Bound on Efficiency of Heat Engine From Uncertainty Relation Viewpoint

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Bound on efficiency of heat engine from uncertainty relation viewpoint

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ABSTRACT

Quantum cycles in established heat engines can be modeled with various quantum systems as working substances. As for example, heat engine can be modeled with an infinite potential well as the working substance to determine the efficiency and work done. However, in this method, the relationship between the quantum observables and the physically measurable parameters, i.e., the efficiency and work done is not well understood from the quantum mechanics approach. A detailed analysis is needed to link the thermodynamical variables (on which the efficiency and work done depends) with the uncertainty principle for better understanding. Here, we present the connection of sum uncertainty relation of position and momentum operators with thermodynamic variables in the quantum heat engine model. We are able to determine the upper and lower bounds on the efficiency of the heat engine through uncertainty relation.

1 Introduction

Thermodynamics is a prominent theory to evaluate the performance of the engines. It stands as a pillar of theoretical physics and even contributes in understanding the modern theories, e.g., black hole entropy and temperature¹, gravity^{2,3}. Though it is successful in the classical regime, the application of thermodynamics need to be re-investigated in quantum system, as the energy is discrete instead of continuous. So, we can expect new thermodynamic effects to come up in the quantum world. However, while dealing with the thermodynamic laws in quantum regime, a striking similarity of the quantum-thermodynamic system with the macroscopic system (which are described by the classical thermodynamics) can be found. For example, in the case of thermal baths, the Carnot efficiency of the engines is equally relevant for quantum system (even with a single particle) also⁴. Now, this raises a question: can all the thermodynamic effects of heat engines of small quantum systems be analyzed and predicted by the known classical thermodynamics? Various works have been performed on the analysis of generic thermodynamic effects and the dynamical behavior which are purely quantum in its nature having no classical counterpart in it⁵.

Quantum thermodynamics explores thermodynamic quantities like temperature, entropy, heat etc. for the microscopic system. It can even analyze the above parameters for a single particle model. The study of quantum thermodynamics comprises of the analysis of heat engines and refrigerators in microscopic regime⁶⁻¹⁴, and also in thermalization mechanism¹⁵. All of the various methods specified so far do not exploit quantum effects in the thermodynamics, i.e., there exists some classical analog in these methods.

Various models for quantum heat-engine realization and its experimental setup have been proposed^{16-19,25}. Heat engines can be discrete or continuous in nature. Two-stroke and four-stroke engines belong to the discrete group whereas a turbine belongs to the continuous engine. The Szilard engine was a seminal work²⁰ to solve the violation of the 2nd law of thermodynamics by Maxwell's demon. The quantum version of this engine was proposed by Kim

et al.²¹. This is an example of the quantum version of a four-stroke engine. During the insertion and deletion of the barrier in quantum Szilard engine, a certain amount of work and heat exchange occurs in the system which doesn't occur in the classical system. Different models and methods to explain the working principle of the Szilard engine has been explored in various works^{26–30}.

Here, in this work we have studied the engines from a more fundamental concept of quantum mechanics and tried to connect the efficiency of engine with the fundamental uncertainty relation of two incompatible operators. We have considered one-dimensional potential well as the working substance for quantum heat engine. Here, we have considered a specific model for our analysis though it is applicable globally to all the engines. We have developed an effective method to analyze the useful work using the uncertainty relation of the position and the momentum of a particle in a box without performing any measurement, but by applying two reservoirs of different temperatures. The through analysis that has been performed in this work is done in the non-relativistic limit. A parallel analysis in the relativistic limit has been analyzed in our work³⁸. During the evolution of quantum information, the essence and importance of uncertainty relation in technology got enriched. It has various applications in quantum technology like quantum cryptography^{37,39,40}, entanglement detection^{41–44}, even in quantum metrology⁴⁵ and quantum speed limit^{46–49}. In recent times, the work^{50–52} have authenticated the uncertainty relation experimentally. The thermal uncertainty relation that we will be applying here is a special case of the general quantum uncertainty relations. The uncertainty relation of two incompatible observable is given by:

$$\Delta a \Delta b \sim \frac{\hbar}{2}, \quad (1)$$

where a and b are any two canonical variable pairs. No better lower bound was known to us until it was explored in the work⁵⁵. They have not only given a better lower bound than the previously known Pati-Maccone uncertainty relation [PMUR], but also developed an upper bound for the uncertainty relation. It is popularly known as the reverse uncertainty relation. Using this principle, we will similarly develop the bound of efficiency and work of the heat engine in terms of uncertainty relation.

We have organized the paper as follows. In Section 2, we develop the thermal uncertainty relation for a one-dimensional potential well of length $2L$. In Section 3, the bound on the sum of variance from the thermal standpoint as well as the traditional one is established. We have devoted Section 4 to develop the correlation between the thermodynamic variables and the sum of variance of position and momentum operators of one-dimensional potential. Section 5 is dedicated to discuss the Stirling cycle and to establish the work done and efficiency in terms of the thermal uncertainty relation. Here, in this section, we have illustrated the bound on the work done and efficiency of the quantum engine. The paper is concluded in Section 6 with some discussions and future prospect of this work in the field of quantum thermodynamics.

2 Thermal uncertainty relation

In the first phase of our analysis, we will evaluate the thermal uncertainty relation (which is one of the special cases of the general uncertainty relation) for a particle in one dimensional potential well. To do so, let us first revisit our textbook problem of the one-dimensional potential well. The one-dimensional potential well is a well-known problem in quantum mechanics^{22,23}. Here, we consider a particle of mass m inside a one-dimensional potential box of length $2L$. The wave-function of this system for the n -th level is $|\psi_n\rangle = \sqrt{\frac{1}{L}} \sin(\frac{n\pi x}{2L})$. So, when the wavefunction of the model under study is known we can calculate the eigenvalue of the system. Eigenvalues of 1-D potential well is $E_n = \frac{n^2 \pi^2 \hbar^2}{2m(2L)^2}$, where $\hbar$ is Planck's constant.

With the wavefunction of the model in our hand, we are all set to derive the uncertainty relation of the position and the momentum for this system. The uncertainty relation for our model is described as^{22,23}

$$\Delta x \Delta p = \frac{\hbar}{2} \sqrt{\left(\frac{(n\pi)^2}{3} - 2\right)} \geq \frac{\hbar}{2}, \quad (2)$$

where $\Delta x^2 = \langle x^2 \rangle - \langle x \rangle^2$ and $\Delta p^2 = \langle p^2 \rangle - \langle p \rangle^2$ and we have $\langle p \rangle = 0$ for all eigenstate. The expectation values of $\langle x \rangle$, $\langle x^2 \rangle$ and $\langle p^2 \rangle$ can be easily evaluated by considering the defined wavefunction of the considered system.

Now, we formulate the uncertainty relation of the system at a certain temperature T from thermodynamics viewpoint. The formulation of the thermal uncertainty relation is performed by the analysis of the partition function of the system. The partition function²⁴, Z , for 1-D potential well is expressed as $Z = \sum_{n=1}^{\infty} e^{-\beta E_n} \approx \frac{1}{2} \sqrt{\frac{\pi}{\beta \alpha}}$, where $\beta = \frac{1}{k_B T}$, k_B is Boltzmann's constant and $\alpha = \frac{\pi^2 \hbar^2}{2m(2L)^2}$. The expression of Z converges to the form mentioned, as the product of β and α is a small quantity. We are using the Gaussian integral as the approximation considering the error in the integration is negligible as the product of β and α is a small quantity. The mean energy for this system evolves to $\langle E \rangle = -\partial \ln Z / \partial \beta = \frac{1}{2\beta}$. The average of the quantum number for the system under study can be conveyed as $\bar{n} = \frac{\sum_n n e^{-\beta E_n}}{\sum_n e^{-\beta E_n}} \approx \frac{1}{\sqrt{\pi \alpha \beta}}$.

Having the mathematical form of the partition function in our hand, we have all the resource to develop the thermal uncertainty relation. Now, we focus on the development of the dispersion relation of the position and the momentum operator at a certain temperature. The dispersion in position can be expressed as

$$\begin{aligned} (\Delta X)_T^2 &= \langle (\Delta X)^2 \rangle_T = \langle X^2 \rangle_T - \langle X \rangle_T^2 \\ &= \frac{L^2}{3} - \frac{2L^2}{\pi^2} \times \frac{e^{-\alpha\beta} - \sqrt{\pi\alpha\beta} \times \text{erfc}(\sqrt{\alpha\beta})}{\frac{1}{2} \sqrt{\frac{\pi}{\alpha\beta}}} \\ &= \frac{L^2}{3} - \frac{4L^2 \sqrt{\alpha\beta}}{\pi^{5/2}} \times (e^{-\alpha\beta} - \sqrt{\pi\alpha\beta}). \end{aligned} \quad (3)$$

Erfc is the complementary error function, which appears while solving $\langle x^2 \rangle$. The dispersion relation of the momentum operator can be analyzed similarly. It is expressed as

$$\begin{aligned} (\Delta P)_T^2 &= \langle (\Delta P)^2 \rangle_T = \langle P^2 \rangle_T - \langle P \rangle_T^2 \\ &= \frac{\pi^3 \hbar^2 \bar{n}^2}{8L^2}. \end{aligned} \quad (4)$$

So, the thermal uncertainty relation for the system at temperature T can be evaluated from Eq. (3), and Eq. (4) as.

$$\begin{aligned} (\Delta X)_T (\Delta P)_T &= \frac{\hbar \bar{n} \pi^{3/2}}{2\sqrt{2}} \left[\frac{1}{3} - \frac{4\sqrt{\alpha\beta}}{\pi^{5/2}} (e^{-\alpha\beta} - \sqrt{\pi\alpha\beta}) \right]^{\frac{1}{2}} \\ &\geq \frac{\hbar}{2}. \end{aligned} \quad (5)$$

The product uncertainty relation loses its importance when the system under consideration is an eigenstate of the observable under study. The sum of uncertainty⁵³ was introduced to capture the uncertainty in the observables when the system happens to be an eigenstate of the observables. The sum of uncertainty for this system at a particular temperature T is expressed as

$$(\Delta X)_T + (\Delta P)_T = L \left[\frac{1}{3} - \frac{4\sqrt{\alpha\beta}}{\pi^{5/2}} (e^{-\alpha\beta} - \sqrt{\pi\alpha\beta}) \right]^{\frac{1}{2}} + \frac{\hbar \bar{n} \pi^{3/2}}{2\sqrt{2}L}. \quad (6)$$

In Fig. 1, the variation of sum uncertainty relation (Eq. (6)) with respect to different temperature is shown. Our thermal uncertainty relation that we have developed (Eq. (6)) for the considered system encounters negligible amount of variation when the length of the potential well is small, whereas the difference is large for higher values of L (the order of the length is considered in nanometer).

The variation of uncertainty relation for different levels is shown in Fig. 2. Similar to the case of temperature analysis the variation is negligible for lower values of L and is large for higher values.

Figure 1. The variation of sum uncertainty relation (Eq. (6)) for different temperature. The dotted line is for lower and the solid line is for higher temperature

3 Bound on sum uncertainty for one dimension potential well

The product of variances is sometimes unable to capture the uncertainty for two incompatible observables. If the state of the system is an eigenstate of one of the observables, then the product of the uncertainties vanishes^{54,55}. To overcome this, the sum of variances is introduced to capture the uncertainty of two incompatible observables. For any quantum model the sum of variance of two incompatible observable which results in the lower bound is defined as

$$\Delta A^2 + \Delta B^2 \geq \frac{1}{2} \sum_n \left(\left| \langle \psi_n | \bar{A} | \psi \rangle \right| + \left| \langle \psi_n | \bar{B} | \psi \rangle \right| \right)^2. \quad (7)$$

For our system, we calculate the lower bound of sum uncertainty for position and momentum operator. So, we replace $A = X$ and $B = P$, which yields to

$$\Delta X^2 + \Delta P^2 \geq \frac{1}{2} \sum_n \left(\left| \langle \psi_n | \bar{X} | \psi \rangle \right| + \left| \langle \psi_n | \bar{P} | \psi \rangle \right| \right)^2. \quad (8)$$

The computation of the reverse uncertainty relation of two observables results to the upper bound of uncertainty relation. To develop the upper bound, we have to utilize the definition of the Dunkl-Williams inequality⁵⁶. The mathematical form of this inequality is

$$\Delta A + \Delta B \leq \frac{\sqrt{2} \Delta(A - B)}{\sqrt{1 - \frac{\text{Cov}(A, B)}{\Delta A \Delta B}}}. \quad (9)$$

Figure 2. Similarly, this shows variation for different values of n .

Squaring both sides of the Eq. (9) we get the upper bound of the sum of variance for two variables as

$$\Delta A^2 + \Delta B^2 \leq \frac{2\Delta(A-B)^2}{1 - \frac{\text{Cov}(A,B)}{\Delta A \Delta B}} - 2\Delta A \Delta B, \quad (10)$$

where $\text{Cov}(A, B)$ is defined as $\text{Cov}(A, B) = \frac{1}{2} \langle \{A, B\} \rangle - \langle A \rangle \langle B \rangle$, and $\Delta(A-B)^2 = \langle (A-B)^2 \rangle - \langle (A-B) \rangle^2$. $\Delta(A-B)^2$ is the variance of the difference of the two incompatible observable.

Now, for our one dimensional potential well system which we have considered as working substance, we calculate the upper bound of the sum of variance for the position and the momentum operator. So, we have to replace $A = X$ and $B = P$ in Eq. (10) and it results to

$$\begin{aligned} \Delta X^2 + \Delta P^2 &\leq \frac{2\Delta(X-P)^2}{1 - \frac{\text{Cov}(X,P)}{\Delta X \Delta P}} - 2\Delta X \Delta P \\ &\leq \frac{L^2}{3} - \frac{2L^2}{(n\pi)^2} + \frac{\pi^2 \hbar^2 n^2}{4L^2}. \end{aligned} \quad (11)$$

In the above equation, i.e., Eq. (11) the upper bound of the system from the standard method is developed using the definition described in Eq. (10). Now, we develop the sum of variance of the same incompatible observables from the thermodynamic standpoint. The expression for the sum of variance of the system at a particular temperature evolves as

$$\Delta X^2 + \Delta P^2 \leq \frac{4L^2}{3} - \frac{8L^2 \sqrt{\alpha\beta}}{\pi^{5/2}} \times (e^{-\alpha\beta} - \sqrt{\pi\alpha\beta}) + \frac{\hbar^2 \bar{n}^2 \pi^3}{4L^2}. \quad (12)$$

The bound of sum uncertainty relation (from thermodynamic perspective developed using Eq. (8) for lower bound and the Eq. (12) describes the upper bound for the considered system) with a particular temperature for

Figure 3. The bound of uncertainty relation (Eq. (8) and (12)) for a particular temperature for different values of n from thermodynamic standpoint.

different levels is shown in Fig. 3. The upper part of the plot is for $n = 1$, and the lower one is for $n = 2$. From Fig. 3, we can infer that the effect of the bounds of uncertainty relation is prominent for higher values of the length of the potential well. The bound are less prominent for lower values of L .

4 Connection of thermodynamic quantities with uncertainty

In the next phase of our analysis, we want to establish a bridge between the thermodynamic quantities with the uncertainty relation. We consider the sum of variance to overcome the flaw that will appear if we consider the product form of uncertainty if the system is an eigenstate of the observables. We will first demonstrate a connection of partition function with our uncertainty relation. The mathematical form of this is given by

$$Z = \frac{\pi \bar{n}}{2} = \frac{L\sqrt{2}}{\hbar\sqrt{\pi}} (\Delta X_T + \Delta P_T + C_T), \quad (13)$$

where $C_T = -\frac{L}{3} + \frac{2L}{\pi^{5/2}\sqrt{\alpha\beta_T}} [\alpha\beta_T - \sqrt{\pi}(\alpha\beta_T)^{3/2} - 1]$ is a constant for a specific temperature, which is derived by expanding the Eq. 6 and neglecting the higher order terms as α and β are small.

Since, we are to able to bridge a relationship between the uncertainty relation and the partition function, we are all set to explain all the thermodynamic variables in terms of uncertainty relations. We develop the Helmholtz free energy²⁴, F , from uncertainty viewpoint which takes the form as

$$\begin{aligned} F &= -\frac{1}{\beta} \ln Z \\ &= -\frac{1}{\beta} \ln \left[\frac{L\sqrt{2}}{\hbar\sqrt{\pi}} (\Delta X_T + \Delta P_T + C_T) \right]. \end{aligned} \quad (14)$$

Entropy is expressed in terms of Helmholtz free energy. So, we uncover the relationship between the variance of position and momentum with entropy. The mathematical form for the entropy from uncertainty viewpoint is

$$S = -\frac{\partial F}{\partial T}$$

$$= k_B \ln \left[\frac{L\sqrt{2}}{\hbar\sqrt{\pi}} (\Delta X_T + \Delta P_T + C_T) \right] + \frac{\hbar\sqrt{\pi}k_B(v + \gamma)}{\sqrt{2}L\beta(\Delta X_T + \Delta P_T + C_T)}, \quad (15)$$

where $v = \frac{\frac{L^2\sqrt{\alpha}}{\sqrt{\beta}\pi^{5/2}}(e^{-\alpha\beta} - \sqrt{\pi\alpha\beta}) - \frac{2L^2\sqrt{\alpha\beta}}{\pi^{5/2}}(\alpha e^{-\alpha\beta} - 1/2\sqrt{\frac{\pi\alpha}{\beta}})}{\left[\frac{L^2}{3} - \frac{4L^2\sqrt{\alpha\beta}}{\pi^{5/2}}(e^{-\alpha\beta} - \sqrt{\pi\alpha\beta})\right]^{\frac{1}{2}}}$ and γ is expressed as $\gamma = -\frac{L}{\pi^{5/2}\sqrt{\alpha\beta}^{3/2}}(\alpha\beta - \sqrt{\pi}(\alpha\beta)^{3/2} - 1) + \frac{2L}{\pi^{5/2}\sqrt{\alpha\beta}}(\alpha - \sqrt{\pi\beta}\alpha^{3/2})$

Figure 4. The variation of entropy (Eq. (15)) for different values of temperature. The scattered plot is higher and solid line is for lower temperature.

In Fig. 4, the variation of entropy in term of uncertainty relation is shown. We can observe an increase in the entropy with an increase in the uncertainty for different temperatures. With an increase in uncertainty, the disorder in the system increases, causing an increase in the entropy.

We know that entropy is a measure of entanglement. So from this relation, we are able to bridge a connection between the uncertainty relation with entanglement. So, we can measure the entanglement property of the system from the uncertainty relation if we are able to model our system with any well-known quantum systems.

Knowing the Helmholtz free energy^{24,31} F , for a given system all the relevant thermodynamic quantities can be computed from it. Here, the correlation of F with the uncertainty in the measure of the position and the momentum is established. Hence this replaces the explicit requirement of computing the internal energy of the system for

deriving the thermodynamic quantities. In addition, it also raises a question whether phase transition (Landau theory and it's multimode coupling) can be analyzed from uncertainty perspective.

5 Stirling cycle and bound on efficiency

A Stirling cycle^{32–35} is composed of four stages, two isothermal processes, and two isochoric processes. During the first stage, we insert a barrier isothermally in the middle of the well. While this quasi-static insertion process is being done, the working medium stays at an equilibrium condition with a hot bath at a temperature T_1 . During the second stage, we perceive an isochoric heat extraction of the working medium by connecting it with a bath at a lower temperature of T_2 . In the next stage of the cycle, there is an isothermal removal of the barrier where we retain the engine in equilibrium at temperature T_2 . Now in the final stage, we bridge the engine to the hot bath at the temperature T_1 and this give rise to isochoric heat absorption. It is represented pictorially in Fig. 5.

Figure 5. The figure shows the four stages(two isothermal and two isochoric process) of the Stirling cycle modeled using potential well.

Similarly to³⁶, we calculate the work done and the efficiency but in terms of uncertainty relation. To develop the work done of the engine a one dimensional well of length $2L$ is considered with a particle of mass m at a temperature of T_1 . The energy of this system is $E_n = \frac{n^2 \pi^2 \hbar^2}{2m(2L)^2}$. The partition function Z_A for the system is $Z \approx \frac{1}{2} \sqrt{\frac{\pi}{\beta \alpha}}$. Now, a

wall is being inserted isothermally which converts the potential well into an infinite double well potential. Due to this insertion of the wall, the energy level for even values of n remain unchanged but the odd one's shifts and overlaps with their nearest neighboring energy level. So the energy of the newly formed partitioned one-dimensional potential box is

$$E_{2n} = \frac{(2n)^2 \pi^2 \hbar^2}{2m(2a)^2}. \quad (16)$$

So, the new partition function stands as

$$Z_B = \sum_n 2e^{-\beta_1 E_{2n}}. \quad (17)$$

The internal energies for the system is calculated from the partition function. The internal energy U_A and U_B is defined as $U_i = -\partial \ln Z_i / \partial \beta_i$ where $i = A, B$ and $\beta_1 = \frac{1}{k_B T_1}$. This results to

$$U_A = U_B = \frac{1}{2\beta_1}. \quad (18)$$

The heat exchanged in this isothermal process can be expressed as

$$Q_{AB} = U_B - U_A + k_B T_1 \ln Z_B - k_B T_1 \ln Z_A. \quad (19)$$

Now the system is connected to a heat bath at a lower temperature T_2 . The partition function for this lower temperature where the energy remains the same is defined as

$$Z_C = \sum_n 2e^{-\beta_2 E_{2n}}. \quad (20)$$

The heat exchanged for this stage of the cycle is the difference of the average energies of the initial and the final states i.e

$$Q_{CB} = U_C - U_B. \quad (21)$$

Where $U_C = -\partial \ln Z_C / \partial \beta_2$ and $\beta_2 = \frac{1}{k_B T_2}$. The system being connected to the heat bath at temperature T_2 we remove the wall isothermally which we call as stage 3. The energy is now of the form $E_n = \frac{n^2 \pi^2 \hbar^2}{2m(2L)^2}$. The corresponding partition function is given by

$$Z_D = \sum_n e^{-\beta E_n}. \quad (22)$$

We can calculate the internal energy U_D similarly as U_C . The heat exchanged during this process is given by

$$Q_{CD} = U_D - U_C + k_B T_2 \ln Z_D - k_B T_2 \ln Z_C. \quad (23)$$

In the fourth stage of the cycle the system is connected back to the heat bath at temperature T_1 . The corresponding energy exchange for this stage can be expressed as

$$Q_{DA} = U_A - U_D. \quad (24)$$

So the total work done for the process in terms of variance of the position and the momentum operator evolves to

$$\begin{aligned} W &= Q_{AB} + Q_{BC} + Q_{CD} + Q_{DA} \\ &= \frac{8L^2 \alpha}{\hbar^2 \pi^2} \left[D \ln \left(\frac{Z_B}{Z_A} \right) + E \ln \left(\frac{Z_D}{Z_C} \right) \right]. \end{aligned} \quad (25)$$

The efficiency of Stirling cycle engine stand as

$$\begin{aligned}
 \eta &= 1 + \frac{Q_{BC} + Q_{CD}}{Q_{DA} + Q_{AB}} \\
 &= \frac{\left(\bar{n}_{T_2}^2 \ln\left(\frac{Z_D}{Z_C}\right) + \bar{n}_{T_1}^2 \ln\left(\frac{Z_B}{Z_A}\right) \right)}{\left(-\bar{n}_{T_2}^2/2 + \bar{n}_{T_1}^2 \left(\ln\left(\frac{Z_B}{Z_A}\right) + 1/2 \right) \right)} \\
 &= \frac{\left[D \ln\left(\frac{Z_B}{Z_A}\right) + E \ln\left(\frac{Z_D}{Z_C}\right) \right]}{\left[-E/2 + D \left(\ln\left(\frac{Z_B}{Z_A}\right) + 1/2 \right) \right]}.
 \end{aligned} \tag{26}$$

Where $D = \frac{8L^2}{\pi^3 \hbar^2} (\Delta X_{T_1} + \Delta P_{T_1} + C_{T_1})^2$ and $E = \frac{8L^2}{\pi^3 \hbar^2} (\Delta X_{T_2} + \Delta P_{T_2} + C_{T_2})^2$.

Figure 6. The bounds on the efficiency by heat engine in term of uncertainty relation. The scattered plot represents the upper bound and the solid line the lower bound of the efficiency.

In Eq. (26), the upper and the lower bound of the efficiency is evaluated in terms of the bound that is being analyzed for the thermal uncertainty relation of the position and the momentum operator. Here, the expression of D and E (for the working model considered for the analysis of the heat engine) gives the required uncertainty relation for the illustration of the bound of the efficiency.

In this paper, we are able to bridge a connection between the efficiency of the heat engine with the variance of the position and the momentum operator. The upper bound of the efficiency for the heat engine is near about constant when the uncertainty is high, whereas it dips a little when uncertainty is less. As shown in Fig. 6, the lower bound of the efficiency is high when the uncertainty in measurement is less and dips gradually with the increase in uncertainty. Thus, with an increase in uncertainty, we can visualize that the lower bound of the efficiency decreases. From Fig. 6, one can infer that the lower and the upper bound of the efficiency is near about the same when the uncertainty in the position and the momentum operator is quite small.

The Carnot efficiency for low temperature limit is expressed as $\left(1 - \frac{T_2}{T_1}\right)$ where T_2 and T_1 are the temperature of the cold and hot bath respectively. The upper bound of the efficiency from uncertainty viewpoint is consistent with the bound given by Carnot cycle. So, we can infer that the position and the momentum of the particle has a direct linkage with the thermodynamic variables. The work⁵⁸ suggests that the efficiency of engines which are powered by non-thermal baths can be higher than the usual convention. This can be testified from uncertainty viewpoint.

In quantum regime, after measurement the system collapses to one of its eigenstate. So, to describe and analyze the efficiency of the engine for different conditions we must have a multi-copy of the system. Whereas, if we know the uncertainty relation of the model considered for analysis, we can describe and analyze different conditions without any measurement being done on the system. So, multiple copies of the system are not required for our analysis. This also reduces the measurement cost for the analysis of the system.

6 Discussion and Conclusion

The quantum heat engine has a predominant role in better understanding of the quantum engines, information, and quantum thermodynamics. This work develops a relationship between the thermodynamic variables with the position and momentum of the particle in the system. We give the analytic formulation of the work and efficiency of the engine in terms of the thermal uncertainty relation. Though we have considered a specific model for our analysis, this analysis has a global effect, i.e., it can be used to explain the efficiency of the various engines with different quantum models as the working substance. Based on these formulations, the physical properties of the heat engine and the thermodynamic variables that we have encountered are as follows.

(a) The total work and the efficiency depends on the position and momentum of the particle. The change in the uncertainty of the position and the momentum has a direct impact on the efficiency rate and the work of the engine. The lower bound of the efficiency of the engine drops gradually when the uncertainty of the observable increases. The upper bound of the efficiency shows a small variation for higher uncertainty relation, which conveys that the conversion rate of work input to output is near about constant for higher uncertainty.

(b) Every quantum thermodynamic variable has a direct connection with the uncertainty relation. Helmholtz free energy shows the dependence of the internal energy of the thermodynamic system with the uncertainty relation of the incompatible observables. The detailed analysis of entropy with the uncertainty relation shows that entropy increases when the uncertainty of any one of the observables increases for a definite temperature. The rate of increase is different for different temperatures.

(c) The uncertainty relation which is the fundamental principle of quantum mechanics is able to predict the efficiency and the total work of the engine without performing any measurement. So the measurement cost for the system gets reduced if we are able to model the system under study with a quantum model for which we can develop the uncertainty relation.

The bridge of the uncertainty relation with the thermodynamic variable raises a question of whether we can analysis the phase transition (Landau theory) from uncertainty perspective.

Most of the known methods for the measurement of entanglement converge to the analysis of entropy⁵⁹. Now, if we can model the system that is being analyzed with a quantum model, we can construct the entanglement from the uncertainty relation for the system. This would be a standard method to measure the entanglement property of the system which might be a solution to the open problem of entanglement measure.

The 1-D problem in the non-relativistic case is a standard problem. The analysis in the relativistic case is not a standard problem. The study of the heat engine with a relativistic particle can be analyzed. The mapping of the entropy with uncertainty to explain the entanglement property for the relativistic system⁶⁰ is an open area to explore. Even the holographic interpretation of entanglement entropy of anti-de Sitter (ADS)/conformal field theory (CFT)⁶¹ can be mapped with uncertainty relation.

This work can be extended in the development of quantum engine in deformed space structures^{62–64} through the correlation of generalized uncertainty relation with the thermodynamic variables. In the paper⁵⁷ they have mentioned the non-commutativity of kinetic and potential energy of quantum harmonic heat engine. Therefore the possibility of a connection between the deformed space structures⁵⁴ and the heat engines can be further explored in the future.

One can even study the anharmonic oscillator models through the uncertainty standpoint.

Even the study of other thermodynamic cycles and to procure the bound for different thermodynamic parameters is a wide open area to explore. As entropy can be mapped with the uncertainty relation, this raises a question of whether all thermodynamic analysis can be mapped with the uncertainty of the observables.

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Additional Information

Author Contributions

The concept was developed through a series of joint discussion among all the authors. P. Chattopadhyay, A. Mitra wrote the initial draft of the manuscript and G. Paul, V. Zarikas performed thorough technical and editorial modification through multiple rounds for better clarity and presentation.

Competing interests

The author(s) declare no competing interests.

Figures

Figure 1

The variation of sum uncertainty relation (Eq. (6)) for different temperature. The dotted line is for lower and the solid line is for higher temperature

Figure 2

Similarly, this shows variation for different values of n .

Figure 3

The bound of uncertainty relation (Eq. (8) and (12)) for a particular temperature for different values of n from thermodynamic standpoint.

Figure 4

The variation of entropy (Eq. (15)) for different values of temperature. The scattered plot is higher and solid line is for lower temperature.

Figure 5

The figure shows the four stages(two isothermal and two isochoric process) of the Stirling cycle modeled using potential well.

Figure 6

The bounds on the efficiency by heat engine in term of uncertainty relation. The scattered plot represents the upper bound and the solid line the lower bound of the efficiency