A dynamic comprehensive mathematical model for Malthusian principles

Yichen Ji (yichen.ji89@gmail.com)
Fermilab https://orcid.org/0000-0002-3398-1752
Ming Ji
Ningxia University

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A dynamic comprehensive mathematical model for Malthusian principles

Yichen Ji[*], Ming Ji

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Abstract

The Malthusian Trap[1, 2, 3] is a mechanism that describes how population growth suppresses the average income in the pre-industrial human history. Studies of the pre-industrial world demonstrated the staggering effect of the average income worldwide[4, 5, 6]. From the original Malthusian literature,"*The perpetual tendency in the race of man to increase beyond the means of subsistence is one of the general laws of animated nature which we can have no reason to expect will change.*", the tendency to "*increase beyond the means of subsistence*" is an axiom of the Malthusian Trap mechanism. The the tendency to "*increase beyond the means of subsistence*" can also be understand as increase the birth rate as high as possible. On the other hand, the birth rate is observed to be decreasing drastically in modern society[7, 8, 9]. The Demographic transition theory[10, 11] demonstrate this transition from high-birth-high-death to low-birth-low-death over economical development. This paper developed a comprehensive mathematical model that tries explain the high-birth tendency and the transition from high-birth-high-death states to low-birth-low-death states. In the model, 3 fundamental elimination processes are demonstrated. 1) The High-Birth-Elimination explains the tendency to increasing birth rate; 2) the Low-Average-Elimination forces decreasing living condition; and 3) the High-Quota-Elimination based on extra resource force gain access to extra resources for creatures or economical developments for human being. The elimination process demonstrated two drastically different situation. Under Malthusian trap, the average income decreased as low as possible while birth rate is pushed as high as possible. Out of Malthusian trap, the average income growth rate is pushed high by decreasing birth rate. The model draws a necessary and sufficient condition between the Malthusian trap and the elimination processes, thus explained the transition from high-birth-high-death to low-birth-low-death based on escaping Malthusian trap.

1 Introduction

In his book *A Farewell to Alms: A Brief Economic History of the World*[2], CLACK demonstrated how birth rate and death rate balance based on average income. However, the book did not explain why people cannot choose to decrease birth rate to increase living standard. In his book, CLACK explained that decreasing birth rate will increase average income, which intuitively suggest that decreasing birth rate is the rational option[12]. However, the rational option was not taken. From the original Malthusian literature[1], the answer to the question is"*The perpetual tendency in the race of man to increase beyond the means of subsistence is*

[*]yichen.ji89@gmail.com

one of the general laws of animated nature which we can have no reason to expect will change." and the tendency to "*increase beyond the means of subsistence*" is a fundamental axiom for the Malthusian trap to function, which is a stronger assumption than the rational choice assumption. This paper provides a comprehensive mathematical model that explain such tendency and the whole Malthusian mechanism. The model is based on the definition of average income and the following 6 assumptions:

1. Creature need resource to reproduce.
2. With unlimited resource, creature can only reproduce as much.
3. The more resource, the more offspring reproduce.
4. All creature eventually die.
5. Creature need resource to survive.
6. The more resource, the less creature die.

Based on the production growth rate, the model predict two states of complete different behavior. The under Malthusian trap state, happens when there is NOT enough production growth, is the situation where 3 fundamental elimination principles DO function thus have high-birth-high-death balance and staggered low average income. The out-of Malthusian trap state, happens when there IS enough production growth rate, is the situation where 3 fundamental elimination principles do NOT function and thus have increasing average income and incentive to decrease birth rate.
The 3 fundamental elimination principles cause the following effect:

1. Creature gain as much resource as possible.
2. Creature exhaust their resource to reproduce.
3. Creature maximize reproduction efficiency by achieve birth-death balance at as low average income as possible.

Based on the model, the process of transition from under Malthusian trap state to out-of Malthusian trap state coincides with the demographic transition theory.

2 Basic Relation Between Resource and Population

Start with the basic definition of average income.

$$A = \frac{P}{N} \tag{1}$$

Where in eq1 P is the total production of resource, N is the population of some creature and A is the average amount of resource for every individual or average income. Take time derivative of eq1 and apply the standard dot notation for time derivative $dA/dt = \dot{A}$ and simplify.

$$\frac{\dot{A}}{A} = \frac{\dot{P}}{P} - \frac{\dot{N}}{N} \tag{2}$$

In eq2, further define the population N as sum of three function: an constant initial population N_0, the population that born $N_b = N_b(t)$ and the population that died $N_d = N_d(t)$ over the period of time t.

$$N = N_0 + N_b - N_d \tag{3}$$
$$\dot{N} = \dot{N}_b - \dot{N}_d \tag{4}$$

Substitute eq4 into eq2

$$\frac{\dot{A}}{A} = \frac{\dot{P}}{P} - \frac{\dot{N}_b}{N} + \frac{\dot{N}_d}{N} \tag{5}$$

Take the simplification $\dot{A}/A = a$, $\dot{P}/P = p$, $\dot{N}/N = r$, $\dot{N}_b/N = b$ and $\dot{N}_d/N = d$ and substitute into eq5

$$p - a = b - d = r \tag{6}$$

In eq 6, p is the percentage economic growth rate or percentage production growth rate; a is the average income percentage growth rate; r is the population percentage growth rate; b and d are percentage birth and death rate.[13]

2.1 r and R_0 from death and birth rate

Based on the death function, the average life span of a population can be calculated as $N/\dot{N}_d$. The average life span is the average time takes for a population to replace every individual.

$$L = \frac{N}{\dot{N}_d} = \frac{1}{d} \tag{7}$$

From the average life span, reproduction rate R_0 can be calculated as

$$R_0 = e^{rL} = e^{\frac{p-a}{d}} \approx 1 + \frac{p-a}{d} \tag{8}$$
$$R_0 = e^{\frac{b}{d}-1} \approx \frac{b}{d} \tag{9}$$

The population is under equilibrium when $R_0 = 1$, $r = 0$ and $b = d$. The population is growing when $R_0 > 1$, $r > 0$ and $b > d$. The population is decreasing when $R_0 < 1$, $r < 0$ and $b < d$.

2.2 A differential equation system describe average income and population for multiple phenotypes

In eq6, every term is a percentage change rate. A relation of percentage change rate between population r, resource production/gathering p and average resource a is obtained. Define kth phenotype as sub-population with different death and birth rate function $b_k = b_k(A)$, $d_k = d_k(A)$ that are only dependent on the average income A. With this definition of phenotype, the time dependence of the death and birth rate function is treated as 1)introduction of new phenotypes and 2) changes in population proportion of phenotypes. Suppose the population consist of l competing phenotypes and all phenotype shares same average income and resource production.

$$p - a = \sum_{k=1}^{l} n_k(b_k - d_k) = \sum_{k=1}^{l} n_k r_k \tag{10}$$

Where $n_k = N_k/N$ gives the proportion of kth phenotype, $\Sigma n_k = 1$. N_k gives the population of kth phenotype and $N = \Sigma N_k$ is the total population. $b_k = \dot{N_{bk}}/N_k$, $d_k = \dot{N_{dk}}/N_k$ and $r_k = \dot{N_k}/N_k$ are the percentage birth, death and population change rate for kth phenotype. $b = \Sigma n_k b_k$, $d = \Sigma n_k d_k$ and $r = \Sigma n_k r_k$ are the percentage birth, death and population change rate for the whole population.

Calculate the time derivative of kth phenotype's proportion in the whole population.

$$\dot{n_k} = r_k - r \tag{11}$$

In eq10 b_k and d_k only depends on average resource A, $b_k = b_k(A)$ and $d_k = d_k(A)$. The resource production function depend on time and population $p = p(N_1 \dots N_k, t)$. If the function $b_k(A)$, $d_K(A)$ and $p(N_1 \dots N_k, t)$ are specified, the population and average resource development will be fully described by the following system of differential equations.

$$\begin{aligned} \frac{1}{A}\frac{dA}{dt} &= p(N_1 \dots N_k, t) - \sum_{k=1}^{l} n_k(b_k(A) - d_k(A)) \\ \frac{1}{N_k}\frac{dN_k}{dt} &= b_k(A) - d_k(A) \\ \frac{dn_k}{dt} &= b_k(A) - d_k(A) - \sum_{k=1}^{l} n_k(b_k(A) - d_k(A)) \\ \frac{1}{N}\frac{dN}{dt} &= \sum_{k=1}^{l} n_k(b_k(A) - d_k(A)) \end{aligned} \tag{12}$$

Though, it is hard to obtain exact $b_k(A)$s, $d_K(A)$s and $p(N_1 \dots N_k, t)$, by determine boundary and general trending of these functions, the general trending of the population and average income can be obtained.

2.3 Elimination and inferior-growth

The differential equation system in eq12 can be used to predict in general what phenotypes are eliminated. For any phenotype, if eq12 predicts

$$lim_{t\to\infty} N_k = 0 \tag{13}$$

then, kth phenotype is eliminated.

One important feature is the differentiation of elimination process and inferior-growth process.

Elimnation

Phenotype k is eliminated if $lim_{t\to\infty} N_k = 0$

Inferior-growth

Phenotype k have an inferior-growth if $lim_{t\to\infty} n_k = 0$

Based on the definition of elimination and inferior-growth:

1. A phenotype is eliminated if $r_k < 0$
2. A phenotype is inferior growth if $r_k < r$

In both case, the phenotype's population proportion goes to $n_k \to 0$. The inferior-growth is a necessary condition for elimination while elimination is sufficient condition for inferior growth. When a phenotype is eliminated the phenotype will disappear overtime, on the other hand when a phenotype is inferior-growth, the phenotype does not disappear. In case of inferior-growth the phenotype's proportion in the whole population drop to 0, but there still is a substantial population of the phenotype existing. In case of elimination, the population of the phenotype drops to 0.

2.4 Birth rate and death rate balancing mechanism for single phenotype under limited resource

Before dive into the elimination mechanism, first demonstrate the birth rate and death death rate balancing mechanism for single phenotype under fixed production P.

Substitute the function of birth rate $b(A)$ and death rate $d(A)$ into eq 6, then the average income exponential change rate $a(A)$ can be solved based on function of production exponential change rate p. At the same time, the population change rate r can be directly solved from b and d. A differential equation system complete describe the population and average income based on birth rate b, death rate d and production change rate p is obtained.

$$\frac{1}{A}\frac{dA}{dt} = p(t) - b(A) + d(A) \tag{14}$$

$$\frac{1}{N}\frac{dN}{dt} = b(A) - d(A) \tag{15}$$

Put the 6 assumptions of birth rate and death rate in mathematical formal way.
In term of percentage birth rate b:

1. There exist a no-birth-average-income $A_{b=0}$ such that when the average income A is less than the no-birth-average-income $A < A_{b=0}$, the birth rate is $b = 0$
2. There is a upper bound on the birth rate $b_{max} > 0$ such that when the average income reaches infinity, the birth rate approaches b_{max}. ($b \to b_{max}$ as $A \to \infty$)
3. The birth b is a monotonic increasing function of average income A.

In term of percentage death rate d:

1. There exist a all-death-average-income $A_{d=\infty}$ such that when the average income A is less than the all-death-average-income $A < A_{d=\infty}$, the death rate is $d = \infty$
2. There is a lower bound on the death rate $d_{min} > 0$ such that when the average income reaches infinity, the death rate approaches d_{min}. ($d \to d_{min}$ as $A \to \infty$)
3. The death b is a monotonic decreasing function of average income A.

From the monotonicity assumption and the boundary of the death and birth rate, the ranges of birth and death rates are d in (d_{min}, ∞) and b in $[0, b_{max})$. Because of basic elimination principle from eq13, all phenotype with $b_{max} \leq d_{min}$, will be eliminated.

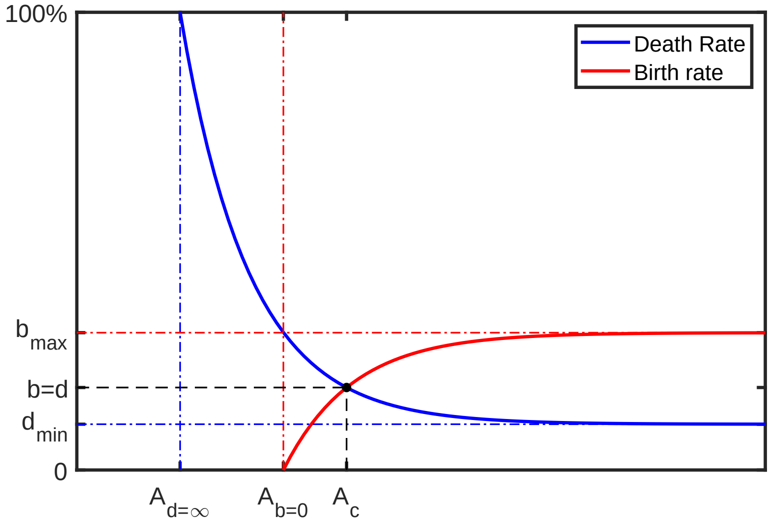

Figure 1: The birth and death plot. The b_{max},d_{min},$A_{b=0}$ and $A_{d=\infty}$ are all marked. When the average income A is large enough, $d \approx d_{min}$, $b \approx b_{max}$ and $r \approx b_{max} - d_{min} = r_{max}$. When $A = A_C$, $b = d$.

Look at the $p = 0$ situation, the differential equation becomes

$$\frac{1}{A}\frac{dA}{dt} = -\frac{1}{N}\frac{dN}{dt} = b(A) - d(A) \tag{16}$$

$$\tag{17}$$

Figure 1 shows a plot the death and birth rate based on these assumptions. Only when the average income A is larger than all-death-average-income and no-birth-average-income $A > A_{d=\infty}, A_{b=0}$, the population survive and give birth. Because the monotonicity of the birth and death rate and the fact that $d_{min} < b_{max}$, there exist one and one only critical average income A_C such that at A_C, $b = d$. If the production exponential change rate $p = 0$, the average income exponential change rate $a = 0$ at A_C. Based on the figure 1 at $p = 0$, when $A > A_C$, $a < 0$; when $A < A_C$,$a > 0$. In other word, $A - A_C$ and a have opposing sign, $a = 0$ at $A = A_C$, which indicate that A approach A_C over time and reaches equilibrium at $A = A_C$.

Start with most simple differential equation with A approach A_C, the famous logistic equation[14] that describe population can be obtained.

$$\dot{A} = k(A_C - A) \tag{18}$$

Eq 18 gives the property of A approach A_C and reach equilibrium at A_C and k is a constant. Substitute eq 18 and $p = 0$ into eq 5.

$$\frac{k(A_C - A)}{A} = -\frac{\dot{N}}{N} \tag{19}$$

Substitute $A = P/N$ into eq 19

$$k(\frac{A_C N}{P} - 1) = -\frac{\dot{N}}{N} \tag{20}$$

With a fixed resource production P, the carrying-capacity K would be given by $K = P/A_C$, by substitute the carrying-capacity into eq 20, the standard logistic equation is obtained.

$$\dot{N} = kN(1 - \frac{N}{K}) \tag{21}$$

3 Fundamental Elimination Mechanism

Even though eq12 assumed time independency on b_k and d_k, time dependency of b_k and d_k can always be treated as introduction of new phenotypes or in other word mutations. Based on the definition of elimination and general assumption of the birth and death rate b_ks and d_ks, rules of elimination can be determined. With these rule of eliminations, one can predict what new phenotype will eliminated old phenotypes. Such a prediction will determined a general direction of evolution.

By assuming $p = 0$, three general elimination mechanism is demonstrated.

3.1 Higher birth rate elimination(HBE) mechanism

A simple elimination mechanism can be formed with two assumptions: 1) the total population stay constant, 2) all different phenotypes share the same percentage death rate. Consider the simple 2 phenotype case, because the total population is fixed, the total death rate and birth is under equilibrium $b = d$. The two phenotypes have different birth rate b_1 and b_2 but have the same death rate d. The birth rate of the whole population is

$$b = n_1 b_1 + n_2 b_2 = d \tag{22}$$

Since the death rates of both phenotype are the same as the full population's death rate $d = d_1 = d_2$, the exponential growth rate for the population of two phenotype is given by

$$\begin{aligned} r_1 &= b_1 - d = b_1 - (n_1 b_1 + n_2 b_2) = n_2(b_1 - b_2) \\ r_2 &= b_2 - d = b_2 - (n_1 b_1 + n_2 b_2) = n_1(b_2 - b_1) \end{aligned} \tag{23}$$

From eq23, the term n_1 and n_2 are always positive; the $b_1 - b_2$ and $b_2 - b_1$ always have the same value and opposing sign. So, the exponential growth rate of the two phenotype are always one positive and one negative, where the positive phenotype have greater birth rate and the negative phenotype have lesser birth rate. Eq23 reviewed the High-Birth-Elimination (HBE) mechanism, where phenotype with greater birth rate will eliminate phenotype with lesser birth. Extending to more than two phenotypes, the phenotype with highest birth rate will eliminate all other phenotypes.

From eq23, a series of differential equation of the proportion of phenotypes in the full population can be derived. Then, an estimation of the time scale of elimination can be calculated. From the definition of the proportion n_1 and n_2 and the fact that total population N stay constant, $r_1 = \dot{n_1}/n_1$ and $r_2 = \dot{n_2}/n_2$ can be derived. Substitute into eq 23.

$$\begin{aligned} \dot{n_1} &= n_1 n_2 (b_1 - b_2) \\ \dot{n_2} &= n_1 n_2 (b_2 - b_1) \end{aligned} \tag{24}$$

Solve the differential equation system.

$$\frac{n_{1f}(1-n_{1i})}{n_{1i}(1-n_{1f})} = \frac{n_{1f}n_{2i}}{n_{1i}n_{2f}} = e^{(b_1-b_2)(t_f-t_i)} \tag{25}$$
$$\frac{n_{2f}(1-n_{2i})}{n_{2i}(1-n_{2f})} = \frac{n_{2f}n_{1i}}{n_{2i}n_{1f}} = e^{(b_2-b_1)(t_f-t_i)} \tag{26}$$

Substitute initial proportion of (99%,1%) and final proportion (1%,99%), obtain the relation between the birth rate difference $\Delta b = |b_1 - b_2|$ and the time takes T.

$$\frac{9.19}{\Delta b\,[\%/year]} = T\,[years] \tag{27}$$

From eq27, two phenotypes with 1% birth rate difference takes 919 years for the population proportion to change from 99% to 1% and 1% to 99%; 2% takes 459.5 years; 0.1% takes 9190 years. Based on the estimation, the time scale for HBE works on the time scale of hundreds years to tens of thousands of years.

The HBE pushed up both birth and death rate with the same population, it does not change the population size but increase the population's turnover rate. For any phenotype, with unlimited resource supply, there is a maximum average life span L_{max}, which only depends of the physiology of the phenotype. Based on eq7, the minimum death rate of a phenotype is $d_{min} = 1/L_{max}$. The HBE pushed up the death rate d, thus decreased the average life span L. So, the HBE suppresses the life span of a population.

Define the accumulated-population N_{Acc} as follow:

$$N_{Acc}(t) = N_b(t + L_{max}) = \int_t^{t+L_{max}} N(t')b(t')dt' \tag{28}$$

The accumulated-population gives all individual born over the period of maximum life span. If we have a population at equilibrium, N and $b = d$ are constant, the integral simplifies to $N_{Acc} = NbL_{max}$. The ratio N/N_{Acc} give the

$$\frac{N}{N_{Acc}} = \frac{d_{min}}{d} = \frac{L}{L_{max}} \tag{29}$$

When every individual live to their maximum life span L_{max} the ratio $N/N_{Acc} = 1$. Table 1 demonstrated the effect of the HBE. The HBE pushes birth and death rates to extreme, the average life span is suppressed and accumulated population is pushed to extreme, while the population stays constant.

Under population limitation ($r = 0$, $R_0 = 1$), any phenotype have $b = d$. However, when phenotypes are competing, the phenotype with greatest birth rate $b_{greater}$ will eliminate phenotypes with lesser birth rate b_{lesser}. The HBE pushes the population to a high-birth-high-death situation and eliminate the low-birth-low-death phenotypes. The historical data of number of children per women[15, 16] generally does not reveal life the difference between accumulated population and the instantaneous population at given time. Since $R_0 = 1$ when the instantaneous population stays constant, overall number of children per women should stay constant at exact 2.1. With a fixed population $R_0 = 1$, the HBE still continuous pushes the birth rate to as high as possible.

Table 1: How high birth rate affect the life span and accumulated population

1 million population with 100 years maximum life span		
birth/death rate	accumulated population	average life span
1%	1 million	100
2%	2 million	50
3%	3 million	33
4%	4 million	25
5%	5 million	20

Which results in pushing up the accumulated population as high as possible or push the turn over rate as high as possible.

The HBE is the reason for the "*increase beyond the means of subsistence*" tendency and the reason why there is no choice in lower birth rate to increase living standard. Choosing lower birth rate leads to elimination.

3.2 Lower average elimination mechanism

The HBE reviews some aspect of the direction of evolution, however there is a fundamental flaw with the logic. A hidden assumption in the logic is that the death rate does not affect birth rate. Because of the hidden assumption, the HBE pushes birth rate to infinity. By relating the average income (in term of resource or monetary currency) to the birth and death rate ($b_k = b_k(A)$ and $d_k = d_k(A)$), the birth rate is limited by the death rate. A more sophisticated elimination mechanism based on average income is obtained.

Suppose there are two different phenotype with different death and birth functions. Continue with the n_1 and n_2 notation from eq 22. $n_1 = N_1/N$ and $n_2 = N_2/N$ are proportion of the two phenotype in the whole population. $N_1 + N_2 = N$ gives the total population of the two phenotypes. The overall death and birth rate are given by

$$b = b_1 n_1 + b_2 n_2 \tag{30}$$
$$d = d_1 n_1 + d_2 n_2 \tag{31}$$

Substitute into eq 6

$$a = p - n_1(b_1 - d_1) - n_2(b_2 - d_2) \tag{32}$$
$$a = p - n_1 r_1 - n_2 r_2 \tag{33}$$

Figure 2 shows the situation with two phenotype. The two phenotype have two different critical average income A_{C1} and A_{C2} with $A_{C1} > A_{C2}$. For constant total production P and $p = 0$, $a = -n_1 r_1 - n_2 r_2$. The population of two phenotypes and the average changes according the following trend:

1. When $A < A_{C2} < A_{C1}$, $r_1, r_2 < 0$ and $a > 0$. Both phenotype's population decrease while the average income increase.
2. When $A > A_{C1} > A_{C2}$, $r_1, r_2 > 0$ and $a < 0$. Both phenotype's population increase while the average income decrease

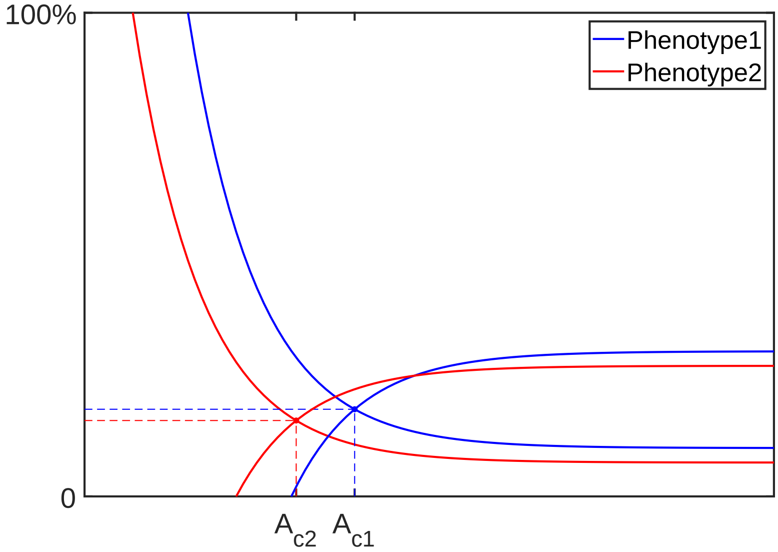

Figure 2: With two phenotypes, and $A_{C1} > A_{C2}$, when $A < A_{C2}$, both phenotype population decrease. When $A > A_{C1}$, both phenotype population increase. When $A_{C2} < A < A_{C1}$, the population of phenotype 2 increase while the population of phenotype 1 decrease. Eventually, the system will reach the $n_1 = 0$,$n_2 = 1$ and $A = A_{C2}$ and phenotype 1 got eliminated.

3. When $A_{C2} < A < A_{C1}$, $r_1 < 0$ and $r_2 > 0$, the sign of a will be dependent on the population proportion n_1 and n_2. So the trend of income change is undetermined, but the population of phenotype 1 increases while the population of phenotype 2 decreases.
4. At $A = A_{C1}$, $r_1 = 0$, $r_2 > 0$. Based on eq 2 ,unless $n_2 = 0$, $a < 0$. Unless phenotype 2 have 0 population ratio, the average income decrease and enter the $A_{C2} < A < A_{C1}$ region. The population of phenotype 2 is increasing while population of phenotype 1 stays same.
5. At $A = A_{C2}$, $r_2 = 0$, $r_1 < 0$. For average income change rate a, $a > 0$ when $n_2 > 0$ and $a = 0$ when $n_2 = 0$. If phenotype 1 have 0 population, average income reaches equilibrium. If there is a non-zero population for phenotype 1, the average income will increase and enter the $A = A_{C2}$, while population of phenotype 2 stays same and population of phenotype 1 decrease.

Base on the analysis of the trend of A's and N's dependency and on A, n_1 and n_2, for any initial A_i, n_{1i}, n_{2i}, A first enter the region $[A_1, A_2]$ if A does not start in the region and then approaches A_{C2}. The population N approaches $N_e = P/A_{C2}$. The population proportion n_1, n_2 approach $n_2 = 1$ and $n_1 = 0$. The population of two phenotype approach $N_2 = N_e$ and $N_1 = 0$. The trend indicate that phenotype 1 will be eliminated because of the higher critical average income.

The analysis demonstrated the lower-average-elimination-mechanism(LAE). This mechanism predicts that with two competing phenotype, the phenotype with lower critical average income A_{Clower} will eliminate the phenotype with greater critical average income $A_{Cgreater}$. Contradict to the instinct of economics prediction that the system always develop toward higher average income, the LAE will push the average income lower. Whenever a now phenotype with lower A_{Clower} appear, it will eliminate phenotypes with higher $A_{Cgreater}$.

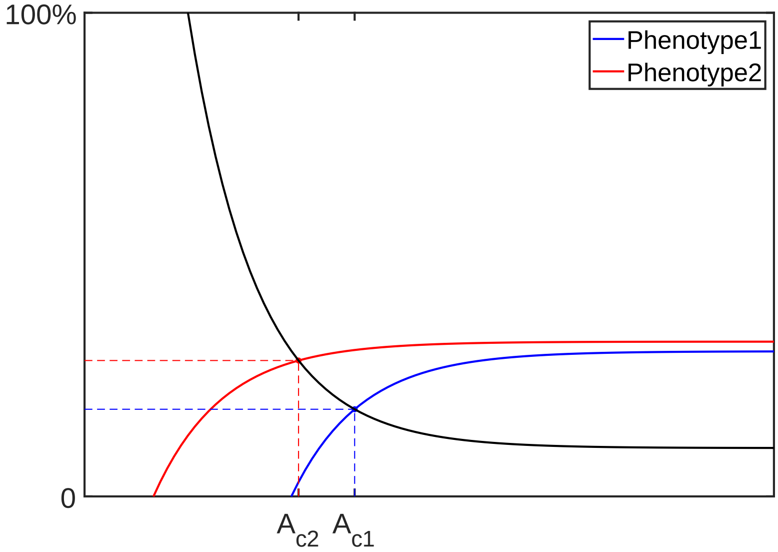

Figure 3: Because of the monotonicity of the death curve, with the same death curve, lower A_C always have higher birth rate.

With the LAE, the HBE can be re-examined. Since the mechanism of LAE does not require a constant population size, only keep the assumption of 2 phenotypes share the same death function. Figure3 shows how HBE function under LAE. When the two phenotypes share the same death rate curve, the lower A_C always have higher birth rate, because of the monotonicity of the death rate curve. If the birth rate curve is considered to be the maximum birth rate of a phenotype at certain average income level A, the HBE predict that the phenotype exhaust their resource to reproduce. The HBE eliminate all other that does not exhaust all their resource to reproduce.

Based on the analysis, it is important to identify that the birth function in the analysis is the physiological limits based on average income rather than actual birth rate based on individuals' choice. The choice of not exhaust their resource and given birth to less off spring will eliminated the individuals who made such choice. Only the individuals exhaust their resource will survive. Thus, the birth function under the analysis is independent of personal choice and only depends on the physiological limits.

If the average income can be equate with living standards, the LAE indicate a constant trend in decreasing the living standards. The LAE will push the living standard or living conditions of any creature as low as possible in exchange of producing as much off-spring as possible. The LAE is the a different way of understanding the Malthusian trap. The average income staggering effect of Malthusian trap is caused by the LAE. Since the LAE is pushing the average income as low as possible.

3.3 High Quota Elimination based on gain access to extra resource

Until this point, the elimination does not consider phenotypes' difference in ability of resource production and distribution. Suppose there are two phenotypes with the total production function $P = P_1 + P_{12}$, where P_{12} is the part of resource accessed by both phenotype while P_1 is only accessed by phenotype 1. So that the two phenotype 1 gain access to extra resource. The average income for two phenotypes are:

$$\begin{aligned} A_1 &= P_{12}/N + P_1/N_1 \\ A_2 &= P_{12}/N \end{aligned} \tag{34}$$

Suppose the two phenotypes have the same death and birth function, $b_1 = b_2$, $d_1 = d_2$ and $A_{C1} = A_{C_2} = A_C$. At $A_2 = A_C$, $A_1 > A_C$, $r_1 > 0$ and $r_2 = 0$. Based on eq 34, with all proportion n_1 and n_2, there is always $A_1 > A_2$. When phenotype 2 reaches the critical average income A_C, phenotype 1 population is still growing. As the population growing, the average income of the two phenotypes enter the region $A_2 < A_C$, $A_1 > A_C$, $r_1 > 0$ and $r_2 < 0$, until eventually a new balance reached at $A_2 < A_C$, $A_1 = A_C$, $r_1 = 0$ and $r_2 < 0$. Over the whole process, the population growth rate of phenotype 2 is always negative, so phenotype 2 is eliminated.

In case the two phenotype have different A_{C1} and A_{C2}, if $A_{C1} > A_{C2}$, phenotype 2 is eliminate because both LAE and HQE will cause its elimination. In case of $A_{C1} < A_{C2}$, two phenotypes will co-exist at

$$N = \frac{P_{12}}{A_{C2}} \tag{35}$$

$$N_1 = \frac{P_1}{A_{C1} - A_{C2}} \tag{36}$$

$$N_2 = \frac{P_{12}A_{C1} - P_{12}A_{C2} - P_1A_{C2}}{A_{C1}A_{C2} - A_{C2}^2} \tag{37}$$

$$n_1 = \frac{P_1}{P_{12}} \frac{A_{C2}}{A_{C1} - A_{C2}} \tag{38}$$

$$n_2 = \frac{P_{12}A_{C1} - P_{12}A_{C2} - P_1A_{C2}}{P_{12}A_{C1} - P_{12}A_{C2}} \tag{39}$$

The High Quota elimination (HQE) based on gain access to extra resource is demonstrated. The mechanism gives a elimination based incentive to develop new technology, explore new resource and develop ability to access new resource. The specific mechanism eliminated based on higher average income and drive the higher average income with exclusive new resources.

There could be other process that exclusively increase average income of one phenotype and cause elimination. To simplify the process, consider a initial state of a balanced population, where $p = 0$, $a = 0$, $b = d$, $A = A_C$ and $N = P/A_C$. Suppose then a small population mutated from the initial state and gain some mechanism to increase their average income, but have the same death and birth function, thus the same critical average income A_C. Suppose phenotype 1 is the mutated phenotype and phenotype 2 is the original phenotype. Because of the mutation, the average income of the two phenotypes are

$$A_1 \geq A_C \tag{40}$$

$$A_2 \leq A_C \tag{41}$$

Because of the difference in average income A, phenotype 1 have a growing population while phenotype 2 have a decreasing population. Based on the mechanism of introducing such average income difference, the dependence of the two phenotypes average income A_k on the population proportion n_1 and n_2 have to be decided case by case. Based on the mechanism, there will be case such that $A_1(n_1, n_2) = A_2(n_1, n_2) = A_C$ at a specific population proportion n_1 and n_2. In such case, mutated phenotype will co-exist with the old phenotype at the specific population proportion n_1 and n_2. There will also be cause such that $A_1 \geq A_C$ and $A_2 \leq A_C$ with all possible population proportion n_1 and n_2. In such case, the mutated phenotype will eliminated the original phenotype. This process is the High Quota Elimination (HQE).

The HQE does not limit phenotypes to have the same death and birth function. In case of phenotypes with different death and birth functions, thus different A_Ck. Based on the different mechanism causing the average income difference for the phenotypes, phenotypes with $A_k \geq A_{Ck}$ at specific combination of n_ks will co-exist while phenotype with $A_k \leq A_{Ck}$ for all combination of n_ks will get eliminated.

The eliminate caused by extra resource indicated that progress in production will cause elimination. If gain access to extra resource indicate advance in technology, the HQE demonstrated that technology advance will cause elimination. The HQE based on extra resource indicated that creatures will maximize resource production. The co-existence of the two phenotype indicate that HQE compete with LAE while HBE is consistent with LAE.

3.4 The direction of evolution and their consequences

Combine all three elimination mechanisms, three general direction of evolution can be derived.

1. Every creature evolve towards gathering or producing as much resource as possible.
2. Every creature evolve towards exhaust all resources in producing offsprings.
3. Every creature evolve towards increasing reproduction efficiency by achieve birth-death balance at as low average income as possible

With the overall direction of evolution, the elimination process imposed harsh consequences. With HBE, the life span of population are suppress to barely have birth-death balance. The population turn-over rate is pushed to extremely higher, so creature dies at relative young age compare to their potential life span. The living condition or average income is pushed as low as possible. On the other hand, the HQE provide an evolutionary advantage in increasing production. Developing economy or increasing resource production is not only an economical incentive but also an evolutionary advantage.

4 Malthusian trap and the economic growth

Until this point, all analysis are based on constant production $p = 0$. What would happen if $p > 0$? Malthusian trap describes a situation where income is staggered because economical growth is lower than population growth. Consider situation with economic growth rate $p > 0$. Assume the HQE based on production increase is always in action and is the reason for positive production production growth rate p.

4.1 Malthusian Trap and escaping the Trap

First, define a maximum population growth rate $r_{max} = b_{max} - d_{min}$. If $p > r_{max}$, $p > r$ all the time. From eq 6, $a > 0$ when $p > r_{max}$, the average income is always increasing, which mean elimination processes based on decreasing average income all shut off. Since the Malthusian trap describes a situation with staggered average income, Malthusian trap describe the situations where $0 < p < r_{max}$.

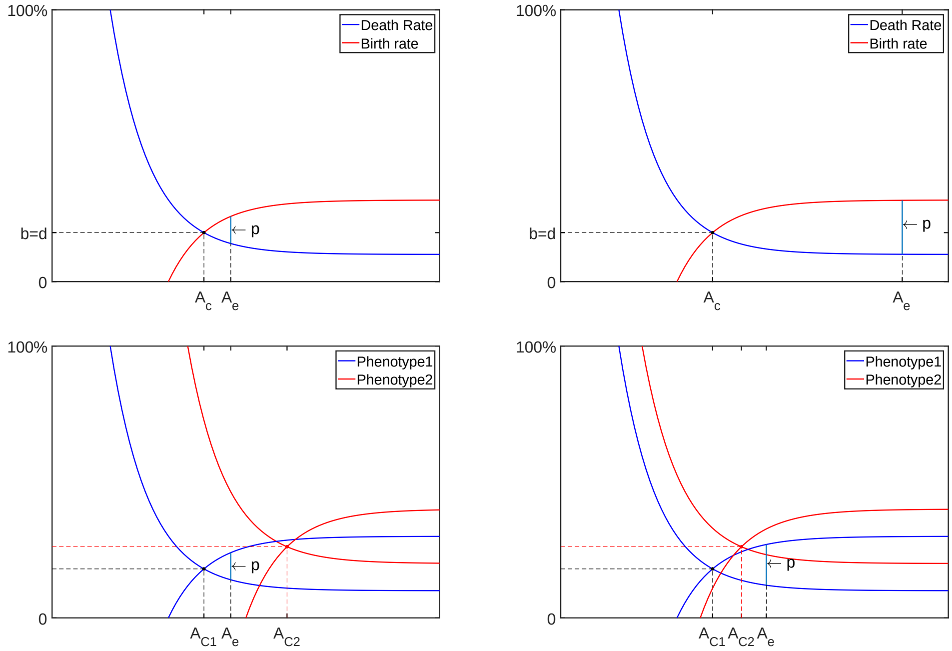

Figure 4: When $0 < p < r_{max}$, the average income reaches equilibrium at equilibrium average income A_e, where $A_e > A_C$. At $A = A_e$, $p = r$ and $a = 0$. The production growth rate p provided a "buffer zone", phenotypes with $A_{Ck} < A_e$ will be inferior-growth but not eliminated.

Figure 4 showed the situation with $0 < p < r_{max}$. Define the equilibrium average income A_e such that at A_e, $r = p$. When $A > A_e$, $r > p$ and $a = p - r < 0$ the average income decrease. When $A < A_e$, $r < p$ and $a = p - r > 0$ the average income increase. When $A = A_e$, $r = p$ and $a = r - p = 0$ the average income stays constant. When $0 < p < r_{max}$, the average income approaches equilibrium average income $A = A_e$ overtime instead of the critical average income A_C, while the population increase at a rate $r = p$.

In case of multiple phenotype, the average income will approach phenotype with lowest equilibrium average income A_{ek}. The phenotype, that have lowest equilibrium average income A_{emin}, have the population growth rate $r = p$. Phenotypes with critical average income $A_{Ck} > A_{emin}$ will be eliminated, as they have $r_k < 0$. Phenotypes with $A_{Ck} < A_{emin}$ will be inferior-growth, as they have $r_k \leq p$. In any case, the population proportion for the phenotype with lowest equilibrium average income A_{emin} will dominate $n_{emin} \rightarrow 1$.

Over all, when the production growth rate is constant in the range $0 < p < r_{max}$, the average income approach $A = A_{emin}$, the population growth rate approaches $r = p$, the proportion of the phenotype with minimum equilibrium average income approaches 1, the proportion of phenotype with their critical average income $A_{Ck} < A_{emin}$ approaches 0 but still have a growing population, the proportion of phenotype with their critical average income $A_{Ck} > A_{emin}$ have their population approaches 0 thus get eliminated. The LAE still works with a production growth rate $0 < p < r_{max}$. Rather than suppress the critical average income A_C, when there is a production growth rate $0 < p < r_{max}$, the mechanism suppresses equilibrium average income A_e. Thus, the HBE still functions with $0 < p < r_{max}$, except when phenotypes' birth rate difference $\Delta b < p$, phenotype with lower birth rate b_{low} will not be eliminated but have inferior-growth. As for HQE, since the LAE is functioning, the average is suppressed at A_e, difference in average income A_k would cause growth rate difference and thus cause inferior-growth or elimination. When the economical growth rate is not high enough $0 < p << r_{max}$, increasing economical growth rate p will not start growth in average income, but rather increase the equilibrium average income A_e, but still have a staggered average income.

Based on the analysis of the elimination process, the two major feature of the Malthusian trap (high-birth-high-death balance and staggering average income) are caused by elimination process rather than economical incentives. The HBE and LAE are forcing high-birth-high-death balance and average income staggering. The HQE provides an elimination based incentive to develop the economy or gain access to new resource. Based on the mechanism for HBE and LAE, the $p << r_{max}$ is necessary and sufficient condition for the HBE and LAE to function.

4.2 Out-of Malthusian Trap

Apparently the escaping the trap happens when $p > r_{max}$. From Figure 4, as $p \to r_{max}$, $A_e \to \infty$. When $p > r_{max}$, there is no equilibrium average income as the average income is increasing. All elimination processes based on average income decrease (HBE and LAE) are shut off with $p > r_{max}$ as there will be no average income decrease. The HQE is shut off after the average income A is high enough so that difference in population growth rate $r(A)$ is negligible. As average income increase, the death rate will continue to decrease and eventually reach $d = d_{min}$. The life span of population reaches the maximum life span L_{max}, ideally every one dies because aging. The accumulated population $N_{acc} = N$ at any given time will be the same a population, as everyone live to their maximum life span. In term of the birth rate b, personal choice on birth become the dominant factor in this regime. Since HBE no longer functions, individuals' choice in how much off spring they produce will no longer cause elimination. Thus, economical incentives start play major role in reproducing.

From eq6, when $p > r_{max}$, the average income is constantly increasing. It is clear from eq6 that increasing in economical growth rate p will cause increasing in a; and decreasing in birth rate b will cause increasing in a. Decreasing birth rate increases living standards. On the other hand, since the average income is always increasing, the HBE no longer function. In case out of the Malthusian trap, decreasing birth rate will no longer cause elimination. So, with $p > r_{max}$, the HBE no longer force high birth rate, at the same time because average income start increasing, there is incentive to decrease birth rate.

With $p > r_{max}$ the HBE is shut off, the trend of decreasing birth rate b start to dominate. The the elimination process only requires that eventually $b \geq d_{min}$, so that phenotypes are not self-eliminated. There is not even a process that maintains a high population size. Eventually, the population change rate and average income change rate will balance at

$$a = p, b = d_{min} \tag{42}$$

Where the average income change rate will be exactly the production growth rate and the birth will match the minimum death rate and the population will eventually balance at some level with mechanism independent of average income variance. The population level will be fully decided by economical mechanisms where the population size is related to the process of maximizing economical growth.

Compare the the population trend under Malthusian trap and out of Malthusian trap, the r_{max} function as a 'phase-changing' point. With $p < r_{max}$, the elimination process are functioning. There is an elimination based "feed-back-loop" in increasing birth rate and decreasing average income. Thus, causing the population to grow as fast as possible and decrease the living standard as low as possible. With $p > r_{max}$ the average income is increasing, the elimination mechanisms are shut off. As the average income start to growing, there is an economical incentive based "feed-back-loop" in decreasing birth rate and increasing average income. The two situation will demonstrate great difference in average income and birth rate.

Even with positive economical growth $0 < p << r_{max}$, the population could be still under Malthusian trap and have staggered extreme low average income and very high birth and death rate. Not until the economical growth rate is high enough $p > r_{max}$ and out-of-trap "feed-back-loop" start functioning, the average income will not start increasing. Once the out-of-trap "feed-back-loop" start functioning, the average income will start grow exponentially and life span will increase drastically. The "phase-change" drastically changes the average income, living standard and even life quality of any population.

4.3 Escaping the Malthusian Trap and the Demographic Transition Theory

The Demographic Transition Theory (DTT)[10, 11] describes the birth rate and death rate change over the course of economy develop. The DTT describe 4 stage of death rate, birth rate change over economical development.

1. Both death rate and birth rate are high, the population stays low.
2. Birth rate stays the same while death rate start to drop, population start to increase.
3. Birth rate drops with death rate, population increasing rate slows.
4. Both birth rate and death rate drop to lower level, population increase stops.

The four stages of the DTT coincide with the transition from Malthusian trap equilibrium ($0 < p < r_{max}$) to out-of-trap equilibrium ($p > r_{max}$). Consider a process where the production growth rate p start with $p_i = 0$ and increase toward a value higher than $p_f > r_{max}$. Ignore the process of reaching equilibrium, only observe the development of the equilibrium average income A_e and related death and birth rates. The top two plots in figure 4 demonstrated the process of p increasing. Correspond to the four stages of DDT

1. Both death rate and birth rate are high, the population stays low. $p = 0$, $A_e = A_C$, the HBE pushed the birth-death balance to high birth high death mode.
2. Death rate start to drop. p start increasing, however still in the $0 < p << r_{max}$ range, as p increase A_e increases. Living standard start to increasing, death rate drop as living standard increase.

3. Birth rate drops with death rate, population increasing rate slows. $p > r_{max}$ is just achieved, the average income A is still at relative low level. The HBE is just turned off and the incentive in decreasing birth rate start to function. Death rate continue drop as living standard increasing.

4. Both birth rate and death rate drop to lower level. $p > r_{max}$, the average income A reach a relative high level. The elimination processes are turned off for a relative long time. Death and birth rate balance again at $b = d_{min}$

The four stage of the DTT stopped as birth rate b decreased to d_{min}. However, there is no mechanism stop the birth rate decrease further. Since with $p > r_{max}$, the population balancing mechanism is unknown, there could be two extra stage.

5. Birth rate dropped below minimum death rate $b < d_{min}$, population start to drop.

6. The population balanced at new level based on other economical mechanisms.

5 Potential Future Studies

There are many potential future studies based on this model and many potential studies to further improve the model. Here is a list of obvious potential studies related to the model.

1. In case of escaping the Malthusian trap, $p > r_{max}$ must achieve. Though, when p is close to r_{max}, the economical incentive in decreasing birth rate b could kick in before $p > r_{max}$. The exact 'phase change' process at the $p \to r_{max}$ require more studies to fully understand the transition process.

2. The only HQE mechanism analyzed in this paper is the HQE based on gain access to new resource. There are many other potential HQE mechanisms. Examine other HQE mechanisms could provide other co-existence situation and other evolutionary directions.

3. Based on the HQE studies, a extra equation system that describe the wealth distribution could be formed. The whole model is based on average income, but does not include information on wealth distribution. A more sophisticated equation system that describe the dynamic of average income, population and income distribution will provide more insight on the whole system.

4. The derivation of eq 12 fully based on the a single resource or production score P. What happen if there are multiple vital resource? If there are multiple vital resource, how do elimination work on different resources? A equation system that describe the dynamic of multiple resource average distribution and population could provide more co-existence conditions.

5. A fundamental assumption about the system is that the death and birth rates functions does not have time dependency. What if there is local time dependency? Apparently, the death and birth rates functions of any creature have age dependency. How could such dependency be included into the equation system? What is the effect of such dependency?

6 Conclusion

Based on simple definition of average income and population, a differential equation system of average income and population development over time is obtained. The input function for the differential equation system is death rate function $d_k(A)$, birth rate function $d_k(A)$ and production growth function $p(N_1 \ldots N_k, t)$. Based on 6 assumption for the death and birth rates functions and the simplification $p = 0$, 3 different elimination process are obtained. The HBE provide a mechanism that maximize the birth rate. The LAE provide a mechanism that minimize average income. The HQE based on extra resource provide a mechanism that maximize the resource production. The brutal consequence of these elimination mechanism is demonstrated. HBE is suppressing average life span and increasing population turn over rate. LAE is suppressing living standard.

Examine the situation $p > 0$, fundamental differences between the under Malthusian trap $p < r_{max}$ and out of Malthusian trap $p > r_{max}$ are demonstrated. Under Malthusian trap $p < r_{max}$, HBE and LAE are functioning, the trend of maximizing birth rate and minimizing average income dominates. The birth-death-balance at a high-birth-high-death situation, population is increasing as production increases while average income stays constant. A positive economical growth rate $p > 0$ will not cause average income growth, but only increase the average income a little bit. Out of Malthusian trap $p > r_{max}$ provide a situation where average income is increasing, HBE and LAE are shut off. The economical incentive to decrease birth rate start functioning. The birth-death-balance balances at low-birth-low-death situation. The average increases as the production increase. The exact population size will be decided by other mechanism based on economical incentives.

By studying the process of escaping the Malthusian trap, the model prediction coincide with the DTT. The 4 stages of the DTT exactly coincide with the situation where $p = 0 \to p > r_{max}$. Then based on the model prediction of $p > r_{max}$ regime, the mechanism of birth rate decrease and even population decrease for developed economical entities are explained.

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