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D'Arcy Wentworth Thompson and two dimensional Möbius image registration

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Abstract

The process for the alignment of two images such that their appearances resemble is called the image registration. Image registration is widely studied in several fields such as remote sensing, computer vision, morphophonemics, medical imaging, etc. The field of image registration is inspired by the pioneer work of D'Arcy Thompson. In his seminal book, *On Growth and form*, he presented the idea of image transformation, between species. Over the past several years his book has been used as justification for the study of diffeomorphic deformations of images of different objects, a field that is now extremely well studied, particularly for the registration of medical images. Thompson's examples are hand-drawn, and consist of a two dimensional outline of an animal (or part of an animal) and a grid superimposed on it. The deformation to another animal form is demonstrated by the picture of the new form and what is intended to be the same grid pushed forward so that the smoothness of the deformations can be clearly seen. This smoothness is one of the essential requirement for the diffeomorphic image registration. Thompson, in his book, mentioned isogonal transformations for his several examples. Nowadays an isogonal transformation is termed as conformal transformation. Thompson's principal point was that the deformations between closely related species should be simple. It has been suggested that this idea of simplicity can be interpreted in modern parlance as being examples of low dimensional groups. Clearly, the infinite dimensional diffeomorphism group or the conformal group would not be simple, but the six dimensional Möbius group (a subgroup of conformal group) would be. His claims of conformally-related change between species were investigated further by Russian scientist Petukhov, who used Thompson's grid method as well as computing the cross-ratio (which is an invariant of the Möbius group, a finite-dimensional subgroup of the group of conformal diffeomorphisms) to check whether sets of points in the images could be related by a Möbius transformation. Thmpson's examples and Petukhov's investigation motivated us to explore Möbius group. This Möbius group is the composition of four simple transformations that are scaling, rotation, translation and inversion. In this paper we present the image registration using Möbius diffeomorphisms. We present the novel idea for the selection of initial guess that plays a key role for the optimisation. The optimisation is based on the gradient descent algorithm with and without the labelled points (landmarks). Numerical examples illustrate the ability of the method to perform Möbius registration.

Keywords: D'Arcy Thmpson's fish, Möbius image registration, Conformal image registration, Landmarks, Optimization

1 Introduction

The process for the alignment of two images such that their appearances resemble is called the image registration. Image registration is widely studied in several fields such as remote sensing [1–3], computer vision [4–6], morphophonemics [7–13], medical imaging [14–26], shapes analysis [27], etc. The field of image registration is inspired by the pioneer work of D'Arcy Thompson. In his seminal book, *On Growth and form* [28] which was published in 1917, he presented the idea of image transformation, between species. Over the past several years his book has been used as justification for the study of diffeomorphic deformations of images of different objects, a field that is now extremely well studied, particularly for the registration of medical images. While there have been many development in image registration since 1942 but the most remarkable and influential approach is to find a diffeomorphism between the two images using a method known as "Large Deformation Diffeomorphic Metric Mapping (LDDMM)" [29–33].

In 1992 Brown [34] published a comprehensive review of image registration. The basic steps have not changed since: Brown suggests that the requirements are (i) a feature space (contains information about the images), (ii) a search space (consists of the set of transformations), (iii) a search strategy (selection of transformations to get an optimal solution), and (iv) a similarity metric (to measure the discrepancy between the images). However, the choice of transformations certainly has changed, since only three sets of transformations are listed by Brown, all finite dimensional: (a) affine transformation (rigid, shearing and aspect ratio, 2D), (b) perspective transformation (3D to 2D, the special case is projective transformation where the scene is a flat plane such as an aerial photograph) and (c) polynomial transformation that can be used when less information is available about the camera geometry [35], [36]. Several methods for image deformation have been developed and applied to the variety of images since 1992 [37–45], the most significant change is the adoption of diffeomorphic image registration, which is now a huge area of research. Diffeomorphisms are smooth functions with smooth inverses [35, 36, 46].

Thompson's examples are hand-drawn, and consist of a two dimensional outline of an animal (or part of an animal) and a grid superimposed on it. The deformation to another animal form is demonstrated by the picture of the new form and what is intended to be the same grid pushed forward so that the smoothness of the deformations can be clearly seen. This smoothness is one of the essential requirement for the diffeomorphic image registration. Thompson's mentioned isogonal transformations for his several examples. Nowadays an isogonal transformation is termed as conformal transformation, i.e., angle-preserving. His claims of conformally-related change between species were investigated further by Petukhov [47], who used Thompson's grid method as well as computing the cross-ratio (which is an invariant of the Möbius group, a finite-dimensional subgroup of the group of conformal diffeomorphisms) to check whether sets of points in the images could be related by a Möbius transformation. His results suggest that there are examples of growth and evolution where a Möbius transformation cannot be ruled out.

Thmpson's examples [28] and Petukhov's investigation [47] motivated us to explore Möbius transformations [48–51]. In this paper, we investigate whether or not this is true by using image registration, rather than a point-based invariant. We register images by minimising the sum-of-squares distance between the pixel intensities [35] (equation (2.9) section 2.2). In this way we can see how close to conformal the image relationships are.

We also present the novel idea for the selection of initial guess (see Example 2, 3) that plays a key role for the optimisation. All the registration using Möbius diffeomorphisms is done over two dimensional grey scale images [35, 36].

2 Möbius registration

We consider image registration by the Möbius group (G), which is a finite-dimensional group of conformal transformations. We identify the plane $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, and likewise the image domain Ω with a domain of $\mathbb{C}$. In fact, Möbius transformations are best regarded as maps of the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \tilde{\infty}$ to itself, where $\tilde{\infty}$ denotes complex infinity. This extended complex plane is also know as the Riemann sphere [52, 53] (section 6.3.2, p. no 83), [54].

A Möbius transformation has the form

$$\mathfrak{M}(z) : \hat{\mathbb{C}} \to \hat{\mathbb{C}} = \frac{az+b}{cz+d};\ a,b,c,d \in \hat{\mathbb{C}} \,\&\, ad-bc \neq 0. \tag{1}$$

Note that if c = 0 in Equation (1) then this transformation will become a rigid transformation [35] (Ch. 2, section 2.2, p. no 25–27) in which the first term $(\frac{a}{d}z)$ represents the scaling and rotation and the second term $\frac{b}{d}$ represents the translation. That is, the rigid group is a subgroup of the Möbius group. A Möbius transformation can be obtained with the composition of the four simple transformations of scaling, rotation, translation and inversion [55]. In the proposition 1 we derive Equation (1) with the composition of these individual transformations.

Proposition 1 *A Möbius transformation is the composition of four simple transformations that are scaling, rotation, translation and inversion.*

Proof For any complex number $z, \forall a,b,c,d \in \hat{\mathbb{C}}$:

$$f_1(z) = z + \frac{d}{c} \qquad \text{(translation).} \tag{2}$$

$$f_1(z) = \frac{1}{z} \qquad \text{(inversion).} \tag{3}$$

$$f_3(z) = \frac{-ad+bc}{c^2} z \quad \text{(scaling and rotation).} \tag{4}$$

$$f_4(z) = z + \frac{a}{c} \qquad \text{(translation).}$$

Now, composition yields:

$$\begin{aligned} f_4 \circ f_3 \circ f_2 \circ f_1 &= f_4 \circ f_3 \circ f_2(z + \frac{d}{c}), && \text{From 2} \\ &= f_4 \circ f_3(\frac{1}{z+d/c}), && \text{From 3} \\ &= f_4(\frac{-ad+bc}{c(cz+d)}), && \text{From 4} \\ &= \frac{az+b}{cz+d} \\ &= \mathfrak{M}(z). \end{aligned}$$

The inverse of Möbius transformation is:

$$\mathfrak{M}^{-1}(z) = \frac{dz-b}{a-cz}.$$

□

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Which can be seen to be another Möbius transformation. Similarly, it is easy to check that the composition of two Möbius transformations is another Möbius transformation. Since the identity transformation is clearly in the set, thus the set of Möbius transformations forms a group. See the proposition 2.

Proposition 2 *If the composition of Möbius diffeomorphisms is defined in the domain $\Omega \subset \mathbb{R}^2$ then the following equation holds:*

$(\varphi_1 \circ \varphi_2).(I(\boldsymbol{x})) = \varphi_1.(\varphi_2.I(\boldsymbol{x})), \forall \varphi_1, \varphi_2 \in Mob(\Omega, \mathbb{R}^2)$
where $\circ$ denotes the composition, '.' indicates the action of φ over the image I, see Section 1.2.4 in [35], and $Mob(\Omega, \mathbb{R}^2)$ denotes the set of rigid diffeomorphisms.

Proof Suppose φ_1 and φ_2 are Möbius diffeomorphisms in Ω such that their composition $\varphi_1 \circ \varphi_2$ is also a diffeomorphism in Ω, then

$$\begin{aligned} (\varphi_1 \circ \varphi_2).I(\mathbf{x}) &= I((\varphi_1 \circ \varphi_2)^{-1}(\mathbf{x})), \\ &= I((\varphi_2^{-1} \circ \varphi_1^{-1})(\mathbf{x})), \\ &= I(\varphi_2^{-1}(\mathbf{x}) \circ \varphi_1^{-1}(\mathbf{x})), \\ &= I \circ \varphi_2^{-1}(\mathbf{x}) \circ \varphi_1^{-1}(\mathbf{x}), \\ &= (I \circ \varphi_2^{-1}(\mathbf{x})) \circ \varphi_1^{-1}(\mathbf{x}), \\ &= (\varphi_2.I(\mathbf{x})) \circ \varphi_1^{-1}(\mathbf{x}), \\ &= \varphi_1.(\varphi_2.I(\mathbf{x})). \end{aligned}$$

□

Although it appears to have 8 real parameters in the 4 complex numbers (a,b,c,d), as the transformation with parameters $(\lambda a, \lambda b, \lambda c, \lambda d)$ is the same as that with parameters (a,b,c,d), for any $\lambda \in \mathbb{C}$, as a transformation group it is 6 dimensional. A popular choice of normalization for (a,b,c,d) is to take $ad-bc=1$. In our work, since Ω is centred on the origin and the singularity of $\mathfrak{M}(z)$ is at $z=-d/c$, which we want to be outside Ω, we take $d=1$ and use the parametrization.

$$\mathfrak{M}(z) = \frac{az+b}{cz+1},$$

Where, to avoid singularities, $-1/c \notin \Omega$.
We now consider how to perform image registration using the Möbius group as the set of allowable transformations.

2.1 Image Registration with the Möbius Group

In theory there are no significant differences between using the rigid group and the Möbius group for image registration. However, in practice the strategy used for rigid registration–a combination of coarse search and gradient descent–needs to be adapted because of the higher dimensionality of the Möbius group. A grid of M points in each parameter would need M^6 function evaluations. One option would be to decrease the value of M. However, instead we develop a new method to get the initial guess.

We will first consider smooth synthetic data, in which the target is generated from the source by a Möbius transformation; then non-smooth synthetic data; then non-smooth data in which the target and the source are not known to be related by a Möbius transformation. A discrete domain 'S' [35] (see section 2.2) is considered in all the examples.

Example 1 Figure 1 shows a pair of smooth images. In this example, the target is generated from the source by implementing a six dimensional Möbius transformation with coefficients $a = \sqrt{3/2} + \mathrm{i}/2, b = 0.2 - 0.3\mathrm{i}, c = 0.1 - 0.2\mathrm{i}$.

We choose the identity as initial guess, i.e, $a_{ini} = 1, b_{ini} = 0, c_{ini} = 0$. The Matlab optimiser *lsqnonlin* ([56], see Ex. 2.4 and Section 2.3.1 in [35]) returns values corresponding numerically to $a_{opt} = (\sqrt{3/2} + \mathrm{i}/2), b_{opt} = 0.2 - 0.3\mathrm{i}, c_{opt} = 0.1 - 0.2\mathrm{i}$. The results of image registration are given in Figure 2; they show a perfect registration. Thus, the optimiser can in this case locate the global minimum starting from the identity

Example 2 In this example, we consider a pair of non-smooth images in which the target is generated with the source by using a Möbius transformation with coefficients $a = \sqrt{3/2}+\mathrm{i}/2, b = 0.2-0.3\mathrm{i}, c = 0.7-0.6\mathrm{i}$. These images are given in Figure 3. We set the initial guess to be the identity and supply these values to the optimiser. After optimisation the set of optimised values, a_{opt} =1.01-0.00i, $b_{opt} = 0.002 - 0.002\mathrm{i}$, $c_{opt} = 0.0038 + 0.0072\mathrm{i}$ are obtained. These values are very near to the identity and we suspect that we will not have a perfect registration. The results of the image registration are given in Figure 4; they are very poor, presumably because the optimiser became stuck in a local minimum.

We therefore develop an alternative method of obtaining an initial guess called landmark or labelled point matching. This method is inspired by landmark-based image registration (see [33, 57, 58]) in which corresponding sets of points $z_i \in \Omega$ in the source and z_i in the target are chosen, and the objective function

$$\sum_i \varphi(z_i) - z_i'^2$$

is minimized over $\varphi \in G$(the Möbius group). As the objective function does not refer to the images at all, the optimisation is significantly easier. On the other hand, the selection of the landmarks is a huge extra task. Automatic selection of landmarks is a major subject of research in the computer science image processing community [59, 60]. Despite extensive research it remains extremely difficult and in practice landmarks are still often selected by hand.

Here we use this approach in a slightly different manner. It is known that there is a Möbius transformation that maps any set of 3 distinct points to any other set of 3 distinct points. We can apply this by limiting ourselves to 3 landmarks. Furthermore, the Möbius transformation in this case can be determined by solving a system of 3 *linear* equations.

Suppose z_i and z_i', $i = 1, 2, 3$, represent three corresponding points on the source and the target grids respectively. The Möbius transformation mapping z_i to $z_i', i = 1, 2, 3$, satisfies

$$\begin{aligned} z_i' &= \frac{az+b}{cz+1}, i = 1, 2, 3 \\ &= az_i + b - cz_i z_i'. \end{aligned} \tag{5}$$

The solution of these three linear equations yields parameters a, b, and c that register the three chosen landmarks (but not the images). We use these parameters as the initial guess for the optimisation. In the present example, the chosen landmarks are shown in Figure 5 and the results of the optimisation are $a_{opt} = \sqrt{3/2} + \mathrm{i}/2$, $b_{opt} = 0.2-0.3\mathrm{i}$, $c_{opt} = 0.7-0.6\mathrm{i}$. The results of image registration are given in Figure 6; a perfect registration is obtained.

Example 3 We now apply the landmark method of obtaining an initial guess for optimisation to a more challenging example. In this example the source and target are non-smooth and there is not known to be a Möbius transformation between them. The source is the same fish as in the previous example, but the target is now the second of Thompson's fish. The original images of Thompson's fish are presented in [28] (p. no. 1064). For the present experiment in image registration the images were scanned and the outline of the boundary of the fish drawn with black marker. Unwanted grid lines were removed digitally. The hand drawn images were scanned and the insides filled with grey using Microsoft Paint. The images were then resized (using MATLAB's `imresize` function, which performs bicubic interpolation) to 100×100. This pair of fish images are given in Figure 7. It can be seen that we have shaded the inside of the fish in grey in order to help the optimisation.

Three landmarks are marked at corresponding locations in the images; these are shown in Figure 8.

Following the method of the previous example, the results of image registration are given in Figure 9. The fins and tail are well matched, but the body shape is not. The final step of optimisation has only made a small change to the landmark registration; this can be seen in the small mismatch of the eyes.

We repeat this example using a different set of landmarks, shown in Figure 10. The results are shown in Figure 11. Once again the optimiser has only made a small correction to the landmark matching, and the body shapes are not well matched.

We now swap the source and the target, see Figure 12. The landmarks are shown in Figure 13 and the results of image registration in Figure 14.

Again, the fins and tail are well registered, but not the body shape. However, this time the optimiser has made a large change to the landmark matching (see the position of the mouths of the fish). It is striking that the conformal map obtained looks very similar to the grid obtained by Thompson and shown in chapter XVII [28] (p. no. 1064).

As a check, we continue with the swapped source and target and try another set of landmarks, shown in Figure 15. The results are given in Figure 16. This time the registration is virtually identical to Figure 14, indicating that in this case the results are not that sensitive to the choice of initial guess and that a global minimum may have been obtained.

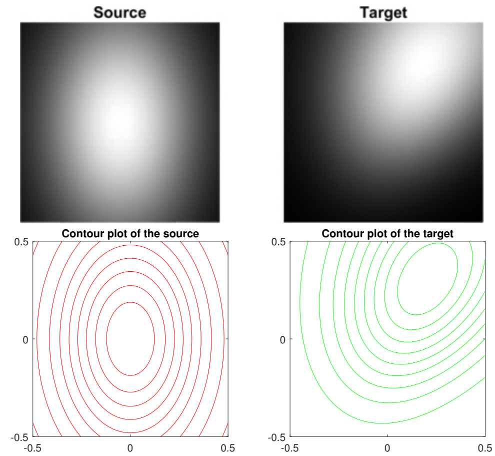

Fig. 1: Data for Example 1. The source is a Gaussian and the target is a Möbius transformation of the source.

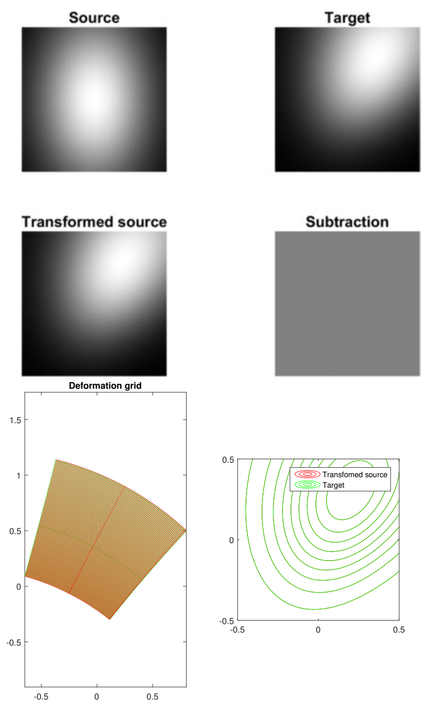

Fig. 2: Results of Möbius registration for Example 1, with *lsqnonlin* and initial guess the identity. A perfect registration[1] is obtained.

[1] A perfect registration means that there are at most small mismatched portion between the transformed source and the target, and the grid looks smooth and conformal.

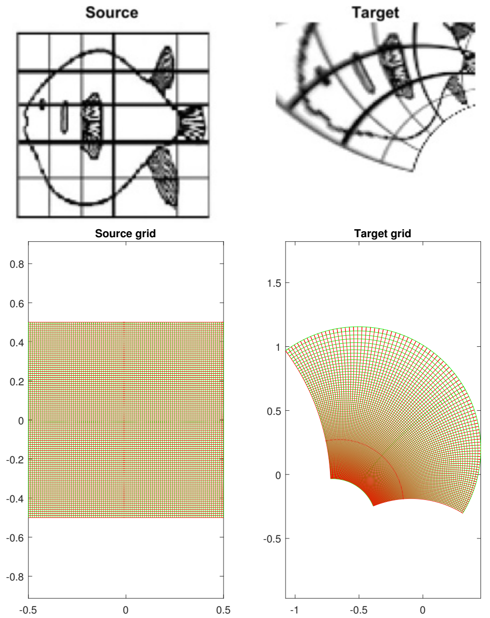

Fig. 3: Data for Example 2. Möbius registration with synthetic non-smooth images. The mapping that generates the source from the target is also shown.

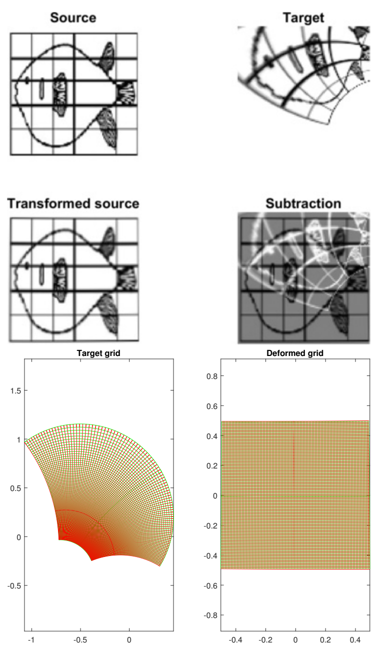

Fig. 4: Results of first attempt at Möbius registration for Example 2. The desired mapping and the mapping computed by *lsqnonlin* are also shown; the results are poor.

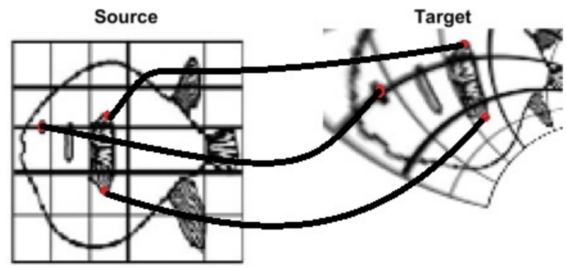

Fig. 5: Selection of corresponding landmarks on both the fish images for Example 2.

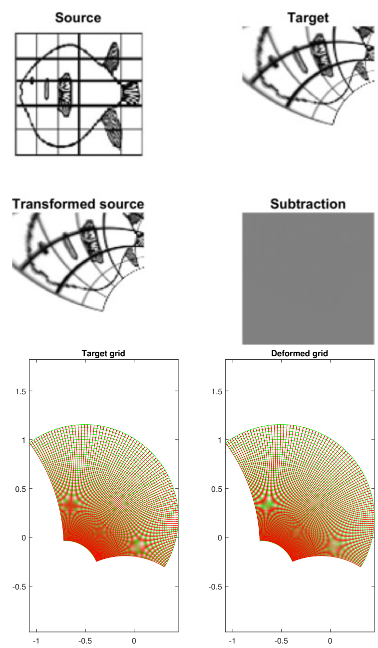

Fig. 6: Results of the second attempt at Möbius registration for Example 2, using an initial guess calculated from landmark matching. A perfect registration is obtained.

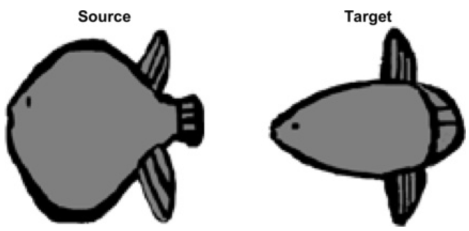

Fig. 7: Data for Example 3 of Möbius registration, a cartoon version of Thompson's fish.

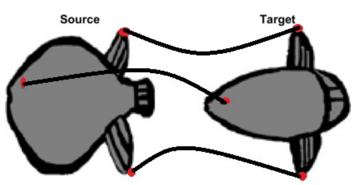

Fig. 8: Selection of three landmarks for Example 3: first set of landmarks.

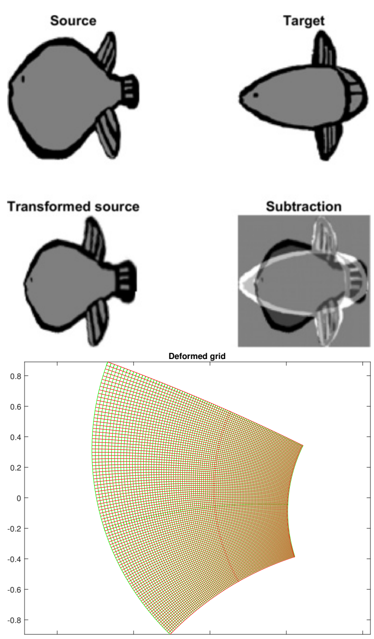

Fig. 9: Results of Möbius registration for Example 3: first set of landmarks.

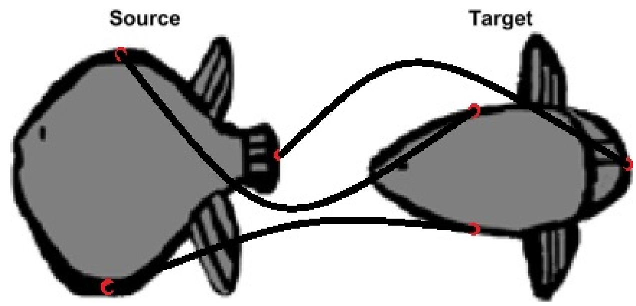

Fig. 10: Selection of a second set of landmarks for Example 3 on both the images.

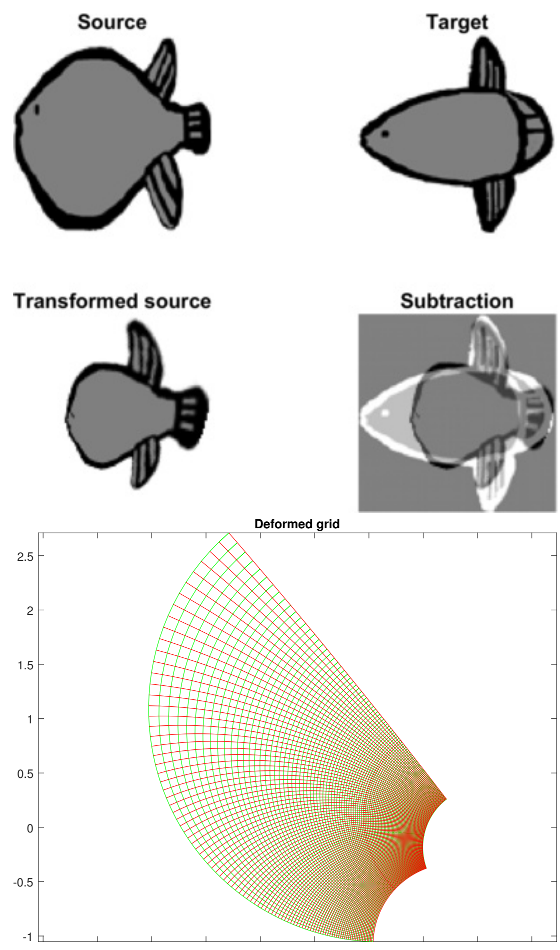

Fig. 11: Results of Möbius registration for Example 3: second set of landmarks.

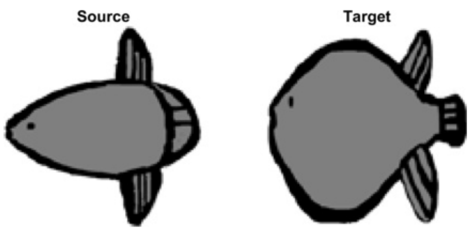

Fig. 12: Second dataset for Example 3: swapped source and target.

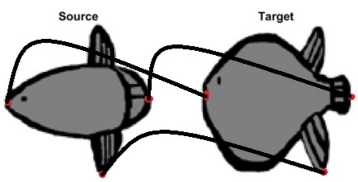

Fig. 13: Selection of a first set of landmarks on both the images for Example 3: second dataset.

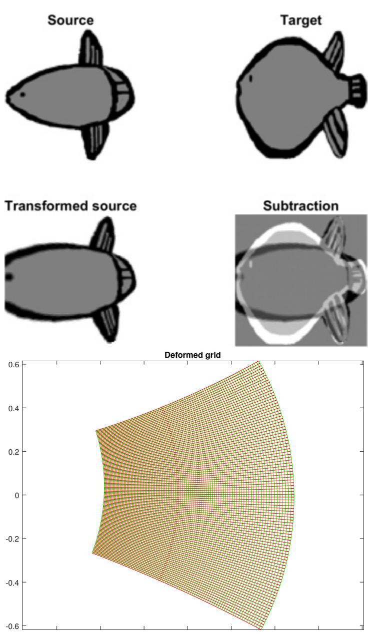

Fig. 14: Results of Möbius registration for Example 3: second dataset, first set of landmarks.

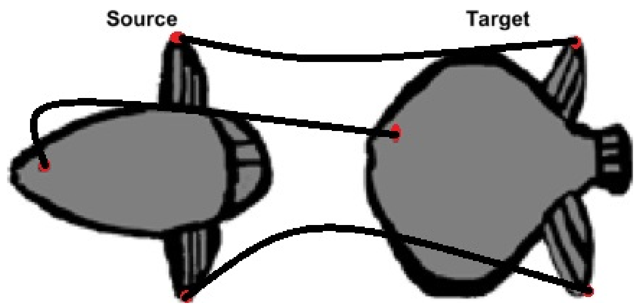

Fig. 15: Selection of corresponding landmarks for Example 3: second set for second dataset.

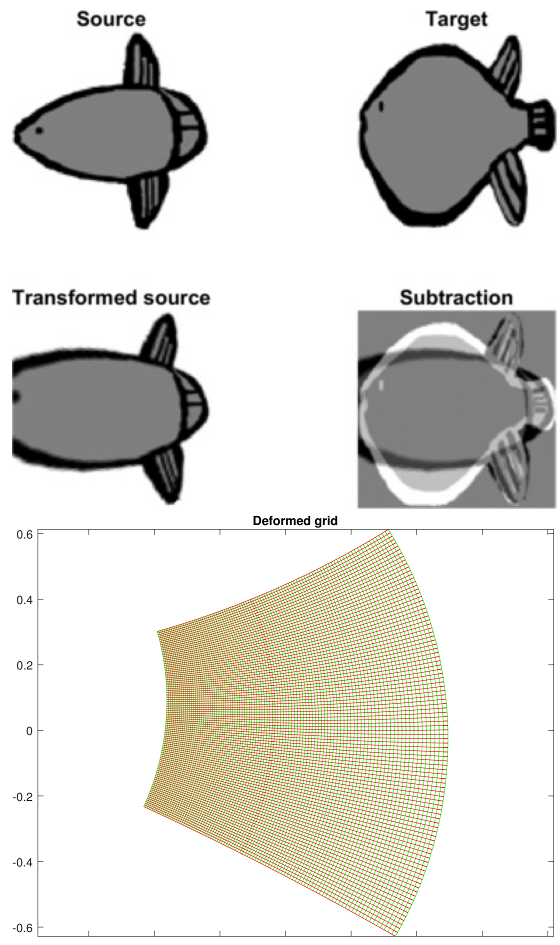

Fig. 16: Results of Möbius registration for Example 3: second dataset, second set of landmarks.

3 Conclusion

To summarize the results of this paper, we have found that we are able to register images in finite-dimensional group, i.e., the Möbius group. We have also seen in the experiments that the choice of initial condition is important to the quality of the registration obtained, and whether or not the algorithm reaches the neighbourhood of the global minimum. Although we have no guarantee, it does appear from the numerical examples that we have reached either the global minimum or something very close to it. The Thompson fish examples are not definitive but it does appear that we have registered the images in the Möbius group and that the poor registrations that we have obtained are evidence that the images are not, in fact, related by a Möbius transformation. Further investigation for the registration of Thompson's fish need to be dealt in the much bigger group that contains the Möbius group.

4 Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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