Entangled Photon Generation From a Three-Level Laser with a Parametric Amplifier

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Entangled Photon Generation From a Three-Level Laser with a Parametric Amplifier

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In this article we study the squeezing and statistical properties of the light produced by a nondegenerate three-level laser, with the parametric amplifier . With the aid of the master equation, we obtain stochastic differential equations. Applying the solutions of the resulting differential equations, we calculate the quadrature variance,the mean and variance of the photon number ,the photon number correlation . However, the two-mode driving light has no effect on the squeezing properties of the cavity modes. On the other hand, the parametric amplifier and the thermal reservoir increase the mean and the variance of the photon number. Furthermore, employing the same solutions, we also obtain the antinormally ordered characteristic function defined in the Heisenberg picture.

Keywords: Master equation ; solution of stochastic differential equations and Langavian Equation.

I. INTRODUCTION

Quantum Optics, the union of quantum field theory and physical optics, undergoing a time of revolutionary change. In recent years, the subject of squeezing of light has received a great deal of attention by several authors [1],[2],[3],[4],[5],[6],[7],[8],[9], and [10]. These non classical states of light (squeezed states) are characterized by a reduction of quantum fluctuations (noise) in one quadrature component of the light below the vacuum level, or below that achievable in a coherent state, at the expense of increased fluctuations in the other component such that the product of these fluctuations still obeys the uncertainty relation. Squeezed light has potential applications in low-noise communications and precision measurements [11, 12]. A parametric oscillator has been considered as an important source of squeezed light. It is one of the most interesting and well characterized optical devices in quantum optics. In a cascade three-level laser, three-level atoms in a cascade configuration are injected into a cavity coupled to a thermal reservoir via a single-port mirror. When a three-level atom in a cascade configuration makes a transition from the top to the bottom level via the intermediate level,the two photons are generated. In this device a pump photon interacts with a nonlinear crystal inside a cavity and is down-converted into two highly correlated photons. If these photons have the same frequency the device is called a degenerate parametric oscillator, otherwise it is called a nondegenerate parametric oscillator. The quantum fluctuations and photon statistics of the signal mode produced by a nondegenerate parametric oscillator coupled to a two-mode thermal reservoir have been analyzed employing the pertinent Fokker Planck equation or the quantum Langevin equations .The quantum dynamics of a non degenerate parametric oscillator coupled to a thermal reservoirs have been analyzed employing the Q function obtained by solving the Fokker-Planck equation using the propagator method [13]. A two mode sub harmonic generator at the lower and above threshold has been theoretically predicted to be a source of light in an entangled state[14]. Recently, the experimental realization of the entanglement in two-mode sub harmonic generator has been demonstrated by Zhang et al.[15].On the other hand, Xiong et al. [16] have recently proposed a scheme for an entanglement based on a non degenerate three -level laser can atoms are injected at the lower level and the top levels are coupled by a strong coherent light. They have found that a non degenerate three level laser can generate light in an entangled state employing the entanglement criteria for bipartite continuous variables states. Moreover, Tan et al. [17]have extended the work of Xiong et al. and examined the generation and evolution of entangled light in the Wigner representation using the sufficient and necessary in separability criteria for a two -mode Gaussian state proposed by Duan et al .[18] and Simon[19]. The generation and manipulation of entanglement has attracted a great deal of interest owing to their wide applications in quantum teleportation [20], quantum dense coding [21], quantum computation [22], quantum error correction [23], and quantum cryptography [24]. The variance of the quadrature operators and the photon number distribution for the signal-idler modes Produced by generation of entanglement from non degenerate three level laser with parametric oscillation have also been studied applying the pertinent Langevin equations . On the other hand, obtaining stochastic differential equations, associated with the normally ordering, for the cavity mode variables appears to involve a relatively less mathematical task. We first obtain stochastic differential equations for the cavity mode variables by applying the pertinent Master equation. With the aid of the resulting equations, we calculate the quadrature variance for the two-mode cavity radiation and the squeezing. In addition, we determine the mean photon number, the photon number entanglement, the variance of the photon number difference, the intensity difference, and the photon number correlation. We also calculate the mean, the variance, and the photon number correlation, in the absence of the parametric amplifier ($\mu = 0$). In the first part we wish to study the squeezing and statistical prop-

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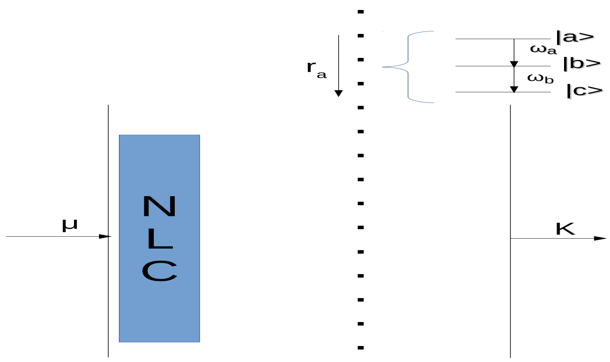

FIG. 1. Schematic representation of a nondegenerate three-level laser with a parametric amplifier and a thermal reservoir.

erties of the light generated by a parametric amplifier nondegenerate three-level laser with in the cavity coupled to a thermal reservoir via a single-port mirror.

II. MASTER EQUATION

We first drive the equation of evolution of the density operator for the three-level laser applying the linear and the adiabatic approximation schemes [4–9]. Then after obtaining the properties of the reservoir submode operators, we drive the time evolution of the reduced density operator for the cavity modes coupled to a two-mode thermal reservoir. Finally, with the help of the two resulting equations, we write the master equation for the system under consideration. We represent the top, intermediate, and bottom levels of a three-level atom in a cascade configuration by $|a\rangle$, $|b\rangle$, and $|c\rangle$, respectively, as shown in Fig. 1. In addition, we assume the two modes a and b to be at resonance with the two transitions $|a\rangle \rightarrow |b\rangle$ and $|b\rangle \rightarrow |c\rangle$ dipole allowed respectively, and direct transition between level $|a\rangle$ and level $|c\rangle$ to be dipole forbidden. The interaction of a non degenerate three-level atom with the cavity modes can be described by the Hamiltonian

$$\hat{H}_1 = ig\big[|a\rangle\langle b|\hat{a} - \hat{a}^\dagger|b\rangle\langle a| + |b\rangle\langle c|\hat{b} - \hat{b}^\dagger|c\rangle\langle b|\big], \quad (1)$$

where g is the coupling constant and $\hat{a}(\hat{b})$ are the annihilation operators for the cavity modes. Moreover, the Hamiltonian describing the parametric interaction, with the pump mode treated classically, can be written as

$$\hat{H}_2 = i\mu(\hat{a}^\dagger\hat{b}^\dagger - \hat{a}\hat{b}), \quad (2)$$

in which μ is proportional to the amplitude of the pump mode. Here, we take the initial state of a single three-level atom to be

$$|\psi_A(0)\rangle = C_a|a\rangle + C_c|c\rangle, \quad (3)$$

and hence, the density operator of a single atom is

$$\hat{\rho}_A(0) = \rho_{aa}^{(0)}|a\rangle\langle a| + \rho_{ac}^{(0)}|a\rangle\langle c| + \rho_{ca}^{(0)}|c\rangle\langle a| + \rho_{cc}^{(0)}|c\rangle\langle c|, \quad (4)$$

where

$$\rho_{aa}^{(0)} = C_a^* C_a, \quad (5)$$

$$\rho_{ac}^{(0)} = C_a C_c^*, \quad (6)$$

$$\rho_{ca}^{(0)} = C_c C_a^*, \quad (7)$$

$$\rho_{cc}^{(0)} = C_c^* C_c. \quad (8)$$

Suppose $\hat{\rho}_{AR}(t, t_j)$ is the density operator for a single atom plus the cavity mode at time t, with the atom injected at time t_j such that $(t-\tau) \leq t_j \leq t$. The density operator for all atoms in the cavity plus the cavity mode at time t can then be written as $\hat{\rho}_{AR}(t) = \sum_j N_j \hat{\rho}_{AR}(t, t_j)$. Then it follows that

$$\hat{\rho}_{AR}(t) = r_a \sum_j \hat{\rho}_{AR}(t, t_j)\Delta t_j, \quad (9)$$

where $N = r_a\Delta t_j$ represents the number of atoms injected into the cavity in a time Δt_j. Moreover, employing Eq. 1, the master equation for the cavity modes coupled to thermal reservior, can be put in the form

$$\begin{aligned}\frac{d}{dt}\hat{\rho}(t) = &-i[\hat{H}_S, \hat{\rho}] + g(\rho_{ab}\hat{a}^\dagger - \hat{a}^\dagger\hat{\rho}_{ab} + \rho_{bc}\hat{b}^\dagger \\ &-\hat{b}^\dagger\rho_{bc} + \hat{a}\rho_{ba} - \rho_{ba}\hat{a} + \hat{b}\rho_{cb} - \rho_{cb}\hat{b}) \\ &+\frac{\kappa}{2}(\bar{n}_{th}+1)(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}) \\ &+\frac{\kappa}{2}\bar{n}_{th}(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger) \\ &+\frac{\kappa}{2}(\bar{n}_{th}+1)(2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{b}^\dagger\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{b}) \\ &+\frac{\kappa}{2}\bar{n}_{th}(2\hat{b}^\dagger\hat{\rho}\hat{b} - \hat{b}\hat{b}^\dagger\rho - \hat{\rho}\hat{b}\hat{b}^\dagger), \end{aligned} \quad (10)$$

in which the matrix element $\rho_{\alpha\beta}$ is defined by

$$\rho_{\alpha\beta} \;=\; \langle\alpha|\hat{\rho}_{AR}|\beta\rangle, \quad (11)$$

with $\alpha, \beta = a, b, c$. Using once more the adiabatic approximation scheme, we see that

$$\hat{\rho}_{ab} = \frac{g r_a}{\gamma^2}(\rho_{ac}^{(0)}\hat{\rho}\hat{b}^\dagger - \rho_{aa}^{(0)}\hat{\rho}\hat{a}), \quad (12)$$

$$\hat{\rho}_{bc} = \frac{g r_a}{\gamma^2}(\rho_{cc}^{(0)}\hat{\rho}\hat{b}^\dagger - \rho_{ac}^{(0)}\hat{a}^\dagger\hat{\rho}). \quad (13)$$

Finally, on account of Eqs. (12), and (13), the equation of evolution of the density operator for the cavity modes given by Eq. (10), takes the form

$$\frac{d}{dt}\rho(t) = -i[\hat{H}_2,\hat{\rho}] + \frac{\kappa}{2}(\bar{n}_{th}+1)\big[2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}\big] + \frac{1}{2}(A\rho_{aa}^{(0)} + \kappa\bar{n}_{th})\big[2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger\big] + \frac{1}{2}(A\rho_{cc}^{(0)} + \kappa(\bar{n}_{th}+1)\big[(2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{b}^\dagger\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{b}\big] + \frac{1}{2}\kappa\bar{n}_{th}\big[2\hat{b}^\dagger\hat{\rho}\hat{b} - \hat{b}\hat{b}^\dagger\rho - \hat{\rho}\hat{b}\hat{b}^\dagger\big] + \frac{1}{2}A\rho_{ac}^{(0)}\big[\hat{a}\hat{b}\hat{\rho} - \hat{a}^\dagger\hat{\rho}\hat{b}^\dagger + \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger - \hat{b}\hat{\rho}\hat{a}\big] + \frac{1}{2}A\rho_{ca}^{(0)}\big[\hat{a}^\dagger\hat{b}^\dagger\hat{\rho} - \hat{a}^\dagger\hat{\rho}\hat{b}^\dagger + \hat{\rho}\hat{a}\hat{b} - \hat{b}\hat{\rho}\hat{a}\big], \quad (14)$$

where

$$A = \frac{2g^2 r_a}{\gamma^2} \quad (15)$$

is linear gain coefficient. The equation of evolution of the density operator associated with the Hamiltonian given by Eq. (2) has the form

$$\frac{d}{dt}\hat{\rho}(t) = \frac{1}{2}\mu\big[\hat{\rho}\hat{a}\hat{b} - \hat{a}\hat{b}\hat{\rho} + \hat{a}^\dagger\hat{b}^\dagger\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger\big] + \frac{\kappa}{2}(\bar{n}_{th}+1)\big[2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}\big] + \frac{1}{2}(A\rho_{aa}^{(0)} + \kappa\bar{n}_{th})\big[2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger\big] + \frac{1}{2}(A\rho_{cc}^{(0)} + \kappa(\bar{n}_{th}+1)\big[2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{b}^\dagger\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{b}\big] + \frac{1}{2}\kappa\bar{n}\big[2\hat{b}^\dagger\hat{\rho}\hat{b} - \hat{b}\hat{b}^\dagger\rho - \hat{\rho}\hat{b}\hat{b}^\dagger\big] + \frac{1}{2}A\rho_{ac}^{(0)}\big[\hat{a}\hat{b}\hat{\rho} - \hat{a}^\dagger\hat{\rho}\hat{b}^\dagger + \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger - \hat{b}\hat{\rho}\hat{a}\big] + \frac{1}{2}A\rho_{ca}^{(0)}\big[\hat{a}^\dagger\hat{b}^\dagger\hat{\rho} - \hat{a}^\dagger\hat{\rho}\hat{b}^\dagger + \hat{\rho}\hat{a}\hat{b} - \hat{b}\hat{\rho}\hat{a}\big], \quad (16)$$

where $A = \frac{2r_a g^2}{\gamma^2}$ is a linear gain coefficient. This is the master equation for the cavity modes of a nondegenerate three-level laser whose cavity contains a non -degenerate para metric amplifier and coupled to a thermal reservoir.

A. The Stochastic Differential equations

Next we seek to determine the solutions of the stochastic differential equations. Thus employing

$$\frac{d}{dt}\langle\hat{A}\rangle = Tr\bigg(\frac{d}{dt}\hat{\rho}(t)\hat{A}\bigg), \quad (17)$$

along with Eq. 17, and applying the cyclic property of the trace operation together with the commutation relations

$$[\hat{a},\hat{a}^\dagger] = [\hat{b},\hat{b}^\dagger] = 1 \quad (18)$$

and

$$[\hat{a},\hat{a}] = [\hat{b},\hat{b}] = [\hat{a},\hat{b}] = 0, \quad (19)$$

we readily obtain

$$\frac{d}{dt}\langle\hat{a}\rangle = -\frac{\mu_a}{2}\langle\hat{a}\rangle + \frac{1}{2}\nu_-\langle\hat{b}^\dagger\rangle, \quad (20)$$

$$\frac{d}{dt}\langle\hat{b}\rangle = -\frac{\mu_c}{2}\langle\hat{b}\rangle + \frac{1}{2}\nu_+\langle\hat{a}^\dagger\rangle, \quad (21)$$

$$\frac{d}{dt}\langle\hat{a}^2\rangle = -\mu_a\langle\hat{a}^2\rangle + \nu_-\langle\hat{b}^\dagger\hat{a}\rangle, \quad (22)$$

$$\frac{d}{dt}\langle\hat{b}^2\rangle = -\mu_c\langle\hat{b}^2\rangle + \nu_+\langle\hat{a}^\dagger\hat{b}\rangle, \quad (23)$$

$$\frac{d}{dt}\langle\hat{a}^\dagger\hat{a}\rangle = -\mu_a\langle\hat{a}^\dagger\hat{a}\rangle + \frac{1}{2}\nu_-\langle\hat{a}^\dagger\hat{b}^\dagger\rangle + \frac{1}{2}\nu_-^*\langle\hat{a}\hat{b}\rangle + A\rho_{aa}^{(0)} + \kappa\bar{n}, \quad (24)$$

$$\frac{d}{dt}\langle\hat{b}^\dagger\hat{b}\rangle = -\mu_c\langle\hat{b}^\dagger\hat{b}\rangle + \frac{1}{2}\nu_+\langle\hat{b}^\dagger\hat{a}^\dagger\rangle + \frac{1}{2}\nu_+^*\langle\hat{a}\hat{b}\rangle + \kappa\bar{n}, \quad (25)$$

$$\frac{d}{dt}\langle\hat{a}^\dagger\hat{b}\rangle = -\frac{1}{2}(\mu_a+\mu_c)\langle\hat{a}^\dagger\hat{b}\rangle + \frac{1}{2}\nu_+\langle\hat{a}^{\dagger 2}\rangle + \frac{1}{2}\nu_-^*\langle\hat{b}^{\dagger 2}\rangle, \quad (26)$$

$$\frac{d}{dt}\langle\hat{a}\hat{b}\rangle = -\frac{1}{2}(\mu_a+\mu_c)\langle\hat{a}\hat{b}\rangle + \frac{1}{2}\nu_+\langle\hat{a}^\dagger\hat{a}\rangle + \frac{1}{2}\nu_-\langle\hat{b}^\dagger\hat{b}\rangle + \frac{1}{2}\nu_+, \quad (27)$$

in which

$$\mu_a = \kappa - A\rho_{aa}^{(0)}, \quad (28)$$

$$\mu_c = \kappa + A\rho_{cc}^{(0)}, \quad (29)$$

$$\nu_\pm = 2\mu \pm A\rho_{ac}^{(0)}. \quad (30)$$

We note that the corresponding c- number are

$$\frac{d}{dt}\langle\alpha\rangle = -\frac{\mu_a}{2}\langle\alpha\rangle + \frac{1}{2}\nu_-\langle\beta^*\rangle, \quad (31)$$

$$\frac{d}{dt}\langle\beta\rangle = -\frac{\mu_c}{2}\langle\beta\rangle + \frac{1}{2}\nu_+\langle\alpha^*\rangle, \quad (32)$$

$$\frac{d}{dt}\langle\alpha^2\rangle = -\mu_a\langle\alpha^2\rangle + \nu_-\langle\beta^*\alpha\rangle, \quad (33)$$

$$\frac{d}{dt}\langle\beta^2\rangle = -\mu_c\langle\beta^2\rangle + \nu_+\langle\alpha^*\beta\rangle, \quad (34)$$

$$\frac{d}{dt}\langle\alpha^*\alpha\rangle = -\mu_a\langle\alpha^*\alpha\rangle + \frac{1}{2}\nu_-\langle\alpha^*\beta^*\rangle + \frac{1}{2}\nu_-^*\langle\alpha\beta\rangle + A\rho_{aa}^{(0)} + \kappa\bar{n}, \quad (35)$$

$$\frac{d}{dt}\langle\beta^*\beta\rangle = -\mu_c\langle\beta^*\beta\rangle + \frac{1}{2}\nu_+\langle\beta^*\alpha^*\rangle + \frac{1}{2}\nu_+^*\langle\alpha\beta\rangle + \kappa\bar{n}, \quad (36)$$

$$\frac{d}{dt}\langle\alpha^*\beta\rangle = -\frac{1}{2}(\mu_a+\mu_c)\langle\alpha^*\beta\rangle + \frac{1}{2}\nu_+\langle\alpha^{*2}\rangle + \frac{1}{2}\nu_-^*\langle\beta^{*2}\rangle, \quad (37)$$

$$\frac{d}{dt}\langle\alpha\beta\rangle = -\frac{1}{2}(\mu_a+\mu_c)\langle\alpha\beta\rangle + \frac{1}{2}\nu_+\langle\alpha^*\alpha\rangle + \frac{1}{2}\nu_-\langle\beta^*\beta\rangle + \frac{1}{2}\nu_+, \quad (38)$$

On basis of Eqs. 31 , 32, we can write

$$\frac{d}{dt}\alpha(t) = -\frac{\mu_a}{2}\langle\alpha\rangle + \frac{1}{2}\nu_-\langle\beta^*\rangle + f_\alpha(t), \quad (39)$$

$$\frac{d}{dt}\beta^*(t) = -\frac{\mu_c}{2}\langle\beta\rangle + \frac{1}{2}\nu_+\langle\alpha^*\rangle + f_\beta^*(t), \quad (40)$$

where $f_\alpha(t)$ and $f_\beta(t)$ are noise forces. We now proceed to determine the properties of the noise force. The expectation value of Eqs. 39 and 40 are found to be

$$\frac{d}{dt}\langle\alpha(t)\rangle = -\frac{\mu_a}{2}\langle\alpha\rangle + \frac{1}{2}\nu_-\langle\beta^*\rangle + \langle f_\alpha(t)\rangle, \quad (41)$$

$$\frac{d}{dt}\langle\beta^*(t)\rangle = -\frac{\mu_c}{2}\langle\beta\rangle + \frac{1}{2}\nu_+\langle\alpha^*\rangle + \langle f_\beta(t)\rangle. \quad (42)$$

Comparison of Eqs. 31 and 41 as well as Eqs. 32 and 42 yields

$$\langle f_\alpha(t)\rangle = \langle f_\beta(t)\rangle = 0. \quad (43)$$

The formal solutions of Eqs. 41 and 42 can be put in the form

$$\alpha(t) = \alpha(0)e^{-\frac{\mu_a t}{2}} + \int_0^t e^{-\frac{\mu_a(t-t')}{2}}[\frac{1}{2}\nu_-\beta^*(t') + f_\alpha(t')]dt', \quad (44)$$

$$\beta^*(t) = \beta(0)e^{-\frac{\mu_c t}{2}} + \int_0^t e^{-\frac{\mu_c(t-t')}{2}}[\frac{1}{2}\nu_+^*\alpha(t') + f_\beta^*(t')]dt'. \quad (45)$$

Moreover, applying the relation

$$\frac{d}{dt}\langle\alpha(t)\alpha(t)\rangle = \langle\alpha(t)(\frac{d}{dt}\alpha(t))\rangle + \langle(\frac{d}{dt}\alpha(t))\alpha(t)\rangle, \quad (46)$$

along with Eq. 41, one can readily verify that

$$\frac{d}{dt}\langle\alpha^2\rangle = -\mu_a\langle\alpha^2(t)\rangle + \nu_-^*\langle\alpha(t)\rangle + 2\langle\alpha(t)f_\alpha(t)\rangle. \quad (47)$$

With aid of Eq. 42, one can readily verify that using the same relation

$$\langle\beta^*(t)f_\alpha(t)\rangle = 0. \quad (48)$$

In view of this result, one can readily get

$$\int_0^t e^{\frac{\mu_a(t-t')}{2}}\langle f_\alpha(t')f_\alpha(t)\rangle dt' = 0 \quad (49)$$

Applying the relation

$$\int_0^t e^{\frac{1}{2}a(t-t')}\langle f(t)g(t')\rangle dt' = D, \quad (50)$$

we assert that

$$\langle f(t)g(t')\rangle = 2D\delta(t-t'),$$

where a and D are a constants or some function of time t. We then see that

$$\langle f_\alpha(t')f_\alpha(t)\rangle = 0. \quad (51)$$

It can also be established in similar manner that

$$\langle f_\beta(t')f_\beta(t)\rangle = \langle f_\alpha^*(t')f_\beta(t)\rangle = 0. \quad (52)$$

Once more using the relation

$$\frac{d}{dt}\langle\alpha^*\alpha\rangle = \langle\frac{d\alpha^*}{dt}\alpha\rangle + \langle\alpha^*\frac{d\alpha}{dt}\rangle, \quad (53)$$

with Eq. 41 and its complex conjugate, we have

$$\frac{d}{dt}\langle\alpha^*\alpha\rangle = -\mu_a\langle\alpha^*\alpha\rangle + \frac{1}{2}\nu_-\langle\alpha^*\beta^*\rangle + \frac{1}{2}\nu_-^*\langle\alpha\beta\rangle + \langle\alpha^* f_\alpha(t)\rangle + \langle f_\alpha^*(t)\alpha(t)\rangle. \quad (54)$$

III. QUADRATURE VARIANCE

Here we seek to analyze the quadrature squeezing properties of the two-mode light in the cavity can be described by two quadratures [10]-[16]

$$\hat{c}_+ = \hat{c}^\dagger + \hat{c}, \quad (55)$$

$$\hat{c}_- = i(\hat{c}^\dagger - \hat{c}), \quad (56)$$

where

$$\hat{c} = \frac{1}{\sqrt{2}}(\hat{a} + \hat{b}), \quad (57)$$

$$\hat{c}^\dagger = \frac{1}{\sqrt{2}}(\hat{a}^\dagger + \hat{b}^\dagger), \quad (58)$$

are the two-mode cavity operators, and $\hat{a}$ and $\hat{b}$ are annihilation operators for cavity modes a and b. In view of Eq.57 and Eq.**??**, one can write Eq.55) as

$$\hat{c}_+ = \frac{1}{\sqrt{2}}[\hat{a} + \hat{b} + \hat{a}^\dagger + \hat{b}^\dagger]. \quad (59)$$

It then follows that

$$\hat{c}_+ = \frac{1}{\sqrt{2}}[\hat{a}_+ + \hat{b}_+]. \quad (60)$$

Following a similar procedure, we get

$$\hat{c}_- = \frac{i}{\sqrt{2}}[\hat{a}_- + \hat{b}_-], \quad (61)$$

where

$$\hat{a}_+ = \hat{a} + \hat{a}^\dagger, \quad \hat{a}_- = i(\hat{a}^\dagger - \hat{a}), \quad (62)$$

$$\hat{b}_+ = \hat{b} + \hat{b}^\dagger, \quad \hat{b}_- = i(\hat{b}^\dagger - \hat{b}). \quad (63)$$

Employing the comuutation relation of the cavity mode operators

$$[\hat{a}, \hat{a}^\dagger] = [\hat{b}, \hat{b}^\dagger] = 1, \tag{64}$$

the quadrature operators $\hat{c}_+$ and $\hat{c}_-$ are Hermitian and satisfy the commutation relation

$$[\hat{c}_+, \hat{c}_-] = 2i. \tag{65}$$

On the basis of these definitions a two-mode light is said to be in a squeezed state if either $\Delta\hat{c}_+^2 < 1$ and $\Delta\hat{c}_-^2 > 1$ or $\Delta\hat{c}_+^2 > 1$ and $\Delta\hat{c}_-^2 < 1$, such that $\Delta\hat{c}_+\Delta\hat{c}_- \geq 1$.

Next we proceed to calculate the quadrature variance of the two-mode cavity light. The variance of the plus and minus quadrature operators of the two-mode cavity light are defined by

$$(\Delta\hat{c}_+)^2 = \langle\hat{c}_+^2\rangle - \langle\hat{c}_+\rangle^2 \tag{66}$$

and

$$(\Delta\hat{c}_-)^2 = \langle\hat{c}_-^2\rangle - \langle\hat{c}_-\rangle^2. \tag{67}$$

On account of Eqs. 55 and 66), the plus quadrature variance can be expressed in terms of the creation and annihilation operators as

$$(\Delta\hat{c}_+)^2 = \langle\hat{c}\hat{c}^\dagger\rangle + \langle\hat{c}^\dagger\hat{c}\rangle + \langle\hat{c}^2\rangle + \langle\hat{c}^{\dagger 2}\rangle - \langle\hat{c}\rangle^2 - \langle\hat{c}^\dagger\rangle^2 - 2\langle\hat{c}\rangle\langle\hat{c}^\dagger\rangle \tag{68}$$

and with the help of Eqs. 56) and 67, we get

$$(\Delta\hat{c}_-)^2 = \langle\hat{c}\hat{c}^\dagger\rangle + \langle\hat{c}^\dagger\hat{c}\rangle - \langle\hat{c}^2\rangle - \langle\hat{c}^{\dagger 2}\rangle + \langle\hat{c}\rangle^2 + \langle\hat{c}^\dagger\rangle^2 - 2\langle\hat{c}\rangle\langle\hat{c}^\dagger\rangle, \tag{69}$$

so that inspection of Eqs. 68 and 69) shows that

$$(\Delta\hat{c}_\pm)^2 = \langle\hat{c}\hat{c}^\dagger\rangle + \langle\hat{c}^\dagger\hat{c}\rangle \pm \langle\hat{c}^2\rangle \pm \langle\hat{c}^{\dagger 2}\rangle \mp \langle\hat{c}\rangle^2 \mp \langle\hat{c}^\dagger\rangle^2 - 2\langle\hat{c}\rangle\langle\hat{c}^\dagger\rangle. \tag{70}$$

This can be expressed interms of c-number variables associated with the normal ordering as

$$(\Delta\hat{c}_\pm)^2 = \langle\gamma(t)\gamma^*(t)\rangle + \langle\gamma^*(t)\gamma(t)\rangle \pm \langle\gamma^2(t)\rangle \pm \langle\gamma^{*2}(t)\rangle$$
$$\mp\langle\gamma(t)\rangle^2 \mp \langle\gamma^*(t)\rangle^2 - 2\langle\gamma(t)\rangle\langle\gamma^*(t)\rangle. \tag{71}$$

where $\gamma(t)$ is the c-number variable corresponding to the operator $\hat{c}(t)$. The c-number equation corresponding to Eq. 57 can be written as

$$\gamma(t) = \frac{1}{\sqrt{2}}[\alpha(t) + \beta(t)] \tag{72}$$

and application of Eq. 72 to Eq. 71 leads to

$$\Delta c_\pm^2 = 1 \pm \bigg[\frac{1}{2}\big[\langle\alpha^2(t)\rangle + \langle\alpha^{*2}(t)\rangle + \langle\beta^2(t)\rangle + \langle\beta^{*2}(t)\rangle\big] + \langle\alpha(t)\beta(t)\rangle$$
$$+\langle\alpha^*(t)\beta^*(t)\rangle \pm \big[\langle\alpha^*(t)\alpha(t)\rangle + \langle\beta^*(t)\beta(t)\rangle + \langle\alpha^*(t)\beta(t)\rangle$$
$$+\langle\beta^*(t)\alpha(t)\rangle\big]\bigg] \mp \frac{1}{2}\langle(\alpha^*(t) + \beta^*(t) \pm \alpha(t) + \beta(t))\rangle^2. \tag{73}$$

Assuming that the cavity modes are initially in vacuum state along with the fact that a noise force at a certain time does not affect the cavity mode variables at earlier time[17]-[19], we easily find

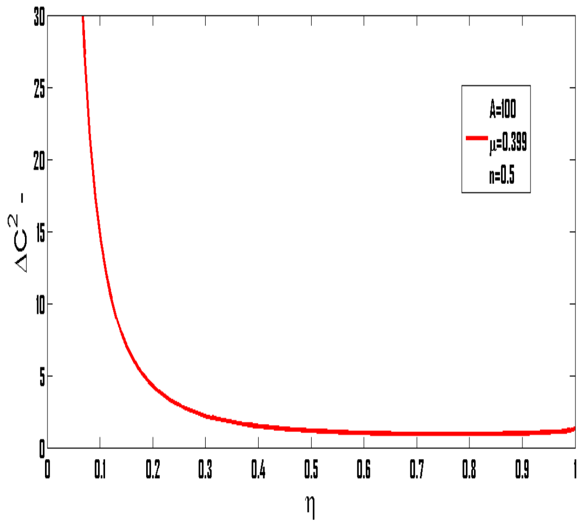

FIG. 2. Plots of the quadrature variances versus η for $A = 100$, $\kappa = 0.8$, $\mu = 0.399$, and $\bar{n}_{th} = 0.5$

$$\langle\alpha^2(t)\rangle = 0. \tag{74}$$

In a similar manner, we see that

$$\langle\beta^2(t)\rangle = 0, \tag{75}$$

$$\langle\alpha^*(t)\beta(t)\rangle = \langle\beta^*(t)\alpha(t)\rangle = 0. \tag{76}$$

Now with the aid of Eqs. 74, 75, and 76, we arrive at

$$\Delta c_\pm^2 = 1 + \frac{1}{2}[\langle\alpha(t)\beta(t)\rangle + \langle\alpha^*(t)\beta^*(t)\rangle]$$
$$\pm\langle\alpha^*(t)\alpha(t)\rangle + \langle\beta^*(t)\beta(t)\rangle. \tag{77}$$

Since $\langle\alpha(t)\beta(t)\rangle = \langle\alpha^*(t)\beta^*(t)\rangle$, we then see that

$$\Delta c_\pm^2 = 1 + \langle\alpha(t)\beta(t)\rangle \pm \langle\alpha^*(t)\alpha(t)\rangle + \langle\beta^*(t)\beta(t)\rangle \tag{78}$$

Which takes the form

$$\Delta c_\pm^2 = 1 + \frac{2\kappa A(1-\eta)(2\kappa + 2A\eta + A) + 16\mu^2 A\eta - 4\kappa A^2\eta^2\bar{n}_{th}}{4[\kappa(\kappa + A\eta) - 4\mu^2](2\kappa + A\eta)}$$
$$\pm\frac{2\kappa(4\mu + A\sqrt{1-\eta^2})(2\kappa + A\eta + A \pm 4\mu)}{4[\kappa(\kappa + A\eta) - 4\mu^2](2\kappa + A\eta)}$$
$$+\frac{4\kappa[(2\kappa + A\eta)(2\kappa + A\eta \pm 4\mu)\bar{n}_n + A^2(1 \pm \sqrt{1-\eta^2})\bar{n}_{th}]}{4[\kappa(\kappa + A\eta) - 4\mu^2](2\kappa + A\eta)} \tag{79}$$

This represents the quadrature variances of the cavity modes for a non degenerate three level laser whose cavity contains a parametric amplifier and coupled to a thermal reservoir. Plot in Fig. 2 indicates that the maximum intracavity squeezing for the above values and with in the parametric amplifier is 50% below the coherent state

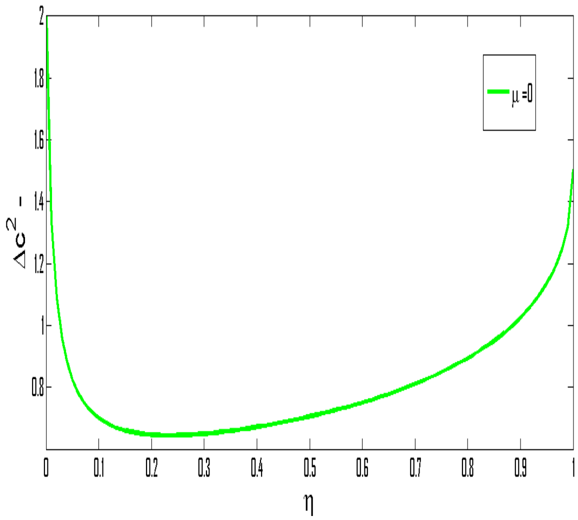

FIG. 3. Plots of the quadrature variances versus η for $A = 100$, $\kappa = 0.8$, $\mu = 0$, and $\bar{n}_{th} = 0.5$

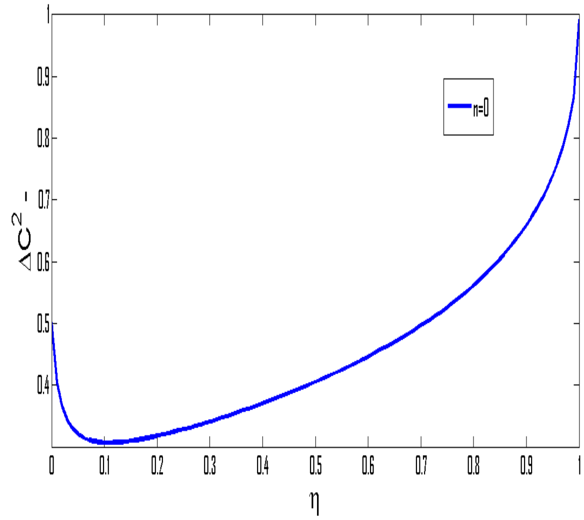

FIG. 4. Plots of the quadrature variances versus η for $A = 100$, $\kappa = 0.8$, $\mu = 0.399$, and $\bar{n}_{th} = 0$

level. Fig. 2 is the plots of the variance of the minus quadrature versus η with parametric amplifier in nondegenerate three-level laser cavity.

We next consider some special cases. We first consider the case in which the parametric amplifier is removed from the cavity. Thus setting $\mu = 0$ in Eq. 79, we see that

$$\Delta c_{\pm}^2 = 1 + \frac{A(1-\eta)(2\kappa + 2A\eta + A) - 2A^2\eta^2\bar{n}_{th}}{2(\kappa + A\eta)(2\kappa + A\eta)}$$

$$\pm \frac{(A\sqrt{1-\eta^2})(2\kappa + A\eta + A)}{2(\kappa + A\eta)(2\kappa + A\eta)}$$

$$+ \frac{4\kappa[(2\kappa + A\eta)^2\bar{n}_{th} + A^2(1 \pm \sqrt{1-\eta^2})\bar{n}_{th}]}{4[\kappa(\kappa + A\eta)](2\kappa + A\eta)}. \qquad (80)$$

This represents the quadrature variances of the cavity modes in the absence of parametric amplifier for a non degenerate three level laser coupled to a thermal reservoir. Fig. 3 the minimum value of the quadrature variance is found to be $\Delta c_{-}^2 = 0.77$ and occurs at $\eta = 0.1$. For $A = 100$, $\kappa = 0.8$, $\mu = 0$, and $\bar{n}_{th} = 0.5$. This indicates that the maximum intracavity squeezing for the above values and in the absence of parametric amplifier is 80% below the coherent state level. Fig. 3 is the plots of the variance of the minus quadrature versus η in the absence of parametric amplifier in nondegenerate three-level laser cavity. This figure shows that the increase of the degree of squeezing due to the parametric amplifier is not significant. Next upon setting $\bar{n}_{th} = 0$ in Eq. 79, we have

$$\Delta c_{\pm}^2 = 1 + \frac{2\kappa A(1-\eta)(2\kappa + 2A\eta + A) + 16\mu^2 A\eta}{4[\kappa(\kappa + A\eta) - 4\mu^2](2\kappa + A\eta)}$$

$$\pm \frac{2\kappa(4\mu + A\sqrt{1-\eta^2})(2\kappa + A\eta + A - 4\mu)}{4[\kappa(\kappa + A\eta) - 4\mu^2](2\kappa + A\eta)}. \qquad (81)$$

This is the quadrature variances of the cavity modes for a nondegenerate three-level laser whose cavity contains a parametric amplifier and whose cavity modes are coupled to a vacuum reservoir. In Fig.4 we plot the variance of the minus quadrature versus η for $A = 100$, $\kappa = 0.8$, $\mu = 0.399$, and $\bar{n}_{th} = 0$. The maximum intrcavity is at 50% and occurs at $\eta = 0.11$. We now consider the case in which the non linear crystal is removed from the cavity and the cavity is coupled to a two-mode vacuum reservoir. Then upon setting $\bar{n}_{th} = \mu = 0$ in Eq. 79, we get

$$\Delta c_{\pm}^2 = 1 + \frac{A(1-\eta)(2\kappa + 2A\eta + A) - A\sqrt{1-\eta^2}(2\kappa + A\eta + A)}{2(\kappa + A\eta)(2\kappa + A\eta)} \qquad (82)$$

This is the quadrature variances of the cavity modes for a nondegenerate three-level laser. In Fig.5 the minimum value of the quadrature variance described by Eq.82 for $A = 100$, $k = 0.8$, and $\bar{n}_{th} = \mu = 0$ is found to be $\Delta c_{-}^2 = 0.45$ and occurs at $\eta = 0.16$. This result implies that the maximum intracavity squeezing for the above values is 40% below the coherent-state level. The plots in Fig.5 represent the variances of the minus quadrature of the cavity modes for a nondegenerate three-level laser alone.

IV. PHOTON STATISTICS

In this section we study the statistical properties of the cavity and output modes produced by a nondegenerate three-level laser whose cavity contains a parametric amplifier.

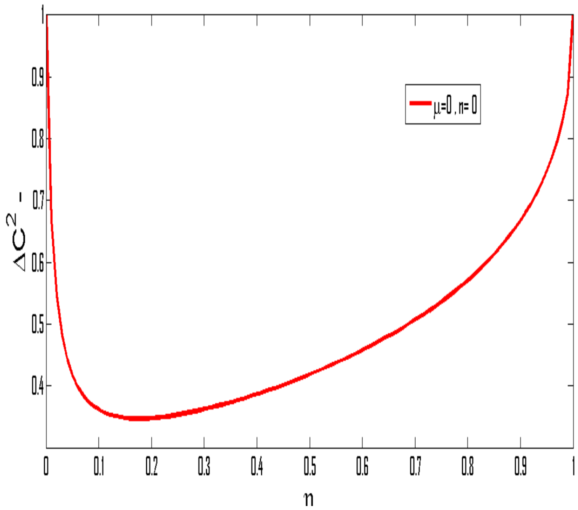

FIG. 5. Plots of the quadrature variances versus η for $A = 100$, $\kappa = 0.8$, $\mu = 0$, and $\bar{n}_{th} = 0$

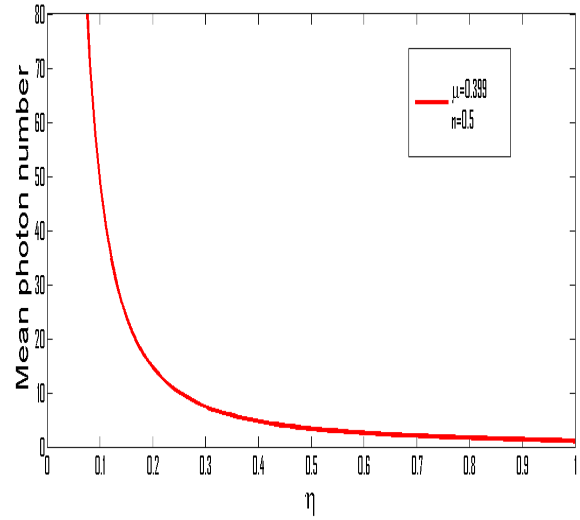

FIG. 6. A plot of the mean photon number versus η for $A = 100$, $\kappa = 0.8$, $\mu = 0.399$, and $\bar{n}_{th} = 0.5$

A. The mean and the variance of the photon number

Here we wish to calculate the mean photon number of the two-mode cavity light coupled to thermal reservoir with parametric amplifier. The mean photon number for the two-modes in terms of density operator can be expressed as

$$\langle \hat{c}^\dagger(t)\hat{c}(t)\rangle = Tr(\rho(t)\hat{c}^\dagger(0)\hat{c}(0)), \tag{83}$$

in which

$$\hat{c} = \hat{a} + \hat{b}, \tag{84}$$

$$\hat{c}^\dagger = \hat{a}^\dagger + \hat{b}^\dagger, \tag{85}$$

where $\hat{a}$, $\hat{b}$ and $\hat{c}$ are the annihilation operators for a light mode a, light mode b, and the two-mode light, respectively [20]-[24]. Employing Eqs. 81 and 82, Eq. 80 can be written as

$$\langle \hat{c}^\dagger(t)\hat{c}(t)\rangle = \langle \hat{a}^\dagger(t)\hat{a}(t)\rangle + \langle \hat{a}^\dagger(t)\hat{b}(t)\rangle + \langle \hat{b}^\dagger(t)\hat{a}(t)\rangle + \langle \hat{b}^\dagger(t)\hat{b}(t)\rangle). \tag{86}$$

Then we have

$$\langle \hat{c}^\dagger(t)\hat{c}(t)\rangle = \langle \hat{a}^\dagger(t)\hat{a}(t)\rangle + \langle \hat{b}^\dagger(t)\hat{b}(t)\rangle. \tag{87}$$

Employing the relation

$$\int d^2\alpha e^{-a\alpha^*\alpha + b\alpha + c\alpha^*} = \frac{\pi}{a} e^{\frac{bc}{a}}, \tag{88}$$

with performing the integration over λ, it yields

$$\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle = \frac{1}{\pi^4}(u^2 - v^2)\frac{d^2}{dxdy}\int d^2\alpha d^2\beta d^2\eta exp(-\eta^*\eta + \eta^*(\alpha + y\beta) + \eta(\alpha^* + v\beta - v\alpha^*))exp(-\alpha^*\alpha + x\alpha + v\alpha^*\beta^* - u\beta^*\beta)|_{x=y=0}, \tag{89}$$

so that carrying out the integration over β and η, there follows

$$\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle = \frac{1}{\pi^4}(u^2 - v^2)\frac{d^2}{dxdy}\int d^2\alpha exp(-\alpha^*\alpha(\frac{u^2 - v^2}{u}) + \alpha^*(uy + -u^2y + v^2y) + x\alpha)|_{x=y=0}. \tag{90}$$

Using Eq. 85 and performing differentiation, by applying the condition, $x = y = 0$, we readily obtain

$$\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle = a - 1. \tag{91}$$

Similarly, following the same procedure, we note that

$$\langle \hat{b}^\dagger(t)\hat{b}(t)\rangle = b - 1. \tag{92}$$

Then in view of Eqs. 88 and 89, Eq. 84 turns out to be

$$\langle \hat{c}^\dagger(t)\hat{c}(t)\rangle = a + b - 2. \tag{93}$$

Now we see that

$$\bar{n} = \frac{2\kappa(4\mu + A\sqrt{1-\eta^2})(2\kappa + A\eta + A - 4\mu)}{4[\kappa(\kappa + A\eta) - 4\mu^2](2\kappa + A\eta)} + \frac{4\kappa[(2\kappa + A\eta)(2\kappa + A\eta + 4\mu)\bar{n}_{th} + A^2(1 - \sqrt{1-\eta^2})\bar{n}_{th}]}{4[\kappa(\kappa + A\eta) - 4\mu^2](2\kappa + A\eta)} \tag{94}$$

The plot on Fig.6 shows that the mean photon number of Eq. 94 for the values $A = 100$, $\kappa = 0.8$, $\mu = 0.399$, and $\bar{n}_{th} = 0.5$. The results show that as η increases the mean photon number decreases.

We next consider some special cases. We first consider the case in which the parametric amplifier is removed from the cavity. Thus setting $\mu = 0$ in Eq. 94 , we see that

$$\bar{n} = \frac{2kA\sqrt{1-\eta^2})(2k + A\eta + A)}{4(k(k + A\eta)(2k + A\eta)} + \frac{4((2k + A\eta)(2k + A\eta)\bar{n}_{th} + A^2(1 - \sqrt{1-\eta^2})\bar{n}_{th}}{4(k(k + A\eta)(2k + A\eta)} \tag{95}$$

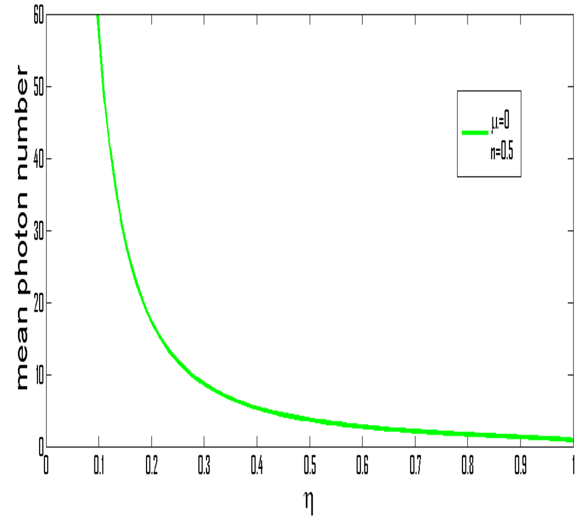

FIG. 7. A plot of the mean photon number versus η for $A = 100$, $\kappa = 0.8$, $\mu = 0$, and $\bar{n}_{th} = 0.5$.

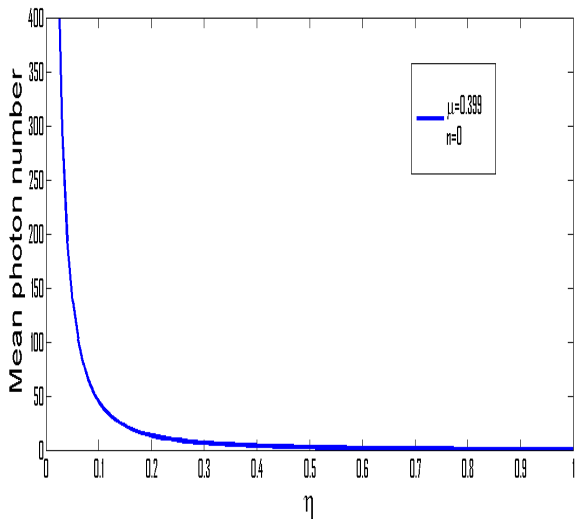

FIG. 8. A plot of the mean photon number versus η for $A = 100$, $\kappa = 0.8$, $\mu = 0.399$, and $\bar{n}_{th} = 0$.

In addition, in the absence of parametric amplifier (when $\mu = 0$) as shown in Eq. 95 together with the plot in Fig.7 shows that the mean photon number for the values $A = 100$, $\kappa = 0.8$, $\mu = 0$, and $\bar{n}_{th} = 0.5$. The results show that as η increases the mean photon number decreases.

Furthermore, in the absence of thermal reservoir, upon setting $\bar{n}_{th} = 0$, it becomes a two-mode vacuum reservoir. Hence in Eq.94 reduces to

$$\bar{n} = \frac{2k(4\mu + A\sqrt{1-\eta^2})(2k + A\eta + A - 4\mu)}{4[k(k + A\eta) - 4\mu^2](2k + A\eta)}. \quad (96)$$

In addition, in the absence of thermal reservoir (when

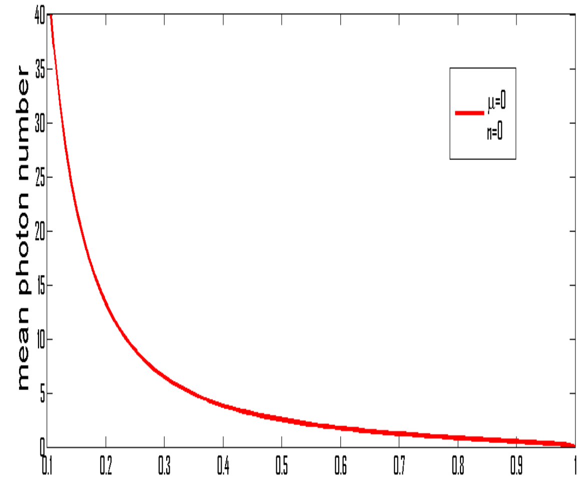

FIG. 9. A plot of the mean photon number versus η for $A = 100$, $\kappa = 0.8$, $\mu = 0$, and $\bar{n}_{th} = 0$.

.

$\bar{n}_{th} = 0$) as shown in Eq. 96 for the values $A = 100$, $\kappa = 0.8$, $\mu = 0.399$, and $\bar{n}_{th} = 0$.The plot in Fig.8shows that the mean photon number decrease as η increases.

Finally, in the absence of both parametric amplifier (when $\mu = 0$) and thermal reservoir (when $\bar{n}_{th} = 0$), the mean photon number of Eq.94 turns out to be

$$\bar{n} = \frac{(A\sqrt{1-\eta^2})(2k + A\eta + A)}{2(k + A\eta)(2k + A\eta)}. \quad (97)$$

Fig.9 shows that the plot of mean photon number in the absence of both parametric amplifier and thermal reservoir for the values $A = 100$, $\kappa = 0.8$, $\mu = 0$, and $\bar{n}_{th} = 0$. The plot in Fig.9 shows that the mean photon number decrease as η increases.

B. The Variance of the Photon Number Difference

We next proceed to calculate the variance of the photon number at steady state. It can be expressed as

$$(\Delta n)^2 = \langle (\hat{c}^\dagger \hat{c})^2 \rangle - \langle \hat{c}^\dagger \hat{c} \rangle^2 \quad (98)$$

where

$$\hat{c} = \hat{a} + \hat{b}$$

$$\hat{c}^\dagger = \hat{a}^\dagger + \hat{b}^\dagger \quad (99)$$

and

$$\gamma = \alpha + \beta$$

$$\gamma^* = \alpha^* + \beta^* \quad (100)$$

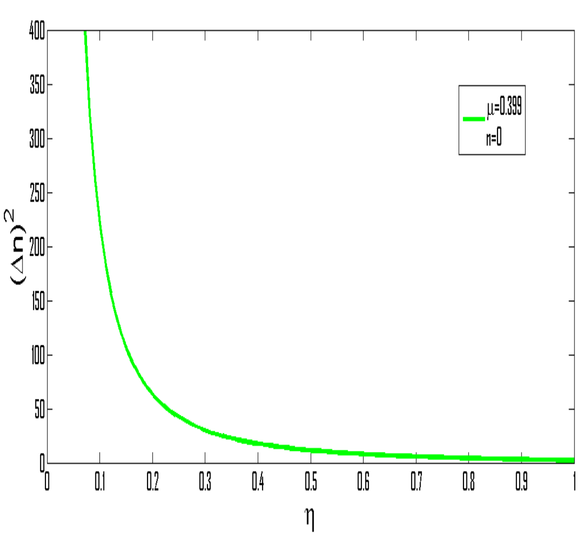

FIG. 10. A plot of the variance of photon number difference versus η for $A = 100$, $\kappa = 0.8$, $\mu = 0.399$, and $\bar{n}_{th} = 0$

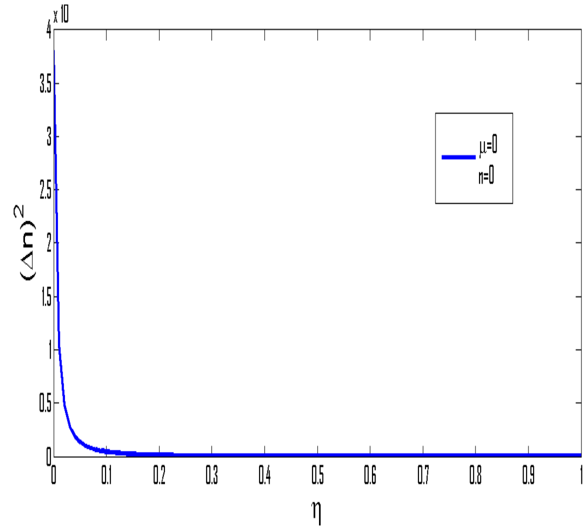

FIG. 11. A plot of the variance of photon number difference versus η for $A = 100$, $\kappa = 0.8$, $\mu = 0$, and $\bar{n}_{th} = 0$

are c-number variables associated with the normal ordering. The Photon number variance takes the form

$$(\Delta n)^2 = \langle \hat{c}^\dagger \hat{c} \rangle \langle \hat{c}\hat{c}^\dagger + \langle \hat{c}^2 \rangle \langle \hat{c}^{\dagger 2} \rangle, \tag{101}$$

from which follows

$$(\Delta n)^2 = 2[1 + \langle \hat{c}^\dagger \hat{c} \rangle] + \langle \hat{c}^2 \rangle \langle \hat{c}^{\dagger 2} \rangle. \tag{102}$$

It is possible to write in c-number as

$$\begin{aligned}(\Delta n)^2 = {} & 2[1 + \langle (\alpha(t) + \beta(t))(\alpha^*(t) + \beta^*(t)) \rangle] + \langle (\alpha(t)\beta(t))^2 \rangle \\ & + \langle (\alpha^*(t) + \beta^*(t)) \rangle + \langle (\alpha^*(t) + \beta^*(t)) \rangle.\end{aligned} \tag{103}$$

With the aid of

$$\langle \alpha^*(t)\beta^*(t) \rangle = \langle \alpha(t)\beta(t) \rangle. \tag{104}$$

One can verify that

$$\begin{aligned}(\Delta n)^2 = {} & 2[1 + \langle \alpha^*(t)\alpha(t) \rangle + \langle \beta^*(t)\beta(t) \rangle \\ & + 2\langle \alpha(t)\alpha(t) \rangle].\end{aligned} \tag{105}$$

Thus the variance of the photon number takes the form

$$\begin{aligned}(\Delta n)^2 = {} & 2 + 2\left[\frac{2\kappa(4\mu + A\sqrt{1-\eta^2})(2\kappa + A\eta + A - 4\mu)}{4[\kappa(\kappa + A\eta) - 4\mu^2](2\kappa + A\eta)}\right] \\ & + 2\left[\frac{4\kappa[(2\kappa + A\eta)(2\kappa + A\eta + 4\mu)\bar{n}_{th} + A^2(1 - \sqrt{1-\eta^2})\bar{n}_{th}]}{4[\kappa(\kappa + A\eta) - 4\mu^2](2\kappa + A\eta)}\right] \\ & + 4\left[\frac{2\kappa A(1-\eta)(2\kappa + 2A\eta + A) + 16\mu^2 A\eta - 4\kappa A^2\eta^2\bar{n}}{4[\kappa(\kappa + A\eta) - 4\mu^2](2\kappa + A\eta)}\right].\end{aligned} \tag{106}$$

This is the photon number variance for a coherently driven three-level laser with parametric amplifier Moreover, in the absence of thermal reservoir ($\bar{n}_{th} = 0$) or this mean the cavity coupled to a two-mode vacuum reservoir, the photon number variance of Eq. 106 takes the form

$$\begin{aligned}(\Delta n)^2 = {} & 2 + 2\left[\frac{2\kappa(4\mu + A\sqrt{1-\eta^2})(2\kappa + A\eta + A - 4\mu)}{4[\kappa(\kappa + A\eta) - 4\mu^2](2\kappa + A\eta)}\right] \\ & + 4\left[\frac{2\kappa A(1-\eta)(2\kappa + 2A\eta + A) + 16\mu^2 A\eta}{4[\kappa(\kappa + A\eta) - 4\mu^2](2\kappa + A\eta)}\right].\end{aligned} \tag{107}$$

Fig.10 shows that the plot of photon number variance in the absence of thermal reservoir for the values $A = 100$, $\kappa = 0.8$, $\mu = 0.399$, and $\bar{n}_{th} = 0$. The plot in Fig.10 shows that the variance of photon number decrease as η increases.

Furthermore, in the absence of both parametric amplifier (when $\mu = 0$) and thermal reservoir (when $\bar{n}_{th} = 0$, the variance of the photon number described by Eq. 106 reduces to

$$\begin{aligned}(\Delta n)^2 = {} & 2 + 2\left[\frac{2\kappa(A\sqrt{1-\eta^2})(2\kappa + A\eta + A)}{4[\kappa(\kappa + A\eta)](2\kappa + A\eta)}\right] \\ & + 4\left[\frac{2\kappa A(1-\eta)(2\kappa + 2A\eta + A)}{4[\kappa(\kappa + A\eta)](2\kappa + A\eta)}\right].\end{aligned} \tag{108}$$

Fig.11 shows that the plot of photon number variance in the absence of both parametric amplifier and thermal reservoir for the values $A = 100$, $\kappa = 0.8$, $\mu = 0$, and $\bar{n}_{th} = 0$. The plot in Fig.11shows that the variance of photon number decrease as η increases.

V. ENTAGLEMENT AMPLIFICATION

Here we seek to study the entanglement condition of the two modes in the cavity. A pair of particles is taken to be entangled in quantum theory, if its states cannot be expressed as a product of the states of its individual constituents. The preparation and manipulation of these entangled states that have nonclassical and nonlocal properties lead to a better understanding of the basic quantum principles. That is, if the density operator for the combined state cannot be described as a combination of the product of density operators of the constituents,

$$\hat{\rho} \neq \sum_j P_j \hat{\rho}_j^{(1)} \bigotimes \hat{\rho}_j^{(2)} \tag{109}$$

$P_j \geq 0$ and $\sum_j P_j = 1$ is set to ensure normalization of the combined density of state. To study the properties of entanglement produced by this quantum optical system, we need an entanglement criterion for the system. According to the criteria set by Duan et al. [20], a quantum state of the system is entangled provided that the sum of the variances of the two EPR(Einstein-Podolsky-Rosen)-type operators (entanglement)$\hat{u}$ and $\hat{v}$ satisfies the condition;

$$(\Delta\hat{u})^2 + (\Delta\hat{v})^2 < 2, \tag{110}$$

where ,

$$\hat{u} = \hat{x}_a - \hat{x}_b, \hat{v} = \hat{p}_a + \hat{p}_b, \tag{111}$$

with

$$\hat{x}_a = \frac{(\hat{a}^\dagger + \hat{a})}{\sqrt{2}}, \hat{x}_b = \frac{(\hat{b}^\dagger + \hat{b})}{\sqrt{2}}, \tag{112}$$

$$\hat{p}_a = \frac{i(\hat{a}^\dagger - \hat{a})}{\sqrt{2}}, \hat{p}_b = \frac{i(\hat{b}^\dagger - \hat{b})}{\sqrt{2}} \tag{113}$$

being the quadrature operators for modes $\hat{a}$ and$\hat{b}$. The total variance of the operators $\hat{u}$ and $\hat{v}$ can be written as

$$(\Delta\hat{u})^2 + (\Delta\hat{v})^2 < 2. \tag{114}$$

This implies that

$$(\Delta\hat{u})^2 = \langle u^2\rangle - \langle u\rangle^2. \tag{115}$$

On account of Eq. 111, we see that

$$(\Delta\hat{u})^2 = \langle(\frac{1}{2}(\hat{a} + \hat{a}^\dagger) - \langle\frac{1}{2}(\hat{b} + \hat{b}^\dagger))^2\rangle, \tag{116}$$

from which follows

$$\begin{aligned}(\Delta\hat{u})^2 = {} & \frac{1}{2}[1 + 2\langle\hat{a}^\dagger\hat{a}\rangle] - \frac{1}{2}[2\langle\hat{a}\hat{b}\rangle] \\ & -\frac{1}{2}[2\langle\hat{a}\hat{b}\rangle] + \frac{1}{2}[1 + 2\langle\hat{b}^\dagger\hat{b}\rangle].\end{aligned} \tag{117}$$

It then follows that

$$(\Delta\hat{u})^2 = 1 + 2\langle\hat{a}^\dagger\hat{a}\rangle + 2\langle\hat{b}^\dagger\hat{b}\rangle - 2\langle\hat{a}\hat{b}\rangle. \tag{118}$$

It is possible to write Eq. 118, in case of c-number variables.

$$\begin{aligned}(\Delta\hat{u})^2 = {} & [1 + 2\langle\alpha^*(t)\alpha(t)\rangle + 2\langle\beta^*(t)\beta(t)\rangle - \\ & 2\langle\alpha(t)\beta(t)\rangle].\end{aligned} \tag{119}$$

Following the same procedure , we easily obtain

$$\begin{aligned}(\Delta\hat{v})^2 = {} & [1 + 2\langle\alpha^*(t)\alpha(t)\rangle + 2\langle\beta^*(t)\beta(t)\rangle \\ & -2\langle\alpha(t)\beta(t)\rangle].\end{aligned} \tag{120}$$

Thus, the sum of the variances of u and v can be expressed as

$$(\Delta\hat{u})^2 + (\Delta\hat{v})^2 = 2(\Delta\hat{u})^2 = 2(\Delta\hat{c}_\pm)^2. \tag{121}$$

We see from this result that the degree of entanglement is directly proportional to the degree of squeezing of the two-mode light. Therefore, we see that

$$\begin{aligned}(\Delta\hat{u})^2 + (\Delta\hat{v})^2 = {} & 2[1 + 2\langle\alpha^*(t)\alpha(t)\rangle + 2\langle\beta^*(t)\beta(t)\rangle \\ & -2\langle\alpha(t)\beta(t)\rangle].\end{aligned} \tag{122}$$

This can be rewritten as

$$\begin{aligned}(\Delta\hat{u})^2 + (\Delta\hat{v})^2 = {} & 2 + 2\langle\alpha^*(t)\alpha(t)\rangle + 2\langle\beta^*(t)\beta(t)\rangle \\ & -4\langle\alpha(t)\beta(t)\rangle.\end{aligned} \tag{123}$$

In view of Eqs. 79, 80, and 81, Eq. 123 takes the form

$$\begin{aligned}(\Delta\hat{u})^2 + (\Delta\hat{v})^2 = {} & 2 + 2\left[\frac{2\kappa(4\mu + A\sqrt{1-\eta^2})(2\kappa + A\eta + A - 4\mu)}{4[\kappa(\kappa + A\eta) - 4\mu^2](2\kappa + A\eta)}\right] \\ & + 2\left[\frac{4\kappa[(2\kappa + A\eta)(2\kappa + A\eta + 4\mu)(\bar{n}_{th}) + A^2(1 - \sqrt{1}}{4[\kappa(\kappa + A\eta) - 4\mu^2](2\kappa + A\eta)}\right. \\ & - 4\left[\frac{2\kappa A(1-\eta)(2\kappa + 2A\eta + A) + 16\mu^2 A\eta - 4\kappa A^2\eta^2}{4[\kappa(\kappa + A\eta) - 4\mu^2](2\kappa + A\eta)}\right.\end{aligned}$$

We next consider some special cases. We first consider the case in which the parametric amplifier is removed from the cavity. Thus setting $\mu = 0$ in Eq.124, one can readily verify that

$$\begin{aligned}(\Delta\hat{u})^2 + (\Delta\hat{v})^2 = 2\Bigg[& 1 + \frac{A(1-\eta)(2\kappa + 2A\eta) - 2A^2\eta^2\bar{n}_{th}}{2(\kappa + A\eta)(2\kappa + A\eta)} \\ & \pm \frac{A\sqrt{1-\eta^2}(2\kappa + A\eta + A + 2A\bar{n}_{th})}{2(\kappa + A\eta)(2\kappa + A\eta)} \\ & + \frac{[(2\kappa + A\eta)^2\bar{n} + A^2\bar{n}_{th}]}{2(\kappa + A\eta)(2\kappa + A\eta)}\Bigg].\end{aligned} \tag{125}$$

This represents the photon entanglement of the cavity modes for a non degenerate three level laser coupled to a two-mode squeezed vacuum reservoir.

The minimum value of the photon entanglement is found to be $\Delta\hat{u}^2 + \Delta\hat{v}^2 = 0.144$ and occurs at $\eta = 0.1$. For $A = 100$, $\kappa = 0.8$, $\mu = 0$, and $\bar{n}_{th} = 0.5$. This indicates that the maximum intracavity squeezing for the above values and in the absence of parametric amplifier is 90% below the coherent state level. Fig. 12is the plots of the photon entanglement versus η in the absence of parametric amplifier in nondegenerate three-level laser cavity. This figure shows that the increase of the degree of squeezing due to the parametric amplifier is not significant.

Next in the absence of a thermal reservoir, upon setting $\bar{n}_{th} = 0$ in Eq. 124, we have

$$\begin{aligned}(\Delta\hat{u})^2 + (\Delta\hat{v})^2 = 2\Bigg[& 1 + \frac{2\kappa A(1-\eta)(2\kappa + 2A\eta + A) + 16\mu^2 A\eta}{4[\kappa(\kappa + A\eta) - 4\mu^2](2\kappa + A\eta)} \\ & \pm \frac{2\kappa(4\mu + A\sqrt{1-\eta^2})(2\kappa + A\eta + A - 4\mu)}{4[\kappa(\kappa + A\eta) - 4\mu^2](2\kappa + A\eta)}\Bigg].\end{aligned} \tag{126}$$

This is the entanglement of the cavity modes for a nondegenerate three-level laser whose cavity contains a parametric amplifier and whose cavity modes are coupled to a two-mode vacuum reservoir.

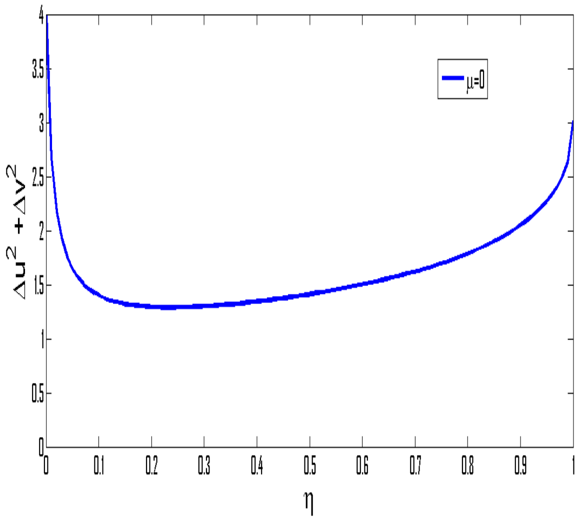

FIG. 12. Plots of $\Delta\hat{u}^2 + \Delta\hat{v}^2$ of two-mode light in the cavity at steady state versus η for $\kappa = 0.8$, $A = 100$, $\mu = 0$, and $\bar{n}_{th} = 0.5$

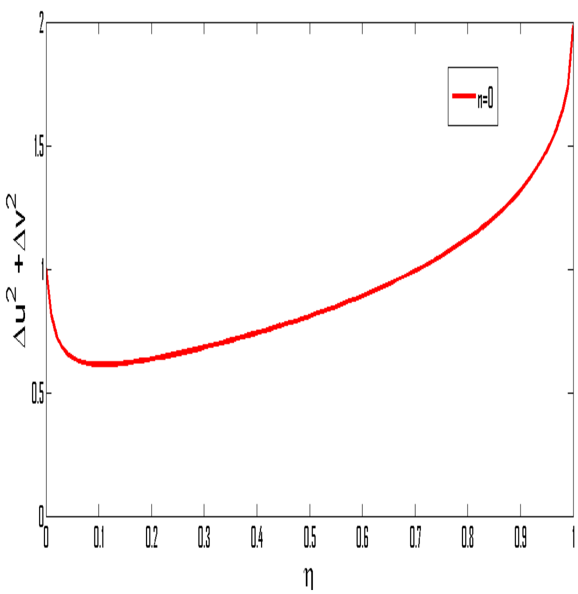

FIG. 13. Plots of $\Delta\hat{u}^2 + \Delta\hat{v}^2$ of two-mode light in the cavity at steady state versus η for $\kappa = 0.8$, $A = 100$, $\mu = 0.399$, and $\bar{n}_{th} = 0$

In Fig.13 we plot the photon entanglement versus η for $A = 100$, $\kappa = 0.8$, $\mu = 0.399$, and $\bar{n}_{th} = 0$. The maximum intracavity is at 65% and occurs at $\eta = 0.11$. We now consider the case in which the non linear crystal is removed from the cavity and the cavity is coupled to a two-mode vacuum reservoir.

Then upon setting $\mu = \bar{n} = 0$ in Eq. 124, we get

$$(\Delta\hat{u})^2 + (\Delta\hat{v})^2 =$$
$$2\left[1 + \frac{A(1-\eta)(2\kappa + 2A\eta + A) - A\sqrt{1-\eta^2}(2\kappa + A\eta + A)}{2(\kappa + A\eta)(2\kappa + A\eta)}\right]. \quad (127)$$

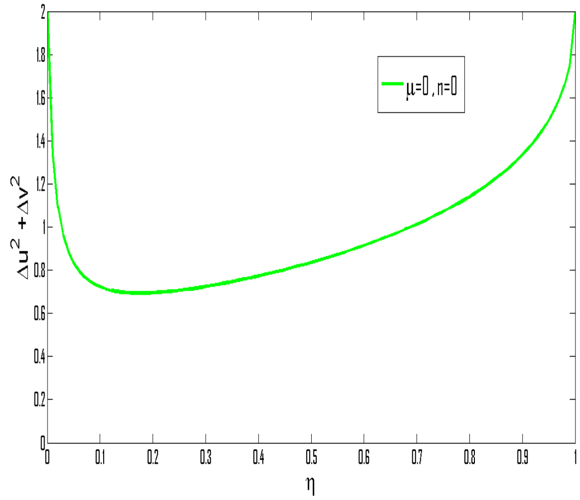

FIG. 14. Plots of $\Delta\hat{u}^2 + \Delta\hat{v}^2$ of two-mode light in the cavity at steady state versus η for $\kappa = 0.8$, $A = 100$, $\mu = 0$, and $\bar{n}_{th} = 0$.

This is the photon entanglement of the cavity modes of a nondegenerate three-level laser with vacuum reservoir.

The minimum value of the photon entanglement described by127 for $A = 100$, $k = 0.8$,$\mu = 0$ and $\bar{n} = 0$ is found to be 70% and occurs at $\eta = 0.16$. This result implies that the maximum intracavity squeezing for the above values is 75% below the coherent-state level. The plots in Fig.14 represent the photon entanglement of the cavity modes for a nondegenerate three-level laser alone.

We immediately notice that, this particular entanglement measure is directly related the two-mode squeezing. This direct relationship shows that whenever there is a two-mode squeezing in the system there will be entanglement in the system as well. It also follows that the degree of entanglement does not depend on the external driving coherent light. This is attributed to the fact that the coherent fields do not introduce additional atomic coherence to the system, as the same is true for the case of squeezing. Using the criterion Eq.110that a significant entanglement between the states of the light generated in the cavity of the non degenerate three-level laser can be manifested due to the strong correlation between the radiation emitted when the atoms decay from the upper energy level to the lower via the intermediate energy level.

Based on the criteria Eq.110, we clearly see from Fig.14 that the two states of the generated light are strongly entangled at steady state. The entanglement disappears when there is no atomic coherence, and it would be stronger for certain values of the atomic coherence for each value of the linear gain coefficient. It can easily be seen that the degree of entanglement increases with the rate at which the atoms are injected into the cavity, A.

VI. CONCLUSION

In this article we have considered a nondegenerate three-level laser , with the parametric amplifier. First we have derived the master equation in the linear and adiabatic approximations. Then using this master equation , we have obtained stochastic differential equations . Applying the solutions of the resulting differential equations , we have calculate the quadrature variance. In addition, using the same solutions we have determined the mean photon number and mean photon Entanglement . We have also seen that the two-mode driving light has no effect on squeezing of cavity modes. Like the squeezing , the parametric amplifier affects the mean photon numbers and the variance of the photon number difference. We have also found that increasing the amplitude of the parametric amplifier increases the mean photon numbers and the variances of the photon numbers. We observe that one effect of the squeezed vacuum is to enhance the degree of squeezing of the signal-idler modes. Furthermore, we have seen that the mean photon number of mode a is greater than that of mode b . We have found that both the mean photon number and the quadrature variance for the two-mode laser light beams is the sum of the mean photon numbers and the quadrature variances of the constituent two-mode laser light beams.

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