Annular structures in perturbed low mass disc-shaped gaseous nebulae I : general and standard models

Vladimir PLETSER (✉ pletservladimir@gmail.com)

Blue Abyss <https://orcid.org/0000-0003-4884-3827>

Research Article

Keywords: Interdisciplinary astronomy, Astrophysical fluid dynamics, Hydrodynamics, Protoplanetary nebulae

Posted Date: March 1st, 2022

DOI: <https://doi.org/10.21203/rs.3.rs-1394295/v1>

License: © ⓘ This work is licensed under a Creative Commons Attribution 4.0 International License.

[Read Full License](#)

Running Title: Annular structures in gaseous nebulae I : general and standard models

Article type: Original article

Title: Annular structures in perturbed low mass disc-shaped gaseous nebulae I : general and standard models

Author: Vladimir Pletser

Affiliation: Institut d'Astronomie et de Geophysique G.Lemaitre, Catholic University of Louvain, Louvain-la-Neuve, Belgium

Present affiliation: Blue Abyss, Newquay, Cornwall, United Kingdom

E-mail address: pletservladimir@gmail.com

ORCID: 0000-0003-4884-3827

Abstract: This is the first of two papers where we study analytical solutions of a bidimensional low mass gaseous disc rotating around a central mass and submitted to small radial perturbations. Hydrodynamics equations are solved for the equilibrium and perturbed configurations. A wave-like equation for the gas perturbed specific mass is deduced and solved analytically for several cases of exponents of the power law distributions of the unperturbed specific mass and sound speed. It is found that, first, the gas perturbed specific mass displays exponentially spaced maxima, corresponding to zeros of the radial perturbed velocity; second, the distance ratio of successive maxima of the perturbed specific mass is a constant depending on disc characteristics and, following the model, also on the perturbation's frequency; and, third, inward and outward gas flows are induced from zones of minima toward zones of maxima of perturbed specific mass, leading eventually to the possible formation of gaseous annular structures in the disc. The results presented may be applied in various astrophysical contexts to thin gaseous discs of negligible relative mass, submitted to small radial perturbations.

Keywords: Interdisciplinary astronomy, Astrophysical fluid dynamics, Hydrodynamics, Protoplanetary nebulae

Statements and Declarations

The author declares that no funds, grants, or other support were received during the preparation of this manuscript.

The author has no relevant financial or non-financial interests to disclose.

The author contributed to the study conception and design. Material preparation, data collection and analysis were performed by Vladimir Pletser. The first draft of the manuscript was written by Vladimir Pletser and the author commented on previous versions of the manuscript. The author read and approved the final manuscript.

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Annular structures in perturbed low mass disc-shaped gaseous nebulae I : general and standard models

Vladimir Pletser

the date of receipt and acceptance should be inserted later

Email: pletservladimir@gmail.com Received: date / Accepted: date

Abstract This is the first of two papers where we study analytical solutions of a bidimensional low mass gaseous disc rotating around a central mass and submitted to small radial perturbations. Hydrodynamics equations are solved for the equilibrium and perturbed configurations. A wave-like equation for the gas perturbed specific mass is deduced and solved analytically for several cases of exponents of the power law distributions of the unperturbed specific mass and sound speed. It is found that, first, the gas perturbed specific mass displays exponentially spaced maxima, corresponding to zeros of the radial perturbed velocity; second, the distance ratio of successive maxima of the perturbed specific mass is a constant depending on disc characteristics and, following the model, also on the perturbation's frequency; and, third, inward and outward gas flows are induced from zones of minima toward zones of maxima of perturbed specific mass, leading eventually to the possible formation of gaseous annular structures in the disc. The results presented may be applied in various astrophysical contexts to thin gaseous discs of negligible relative mass, submitted to small radial perturbations.

Keywords: Interdisciplinary astronomy, Astrophysical fluid dynamics, Hydrodynamics, Protoplanetary nebulae

1 Introduction

Discs play an important role in astrophysics. Protostellar discs are found around certain categories of young stars. Dynamical accretion discs intervene in the feeding process of massive stars by less massive ones in some binary systems. Galaxies have often the shape of a disc surrounding a central bulge. Planetary rings form discs around giant planets. Furthermore, it is generally believed that the planetary system and the regular satellites systems originate from disc-shaped nebulae surrounding the proto-Sun and the giant proto-planets. The disc stage is thus an important step in some systems

Institut d'Astronomie et de Geophysique G.Lemaitre, Catholic University of Louvain, Louvain-la-Neuve, Belgium

Present address: Blue Abyss, Newquay, Cornwall, United Kingdom

evolution. Depending on disc and central mass characteristic and on their mutual relative importance, different kind of structures may appear in discs: bars, spiral arms, rings. Theory of disc dynamical evolution in an astrophysical context may be applied to other situations, for example the theory of spiral density waves of galactic arms was successfully applied to models of planetary rings (see e.g., [1]). Characteristics of a disc that may influence its evolution are self-gravity, thermal pressure, interaction with magnetic fields, rotation and viscosity. In this paper, we show that annular rings may appear under certain circumstances in thin low mass gaseous discs, where self-gravitation, viscous and magneto-hydrodynamics effects can be neglected. We study the behaviour of a nebular disc taken away from equilibrium by small perturbations, extending the classical Jeans' model of an incompressible isothermal stationary nebula submitted to perturbations. Although initially intended for protoplanetary discs, the results of the investigations presented here can be applied to any thin gaseous disc that can be described by the model considered. Hypotheses on the model are discussed in section 2. We consider in section 3 a differentially rotating thin gaseous axisymmetric nebula undergoing polytropic transformations of index γ and departing from equilibrium because of small radial perturbations. Physical characteristics of the nebular disc are supposed continuous and have power law dependencies on the radial distance r , in particular for the specific mass $\rho \sim r^d$ and sound speed $c \sim r^{\frac{s}{2}}$. Equations describing the hydrodynamic model are solved for the equilibrium and perturbed configurations, where the perturbations are assumed small enough for the equations to be linearized. A wave-like equation is deduced for the nebula perturbed gas specific mass and expressions of the gas radial velocity and specific mass flux momentum are found in function of the gas specific mass. Looking for solutions yielding annular gaseous structures to appear in the disc, these equations are solved analytically in Section 4 for two particular models ($d = 0$ and $d < 2(2\gamma - 1)$; $s = 2$) and for a third general case ($d = (s - 2)$; $s < 2$) for small frequencies. A particular case of the latter with $s = -1$ and $d = -3$, called the "standard model", is briefly introduced. Expressions of the distance ratios β of the maxima of the perturbed gas specific mass are also deduced. Profiles of the perturbed specific mass and velocity are presented in Section 5 and the possible formation of annular structures are discussed. We are not aware of previous similar general analytical resolutions, although particular cases were treated in [2]. In a second paper, we explore analytical solutions for two other general models, including a polytropic case. Both papers are reworked excerpts of [3].

2 Hypotheses on the disc model

The mass of a primeval nebula is a key factor in deciding on its later evolution: either the nebula mass is large, typically greater than or close to the central mass, and sub-regions of the nebula of large specific mass may undergo local collapse, or the primeval nebula mass is low, typically a few percents of the central mass, and gravitational instabilities may never develop in this case [4]. In some theories of protoplanetary nebula formation (see e.g., [5]), viscous friction plays an important role in inducing an inward flow of accretion material onto the central primary and causing the conversion of kinetic into thermal energy to be the dominant heat source [6]. However, the epoch at which the viscous friction becomes the predominant effect is critical in a nebula history. After an initial collapse phase, a low mass rotating nebula can achieve a stationary equilibrium without considering turbulent friction [7]. On the other hand, an axisymmetric

equilibrium configuration was shown to be unstable against non-axisymmetric perturbations, the result being a binary system [8, 9]. Furthermore, friction processes are not always able to produce a central object surrounded by a disc-shaped nebula [6]. Therefore, it is reasonable to assume, within the low mass nebula hypothesis, that there was a period in a nebula history during which the viscous friction may not have been the predominant process governing the disc evolution, independently of further evolution where the viscous effects may have become predominant. The problem of transfer of angular momentum from the central mass to outer parts of the disc is not addressed here, as it depends on viscous processes (see e.g., [10]). We consider the model of a disc-shaped gaseous nebular disc of mass M_d in rotation around a central mass M^* , supposed spherical. The disc mass is negligible in front of the central mass $M_d \ll M^*$ and the disc thickness is small compared to its radius. The nebula is assumed to be composed of gas only, the presence of nebular dust being neglected. Self-gravitation, magneto-hydrodynamics and viscous effects in the disc are not considered (although, the viscous force is included in the general equations of section 3, but neglected further in section 4). The effects of small radial perturbations on the disc are studied, without any coupling to non-radial perturbations.

3 Disc hydrodynamic model

The motion of the gas of specific mass ρ is described in spatial Eulerian coordinates by the vectorial Navier-Stokes equation, which relates for an unit volume of gas, the inertia force (sum of the time derivative, denoted by an upper dot, of the vectorial velocity $\mathbf{v}$ and the advection term), the gradients of the pressure p and the gravitational potential V , and the viscous forces

$$\rho \dot{\mathbf{v}} + \rho (\mathbf{v} \cdot \nabla) \mathbf{v} + \nabla p + \rho \nabla V = \rho \nu \Delta \mathbf{v} \quad (1)$$

where ν is the kinematic viscosity, ∇ and Δ the gradient and vectorial Laplacian operators. This equation is complemented by the continuity and Poisson equations

$$\dot{\rho} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad (2)$$

$$\Delta V = 4\pi G \rho \quad (3)$$

where $\nabla \cdot$ and Δ are the divergence and scalar Laplacian operators and G the gravitational constant. The disc gravitational potential is neglected in front of the central mass gravitational potential and the viscosity ν is assumed constant in the disc.

At dynamic equilibrium, the stationary model is described by

$$\rho_0 (\mathbf{v}_0 \cdot \nabla) \mathbf{v}_0 + \nabla p_0 + \rho_0 \nabla V_0 = \rho_0 \nu \Delta \mathbf{v}_0 \quad (4)$$

$$\nabla \cdot (\rho_0 \mathbf{v}_0) = 0 \quad (5)$$

$$\Delta V_0 = 4\pi G \rho_0 \quad (6)$$

where the index 0 denotes the equilibrium characteristics. Allowing for small radial periodic perturbations to take the model away from equilibrium, the linearized perturbed equations read, after simplification by the equilibrium equations (4) to (6),

$$\begin{aligned} \rho_0 \dot{\mathbf{v}}_1 + \rho_0 ((\mathbf{v}_1 \cdot \nabla) \mathbf{v}_0 + (\mathbf{v}_0 \cdot \nabla) \mathbf{v}_1) + \rho_1 (\mathbf{v}_0 \cdot \nabla) \mathbf{v}_0 + \nabla p_1 + \rho_1 \nabla V_0 \\ = \rho_0 \nu \Delta \mathbf{v}_1 + \rho_1 \nu \Delta \mathbf{v}_0 \end{aligned} \quad (7)$$

$$\dot{\rho}_1 + \nabla \cdot (\rho_1 \mathbf{v}_0) + \nabla \cdot (\rho_0 \mathbf{v}_1) = 0 \quad (8)$$

$$\Delta V_1 = 4\pi G \rho_1 \quad (9)$$

where indexes 1 denote the perturbed characteristics. As the model is plane and axisymmetric, these equations are solved in a cylindrical polar reference frame. Considering that the equilibrium characteristics depend only on the radial distance r and that the perturbed characteristics depend on r and on the time t , the equilibrium and perturbed gas vectorial velocities are written respectively

$$\mathbf{v}_0 = (0, v_0(r), 0)$$

$$\mathbf{v}_1 = (v_1(r, t), 0, 0)$$

At dynamical equilibrium, the radial and azimuthal components of the Navier-Stokes equation (4) and the Poisson equation (6) read, with the prime sign ' denoting $\partial/\partial r$,

$$\frac{\rho_0 v_0^2}{r} - p'_0 - \rho_0 V'_0 = 0 \quad (10)$$

$$\rho_0 \nu \left(v''_0 + \frac{v'_0}{r} - \frac{v_0}{r^2} \right) = 0 \quad (11)$$

$$V''_0 + \frac{V'_0}{r} = 4\pi G \rho_0 \quad (12)$$

The perturbed equations (7) to (9) become

$$\rho_0 v_1 - \frac{\rho_1 v_0^2}{r} + p'_1 + \rho_0 V'_1 + \rho_1 V'_0 = \rho_0 \nu \left(v''_1 + \frac{v'_1}{r} - \frac{v_1}{r^2} \right) \quad (13)$$

$$\rho_0 v_1 \left(v'_0 + \frac{v_0}{r} \right) = \rho_1 \nu \left(v''_0 + \frac{v'_0}{r} - \frac{v_0}{r^2} \right) \quad (14)$$

$$\dot{\rho}_1 + \frac{\rho_0 v_1}{r} + \rho'_0 v_1 + \rho_0 v'_1 = 0 \quad (15)$$

$$V''_1 + \frac{V'_1}{r} = 4\pi G \rho_1 \quad (16)$$

This set of equations is completed by a gas state equation. The nebula gas is approximated by a perfect gas undergoing polytropic transformations of index γ , assumed to be constant throughout the disc. Denoting the local sound speed by c , the pressure at equilibrium reads

$$p_0 = \frac{c_0^2 \rho_0}{\gamma} \quad (17)$$

Using the gas polytropic relation, $p/\rho^\gamma = \text{constant}$, the linearized perturbed pressure reads

$$p_1 = \frac{c_0^2 \rho_1}{\gamma} + 2 \frac{c_0 c_1 \rho_0}{\gamma} = c_0^2 \rho_1 \quad (18)$$

Expressions of the gas circular velocity at equilibrium are found from the radial and azimuthal components of the Navier-Stokes equation (10) and (11) and are given in Appendix A.

Solving for the gas perturbed specific mass ρ_1 and velocity v_1 , the equation (13), with (10), (17) and (18), reads

$$\dot{v}_1 + \frac{c_0^2}{\rho_0} \left(\rho_1' + \rho_1 \left(\left(\frac{\gamma-1}{\gamma} \right) \frac{(c_0^2)'}{c_0^2} - \frac{\rho_0'}{\gamma\rho_0} \right) \right) + V_1' = \nu \left(v_1'' + \frac{v_1'}{r} - \frac{v_1}{r^2} \right) \quad (19)$$

Taking the time derivative of (15) and introducing (16) and (19) yield

$$\begin{aligned} \ddot{\rho}_1 - c_0^2 \left(\rho_1'' + \rho_1' \left(\left(\frac{2\gamma-1}{\gamma} \right) \frac{(c_0^2)'}{c_0^2} - \frac{\rho_0'}{\gamma\rho_0} + \frac{1}{r} \right) \right. \\ \left. + \rho_1 \left(\left(\frac{\gamma-1}{\gamma} \right) \frac{(c_0^2)''}{c_0^2} + \frac{(c_0^2)'}{c_0^2} \left(\left(\frac{\gamma-1}{\gamma} \right) \frac{1}{r} - \frac{\rho_0'}{\gamma\rho_0} \right) \right. \right. \\ \left. \left. - \frac{1}{\gamma\rho_0} \left(\rho_0'' + \rho_0' \left(\frac{1}{r} - \frac{\rho_0'}{\rho_0} \right) \right) + \frac{4\pi G\rho_0}{c_0^2} \right) \right) \\ = \rho_0' V_1' - \frac{1}{r} \frac{\partial}{\partial r} \left(r\rho_0\nu \left(v_1'' + \frac{v_1'}{r} - \frac{v_1}{r^2} \right) \right) \end{aligned} \quad (20)$$

The specific mass flux radial momentum Φ is defined as

$$\Phi = r\rho_0 v_1$$

and its behaviour is given by the continuity equation (15)

$$\dot{\rho}_1 + \frac{1}{r} \frac{\partial}{\partial r} (r\rho_0 v_1) = \dot{\rho}_1 + \frac{\Phi'}{r} = 0 \quad (21)$$

4 Solutions for homogeneous equations

4.1 Time and space dependent separated equations

It seems hopeless to try to find an analytical solution to the third order differential equation (20) in v_1 and ρ_1 . However, a wave equation in ρ_1 with a mass term can be found if one neglects the right side of (20): the gas is assumed of low viscosity such as the viscous friction can be neglected in front of the pressure gradient and of the central mass gravitational gradient and secondly, the product of the radial derivatives of the unperturbed specific mass ρ_0 and of the perturbed gravitational potential is shown to be small (see Appendix B) and can be neglected $\rho_0' V_1' \approx 0$. Using notations of [2], the equilibrium characteristics are written with power law dependencies on the radial distance r . With the dimensionless variable R , one defines

$$R = \frac{r}{r_c} ; \rho_0 = \rho_c R^d ; c_0^2 = c_c^2 R^s \quad (22)$$

where r_c is a reference distance corresponding to the disc inner radius, ρ_c and c_c are the nebula reference specific mass and sound speed at the disc inner edge. The exponents d and s depend on the nebula physical models and are addressed further. The homogeneous equation (20) becomes

$$\ddot{\rho}_1 - \frac{R^s}{A^2} \left(\rho_1'' + \left(2s + 1 - \frac{d+s}{\gamma} \right) \frac{\rho_1'}{R} + \left(B^2 R^{d+2-s} + s \left(s - \frac{d+s}{\gamma} \right) \right) \frac{\rho_1}{R^2} \right) = 0 \quad (23)$$

with, from now on, the prime sign $'$ denoting $\partial/\partial R$ and where

$$A^2 = \frac{r_c^2}{c_c^2} ; B^2 = \frac{4\pi G \rho_c r_c^2}{c_c^2}$$

are constants. Posing

$$\rho_1(R, t) = D(R) \Theta(t) ; v_1(R, t) = U(R) \Xi(t) ; \Phi(R, t) = \Phi(R) \Psi(t) \quad (24)$$

and choosing $-\omega^2$ as separating constant (ω real), for periodic perturbations that do not grow exponentially with time, (23) yields

$$\ddot{\Theta}(t) + \omega^2 \Theta(t) = 0 \quad (25)$$

$$D'' + \left(2s + 1 - \frac{d+s}{\gamma}\right) \frac{D'}{R} + \left(B^2 R^{d+2-s} + \omega^2 A^2 R^{2-s} + s \left(s - \frac{d+s}{\gamma}\right)\right) \frac{D}{R^2} = 0 \quad (26)$$

The perturbed continuity equation (15) yields, with κ as a separating constant

$$\dot{\Theta}(t) - \kappa \Xi(t) = 0 ; \Psi(t) = \Xi(t) \quad (27)$$

$$U(R) = -\kappa \frac{r_c}{\rho_c} R^{-(d+1)} \int D(R) R dR \quad (28)$$

$$\Phi(R) = r_c \rho_c R^{d+1} U(R) = -\kappa r_c^2 \int D(R) R dR \quad (29)$$

showing that $\Phi(R)$ is strongly dependent on the behaviour of the radial perturbed velocity.

The solutions of (24) and (26) for the time-dependent part of ρ_1 and v_1 are

$$\Theta(t) = C \sin(\omega t + \varphi) ; \Xi(t) = \Psi(t) = \frac{C}{\kappa} \omega \cos(\omega t + \varphi) \quad (30)$$

with C and φ constants to be determined by initial conditions. The solutions (30) show that the time dependent parts of the gas perturbed specific mass $\Theta(t)$ and velocity $\Xi(t)$ have the same frequency and the same initial phase but they are out of phase by $\pi/2$, for κ positive, while the time dependent part of the specific mass flux radial momentum $\Psi(t)$ is identical to the one of the gas perturbed velocity $\Xi(t)$. The type of solution of equation (26) and hence the radial behaviour of ρ_1 , v_1 and Φ depend on the exponents d and s of the ρ_0 and c_0 radial distributions. Searching for analytical solutions of the equation (26) for annular structures to appear in the disc, we solve these equations (26), (28) and (29) for certain values of d and s . Two boundary conditions are given: first, at the disc inner edge, for $R = 1$, the nebula perturbed specific mass must equal a parameter $\rho_{c1}^*(t)$ independent of disc physical characteristics, but that can depend on the time t , and second, for increasing R , the nebula perturbed specific mass must decrease and vanish far away from the central mass, for $R \gg 1$, for all time t .

4.2 Solutions for $d = 0$ and $s = 2$

We consider first the unrealistic case of an incompressible nebula ($d = 0$) with a sound speed increasing linearly with the distance ($s = 2$). This first case is purely theoretical, as for a nebula with constant specific mass undergoing polytropic transformations, the sound speed should be constant. The equation (26) becomes then a simple Euler type equation

$$D'' + \left(\frac{5\gamma - 2}{\gamma} \right) \frac{D'}{R} + \left(B^2 + \omega^2 A^2 + 4 \left(\frac{\gamma - 1}{\gamma} \right) \right) \frac{D}{R^2} = 0 \quad (31)$$

Under the condition

$$B^2 + \omega^2 A^2 + 4 \left(\frac{\gamma - 1}{\gamma} \right) > 1$$

yielding

$$\omega^2 > \frac{c_c^2}{r_c^2} \left(\frac{4 - 3\gamma}{\gamma} \right) - 4\pi G \rho_c \quad (32)$$

and with the first boundary condition and posing

$$y = \sqrt{B^2 + \omega^2 A^2 + \frac{3\gamma - 1}{\gamma}}$$

the solution of (31) reads

$$D = \frac{\rho_{c1}^*}{R} \cos(y \ln(R)) \quad (33)$$

where $\ln$ is the Napier logarithm function. The radial terms of the perturbed velocity and of the specific mass flux radial momentum are found from (28) and (29)

$$U = -\kappa \frac{\rho_{c1}^*}{\rho_c} \frac{r_c}{y^2 + 1} R \cos(y \ln(R) - \arctan(y)) \quad (34)$$

$$\Phi = -\kappa \rho_{c1}^* \frac{r_c^2}{y^2 + 1} R^2 \cos(y \ln(R) - \arctan(y)) \quad (35)$$

The extrema (maxima and minima) of D are found from

$$D' = -\frac{\rho_{c1}^* \sqrt{y^2 + 1}}{R^2} \cos(y \ln(R) - \arctan(y)) = 0 \quad (36)$$

The zeros of D (33), U (34), Φ (35) and D' (36) are given by

$$R = \alpha_1 (\beta_1^\lambda)^n ; \quad \alpha_1 = \exp\left(\frac{\pi/2 + \varphi_1}{y}\right) ; \quad \beta_1^\lambda = \exp\left(\frac{\pi}{y}\right) \quad (37)$$

and n non-negative integers, $\varphi_1 = 0$ for D and $\varphi_1 = \arctan(y)$ for U , Φ and D' . The initial spatial phase between D and U is $\arctan(y) = \pi/2$, provided that y is large enough within the condition (32), while there is no initial phase between U (or Φ) and D' . The distance ratio of two successive maxima of D , for $D'' < 0$, is

$$\beta_1 = (\beta_1^\lambda)^2 = \exp\left(\frac{2\pi}{\frac{r_c}{c_c} \sqrt{\omega^2 + 4\pi G \rho_c + \frac{c_c^2}{r_c^2} \left(\frac{3\gamma - 4}{\gamma} \right)}}\right) \quad (38)$$

which, from (32), is a real constant depending on the nebula characteristics r_c , c_c , ρ_c , γ and on the perturbations circular frequency ω . Note that the condition (32) is equivalent to the dispersion relation in the classical Jeans problem (see e.g., [11]) with, for $\omega^2 = 0$, critical wave number and wavelength

$$k_{crit} = \frac{\sqrt{4\pi G \rho_c}}{c_c} = \frac{\sqrt{\frac{4-3\gamma}{\gamma}}}{r_c} ; \quad \lambda_{crit} = 2\pi r_c \sqrt{\frac{\gamma}{4-3\gamma}} \quad (39)$$

The relation (32) ensures that the perturbations do not grow exponentially with time.

4.3 Solutions for $s = 2$ and $d < 2(2\gamma - 1)$, $d \neq 0$

In this second case, the sound speed increases linearly with the radial distance and the specific mass depends on the radial distance, with the conditions that d must be non-null and smaller than $2(2\gamma - 1)$. The equation (26) becomes

$$D'' + \left(\frac{5\gamma - (d+2)}{\gamma} \right) \frac{D'}{R} + \left(B^2 R^d + \omega^2 A^2 + 2 \left(\frac{2\gamma - (d+2)}{\gamma} \right) \right) \frac{D}{R^2} = 0 \quad (40)$$

which is a Bessel type equation, whose general solution reads

$$D = K_1 R^{((d+2)/2\gamma)-2} Z_\nu(z) \quad (41)$$

where $Z_\nu(z)$ is the Bessel function of first kind with z the argument and ν , from now on, the order

$$z = \frac{2}{d} B R^{d/2} ; \quad \nu = \frac{2}{d} \sqrt{\left(\frac{d+2}{2\gamma} \right)^2 - \omega^2 A^2} \quad (42)$$

and K_1 is a constant determined by the first boundary condition

$$K_1 = \frac{\rho_{c1}^*}{Z_\nu\left(\frac{2}{d}B\right)}$$

For circular frequencies ω such that

$$\omega > \frac{d+2}{2\gamma A} = \left(\frac{d+2}{2\gamma} \right) \frac{c_c}{r_c} \quad (43)$$

the order ν is a pure imaginary, $\nu = jy$ with $j = \sqrt{-1}$ and, from now on,

$$y = \frac{2}{d} \sqrt{\omega^2 A^2 - \left(\frac{d+2}{2\gamma} \right)^2}$$

The function $Z_\nu(z)$ takes complex values and reads generally [12]

$$Z_\nu(z) = \left(\frac{z}{2} \right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2} \right)^{2k}}{k! \Gamma(\nu + k + 1)} \quad (44)$$

where Γ is the Legendre Gamma function. Writing

$$\begin{aligned} \Gamma(k+1+jy) &= h_k \exp(j\eta_k) \\ h_k &= k! \prod_{n=0}^{\infty} \frac{1}{\sqrt{\frac{y^2}{(k+1+n)^2} + 1}} \\ \eta_k &= y\Psi(k+1) + \sum_{n=0}^{\infty} \left(\frac{y}{(k+1+n)} - \arctan\left(\frac{y}{(k+1+n)}\right) \right) \end{aligned} \quad (45)$$

where Ψ is the digamma function, the Bessel function of imaginary order reads

$$Z_{\nu}(z) = \sum_{k=0}^{\infty} C_{1k} \left(\frac{z}{2}\right)^{2k} \exp\left(j\left(y \ln\left(\frac{z}{2}\right) - \eta_k\right)\right) \quad (46)$$

with

$$C_{1k} = \exp(q) \frac{(-1)^k}{k! h_k}$$

where $q = 0$ if $d > 0$ and $q = -\pi y$ if $d < 0$ and where, from now on, z has to be replaced by its absolute value

$$|z| = \frac{2}{|d|} B R^{d/2}$$

Taking the real part of (46), the relation (41) reads

$$D = K_1 R^{((d+2)/2\gamma)-2} \sum_{k=0}^{\infty} C_{1k} \left(\frac{z}{2}\right)^{2k} \cos\left(y \ln\left(\frac{z}{2}\right) - \eta_k\right) \quad (47)$$

The second boundary condition, decreasing D for increasing R , restricts the exponent of R , giving the initial condition on d , $d < 2(2\gamma - 1)$, $d \neq 0$.

The radial terms of the perturbed velocity and of the specific mass flux momentum become, from (28) and (29),

$$U = -\kappa K_1 \frac{r_c}{\rho_c} R^{((d+2)/2\gamma)-(d+1)} \sum_{k=0}^{\infty} C_{2k} \left(\frac{z}{2}\right)^{2k} \sin\left(y \ln\left(\frac{z}{2}\right) - \eta_k + \tau_k\right) \quad (48)$$

$$\Phi = -\kappa K_1 r_c^2 R^{(d+2)/2\gamma} \sum_{k=0}^{\infty} C_{2k} \left(\frac{z}{2}\right)^{2k} \sin\left(y \ln\left(\frac{z}{2}\right) - \eta_k + \tau_k\right) \quad (49)$$

with

$$C_{2k} = \frac{2C_{1k}}{\sqrt{\left(kd + \frac{d+2}{2\gamma}\right)^2 + \left(\frac{yd}{2}\right)^2}} ; \tau_k = \arctan\left(\frac{2}{yd} \left(kd - \frac{d+2}{2\gamma}\right)\right)$$

The extrema of D are solutions of

$$D' = -K_1 R^{((d+2)/2\gamma)-3} \sum_{k=0}^{\infty} C_{3k} \left(\frac{z}{2}\right)^{2k} \sin\left(y \ln\left(\frac{z}{2}\right) - \eta_k + \mu_k\right) = 0 \quad (50)$$

$$\tan\left(y \ln\left(\frac{z}{2}\right)\right) = \frac{\sum_{k=0}^{\infty} C_{3k} \left(\frac{z}{2}\right)^{2k} \sin(\eta_k - \mu_k)}{\sum_{k=0}^{\infty} C_{3k} \left(\frac{z}{2}\right)^{2k} \cos(\eta_k - \mu_k)} \quad (51)$$

with

$$C_{3k} = C_{1k} \sqrt{\left(2 - kd - \frac{d+2}{2\gamma}\right)^2 + \left(\frac{yd}{2}\right)^2}$$

$$\mu_k = \arctan\left(\frac{2}{yd} \left(2 - kd - \frac{d+2}{2\gamma}\right)\right)$$

The zeros of U and Φ are found like in (51) with C_{3k} and μ_k replaced by C_{2k} and τ_k . It seems that there are no simple analytical solutions to (51). However, for small arguments $(z/2) \ll 1$, i.e.,

$$\frac{4\pi G \rho_c r_c^2}{dc_c^2} R^d \ll 1 \quad (52)$$

one finds similar solutions for D (47), U (48), Φ (49) and D' (50), in the form

$$\tan\left(y \ln\left(\frac{z}{2}\right)\right) \approx \tan(\kappa)$$

with κ constant, as the first term for $k = 0$ in the series of (51) predominates, yielding $\kappa = \eta_0$ for D , $\kappa = (\eta_0 - \tau_0)$ for U and Φ , and $\kappa = (\eta_0 - \mu_0)$ for D' .

The zeros of D (47), U (48), Φ (49) and D' (50) are then given by

$$R = \alpha_2 (\beta_2^\vee)^n ; \quad \alpha_2 = \left(\frac{|d|}{B}\right)^{2/d} \exp\left(\frac{2(\eta_0 + \phi_2)}{dy}\right) ; \quad \beta_2^\vee = \exp\left(\frac{2\pi}{dy}\right) \quad (53)$$

n being non-negative integers and $\phi_2 = \pi/2$ for D , $\phi_2 = -\tau_0$ for U and Φ , and $\phi_2 = -\mu_0$ for D' . Provided that y is large enough within the condition (43), one has $\tau_0 \ll 1$ and $\mu_0 \ll 1$. The initial phase between D and U is $(\pi/2) - \tau_0 \approx (\pi/2)$, while the initial phase between U (or Φ) and D' is $(\mu_0 - \tau_0) \approx 0$. The distance ratio of two successive maxima of D is

$$\beta_2 = (\beta_2^\vee)^2 = \exp\left(\frac{2\pi}{\sqrt{\omega^2 \frac{r_c^2}{c_c^2} - \frac{c_c^2}{r_c^2} \left(\frac{d+2}{2\gamma}\right)^2}}\right) \quad (54)$$

which, from (43), is a real constant depending on nebula reference characteristics and on ω .

4.4 Solutions for $d = s - 2$ with $d > (2\gamma - 1)/(1 - \gamma)$, $d \neq 0$

The third case is more general and considers the two exponents linked by the relation $d = s - 2$ with the restrictions $d \neq 0$ ($s \neq 2$) and $d > (2\gamma - 1)/(1 - \gamma)$. The equation (26) becomes

$$D'' + \left(2d + 5 - \frac{2(d+1)}{\gamma}\right) \frac{D'}{R}$$

$$+ \left(B^2 + \frac{\omega^2 A^2}{R^d} + (d+2) \left(d + 2 - \frac{2(d+1)}{\gamma}\right)\right) \frac{D}{R^2} = 0 \quad (55)$$

which is another Bessel type differential equation, whose solutions are

$$D = K_2 R^{((d+1)/\gamma)-(d+2)} Z_\nu(z) \quad (56)$$

where the argument z and the order ν are now

$$z = \frac{2}{|d|} \omega A R^{|d|/2} ; \quad \nu = \frac{2}{|d|} \sqrt{\left(\frac{d+1}{\gamma}\right)^2 - B^2} \quad (57)$$

with K_2 a constant determined by the first boundary condition

$$K_2 = \frac{\rho_{c1}^*}{Z_\nu\left(\frac{2}{|d|} \omega A\right)}$$

Under the condition

$$B^2 > \left(\frac{d+1}{\gamma}\right)^2$$

yielding

$$\frac{4\pi G \rho_c r_c^2}{c_c^2} > \left(\frac{d+1}{\gamma}\right)^2 \quad (58)$$

the order ν is a pure imaginary, $\nu = jy$, with from now on

$$y = \frac{2}{|d|} \sqrt{B^2 - \left(\frac{d+1}{\gamma}\right)^2}$$

Writing the Bessel functions of imaginary order as in (46), with $q = 0$ in C_{1k} , the solution (56) becomes

$$D = K_2 R^{((d+1)/\gamma)-(d+2)} \sum_{k=0}^{\infty} C_{1k} \left(\frac{z}{2}\right)^{2k} \cos\left(y \ln\left(\frac{z}{2}\right) - \eta_k\right) \quad (59)$$

The second boundary condition is fulfilled by the restriction on the exponent of R (with $\gamma > 1$).

The radial parts of the perturbed velocity and of the specific mass flux momentum read, from (28) and (29),

$$U = -\kappa K_2 \frac{r_c}{\rho_c} R^{((d+1)/\gamma)-(2d+1)} \sum_{k=0}^{\infty} C_{4k} \left(\frac{z}{2}\right)^{2k} \sin\left(y \ln\left(\frac{z}{2}\right) - \eta_k + \theta_k\right) \quad (60)$$

$$\Phi = -\kappa K_2 r_c^2 R^{((d+1)/\gamma)-d} \sum_{k=0}^{\infty} C_{4k} \left(\frac{z}{2}\right)^{2k} \sin\left(y \ln\left(\frac{z}{2}\right) - \eta_k + \theta_k\right) \quad (61)$$

Like in the previous section, the extrema of D are solutions of

$$D' = -K_2 R^{((d+1)/\gamma)-(d+3)} \sum_{k=0}^{\infty} C_{5k} \left(\frac{z}{2}\right)^{2k} \sin\left(y \ln\left(\frac{z}{2}\right) - \eta_k + \sigma_k\right) = 0 \quad (62)$$

$$\tan\left(y \ln\left(\frac{z}{2}\right)\right) = \frac{\sum_{k=0}^{\infty} C_{5k} \left(\frac{z}{2}\right)^{2k} \sin(\eta_k - \sigma_k)}{\sum_{k=0}^{\infty} C_{5k} \left(\frac{z}{2}\right)^{2k} \cos(\eta_k - \sigma_k)} \quad (63)$$

with

$$C_{5k} = C_{1k} \sqrt{\left(d + 2 - \frac{d+1}{\gamma} - k|d|\right)^2 + \left(\frac{yd}{2}\right)^2}$$

$$\sigma_k = \arctan\left(\frac{2}{y|d|} \left(d + 2 - \frac{d+1}{\gamma} - k|d|\right)\right)$$

The zeros of U and Φ are found like in (63) with C_{4k} and θ_k replacing C_{5k} and σ_k .

For small arguments $(z/2) \ll 1$, i.e.,

$$\omega \ll |d| \frac{c_c}{r_c} R^{-|d|/2} \quad (64)$$

one finds similar solutions for D (59), U (60), Φ (61) and D' (62) like in the previous case, as (63) is equal to a constant, $\tan(\kappa)$, with $\kappa = \eta_0$ for D , $\kappa = (\eta_0 - \theta_0)$ for U and Φ , and $\kappa = (\eta_0 - \sigma_0)$ for D' .

The zeros of D (59), U (60), Φ (61) and D' read

$$R = \alpha_3 (\beta_3^*)^n \quad ; \quad \alpha_3 = \left(\frac{|d|}{\omega A}\right)^{2/|d|} \exp\left(\frac{2(\eta_0 + \phi_3)}{|d|y}\right) \quad ; \quad \beta_3^* = \exp\left(\frac{2\pi}{|d|y}\right) \quad (65)$$

with n non-negative integers, $\phi_3 = \pi/2$ for D , $\phi_3 = -\theta_0$ for U and Φ , and $\phi_3 = -\sigma_0$ for D' . Provided that y is large enough within the condition (58), one has $\theta_0 \ll 1$ and $\sigma_0 \ll 1$. The initial phase between D and U (or Φ) is $\pi/2 - \theta_0 \approx \pi/2$, while the initial phase between U (or Φ) and D' is $(\sigma_0 - \theta_0) \approx 0$.

The distance ratio of two successive maxima of D is

$$\beta_3 = (\beta_3^*)^2 = \exp\left(\frac{2\pi}{\sqrt{4\pi G \rho_c \frac{r_c^2}{c_c^2} - \left(\frac{d+1}{\gamma}\right)^2}}\right) \quad (66)$$

which, from (58), is a real constant depending on the reference characteristics but independent of ω . The period of the small perturbations must be larger than a minimum value

$$P_m = \frac{2\pi}{|d|} \frac{r_c}{c_c} (R_{max})^{|d|/2} \quad (67)$$

deduced from the condition (64) applied to the whole range of radial distances of a nebula (R_{max} is the ratio of the disc outer to inner radii).

4.5 Standard model

We mention an interesting particular case, called the "standard model", of the general case $d = (s - 2)$ above. One writes the gravitational potential in the unperturbed disc as a power law distribution in R ($= r/r_c$)

$$V_0 = V_c R^u \quad (68)$$

where V_c is the gravitational potential of the central mass M^* (the gravitational potential of the disc is neglected as $M_d \ll M^*$) and u is an exponent to be defined by

physical models. Replacing in the Poisson equation at equilibrium (12) with (22) yields successively

$$\frac{u^2 V_c}{r_c^2} R^{u-2} = 4\pi G \rho_c R^d \quad (69)$$

$$V_c = \frac{4\pi G \rho_c r_c^2}{u^2} = \frac{3GM_c}{u^2 r_c} \quad (70)$$

for $d = u - 2$ and with $M_c = (4\pi/3) r_c^3 \rho_c$, the mass of the homogeneous sphere of specific mass ρ_c and radius r_c .

We make the hypothesis for the "standard model" that the reference distance r_c of the disc inner edge can be approximated by the central body unperturbed external radius r_c^*

$$r_c \approx r_c^* \quad (71)$$

(superscript * denotes central body characteristics). Noting the central body mean specific mass by ρ^* , identifying V_c in (70) with the gravitational potential of the central mass M^* yields

$$\rho_c = \frac{u^3}{3} \rho^* \quad (72)$$

In the simplest case, the gravitational potential of a spherical body is given by (68), with $u = -1$. The condition (70) yields then $d = -3$ and, from (72), the nebula reference specific mass ρ_c is one third of the mean specific mass of the central body.

On the other hand, within the perfect gas approximation, the sound speed distribution (22) follows the gas temperature radial distribution in the disc, which can be represented by a power law relation of exponent ζ

$$c_c^2 R^s = \frac{\gamma \Re}{\mu} T_c R^\zeta \quad (73)$$

with $\Re$ the perfect gas constant, μ the gas molecular mass and T_c a reference temperature at the disc inner edge, that can be approximated for example by the central body effective temperature. The radial behaviour of the temperature in a nebula is model dependent. Considering only the central body luminosity as the dominant source of energy heating the nebula (the gas viscosity is neglected), the temperature gradient is adiabatic with $\zeta = -1$ for an optically thick nebula [13], yielding $s = -1$.

We define then the "standard model" of a disc as the case with $u = -1$, $d = -3$ and $s = -1$, and it can be solved with these values by the general case $d = (s - 2)$ above. The distance ratio of maxima of the gas perturbed specific mass distribution writes then, from (66),

$$\beta_{st.mod.} = \exp \left(\frac{2\pi c_c}{\sqrt{\frac{GM^*}{r_c} - \left(\frac{2c_c}{\gamma}\right)^2}} \right) \quad (74)$$

The condition (58) ensures that this ratio is a real constant.

This simple "standard model" can be useful as a first approximation model, provided that the disc mass M_d calculated with the value (72) of ρ_c fulfills the initial condition $M_d \ll M^*$. Let's note also that in the above approximation, the value $\rho_{c1}^*(t)$ that the nebula perturbed specific mass has to match at the disc inner edge

(first boundary condition) can be approximated by the perturbed specific mass of the central body at its outer edge, for $r = r_c^* = r_c$ or $R = 1$, at the epoch t . (Strictly speaking, one should consider the central body external perturbed radius $r_{c1}^* = r_c^* + \xi(r_c^*, t)$, where $\xi(r_c^*, t)$ is the radial displacement of the central body outer edge at $r = r_c^*$ due to small perturbations at the epoch t , yielding $R_{c1} = r_{c1}^*/r_c^* = 1 + \xi/r_c^*$; but if the displacements are small in front of the central body unperturbed radius ($\xi \ll r_c^*$), one has $R_{c1} \approx R_c = 1$).

5 Formation of annular structures

For all the cases considered, the spatial part of the perturbed specific mass D has a sign opposite to the signs of its radial derivative D' , of the radial perturbed velocity U and of the specific mass flux radial momentum Φ . The functions U and Φ have an initial phase difference of approximately $\pi/2$ with respect to the function D . The zeros of U correspond to the extrema of D and vice-versa. For increasing R , U and Φ are positive (respectively negative) between successive minima and maxima (respectively successive maxima and minima) of D , as shown in Figure 1. This configuration yields radial outward flows of gas between successive minima and maxima of D and radial inward flows of gas between successive maxima and minima. The extrema amplitudes of D and D' decrease for increasing R , while the extrema amplitudes of U and Φ increase for increasing R , although less for Φ than for U in the case $d = (s - 2)$. The nebular gas, flowing outward (respectively inward) with a positive (respectively negative) radial velocity U , may accumulate in annular rings centered on circular orbits with radii corresponding to the distances of the maxima of the gas perturbed specific mass, depleting the zones of minima of perturbed specific mass.

In a rotating nebula containing "dust", the solid particles experience an inward drift due to the gas drag caused by the difference of the gas circular velocity and the Keplerian orbital velocity, the former being less than the latter [14]. Smaller particles are more affected by the gas drag than larger ones. Particles on eccentric orbits encounter gas of variable density, causing a circularization of their orbit. If a radial velocity is superimposed onto the gas circular velocity, solid particles experience an additional radial drag causing the orbit of smaller particles to decay more (respectively less) rapidly in the case of inward (respectively outward) gas flow, larger particles being less affected. The nebular "dust" is dragged along with the gas, causing the orbits eccentricity of particles to change, favouring collision and accretion (see e.g., [15]). This process would eventually result in an accumulation of solid particles dragged along with the gas, near zones of maxima of gas perturbed specific mass. A more detailed analysis of the dynamical gas/particle interactions would confirm this, but is outside the scope of this paper.

6 Conclusions

It was shown that, when under small radial periodic perturbations and disregarding non-radial perturbations, thin rotating low mass gaseous discs, described by a simple two-dimensional axisymmetric model, evolve such as the perturbed part of the gas specific mass displays exponentially spaced maxima, two successive maxima being separated by a minimum. The gas flows from locations of specific mass minima inward to

Fig. 1 Plots of gas perturbed specific mass D and radial velocity U vs radial distance R ("standard model", $d = -3$, $s = -1$ with $\gamma = 3/2$; D and U from (59) and (62) in text, first term only ($k = 0$) in the series). The direction of the gas flow from minima toward maxima of D is indicated by arrows: left arrows for inward gas flow with a negative velocity U , right arrows for outward gas flow with a positive velocity U ; the middle line represent the unperturbed gas specific mass ρ_0 for D and the unperturbed radial velocity (nil in this model) for U .

the preceding maximum or outward to the next maximum, as the gas radial velocity is negative (inward flow) or positive (outward flow). This mechanism would eventually form gaseous annular structures.

Furthermore, the distance ratio of two successive maxima is found to be a constant depending on disc characteristics (and on the perturbations frequency for the first two cases). The nature and origin of the perturbations are not discussed here. However, one can make the hypothesis that the origin of the perturbations may lie within the central mass or at the interface disc/central mass, due to periodic radial motions.

Lower limit on orders of magnitudes of time scales can be deduced for the case $d = (s - 2)$ from the condition (67) on the period of the perturbations. For the "standard model", minimum periods depend on dimensions of the central mass and are in the order of several 10^3 years for protostellar discs similar to what the protoplanetary disc around the proto-Sun may have been and in the order of several 10^{-1} year for giant planets proto-satellite discs. In a second paper, we explore analytical solutions of the perturbed specific mass wave-like propagation by considering two other general cases.

Acknowledgements We wish to thank Prof. O. Godart, Catholic University of Louvain, Louvain-la-Neuve, Belgium, for early discussions on the subject of this paper.

Appendix A

Solving for the gas velocity v_0 at equilibrium within the hypothesis that the kinematic viscosity is negligible ($\nu = 0$), the azimuthal component of the Navier-Stokes equation (11) yields

$$v_0'' + \frac{v_0'}{r} - \frac{v_0}{r^2} = f_v(r) \quad (75)$$

with the notation $'' = \partial/\partial r$ and where $f_v(r)$ is an unspecified function of r , giving in general

$$v_0 = C_1 \frac{1}{r} + C_2 + F_v(r) \quad (76)$$

with

$$F_v(r) = \frac{2}{r^2} \int \left(\int f_v(r) dr \right) r dr$$

and C_1 and C_2 constants determined by boundary conditions. If the viscosity ν is non null, then obviously $f_v(r)$ and $F_v(r)$ have to be nil in (75) and (76). For $r \rightarrow \infty$, the gas circular velocity has to stay within finite values, yielding theoretically $C_2 = 0$. Another expression of the gas circular velocity v_0 at equilibrium is found from the radial component of the Navier-Stokes equation (10). Using (17) and (22), it yields

$$v_0 = \sqrt{\frac{GM^*}{r} + \frac{c_c^2}{r_c^s} \left(\frac{s+d}{\gamma} \right) r^s} \quad (77)$$

where s and d are usually negative. For the gas circular velocity v_0 to be real, the Keplerian velocity has to be greater than the velocity induced by the gas gradient pressure, which is usually the case in real nebulae [14]. The relations (76) and (77) are complementary in describing the radial profile of the circular gas velocity at equilibrium. For $\nu \neq 0$ ($F_v(r) = 0$) and noting generally $v_0(r) \sim r^p$, the value $p = +1$ ($C_1 = 0$) gives the rotation velocity of a solid, and approximately of a fluid with high viscosity, at a constant angular speed. A value $p = -1$ ($C_2 = 0$) describes the rotation of a perfect gas, and approximately of a fluid with low viscosity. The value $p = -1/2$ corresponds to a Keplerian rotation. A value $p = s/2$ describes the rotation of a gas dominated by thermal pressure. The gas circular velocity profile in a real nebula is at least a combination of the three first cases, as observed in the galaxies' rotation curves [16, 17]: highly viscous fluid near the central mass ($v_0(r) \approx$ linear relation), lesser viscous fluid further from the centre ($v_0(r) \approx$ inverse linear relation) and, after a transition region, approximate Keplerian rotation in the external regions ($v_0(r) \approx$ inverse root square relation).

Appendix B

One can neglect $\rho_0' V_1'$ in (20) if small displacements occur due to small radial perturbations. Assuming that a fluid element is displaced from vectorial positions x to $x + \xi(x)$, where $\xi(x)$ is a small displacement, vectorial function of x , the perturbed specific mass at x reads

$$\rho_1(x) = -\nabla \cdot (\rho_0 \xi) \quad (78)$$

where the specific mass in the divergence operator is replaced by the unperturbed specific mass as it is multiplied by the small quantity ξ [18, 19]. Assuming that ρ_1 and ξ depend only on r in a cylindrical polar referential, $\xi = (\xi(r), 0, 0)$, the relation (78) reads

$$\rho_1(r) = \frac{-1}{r} \frac{\partial(r\rho_0\xi)}{\partial r} \quad (79)$$

and from (9) and (6), with the notation $\prime = \partial/\partial r$,

$$V_1' = \frac{4\pi G}{r} \int \rho_1 r dr = -4\pi G \rho_0 \xi \quad (80)$$

The product $\rho_0' V_1'$ in (20) reads then, with (22),

$$\rho_0' V_1' = -4\pi G d \rho_c^2 R^{2d-1} \left(\frac{\xi}{r_c} \right) \quad (81)$$

showing that it can be neglected if the small displacement ξ is small enough in comparison with the central mass radius r_c .

References

- [1] Shu, F.H., 1984, in "Planetary rings", eds. R.Greenberg and A.Brahic, Univ. Arizona Press, Tucson, p. 513.
- [2] Nowotny, E., 1979, The Moon and the Planets, 21, p. 257.
- [3] Pletser V., 1990, "On exponential distance relations in planetary and satellite systems, observations and origin", PhD Thesis, Physics Dept, Faculty of Sciences, Catholic University of Louvain, Louvain-la-Neuve, Belgium (available at <https://www.researchgate.net/publication/257927392>).
- [4] Lin, D.N.C., 1986, in "The Solar System, Observations and Interpretations", ed. M.G. Kivelson, Prentice-Hall, New Jersey, p. 28.
- [5] Black, D.C. & Matthews, M.S., eds, 1985, "Protostars and Planets II", Univ. Arizona Pr.
- [6] Morfill, G.E., Tschamuter, W. & Volk, H.J., 1985, in "Protostars and Planets II", D.C. Black and M.S. Matthews eds., Univ. Arizona Pr., p. 493.
- [7] Bodenheimer, P. & Tscharnuter, W.M., 1979, Astron. Astrophys., 74, p. 288.
- [8] Rozyczka, M., Tscharnuter, W.M., Winkler, K.H. & Yorke, H.W., 1980 a, Astron. Astrophys., 83, p. 118.
- [9] Rozyczka, M., Tschamuter, W.M. & Yorke, H.W., 1980 b, Astron. Astrophys., 83, p. 347.
- [10] Cassen, P., Shu, F.H. & Terebey, S., 1985, in "Protostars and Planets II", D.Black and M.Matthews eds., Univ. Arizona Pr., p. 448.
- [11] Tscharnuter, W.M., 1985, in "Birth and infancy of stars", eds. R.Lucas, A.Omont and R.Stora, Elsevier Sc.Publ.B.V., North-Holland, p. 601.
- [12] Jahnke-Emde-Losch, 1966, "Tafeln höherer Funktionen", B.G. Teubner Verlagsges, Stuttgart.
- [13] Pollack, J.B., Grossman, A.S., Moore, R. & Grasboke, H.C., 1977, Icarus, 30, p. 111.
- [14] Weidenschilling, S.J., 1977, MNRAS, 180, p. 57.
- [15] Brahic, A., 1977, Astron. Astrophys. 54, p. 895.
- [16] Shapley, H., 1972, "Galaxies", Harvard Univ. Press, Cambridge, USA, p. 151.
- [17] Bowers, R. & Deeming, T., eds, 1984, "Astrophysics vol. II: Interstellar Matter and Galaxies", Jones and Bartlett Publ., Boston, p. 542.
- [18] Chandrasekhar, S., 1969, "Ellipsoidal Figures of Equilibrium", Yale Univ. Pr., New Haven, Connecticut.
- [19] Binney, J. & Tremaine, S., 1987, Galactic Dynamics, Princeton Univ. Pr., New Jersey, p. 283.

Statements and Declarations

The author declares that no funds, grants, or other support were received during the preparation of this manuscript.

The author has no relevant financial or non-financial interests to disclose.

The author contributed to the study conception and design. Material preparation, data collection and analysis were performed by Vladimir Pletser. The first draft of the manuscript was written by Vladimir Pletser and the author commented on previous versions of the manuscript. The author read and approved the final manuscript.

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.