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Time Operator in Quantum Mechanics; Fourier Transform

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We can not only bring time operator in quantum mechanics (non-relativistic) but also determine its Eigen value, commutation relation of its square with energy and some of the properties of time operator like either it is Hermitian or not, either its expectation value is real or complex for a wave packet etc. Exactly these are what I have done.

Keywords: Quantum Mechanics, Quantum physics, Quantum Operators, and Time Operators.

I. INTRODUCTION

Any kind of measurement is organized in space and time. Normally physical quantities are either function of space or function of time and sometimes function of both. In non-relativistic quantum mechanics, though wave function depends on both space and time, only space enters quantum mechanics as operator, time does not. There is no time operator in quantum mechanics. All quantum mechanics textbooks introduce only space as operator, not time. But it is not only possible to produce time operator in quantum mechanics but also we can find out Eigen value of time operator, determines its squares commutation relation with energy, and prove that it is Hermitian operator; the expectation value of time operator is real for a wave packet and any two wave functions $\psi(r, t)$ and $\Phi(r, E)$ are Fourier transform of each other, where r, t, E represents respectively position, time and energy of any quantum mechanical particle.

II. ORIGIN OF TIME OPERATOR

If wave nature of some quantum mechanical free particle is described by de Broglie wave function [1]

$$\Psi(r, t) = A.e^{i(k.r - \omega t)} \quad (1)$$

where, K is the wave number, ω is angular frequency, t is the time and r is position .

$$\Rightarrow \Psi(r, t) = A.e^{i(\frac{p.r - E.t}{\hbar})} [Momentum \mathbf{p} = \hbar k, E = \hbar \omega]$$

$$\Rightarrow \frac{\partial \psi(r, t)}{\partial E} = \frac{-i}{\hbar} t A.e^{i(\frac{p.r - E.t}{\hbar})}$$

$$\frac{\partial \psi(r, t)}{\partial E} = \frac{-i}{\hbar} t \psi(r, t)$$

$$\frac{\partial}{\partial E} = -\frac{i}{\hbar} t$$

$$\hat{t} = i\hbar \frac{\partial}{\partial E} \quad (2)$$

This is time operator. Though in Zhi-Yong Wang's paper [2] there was a minus sign in front of $\hat{t} = i\hbar \frac{\partial}{\partial E}$. We can omit that.

III. EIGEN VALUE OF TIME OPERATOR

Now we will see how to find out its Eigen value. We have known, $\hat{t} = i\hbar \frac{\partial}{\partial E}$

$$\Rightarrow \hat{t}\psi = i\hbar \frac{\partial}{\partial E} \psi$$

$$\Rightarrow \hat{t}\psi = i\hbar \frac{\partial}{\partial E} \left(A.e^{i(\frac{p.r - E.t}{\hbar})} \right)$$

$$\Rightarrow \hat{t}\psi = i\hbar A \left(\frac{-i}{\hbar} t \right) e^{-iEt}$$

$$\Rightarrow \hat{t}\psi = t\psi$$

$$\Rightarrow \hat{t} = t$$

This is the Eigen value of time operator.

We know that the commutation relation between time operator and energy operator is $i\hbar$ [3].

IV. COMMUTATION RELATION BETWEEN SQUARE OF TIME OPERATOR AND ENERGY

Now we will determine the commutation relation between square of time operator and energy. $[\hat{t}^2, E]\psi = (\hat{t}^2 E - E \hat{t}^2)\psi$

$$\Rightarrow [\hat{t}^2, E]\psi = i\hbar \frac{\partial}{\partial E^2} (E\psi) - i\hbar E \frac{\partial \psi}{\partial E^2}$$

$$\Rightarrow [\hat{t}^2, E]\psi = i\hbar \left(\frac{\partial}{\partial E^2} \right) \psi + i\hbar E \frac{\partial \psi}{\partial E^2} - i\hbar E \frac{\partial \psi}{\partial E^2}$$

$$\Rightarrow [\hat{t}^2, E]\psi = 0$$

$$\Rightarrow [\hat{t}^2, E] = 0.$$

This is the commutation relation between square of time operator and energy and it is zero. This means, square of time operator and energy commutes.

V. WAVE FUNCTIONS WHICH DEPENDS ON POSITION AND TIME IS FOURIER TRANSFORM OF WAVE FUNCTION DEPENDENT ON POSITION AND ENERGY

We can show that two wave functions $\psi(r, t)$ and $\phi(r, E)$ are Fourier transform of each other. If wave function $\psi(r, t)$ is given by that,

$$\psi(r, t) = A\psi(r) e^{\frac{-iEt}{\hbar}} \quad (3)$$

$$\Rightarrow \psi_E(t) = A e^{\frac{-iEt}{\hbar}}$$

$$\langle t | \psi_E \rangle = A e^{\frac{-iEt}{\hbar}} \quad (4)$$

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$$\langle \psi_E | t' \rangle = A^* e^{-\frac{iEt'}{\hbar}} \quad (5)$$

By multiplying Equations 4 and 5,

$$\begin{aligned} \langle t | \psi_E \rangle \langle \psi_E | t' \rangle &= |A|^2 e^{\frac{iE(t'-t)}{\hbar}} \\ |A|^2 e^{\frac{iE(t'-t)}{\hbar}} dE &= \int \langle t | \psi_E \rangle \langle \psi_E | t' \rangle dE \\ |A|^2 e^{\frac{iE(t'-t)}{\hbar}} dE &= \int \langle t | E \rangle \langle E | t' \rangle dE \\ |A|^2 e^{\frac{iE(t'-t)}{\hbar}} dE &= \langle t | \int E | \rangle \langle E | t' \rangle dE \\ |A|^2 e^{\frac{iE(t'-t)}{\hbar}} dE &= \langle t | t' \rangle \\ |A|^2 e^{\frac{iE(t'-t)}{\hbar}} dE &= \delta(t - t') \\ |A|^2 e^{\frac{iE(t'-t)}{\hbar}} dE &= \frac{1}{2\pi} \int e^{(t-t')d(\hbar\omega)} dE \\ \text{In the left hand side, We interchange } t \text{ and } t' \text{ and get} \\ |A|^2 e^{\frac{iE(t-t')}{\hbar}} dE &= \frac{1}{2\pi\hbar} \int e^{(t-t')d(\hbar\omega)} dE \\ |A|^2 &= \frac{1}{2\pi\hbar} \\ A &= \frac{1}{\sqrt{2\pi\hbar}} \end{aligned}$$

Substituting the value of A into Equation 3,

$$\langle t | \psi_E \rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{-\frac{iEt}{\hbar}} \quad (6)$$

$$\langle t | \psi_E \rangle = (\sqrt{2\pi\hbar})^{-\frac{1}{2}} e^{\frac{iEt}{\hbar}} \quad (7)$$

Now,

$$\begin{aligned} \langle t | \alpha \rangle &= \int d^3E \langle t | E \rangle \langle E | \alpha \rangle \\ \langle t | \alpha \rangle &= \int d^3E (\sqrt{2\pi\hbar})^{-\frac{1}{2}} e^{-\frac{iEt}{\hbar}} \langle E | \alpha \rangle \quad [\text{From 6}] \\ \langle t | \alpha \rangle &= (\sqrt{2\pi\hbar})^{-\frac{1}{2}} \int \langle E | \alpha \rangle e^{-\frac{iEt}{\hbar}} d^3E \\ \langle t | \psi_\alpha \rangle &= (\sqrt{2\pi\hbar})^{-\frac{1}{2}} \int \langle E | \varphi_\alpha \rangle e^{-\frac{iEt}{\hbar}} d^3E \end{aligned}$$

$$\psi_\alpha(t) = (\sqrt{2\pi\hbar})^{-\frac{1}{2}} \varphi_\alpha(E) e^{-\frac{iEt}{\hbar}} d^3E \quad (8)$$

Again,

$$\begin{aligned} \langle E | \alpha \rangle &= \int d^3t \langle E | t \rangle \langle t | \alpha \rangle \\ \langle E | \alpha \rangle &= (\sqrt{2\pi\hbar})^{-\frac{1}{2}} \int e^{\frac{iEt}{\hbar}} d^3t \langle t | \alpha \rangle \quad [\text{From 7}] \\ \langle E | \alpha \rangle &= (\sqrt{2\pi\hbar})^{-\frac{1}{2}} \langle t | \alpha \rangle \int e^{\frac{iEt}{\hbar}} d^3t \\ \langle E | \varphi_\alpha \rangle &= (\sqrt{2\pi\hbar})^{-\frac{1}{2}} \langle t | \psi_\alpha \rangle \int e^{\frac{iEt}{\hbar}} d^3t \end{aligned}$$

$$\varphi_\alpha(E) = (\sqrt{2\pi\hbar})^{-\frac{1}{2}} \psi(t) \int e^{\frac{iEt}{\hbar}} d^3t \quad (9)$$

We can generalize 8 and 9 respectively and write these equations,

$$\psi_\alpha(t) = (\sqrt{2\pi\hbar})^{-\frac{1}{2}} \varphi_\alpha(E) e^{-\frac{iEt}{\hbar}} d^3E \quad (10)$$

$$\varphi_\alpha(E) = (\sqrt{2\pi\hbar})^{-\frac{1}{2}} \psi(t) \int e^{\frac{iEt}{\hbar}} d^3t \quad (11)$$

We can rewrite Equation 3 into this form,

$$\psi(r, t) = \psi(t) e^{\frac{ip \cdot r}{\hbar}} \quad (12)$$

$$\begin{aligned} \psi(r, t) &= e^{\frac{ip \cdot r}{\hbar}} (\sqrt{2\pi\hbar})^{-\frac{1}{2}} \int \varphi(E) e^{-\frac{iEt}{\hbar}} d^3E \quad [\text{From 10}] \\ \psi(r, t) &= (\sqrt{2\pi\hbar})^{-\frac{1}{2}} \int \varphi(E) e^{\frac{ip \cdot r}{\hbar}} e^{-\frac{iEt}{\hbar}} d^3E \end{aligned}$$

$$\psi(r, t) = (\sqrt{2\pi\hbar})^{-\frac{1}{2}} \int \varphi(E, r) e^{-\frac{iEt}{\hbar}} d^3E \quad (13)$$

where, $\varphi(E, r) = \varphi(E) e^{\frac{ip \cdot r}{\hbar}}$

Like Equation 12 we can write energy dependent wave function like this,

$$\begin{aligned} \varphi(r, E) &= \varphi(E) e^{\frac{ip \cdot r}{\hbar}} \\ \varphi(r, E) &= (\sqrt{2\pi\hbar})^{-\frac{1}{2}} \int \psi(t) e^{\frac{iEt}{\hbar}} d^3t e^{\frac{ip \cdot r}{\hbar}} \quad [\text{From 11}] \\ \varphi(r, E) &= (\sqrt{2\pi\hbar})^{-\frac{1}{2}} \int \psi(t) e^{\frac{ip \cdot r}{\hbar}} e^{\frac{iEt}{\hbar}} d^3t. \end{aligned}$$

$$\varphi(r, E) = (\sqrt{2\pi\hbar})^{-\frac{1}{2}} \int \psi(r, t) e^{\frac{iEt}{\hbar}} d^3t \quad (14)$$

Where, $\psi(r, t) = \psi(t) e^{\frac{ip \cdot r}{\hbar}}$.

From Equations 12 and 14 we can see that wave function.

$\psi(r, t)$ and $\varphi(r, E)$ are each others Fourier transform. Until now, wave functions only depend on position, time, and momentum. For the first time, I have showed that wave also depend on energy.

VI. EXPECTATION VALUE OF TIME OPERATOR

Now we will see that the expectation value of time operator is real.

$$\begin{aligned} \langle t \rangle &= \int_{-\infty}^{\infty} \psi^* t \psi dE \\ \langle t \rangle &= i\hbar \int_{-\infty}^{\infty} \psi^* \frac{\partial}{\partial E} \psi dE \\ \langle t \rangle &= i\hbar [\psi^* \int_{-\infty}^{\infty} \frac{\partial \psi}{\partial E} \psi dE]_{-\infty}^{\infty} \\ &\quad - i\hbar \int_{-\infty}^{\infty} \left(\frac{\partial \psi^*}{\partial E} \int \frac{\partial \psi}{\partial E} \psi dE \right) dE \\ \langle t \rangle &= 0 - i\hbar \int_{-\infty}^{\infty} \left(\frac{\partial \psi^*}{\partial E} \psi \right) dE \\ \langle t \rangle &= \int_{-\infty}^{\infty} \left(i\hbar \frac{\partial \psi^*}{\partial E} \psi \right)^* dE \\ \langle t \rangle &= \int_{-\infty}^{\infty} \psi t \psi^* dE \\ \langle t \rangle &= \int_{-\infty}^{\infty} (\psi^* t \psi)^* dE \\ \langle t \rangle &= \langle t^* \rangle \end{aligned}$$

This is possible only when time operators expectation value is real number. So $\langle t \rangle$ is real. In other papers [2] [3], it has proved that time operator is a Hermitian operator. If any operator is Hermitian then its expected value is real. I dont prove that time operators self-adjointness but directly prove that time operators expectation value is real.

VII. CONCLUSION

Definition of time operator and its commutation relation with energy are given in many papers [2] [3], but

we have shown that its square commutes with energy; $\psi(r, t)$ and $\varphi(r, E)$ are each others Fourier transform and expectation value of time operator is real. We also have determined its Eigen value. Readers of this paper can

use time operator and create a new kind of equation of motion which will become alternate of Schrodinger equation.

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