} N_{ii} \\ W_{ii} \\ V_{ii} \\ P_{ii} \end{pmatrix} e^{i2\vec{k}_i \cdot \vec{r}} \\
&+ \begin{pmatrix} N_{12} \\ W_{12} \\ V_{12} \\ P_{12} \end{pmatrix} e^{i(\vec{k}_1 - \vec{k}_2) \cdot \vec{r}} + \begin{pmatrix} N_{23} \\ W_{23} \\ V_{23} \\ P_{23} \end{pmatrix} e^{i(\vec{k}_2 - \vec{k}_3) \cdot \vec{r}} + \begin{pmatrix} N_{31} \\ W_{31} \\ V_{31} \\ P_{31} \end{pmatrix} e^{i(\vec{k}_3 - \vec{k}_1) \cdot \vec{r}} + c.c.,
\end{aligned} \tag{3.19}$$

where

$$\begin{aligned}
N_0 &= n_0(|\Theta_1|^2 + |\Theta_2|^2 + |\Theta_3|^2), \\
W_0 &= w_0(|\Theta_1|^2 + |\Theta_2|^2 + |\Theta_3|^2), \\
V_0 &= v_0(|\Theta_1|^2 + |\Theta_2|^2 + |\Theta_3|^2), \\
P_0 &= p_0(|\Theta_1|^2 + |\Theta_2|^2 + |\Theta_3|^2), \\
N_i &= l_1 P_i, W_i = l_2 P_i, V_i = l_3 P_i, \\
N_{ii} &= n_{11} \Theta_i^2, W_{ii} = w_{11} \Theta_i^2, V_{ii} = v_{11} \Theta_i^2, P_{ii} = p_{11} \Theta_i^2, \\
\begin{pmatrix} N_{ij} \\ W_{ij} \\ V_{ij} \\ P_{ij} \end{pmatrix} &= \begin{pmatrix} n_{12} \\ w_{12} \\ v_{12} \\ p_{12} \end{pmatrix} \Theta_i \bar{\Theta}_j,
\end{aligned}$$

where $n_0, w_0, v_0, p_0, n_{11}, w_{11}, v_{11}, p_{11}, n_{12}, w_{12}, v_{12}, p_{12}$ are known.

For ε^3 , we get

$$\begin{aligned}
L_T \begin{pmatrix} n_3 \\ w_3 \\ v_3 \\ p_3 \end{pmatrix} &= \frac{\partial}{\partial T_1} \begin{pmatrix} n_2 \\ w_2 \\ v_2 \\ p_2 \end{pmatrix} + \frac{\partial}{\partial T_2} \begin{pmatrix} n_1 \\ w_1 \\ v_1 \\ p_1 \end{pmatrix} - \tau_1 \begin{pmatrix} m_{11}n_2 + m_{12}w_2 + m_{13}v_2 + m_{14}p_2 \\ m_{21}n_2 + m_{22}w_2 + m_{23}v_2 + m_{24}p_2 \\ m_{31}n_2 + m_{32}w_2 + m_{33}v_2 + m_{34}p_2 \\ m_{41}n_2 + m_{42}w_2 + m_{43}v_2 + m_{44}p_2 \end{pmatrix} \\
&- \tau_2 M \begin{pmatrix} m_{11}n_1 + m_{12}w_1 + m_{13}v_1 + m_{14}p_1 \\ m_{21}n_1 + m_{22}w_1 + m_{23}v_1 + m_{24}p_1 \\ m_{31}n_1 + m_{32}w_1 + m_{33}v_1 + m_{34}p_1 \\ m_{41}n_1 + m_{42}w_1 + m_{43}v_1 + m_{44}p_1 \end{pmatrix} - \begin{pmatrix} n_1^2 w_1 \\ -n_1^2 w_1 \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} E_n \\ E_w \\ E_v \\ E_p \end{pmatrix}.
\end{aligned} \tag{3.20}$$

Therefore, the following expression can be derived by the above equation:

$$\begin{aligned}
\begin{pmatrix} E_n^1 \\ E_w^1 \\ E_v^1 \\ E_p^1 \end{pmatrix} &= \begin{pmatrix} l_1 \frac{\partial P_1}{\partial T_1} \\ l_2 \frac{\partial P_1}{\partial T_1} \\ l_3 \frac{\partial P_1}{\partial T_1} \\ \frac{\partial P_1}{\partial T_1} \end{pmatrix} + \begin{pmatrix} l_1 \frac{\partial \Theta_1}{\partial T_2} \\ l_2 \frac{\partial \Theta_1}{\partial T_2} \\ l_3 \frac{\partial \Theta_1}{\partial T_2} \\ \frac{\partial \Theta_1}{\partial T_2} \end{pmatrix} - \tau_1 \begin{pmatrix} l_1 m_{11} + l_2 m_{12} + l_3 m_{13} + m_{14} \\ l_1 m_{21} + l_2 m_{22} + l_3 m_{23} + m_{24} \\ l_1 m_{31} + l_2 m_{32} + l_3 m_{33} + m_{34} \\ l_1 m_{41} + l_2 m_{42} + l_3 m_{43} + m_{44} \end{pmatrix} P_1 \\
&\quad - \tau_2 \begin{pmatrix} l_1 m_{11} + l_2 m_{12} + l_3 m_{13} + m_{14} \\ l_1 m_{21} + l_2 m_{22} + l_3 m_{23} + m_{24} \\ l_1 m_{31} + l_2 m_{32} + l_3 m_{33} + m_{34} \\ l_1 m_{41} + l_2 m_{42} + l_3 m_{43} + m_{44} \end{pmatrix} \Theta_1 + \begin{pmatrix} G_{11}|\Theta_1|^2 + G_{12}|\Theta_2|^2 + |\Theta_3|^2 \\ G_{21}|\Theta_1|^2 + G_{22}|\Theta_2|^2 + |\Theta_3|^2 \\ 0 \\ 0 \end{pmatrix} \Theta_1,
\end{aligned} \tag{3.21}$$

$$\begin{aligned}
\begin{pmatrix} E_n^2 \\ E_w^2 \\ E_v^2 \\ E_p^2 \end{pmatrix} &= \begin{pmatrix} l_1 \frac{\partial P_2}{\partial T_1} \\ l_2 \frac{\partial P_2}{\partial T_1} \\ l_3 \frac{\partial P_2}{\partial T_1} \\ \frac{\partial P_2}{\partial T_1} \end{pmatrix} + \begin{pmatrix} l_1 \frac{\partial \Theta_2}{\partial T_2} \\ l_2 \frac{\partial \Theta_2}{\partial T_2} \\ l_3 \frac{\partial \Theta_2}{\partial T_2} \\ \frac{\partial \Theta_2}{\partial T_2} \end{pmatrix} - \tau_1 \begin{pmatrix} l_1 m_{11} + l_2 m_{12} + l_3 m_{13} + m_{14} \\ l_1 m_{21} + l_2 m_{22} + l_3 m_{23} + m_{24} \\ l_1 m_{31} + l_2 m_{32} + l_3 m_{33} + m_{34} \\ l_1 m_{41} + l_2 m_{42} + l_3 m_{43} + m_{44} \end{pmatrix} P_2 \\
&\quad - \tau_2 \begin{pmatrix} l_1 m_{11} + l_2 m_{12} + l_3 m_{13} + m_{14} \\ l_1 m_{21} + l_2 m_{22} + l_3 m_{23} + m_{24} \\ l_1 m_{31} + l_2 m_{32} + l_3 m_{33} + m_{34} \\ l_1 m_{41} + l_2 m_{42} + l_3 m_{43} + m_{44} \end{pmatrix} \Theta_2 + \begin{pmatrix} G_{11}|\Theta_1|^2 + G_{12}|\Theta_2|^2 + |\Theta_3|^2 \\ G_{21}|\Theta_1|^2 + G_{22}|\Theta_2|^2 + |\Theta_3|^2 \\ 0 \\ 0 \end{pmatrix} \Theta_2,
\end{aligned} \tag{3.22}$$

$$\begin{aligned}
\begin{pmatrix} E_n^3 \\ E_w^3 \\ E_v^3 \\ E_p^3 \end{pmatrix} &= \begin{pmatrix} l_1 \frac{\partial P_3}{\partial T_1} \\ l_2 \frac{\partial P_3}{\partial T_1} \\ l_3 \frac{\partial P_3}{\partial T_1} \\ \frac{\partial P_3}{\partial T_1} \end{pmatrix} + \begin{pmatrix} l_1 \frac{\partial \Theta_3}{\partial T_2} \\ l_2 \frac{\partial \Theta_3}{\partial T_2} \\ l_3 \frac{\partial \Theta_3}{\partial T_2} \\ \frac{\partial \Theta_3}{\partial T_2} \end{pmatrix} - \tau_1 \begin{pmatrix} l_1 m_{11} + l_2 m_{12} + l_3 m_{13} + m_{14} \\ l_1 m_{21} + l_2 m_{22} + l_3 m_{23} + m_{24} \\ l_1 m_{31} + l_2 m_{32} + l_3 m_{33} + m_{34} \\ l_1 m_{41} + l_2 m_{42} + l_3 m_{43} + m_{44} \end{pmatrix} P_3 \\
&\quad - \tau_2 \begin{pmatrix} l_1 m_{11} + l_2 m_{12} + l_3 m_{13} + m_{14} \\ l_1 m_{21} + l_2 m_{22} + l_3 m_{23} + m_{24} \\ l_1 m_{31} + l_2 m_{32} + l_3 m_{33} + m_{34} \\ l_1 m_{41} + l_2 m_{42} + l_3 m_{43} + m_{44} \end{pmatrix} \Theta_3 + \begin{pmatrix} G_{11}|\Theta_1|^2 + G_{12}|\Theta_2|^2 + |\Theta_3|^2 \\ G_{21}|\Theta_1|^2 + G_{22}|\Theta_2|^2 + |\Theta_3|^2 \\ 0 \\ 0 \end{pmatrix} \Theta_3,
\end{aligned} \tag{3.23}$$

where,

$$G_{11} = G_{12} = G_{13} = l_1^2 l_2, G_{21} = G_{22} = G_{23} = -l_1^2 l_2.$$

By the Fredholm solvability condition, then it is combined with (3.21)-(3.23) allows us to

deduce the following expression:

$$\left\{ \begin{aligned} & (l_1 + l_2 l_2^+ + l_3 l_3^+ + l_4^+) \left(\frac{\partial P_1}{\partial T_1} + \frac{\partial \Theta_1}{\partial T_2} \right) \\ & = (\tau_1 P_1 + \tau_2 \Theta_1) [(l_1 m_{11} + l_2 m_{12} + l_3 m_{13} + m_{14}) + l_2^+ (l_1 m_{21} + l_2 m_{22} + l_3 m_{23} + m_{24}) \\ & \quad + l_3^+ (l_1 m_{31} + l_2 m_{32} + l_3 m_{33} + m_{34}) + l_4^+ (l_1 m_{41} + l_2 m_{42} + l_3 m_{43} + m_{44})] \\ & \quad - (G_{11} |\Theta_1|^2 + G_{12} |\Theta_2|^2 + |\Theta_3|^2) \Theta_1 - l_2^+ (-G_{11} |\Theta_1|^2 - G_{12} |\Theta_2|^2 + |\Theta_3|^2) \Theta_1, \\ & (l_1 + l_2 l_2^+ + l_3 l_3^+ + l_4^+) \left(\frac{\partial P_2}{\partial T_1} + \frac{\partial \Theta_2}{\partial T_2} \right) \\ & = (\tau_1 P_2 + \tau_2 \Theta_2) [(l_1 m_{11} + l_2 m_{12} + l_3 m_{13} + m_{14}) + l_2^+ (l_1 m_{21} + l_2 m_{22} + l_3 m_{23} + m_{24}) \\ & \quad + l_3^+ (l_1 m_{31} + l_2 m_{32} + l_3 m_{33} + m_{34}) + l_4^+ (l_1 m_{41} + l_2 m_{42} + l_3 m_{43} + m_{44})] \\ & \quad - (G_{11} |\Theta_1|^2 + G_{12} |\Theta_2|^2 + |\Theta_3|^2) \Theta_2 - l_2^+ (-G_{11} |\Theta_1|^2 - G_{12} |\Theta_2|^2 + |\Theta_3|^2) \Theta_2, \\ & (l_1 + l_2 l_2^+ + l_3 l_3^+ + l_4^+) \left(\frac{\partial P_3}{\partial T_1} + \frac{\partial \Theta_3}{\partial T_2} \right) \\ & = (\tau_1 P_3 + \tau_2 W_3) [(l_1 m_{11} + l_2 m_{12} + l_3 m_{13} + m_{14}) + l_2^+ (l_1 m_{21} + l_2 m_{22} + l_3 m_{23} + m_{24}) \\ & \quad + l_3^+ (l_1 m_{31} + l_2 m_{32} + l_3 m_{33} + m_{34}) + l_4^+ (l_1 m_{41} + l_2 m_{42} + l_3 m_{43} + m_{44})] \\ & \quad - (G_{11} |\Theta_1|^2 + G_{12} |\Theta_2|^2 + |\Theta_3|^2) \Theta_3 - l_2^+ (-G_{11} |\Theta_1|^2 - G_{12} |\Theta_2|^2 + |\Theta_3|^2) \Theta_3. \end{aligned} \right. \quad (3.24)$$

Let $A_i = A_i^n = l_3 A_i^w = l_2 A_i^v = l_1 A_i^p$ be the coefficient of $e^{i\vec{k}_j \cdot \vec{r}}$ ($j = 1, 2, 3$), then

$$\begin{pmatrix} A_i^n \\ A_i^w \\ A_i^v \\ A_i^p \end{pmatrix} = \varepsilon \begin{pmatrix} l_1 \\ l_2 \\ l_3 \\ 1 \end{pmatrix} W_i + \varepsilon^2 \begin{pmatrix} l_1 \\ l_2 \\ l_3 \\ 1 \end{pmatrix} P_i + o(\varepsilon^3), i = 1, 2, 3. \quad (3.25)$$

Multiply (3.18) by ε and multiply (3.24) by ε^2 , and merge variables using (3.8) and (3.25). Therefore, the following equation is obtained:

$$\begin{cases} \epsilon_0 \frac{\partial A_1}{\partial t} = \xi A_1 + h \bar{A}_2 \bar{A}_3 - [\alpha_1 |A_1|^2 + \alpha_2 (|A_2|^2 + |A_3|^2)] A_1, \\ \epsilon_0 \frac{\partial A_2}{\partial t} = \xi A_2 + h \bar{A}_1 \bar{A}_3 - [\alpha_1 |A_2|^2 + \alpha_2 (|A_1|^2 + |A_3|^2)] A_2, \\ \epsilon_0 \frac{\partial A_3}{\partial t} = \xi A_3 + h \bar{A}_1 \bar{A}_2 - [\alpha_1 |A_3|^2 + \alpha_2 (|A_1|^2 + |A_2|^2)] A_3, \end{cases} \quad (3.26)$$

where,

$$\epsilon_0 = -\frac{l_1 + l_2 l_2^+ + l_3 l_3^+ + l_4^+}{q \tau_T}, \xi = -\frac{\tau_T - \tau}{\tau_T}, h = \frac{l_2 l_3}{q \tau_T}, \\ \alpha_1 = -\frac{G_{11} + l_2^+ G_{21}}{q \tau_T}, \alpha_2 = -\frac{G_{21} + l_2^+ G_{22}}{q \tau_T},$$

$$q = (l_1 m_{11} + l_2 m_{12} + l_3 m_{13} + m_{14}) + l_2^+ (l_1 m_{21} + l_2 m_{22} + l_3 m_{23} + m_{24}) \\ + l_3^+ (l_1 m_{31} + l_2 m_{32} + l_3 m_{33} + m_{34}) + l_4^+ (l_1 m_{41} + l_2 m_{42} + l_3 m_{43} + m_{44}).$$

Substituting $A_i = \vartheta_i e^{i\varsigma_i}$ into (3.26), we can obtain the following form:

$$\begin{cases} \epsilon_0 \frac{\partial \varsigma}{\partial t} = -h \frac{\vartheta_1^2 \vartheta_2^2 + \vartheta_1^2 \vartheta_3^2 + \vartheta_2^2 \vartheta_3^2}{\vartheta_1 \vartheta_2 \vartheta_3}, \\ \epsilon_0 \frac{\partial \vartheta_1}{\partial t} = \sigma \vartheta_1 + h \vartheta_2 \vartheta_3 \cos \varsigma - \alpha_1 \vartheta_1^3 - \alpha_2 (\vartheta_2^2 + \vartheta_3^2) \vartheta_1, \\ \epsilon_0 \frac{\partial \vartheta_2}{\partial t} = \sigma \vartheta_2 + h \vartheta_1 \vartheta_3 \cos \varsigma - \alpha_1 \vartheta_2^3 - \alpha_2 (\vartheta_1^2 + \vartheta_3^2) \vartheta_2, \\ \epsilon_0 \frac{\partial \vartheta_3}{\partial t} = \sigma \vartheta_3 + h \vartheta_1 \vartheta_2 \cos \varsigma - \alpha_1 \vartheta_3^3 - \alpha_2 (\vartheta_1^2 + \vartheta_2^2) \vartheta_3, \end{cases} \quad (3.27)$$

where $\varsigma = \varsigma_1 + \varsigma_2 + \varsigma_3$.

System (3.27) correspond to four kinds of different pattern structures. Table 1 shows that the corresponding generation conditions of four different pattern structures.

Table 1: Four different pattern structures corresponding to the generation conditions

Pattern structure	Expression	Generation conditions
mixed state	$\vartheta_1 = \frac{ h }{\alpha_2 - \alpha_1}, \vartheta_2 = \vartheta_3 = \sqrt{\frac{\sigma - \alpha_1 \vartheta_1^2}{\alpha_2 + \alpha_1}}$	$\alpha_2 > \alpha_1, \sigma > \sigma_3 = \frac{h^2 \alpha_1}{(\alpha_2 - \alpha_1)^2}$
spots pattern	$\vartheta_1 = \vartheta_2 = \vartheta_3 = \frac{ h \pm \sqrt{h^2 + 4(\alpha_1 + 2\alpha_2)\sigma}}{2(\alpha_1 + 2\alpha_2)}$	$\sigma > \sigma_1 = \frac{-h^2}{4(\alpha_1 + 2\alpha_2)}$
stripes pattern	$\vartheta_1 = \sqrt{\frac{\sigma}{\alpha_1}}, \vartheta_2 = \vartheta_3 = 0$	$\sigma > 0$
stationary state	$\vartheta_1 = \vartheta_2 = \vartheta_3 = 0$	always

4 Numerical results

We conduct numerical simulation according to the results of the above theoretical analysis in this part. The boundary condition is the Neumann boundary condition. The space region studied is a two-dimensional space region with a size of $[0, 100] \times [0, 100]$, the time interval is 500. The space and time steps are $\Delta x = 1$ and $\Delta t = 0.001$, respectively. Several types of vegetation patterns are obtained by using multi-scale theory. The following is mainly to study the effect of intensity of nonlocal effect τ and diffusion coefficient β on vegetation pattern.

Selecting the following parameter values: $\eta = 2.6, \gamma = 1.2$. Through calculation, the values of parameters $h = \frac{l_2 l_3}{q \tau T}, \alpha_1 = -\frac{G_{11} + l_2^+ G_{21}}{q \tau T}, \alpha_2 = -\frac{G_{21} + l_2^+ G_{22}}{q \tau T}, \sigma_1 = \frac{-h^2}{4(\alpha_1 + 2\alpha_2)}, \sigma_2 = 0, \sigma_3 = \frac{h^2 \alpha_1}{(\alpha_2 - \alpha_1)^2}, \sigma_4 = \frac{(2\alpha_1 + \alpha_2)h^2}{(\alpha_2 - \alpha_1)^2}$ can be given. According to reference [33], the pattern structure corresponding to different control parameters is shown in Table 2:

Table 2: The pattern structure corresponding to different control parameters

Interval	Pattern structure
$\sigma \in (\sigma_2, \sigma_3)$	spot pattern
$\sigma \in (\sigma_3, \sigma_4)$	mixed pattern
$\sigma \in (\sigma_4, +\infty)$	stripe pattern

We choose $\tau = 0.22$ and other parameters can be obtained $h = \frac{l_2 l_3}{q \tau T}, \sigma = 0.6273504, \sigma_3 = 4.3764255$. Obviously, inequality $0 = \sigma_2 < \sigma < \sigma_3$ holds and system (2.3) will present spot pattern. Fig. 3 shows that the temporal succession of vegetation pattern when $\tau = 0.22, \beta = 30$. The vegetation is uniformly distributed at the start. With the increase of time, the vegetation

Figure 3: When $\tau = 0.22$, the evolution of vegetation density over time. The pattern finally presents spot structure. Other parameters are fixed as: $\eta = 2.6, \gamma = 1.2, \beta = 30$. The evolution process of pattern is $(a) \rightarrow (b) \rightarrow (c) \rightarrow (d)$.

Figure 4: When $\tau = 0.22$, the evolution of water density over time. Other parameters are fixed as: $\eta = 2.6, \gamma = 1.2, \beta = 30$. The evolution process of pattern is $(a) \rightarrow (b) \rightarrow (c) \rightarrow (d)$.

pattern becomes spot pattern, and the vegetation gathers together in the form of clusters. Fig. 4 shows that the evolution process of the corresponding pattern of water density at different time.

In order to explore the influence of diffusion coefficient β on vegetation pattern, the simulation results are shown in Figure 5. Obviously, the pattern structure changes with the change of β . When $\beta = 8$, vegetation pattern shows the stripe structure and control parameter σ is greater σ_4 (Fig. 5(a)). With the increase of β , strip pattern begins to lose stability and spot pattern appears gradually, σ is between σ_3 and σ_4 , the mixed patterns appear in Fig. 5(b) and Fig. 5(c). When β increases further, pattern shows spot structure in Fig. 5(d), σ is between σ_2 and σ_3 . In the process, the band pattern disappears gradually and the whole space presents spot pattern structure [33]. One can conclude that with the increase of diffusion coefficient, the change of pattern structure is as follows: stripe pattern $\rightarrow$ mixed pattern $\rightarrow$ spot pattern, and vegetation forms higher density clusters.

Figure 5: Different β corresponds to the vegetation patterns. With the increase of β , the pattern changes from stripe structure to spot structure. Other parameters are fixed as: $\eta = 2.6, \gamma = 1.2, \tau = 0.22$. (a) $\beta = 8$; (b) $\beta = 10$; (c) $\beta = 12$; (d) $\beta = 14$.

Moreover, we run another set of simulations for different τ in Fig. 6 with $\eta = 2.6, \gamma = 1.2, \beta = 30$. As can be seen from Fig. 6, with the increase of τ , the spot patterns grow smaller in size and become far more in number which is as shown in Fig. 7. This means that vegetation forms smaller clusters as the intensity of nonlocal effect increases, and the number of vegetation clusters formed increases. One can conclude that the isolation degree of vegetation pattern is increasing and the increased intensity of nonlocal effect is not conducive to the robustness of the ecosystem.

The spatial distribution of vegetation is shown in Fig. 8 which is more intuitively. Fig. 9 shows the impact of τ on vegetation density. We can find that the vegetation density increases with the increase of τ . That is to say, τ is positively correlated with average vegetation density.

Figure 6: Different τ corresponds to the vegetation patterns. With the increase of τ , the pattern changes from stripe structure to spot structure. Other parameters are fixed as: $\eta = 2.6, \gamma = 1.2, \beta = 30$. (a) $\tau = 0.01$; (b) $\tau = 0.1$; (c) $\tau = 0.15$; (d) $\tau = 0.24$.

With the increase of time, the density of vegetation tends to a constant. As a result, one can conclude that the nonlocal effect is beneficial to the growth of vegetation.

In order to further study the influence of parameter values on pattern formation, we carry out relevant numerical simulation. In Fig. 10, we plot the effect τ on the recovery rate and recovery time. Select the initial state of steady-state pattern at $\tau = 0.24$, the time required for different τ to reach the initial state under small perturbation is simulated which is called recovery time. The recovery rate is the rate at which the initial state is reached and is as shown in Figure 10(b). The results show that the recovery time decrease gradually with the increase of τ . On the contrary, the recovery rate increases as τ increases, that is, τ is more closer to 2.4, the recovery rate is more higher.

5 Conclusions and discussion

In this paper, the nonlocal delay term with strong kernel is introduced into the vegetation-water model which is considered the nonlocal effect of water uptake by vegetation roots. A two-variable model with nonlocal delay is transformed into a four-variable reaction diffusion equation. Through mathematical analysis, we deduce the conditions for stability of ODE model and Turing instability of the model with diffusion. The amplitude equation is derived from the theory of multi-scale analysis. According to the coefficients of the amplitude equation, the four structures and stability of the pattern can be determined.

By numerical simulation, the relation between intensity of nonlocal effect τ and vegetation

Figure 7: The number of patterns at different parameter τ with $\eta = 2.6, \gamma = 1.2, \beta = 30$.

Figure 8: Different τ corresponds to the spatial distribution of vegetation. Other parameters are fixed as: $\eta = 2.6$, $\gamma = 1.2$, $\beta = 30$. (a) $\tau = 0.01$; (b) $\tau = 0.1$; (c) $\tau = 0.15$; (d) $\tau = 0.24$.

Figure 9: The change process of the average density of the vegetation with parameter τ .

Figure 10: Different τ corresponds to recovery time and recovery rate. Other parameters are fixed as: $\eta = 2.6, \gamma = 1.2, \beta = 30$. (a) Recovery time; (b) Recovery rate. The recovery time is negatively correlated with τ and the recovery rate is positively correlated with τ .

density is obtained. The results show that vegetation density is positively correlated with intensity of nonlocal effect τ , that is, the greater the intensity of nonlocal effect is, the higher the vegetation density will be. Besides, we can obtain that with the change of τ , the vegetation pattern always presents spot structure, which means that τ will not cause phase change of vegetation pattern. However, the isolation degree of vegetation pattern is increasing and the number of patterns is increasing as τ increase, the increased intensity of nonlocal effect is not conducive to improving the robustness of the ecosystem which provides a strong theoretical basis for vegetation protection. Our results point to the recovery time is negatively correlated with τ , and the recovery rate is positively correlated with τ which implies that the effect of τ on ecosystem resilience.

We next explore the effect of diffusion coefficient of water on pattern structure. The results show that the increase of β leads to the change of pattern structure: stripe pattern $\rightarrow$ mix pattern $\rightarrow$ spot pattern. This suggests that water diffusion has an important influence on vegetation distribution and may lead to desertification in this area.

It is worth noting that strong kernel are studied in this paper, which is very different from the weak Kernel. In fact, the roots of vegetation not only absorb water at the current location, but also absorb water near the roots over a certain period of time, which shows that it is necessary to introduce nonlocal effect into the vegetation model. We expands on the existing work and specifically study the effect of strong kernel on vegetation distribution. According to the numerical simulation of strong kernel, we can see that the intensity of water absorption by roots of vegetation increases firstly and then decreases which is more consistent with the characteristics of water absorption of vegetation in arid and semi-arid regions. It can be seen from our conclusion that the nonlocal delay plays a great role in vegetation growth. Our findings provide a theoretical basis for vegetation protection and early warning of desertification.

It is rewarding mentioning that our methods and results can also be applied to other fields, such as infectious disease models, population models [39,40] and so on. Meanwhile, randomness and climatic factors have important effects on vegetation [41,42]. An interesting topic for future work will be to we will add noise factor and meteorological factor into the vegetation model. For example, temperature, light and other meteorological factors are coupled into the vegetation model and the future vegetation distribution is predicted based on the actual data.

Data Availability Statement

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declaration of Interests

The authors report no conflicts of interest. The authors alone are responsible for the content and writing of this article.

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References

- [1] E. Gilad, J. von Hardenberg, A. Provenzale, M. Shachak, E. Meron, A mathematical model of plants as ecosystem engineers, *Journal of Theoretical Biology* **244** (4) (2007) 680-691.
- [2] F. I. Pugnaire, M. T. Luque, Changes in plant interactions along a gradient of environmental stress. *Oikos* **93** (1) (2001) 42-49.
- [3] R. Lefever, O. Lejeune, On the origin of tiger bush, *Bulletin of Mathematical Biology* **59** (1997) 263-294.
- [4] C. A. Klausmeier, Regular and irregular patterns in semiarid vegetation, *Science* **284** (1999) 1826-1828.
- [5] R. HilleRisLambers, M. Rietkerk, F. van den Bosch, H. H. T. Prins, H. de Kroon, Vegetation pattern formation in semi-arid grazing systems, *Ecology* **82**(1) (2001) 50-61.
- [6] J. A. Bonachela, R. M. Pringle, E. Sheffer, T. C. Coverdale, et al., Termite mounds can increase the robustness of dryland ecosystems to climatic change, *Science* **347** (6222) (2015) 651-655.
- [7] F. Borgogno, P. Dodorico, F. Laio, L. Ridolfi, Mathematical models of vegetation pattern formation in ecohydrology, *Reviews of Geophysics* **47**(1) (2009) RG1005.
- [8] S. Getzin, H. Yizhaq, B. Belld, T. E. Erickson, et al., Discovery of fairy circles in Australia supports self-organization theory, *Proceedings of the National Academy of Sciences* **113**(13) (2016) 3551-3556.
- [9] S. Kéfi, M. Rietkerk, G. G. Katul, Vegetation pattern shift as a result of rising atmospheric CO₂ in arid ecosystems, *Theoretical Population Biology* **74**(4) (2008) 332-344.
- [10] K. Gowda, H. Riecke, M. Silber, Transitions between patterned states in vegetation models for semiarid ecosystems, *Physical review E* **89**(2) (2014) 022701.
- [11] J. A. Sherratt, An analysis of vegetation stripe formation in semi-arid landscapes, *Journal of Mathematical Biology* **51** (2005) 183-197.
- [12] L. Eigentler, J. A. Sherratt, Analysis of a model for banded vegetation patterns in semi-arid environments with nonlocal dispersal, *Journal of Mathematical Biology* **77** (2018) 739-763.

- [13] B. J. Kealy, D. J. Wollkind, A nonlinear stability analysis of vegetative turing pattern formation for an interaction-diffusion plant-surface water model system in an arid flat environment, *Bulletin of Mathematical Biology* **74**(4) (2012) 803-833.
- [14] O. Lejeune, M. Tildi, R. Lefever, Vegetation spots and stripes: Dissipative structures in arid landscapes, *International Journal of Quantum Chemistry* **98**(2) (2004) 261-271.
- [15] S. Kéfi, M. Rietkerk, C. L. Alados, Y. Pueyo, et al., Spatial vegetation patterns and imminent desertification in mediterranean arid ecosystems, *Nature* **449**(7159) (2007) 213-217.
- [16] A. Kletter, J. von Hardenberg, E. Meron, A. Provenzale, Patterned vegetation and rainfall intermittency, *Journal of Theoretical Biology* **256**(4) (2009) 574-583.
- [17] G. Q. Sun, M. Jusup, Z. Jin, Y. Wang, Z. Wang, Pattern transitions in spatial epidemics: Mechanisms and emergent properties, *Physics of Life Reviews* **19** (2016) 43-73.
- [18] A. Doelman, H. van der Ploeg, Homoclinic stripe patterns, *SIAM Journal on Applied Dynamical Systems* **1**(1) (2002) 65-104.
- [19] J. A. Sherratt, Pattern solutions of the Klausmeier model for banded vegetation in semi-arid environments I, *Nonlinearity* **23**(10) (2010) 2657-2675.
- [20] E. Siero, A. Doelman, M. B. Eppinga, J. D. M. Rademacher, M. Rietkerk, K. Siteur, Striped pattern selection by advective reaction-diffusion systems: Resilience of banded vegetation on slopes, *Chaos: An Interdisciplinary Journal of Nonlinear Science* **25**(3) (2015) 036411.
- [21] J. von Hardenberg, E. Meron, M. Shachak, Y. Zarmi, Diversity of vegetation patterns and desertification, *Physical Review Letters* **87**(19) (2001) 198101.
- [22] J. van de Koppel, M. Rietkerk, F. J. Weissing, Catastrophic vegetation shifts and soil degradation in terrestrial grazing systems, *Trends in Ecology Evolution* **12** (1997) 352-356.
- [23] R. M. May, Thresholds and breakpoints in ecosystems with a multiplicity of stable states, *Nature* **269** (1977) 471-477.
- [24] S. Kéfi, M. Rietkerk, M. van Baalen, M. Loreau, Local facilitation, bistability and transitions in arid ecosystems, *Theoretical Population Biology* **71** (2007) 367-379.
- [25] G. Q. Sun, C. H. Wang, L. L. Chang, Y. P. Wu, L. Li, Z. Jin, Effects of feedback regulation on vegetation patterns in semi-arid environments, *Applied Mathematical Modelling* **61** (2018) 200-215.
- [26] P. W. Bates, P. C. Fife, X. Ren, X. Wang, Traveling waves in a convolution model for phase transitions, *Archive for Rational Mechanics and Analysis* **138** (1997) 105-136.
- [27] B. S. Han, Z. C. Wang, Traveling wave solutions in a nonlocal reaction-diffusion population model, *Communications on Pure and Applied Analysis* **15**(3) (2016) 1057-1076.

- [28] M. Alfaro, J. Coville, G. Raoul, Travelling Waves in a Nonlocal Reaction-Diffusion Equation as a Model for a Population Structured by a Space Variable and a Phenotypic Trait, *Journal Communications in Partial Differential Equations* **38**(12) (2013) 2126-2154.
- [29] S. B. Ai, Traveling wave fronts for generalized Fisher equations with spatio-temporal delays, *Journal of Differential Equations* **232** (2007) 104-133.
- [30] H. Berestycki, G. Nadin, B. Perthame, L. Ryzhik, The non-local Fisher-KPP equation: Traveling waves and steady states, *Nonlinearity* **22** (2009) 2813-2844.
- [31] S. A. Gourley, Traveling front solutions of a nonlocal Fisher equation, *Journal of Mathematical Biology* **41** (2000) 272-284.
- [32] J. Al-Omari, S. A. Gourley, Monotone traveling fronts in age-structured reaction-diffusion model of a single species, *Journal of Mathematical Biology* **45** (2002) 294-312.
- [33] B. S. Han, Z. C. Wang, Turing patterns of a LotkaCvolterra competitive system with non-local delay, *International Journal of Bifurcation and Chaos* **28**(7) (2018) 1830021.
- [34] S. Zaytseva, J. P. Shi, L. B Shaw, Model of pattern formation in marsh ecosystems with nonlocal interactions, *Journal of Mathematical Biology* **86** (2019) 655-686.
- [35] S. Thompson, G. Katul, J. Terborgh, P. Alvarez-Loayza, Spatial organization of vegetation arising from non-local excitation with local inhibition in tropical rainforests, *Physica D* **238** (2009) 1061-1067.
- [36] Z. G. Guo, G. Q. Sun, Z. Wang, Z. Jin, L. Li, C. Li, Spatial dynamics of an epidemic model with nonlocal infection, *Applied Mathematics and Computation* **377** (2020) 125158.
- [37] Q. Xue, C. Liu, L. Li, G. Q. Sun, Z. Wang, Interactions of diffusion and nonlocal delay give rise to vegetation patterns in semi-arid environments, *Applied Mathematics and Computation* **399** (2021) 126038.
- [38] S. A. Gourley, J. W. H. So, Dynamics of a food-limited population model incorporating nonlocal delays on a finite domain, *Journal of Mathematical Biology* **44** (2002) 49-78.
- [39] S. Wu, J. Shi, B. Wu, Global existence of solutions and uniform persistence of a diffusive predator-prey model with prey-taxis, *Journal of Differential Equations* **260**(7) (2016) 5847-5874.
- [40] B. E. Ainseba, M. Bendahmane, A. Noussair, A reactionCdiffusion system modeling predator-prey with prey-taxis, *Nonlinear Analysis-Real World Applications* **9**(5) (2008) 2086-2105
- [41] M. J. Steinbauer, J. A. Grytnes, G. Jurasinski, A. Kulonen, et al., Accelerated increase in plant species richness on mountain summits is linked to warming, *Nature* **556** (2018) 231-250.
- [42] C. J. Tucker, I. Y. Fungt, C. D. Keeling, R. H. Gammon, Relationship between atmospheric CO2 variations and a satellite-derived vegetation index, *Nature* **319** (1986) 195-198.