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Weights Allocation Based on Optimal Uncertainty Evaluation

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Abstract: In the traditional measurement theory, precision is defined as the dispersion of measured value, and is used as the basis of weights calculation in the adjustment of measurement data with different qualities, which leads to the trouble that trueness is completely ignored in the weight allocation. In this paper, following the pure concepts of probability theory, the measured value (observed value) is regarded as a constant, the error as a random variable, and the variance is the dispersion of all possible values of an unknown error. Thus, a rigorous formula for weights calculation and variance propagation is derived, which solves the theoretical trouble of determining the weight values in the adjustment of multi-channel observation data with different qualities. The results show that the optimal weights are not only determined by the covariance array of observation errors, but also related to the model of adjustment.

Keywords: measurement error, random variable, uncertainty, survey adjustment, weight.

1. Introduction

The traditional measurement error theory is interpreted following the logic of error classification, in which the measured value (observed value) and random error are regarded as random variables with variance, while the systematic error is regarded as constant without variance^[1,2,3,4,5,6]. In this way, when multiple observations with different qualities are obtained from a single measurand, according to this conceptual logic, the weight of each observed value is calculated by its precision, and the final measured value is equal to the weighted average value of all observed values.

However, the biggest trouble faced by this method is that the systematic error of each observation is different from each other. Some observations with large systematic error but small random error get larger weights, which can make the final measured value have better precision, but its trueness will be worse. Although we can explain the existing theory based on the assumption that the systematic error does not exist or can be ignored, this assumption is usually not tenable in actual measurement.

For example, in the field of geodesy, the multiplicative constant error of a geodimeter is considered as systematic error, its measured value given by metrology field is usually used to correct the observed value, and people think that the residual systematic error after correction can be ignored. However, in the field of instrument manufacturing, the multiplicative constant error of geodimeter is originally the residual error after correction, which comes from the residual error of quartz crystal frequency after temperature correction, and still varies with temperature. In the field of instrument, the limited range of this error is also given. Therefore, it is actually meaningless to use a test sample given by metrology field to correct the observed value.

Therefore, this concept of systematic error often makes the measurement practice in trouble.

For example, in 2020, the National Bureau of surveying, mapping and geographic information of China announced that the elevation of Mount Everest is 8848.86m, which is actually an adjustment value obtained from the trigonometric leveling data of geodimeter and GNSS satellite survey data according to a certain weight proportion. However, at present, we have not seen the

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public information about weight calculation, nor the official data about error evaluation.

In recent years, literature [7,8,9,10,11,12,13] proposed an interpretation method of measurement theory based new concepts, which clearly pointed out that the traditional measurement theory's understanding of constant and random variable concepts is inconsistent with probability theory. Because the constant in probability theory refers to a numerical value, and the random variable refers to a quantity whose numerical value is unknown, this theory regards the observed value (measured value) as a constant, the error and the true value as random variables, and the variance of error is the dispersion of all possible values of the error. In this conceptual system, errors cannot be classified as systematic error and random error.

Although these documents have been published formally, they do not deal with the problem of weight calculation in multi-channel data adjustment, so this paper will complete the mathematical deduction of this calculation principle.

2. Basic concepts

2.1. Constant and random variable

In probability theory, a constant is a numerical value, but a random variable is an unknown quantity without numerical value. To study the random variable, people can only use all its possible values (sample space) to describe its probability interval.

2.2. Mathematical expectation and variance

For a random variable X with sample space $\{x_i\}$, there is $X \in \{x_i\}$, p_i is the probability of occurrence of event $\{X = x_i\}$, that is, $p_i = P\{X = x_i\}$ (the probability density function $p(x)$ is used to express continuous random variable), and its mathematical expectation and variance are respectively defined as:

$$\begin{cases} E(X) = \sum_{i=1}^n p_i x_i & \text{or} & E(X) = \int_{-\infty}^{+\infty} x p(x) dx \\ \sigma^2(X) = E[X - E(X)]^2 \end{cases} \quad (2-1)$$

Its meaning is that all possible values of random variable X constitute a density distribution interval (Frequency school) with $E(X)$ as the center and $\sigma^2(X)$ as the width evaluation, or the random variable X exists within a probability interval (Bayesian school) described by $E(X)$ and $\sigma^2(X)$. In other words, the mathematical expectation $E(X)$ and the variance $\sigma^2(X)$ are the evaluation of the probability interval of X .

Because all possible values of a constant C are itself, there are:

$$\begin{cases} E(C) = C \\ \sigma^2(C) = E[C - E(C)]^2 = 0 \end{cases} \quad (2-2)$$

For example, when there is $x = 8848.86$, there must be $E(x) = E(8848.86) = 8848.86$ and $\sigma^2(x) = \sigma^2(8848.86) = 0$.

2.3. The origin of the conceptual trouble in traditional measurement theory

In Section 4.2 of literature [2] and Section 5.17 of literature [3], a method process is described as follows:

When obtaining n observed values q_k , the final measured value $\bar{q}$ is

$$\bar{q} = \frac{1}{n} \sum_{k=1}^n q_k \quad (2-3)$$

The variance of each observed value q_k is

$$\sigma^2(q_k) \approx \frac{1}{n-1} \sum_{j=1}^n (q_j - \bar{q})^2 \quad (2-4)$$

The variance of the final measured value $\bar{q}$ is

$$\sigma^2(\bar{q}) = \frac{\sigma^2(q_k)}{n} \quad (2-5)$$

Obviously, in formula (2-3), $\bar{q}$ and each q_k respectively represent a numerical value; but in formulas (2-4) and (2-5), these numerical values $\bar{q}$ and q_k are given a variance that is not zero, which obviously violates the formula (2-2).

For example, the following is the result expression of the thermometer calibration case on page 90 of reference [2]:

H.3.3 Calculation of results

The data to be fitted are given in the second and third columns of Table H.6. Taking $t_0 = 20^\circ\text{C}$ as the reference temperature, application of Equations (H.13a) to (H.13g) yields

$$\begin{aligned} y_1 &= -0,171\,2^\circ\text{C} & s(y_1) &= 0,002\,9^\circ\text{C} \\ y_2 &= 0,002\,18 & s(y_2) &= 0,000\,67 \\ r(y_1, y_2) &= -0,930 & s &= 0,003\,5^\circ\text{C} \end{aligned}$$

The fact that the slope y_2 is more than three times larger than its standard uncertainty provides some indication that a calibration curve and not a fixed average correction is required.

However, obtained by equivalent substitutions, the actual mathematical meanings of these expressions are as follows.

$$\left. \begin{aligned} y_1 &= -0.1712 \\ s(y_1) &= 0.0029 \end{aligned} \right\} \Rightarrow s(-0.1712) = 0.0029$$

$$\left. \begin{aligned} y_2 &= 0.00218 \\ s(y_2) &= 0.00067 \end{aligned} \right\} \Rightarrow s(0.00218) = 0.00067$$

$$\left. \begin{aligned} y_1 &= -0.1712 \\ y_2 &= 0.00218 \\ r(y_1, y_2) &= -0.930 \end{aligned} \right\} \Rightarrow r(-0.1712, 0.00218) = -0.930$$

Obviously, these expressions violate the formula (2-2).

Because the numerical values (observed value and measured value) are regarded as random variables, the traditional measurement theory uses Fig 1 to interpret its basic measurement concepts, which further leads to the philosophy of error classification and misleads the conceptual difference between trueness and precision. Formulas (2-4) and (2-5) are the mathematical expressions of the so-called precision concept.

Fig 1. Schematic diagram in traditional measurement theory

In a word, the concept of error classification in traditional measurement theory is obtained by misinterpreting the concept of random variable, so the classification concept of precision and trueness is actually not correct^{[7][8][9][10][11][12][13]}.

2.4. The probability expressions of measured value, error and true value

In references^[7,8,9,10,11,12,13], Fig 2 is used to interpret its basic measurement concepts, which is different from the traditional measurement theory. Among them, the final measured value is an observed value x_0 which is a sample within all possible observed values $\{x_i\}$, and $\{x_i\}$ is the sample space of random variable X .

Fig 2. Conceptual sketch of new concept theory

Because x_0 is a numerical value, according to the formula (2-2), there must be:

$$\begin{cases} E(x_0) = x_0 \\ \sigma^2(x_0) = 0 \end{cases} \quad (2-6)$$

For the unknown deviation $\Delta_A = x_0 - E(X)$, because it has the same sample space $\{x_i - E(X)\}$ as the random variable $\Delta X = X - E(X)$, we can use $\Delta X = X - E(X)$ to express the deviation Δ_A , that is, $\Delta_A = X - E(X)$. Therefore, there are

$$\begin{cases} E(\Delta_A) = E[X - E(X)] = 0 \\ \sigma^2(\Delta_A) = E(\Delta_A^2) = E[X - E(X)]^2 \end{cases} \quad (2-7)$$

Its concept meaning is that variance $\sigma^2(\Delta_A)$ is the dispersion of all possible values $\{x_i - E(X)\}$ of deviation Δ_A (Frequency School), and is also the evaluation value of the probability interval of deviation Δ_A (Bayesian School).

For the unknown deviation $\Delta_B = E(X) - X_T$, because it comes from the previous measurement, its formation principle is similar to that of deviation Δ_A , and it also has all its possible values, so there are

$$\begin{cases} E(\Delta_B) = 0 \\ \sigma^2(\Delta_B) = E(\Delta_B^2) \end{cases} \quad (2-8)$$

For example, the field of geodesy generally considers that the multiplicative constant error of geodimeter is a systematic error without variance, but from the perspective of instrument manufacturing, this error is the random error, and the instrument manufacturer also uses all its possible values to count its probability interval (usually expressed by the maximum permissible error MPE). That is, the multiplicative constant error of geodimeter also has its variance like the output deviation in the field of geodesy.

That is to say, both Δ_A and Δ_B are unknown deviations, have their own variances, and cannot be classified as random error and systematic error respectively. That is, the concepts of precision and trueness were abolished.

$$\text{In this way, there are} \quad \Delta = \Delta_A + \Delta_B \quad (2-9)$$

$$\begin{aligned} E(\Delta) &= E(\Delta_A) + E(\Delta_B) \\ &= 0 \end{aligned} \quad (2-10)$$

$$\begin{aligned} \sigma^2(\Delta) &= E[\Delta - E(\Delta)]^2 \\ &= E(\Delta^2) \\ &= E(\Delta_A + \Delta_B)^2 \\ &= \sigma^2(\Delta_A) + \sigma^2(\Delta_B) \end{aligned} \quad (2-11)$$

The variance $\sigma^2(\Delta)$ is the evaluation value of the probability range of the total error Δ (Bayesian School), is also the dispersion of all possible values of the total error Δ (Frequency School), and is called as the uncertainty of the error Δ . Please note that this uncertainty is not the dispersion of the measured value interpreted by the traditional measurement theory.

Therefore, for the true value X_T , there are:

$$X_T = x_0 - \Delta \quad (2-12)$$

$$\begin{aligned} E(X_T) &= E(x_0) - E(\Delta) \\ &= x_0 \end{aligned} \quad (2-13)$$

$$\begin{aligned} \sigma^2(X_T) &= E[X_T - E(X_T)]^2 \\ &= E(x_0 - \Delta - x_0)^2 \\ &= \sigma^2(\Delta) \end{aligned} \quad (2-14)$$

In conclusion, the above concepts are summarized in Table 2 ^{[10][11][12]}.

Table 2. The probability expressions of measured value, error and true value

Measured value x_0	Error Δ	True value X_T
$\begin{cases} E(x_0) = x_0 \\ \sigma^2(x_0) = 0 \end{cases}$	$\begin{cases} E(\Delta) = 0 \\ \sigma^2(\Delta) = E(\Delta^2) \end{cases}$	$\begin{cases} E(X_T) = x_0 \\ \sigma^2(X_T) = \sigma^2(\Delta) \end{cases}$

The above conclusion is derived by using an observed value x_0 as the final measured value. When obtaining multiple observations x_i , the submission method of the best measured value is shown in Section 3.

2.5. Regularity and randomness of error

Any error has variance to evaluate its probability range, and the regular error is of course the same, because the variance of error is the dispersion of all its possible values. Fig 3 is a conceptual diagram describing the regularity and randomness of the periodic error of the geodimeter. When

observing the corresponding relationship between the error value and the range, we can see the regularity. When observing the density distribution of the error values, we can see the randomness.

Fig 3. Regularity and randomness of sine error

2.6. The contribution form of error to repeated measurement

Because errors vary with various measurement conditions, the variation of measurement conditions in repeated measurement determines the contribution form of errors to repeated observations (deviation, dispersion, outlier or both). It is precisely because errors can lead to the dispersion of repeated observations, so the variance of any error can be obtained through experimental statistics, and these data exist in the instrument specifications or instructions.

2.7. Contribution form of error to final measured value

The contribution form of error to the final measured value is only to make it deviate from the true value. This is because, after the measurement adjustment, the final measured value is unique, and the difference between it and the true value is a unique unknown deviation.

2.8. The law of covariance propagation

Because every error has its variance, the concept of variance can be extended to any two errors, which is called covariance. It comes from the common component of the two errors.

$$\sigma(\Delta x_j \Delta x_k) = E(\Delta x_j \Delta x_k) \quad (2-15)$$

Thus, for an error array $\Delta \mathbf{X} = (\Delta x_1, \Delta x_2, \dots, \Delta x_n)^T$, the definition of variance is as follows:

$$\mathbf{D}(\Delta \mathbf{X}) = E(\Delta \mathbf{X} \Delta \mathbf{X}^T) = \begin{bmatrix} \sigma_{11}^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22}^2 & \cdots & \sigma_{2n} \\ \vdots & \vdots & & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn}^2 \end{bmatrix} \quad (2-16)$$

Therefore, according to the definition of variance (2-16), when there is an error propagation equation $\Delta \mathbf{Z} = \mathbf{K} \cdot \Delta \mathbf{X}$, we can get:

$$\mathbf{D}(\Delta \mathbf{Z}) = \mathbf{K} \cdot \mathbf{D}(\Delta \mathbf{X}) \cdot \mathbf{K}^T \quad (2-17)$$

This is the law of covariance propagation, which is the propagation relation of probability interval between errors, or the propagation relation of the dispersion of all possible values of errors.

After the data processing, the difference between the measured value and the true value is an unknown deviation. Therefore, the task of measurement is to adjust the measured value to minimize the variance of the deviation and submit its value.

3. The solution of weight values

The smaller the error is, the better the quality of the measured value is. However, we don't know the error value, and can only use the variance of the error to describe its probability interval, so we can only think that the quality of the measured value is the best when the variance of the error is the smallest. Therefore, the mathematical problem we are faced with is to minimize the variance of the error of the final measured value by assigning the weights of each observed value.

3.1. Direct measurement model for single measurand

As shown in Fig 2, when n different observed values x_i are obtained by repeated observation of a measurand, assuming that the true value is Y , the error equation is as follows:

$$\begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} [Y] \quad (3-1)$$

The error sample set $\{V_1, V_2, \dots, V_n, \dots\}$ constitutes a sample space of error V . Therefore, according to the formula in Table 2, there are:

$$E(V) = 0 \quad (3-2)$$

$$\begin{aligned} \sigma^2(V) &= E(V^2) \\ &\approx \frac{\sum V_i^2}{n} \\ &= \frac{1}{n} (x_1^2 + x_2^2 + \dots + x_n^2 - 2x_1Y - 2x_2Y - \dots - 2x_nY + nY^2) \end{aligned} \quad (3-3)$$

The best measured value y is the value when $\sigma^2(V)$ or $\sum V_i^2$ takes the minimum value, and this is a minimum problem of quadratic function, so making $\left. \frac{d(\sigma^2(V))}{dY} \right|_{Y=y} = 0$, the best measured value is obtained as follows:

$$y = \frac{1}{n} \sum_{i=1}^n x_i \quad (3-3)$$

This is the least square method. As you can see, in the above deduction process, each x_i and y do not represent a random variable, which is consistent with the conceptual principle in Section 2.

Of course, formula (3-3) is actually an equal weight processing method, that is, the weight of each observation x_i is $\frac{1}{n}$. However, if n observation errors Δx_i have different variances and covariances, that is, the quality of each observation x_i is different from each other, then the equal weight processing is not scientific. At this time, different weights must be given to each observed value to ensure that the quality of the final measured value is the best. Therefore, in order to study each weight p_i when $\sum P_i V_i^2$ takes the minimum value, the equation (3-1) is modified as:

$$\begin{bmatrix} P_1^{1/2} V_1 \\ P_2^{1/2} V_2 \\ \vdots \\ P_n^{1/2} V_n \end{bmatrix} = \begin{bmatrix} P_1^{1/2} x_1 \\ P_2^{1/2} x_2 \\ \vdots \\ P_n^{1/2} x_n \end{bmatrix} - \begin{bmatrix} P_1^{1/2} & 0 & \dots & 0 \\ 0 & P_2^{1/2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & P_n^{1/2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} [Y] \quad (3-4)$$

Similarly, by using the least square method, there is

$$y = (P_1 x_1 + P_2 x_2 + \dots + P_n x_n) / \sum_{i=1}^n P_i \quad (3-5)$$

Please note that in the above derivation process, it is impossible to obtain the best weights p_i by making $\left. \frac{\partial(\sigma^2(V))}{\partial P_i} \right|_{P_i=p_i} = 0$, because the best weight p_i actually depends on the evaluation

$\sigma^2(\Delta x_i)$ of error $\Delta x_i = x_i - X_T$ rather than the evaluation $\sigma^2(V)$ of error $v_i = x_i - y$.

For the convenience of calculation, making $\sum_{i=1}^n P_i = 1$, the formula (3-5) becomes:

$$y = P_1 x_1 + P_2 x_2 + \cdots + P_n x_n \quad (3-6)$$

Now, using the quality of each x_i , we begin to solve the general calculation method of each weight p_i .

The error propagation equation is obtained by taking the total differential of equation (3-6), that is:

$$\Delta y = P_1 \Delta x_1 + P_2 \Delta x_2 + \cdots + P_n \Delta x_n \quad (3-7)$$

Applying the formula (2-17) to equation (3-7), there is:

$$\begin{aligned} \sigma^2(\Delta y) &= \begin{bmatrix} P_1 & P_2 & \cdots & P_n \end{bmatrix} \mathbf{D}(\Delta \mathbf{X}) \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{bmatrix} \\ &= P_1^2 \sigma_1^2 + P_2^2 \sigma_2^2 + \cdots + P_n^2 \sigma_n^2 + 2P_1 P_2 \sigma_{12} + 2P_1 P_3 \sigma_{13} + \cdots \end{aligned} \quad (3-8)$$

Please note that the variance σ_i^2 is the dispersion $\sigma^2(\Delta x_i)$ of all possible values of the error $\Delta x_i = x_i - X_T$, which expresses the evaluation of the probability interval of the error Δx_i . This concept of variance is consistent with probability theory, but it is different from the concept of dispersion $\sigma^2(x_i)$ of observation x_i in traditional measurement theory.

When the value of the function $\sigma^2(\Delta y)$ is the minimum, the quality of the measured value y is the best, which is a problem of finding the minimum value of the function $\sigma^2(\Delta y)$ under the restriction condition $\sum_{i=1}^n P_i = 1$. Therefore, according to the Lagrange multiplier method, the following functions are formed:

$$\begin{aligned} La &= \sigma^2(\Delta y) + 2\lambda(P_1 + P_2 + \cdots + P_n - 1) \\ &= P_1^2 \sigma_1^2 + P_2^2 \sigma_2^2 + \cdots + P_n^2 \sigma_n^2 + 2P_1 P_2 \sigma_{12} + 2P_1 P_3 \sigma_{13} + \cdots \\ &\quad + 2\lambda(P_1 + P_2 + \cdots + P_n - 1) \end{aligned} \quad (3-9)$$

Taking the partial derivative of each P_i and making it equal to 0, the Lagrange equation is obtained as follows:

$$\left. \begin{aligned} (P_1 \sigma_1^2 + P_2 \sigma_{12} + P_3 \sigma_{13} + \cdots + P_n \sigma_{1n} + \lambda) \Big|_{P_i=p_i, \lambda=\lambda_0} &= 0 \\ (P_1 \sigma_{21} + P_2 \sigma_2^2 + P_3 \sigma_{23} + \cdots + P_n \sigma_{2n} + \lambda) \Big|_{P_i=p_i, \lambda=\lambda_0} &= 0 \\ &\vdots \\ (P_1 \sigma_{n1} + P_2 \sigma_{n2} + P_3 \sigma_{n3} + \cdots + P_n \sigma_n^2 + \lambda) \Big|_{P_i=p_i, \lambda=\lambda_0} &= 0 \\ (P_1 + P_2 + \cdots + P_n - 1) \Big|_{P_i=p_i, \lambda=\lambda_0} &= 0 \end{aligned} \right\} \quad (3-10)$$

That is:

$$\begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} & 1 \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2n} & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_n^2 & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \\ \lambda_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad (3-11)$$

Because the covariance array of error array $\Delta \mathbf{X} = [\Delta x_1, \Delta x_2, \dots, \Delta x_n]^T$ is:

$$\mathbf{D}(\Delta \mathbf{X}) = E \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \vdots \\ \Delta x_n \end{bmatrix} \begin{bmatrix} \Delta x_1 & \Delta x_2 & \cdots & \Delta x_n \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_n^2 \end{bmatrix} \quad (3-12)$$

So the equation (3-11) can be written as follows:

$$\begin{bmatrix} & & & & 1 \\ & & & & 1 \\ & & & & \vdots \\ & & & & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \\ \lambda_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad (3-13)$$

Therefore

$$\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \\ \lambda_0 \end{bmatrix} = \begin{bmatrix} & & & & 1 \\ & & & & 1 \\ & & & & \vdots \\ & & & & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad (3-14)$$

In this way, according to formula (3-6), the optimal measured value is

$$\begin{aligned} y_0 &= y|_{p_i=p_i} \\ &= p_1 x_1 + p_2 x_2 + \cdots + p_n x_n \end{aligned} \quad (3-15)$$

And according to formula (3-13), there is

$$\mathbf{D}(\Delta \mathbf{X}) \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \lambda_0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (3-16)$$

Therefore

$$\begin{bmatrix} p_1 & p_2 & \cdots & p_n \end{bmatrix} \mathbf{D}(\Delta \mathbf{X}) \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} = -\lambda_0 \quad (3-17)$$

According to equation (3-8), we can get:

$$\begin{aligned}\sigma^2(\Delta y_0) &= \sigma^2(\Delta y) \Big|_{p_i=p_i} \\ &= \begin{bmatrix} p_1 & p_2 & \cdots & p_n \end{bmatrix} \mathbf{D}(\Delta \mathbf{X}) \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} = -\lambda_0\end{aligned}\quad (3-18)$$

Combining formulas (3-16) and (3-18), we can get

$$\mathbf{D}(\Delta \mathbf{X}) \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} = \begin{bmatrix} \sigma^2(\Delta y_0) \\ \sigma^2(\Delta y_0) \\ \vdots \\ \sigma^2(\Delta y_0) \end{bmatrix}\quad (3-19)$$

This is the rigorous mathematical expression that the weight is inversely proportional to the variance.

In formula (3-11), if each $\sigma_{ij} = 0$, we can get the following results:

$$p_1 : p_2 : \cdots : p_n = \frac{1}{\sigma_1^2} : \frac{1}{\sigma_2^2} : \cdots : \frac{1}{\sigma_n^2}\quad (3-20)$$

This is consistent with the fact that we used to use a single $\frac{1}{\sigma_i^2}$ to determine the weight ratio.

Obviously, the formula (3-20) does not hold when the hypothesis $\sigma_{ij} = 0$ does not exist.

Next, let's see the actual cases.

Example 1, as shown in Fig 2, when n observations x_i are obtained by using the same sensor to directly repeat the measurement, please calculate the final measured value according to formula (3-6).

Because $\Delta x_i = \Delta_B + \Delta_{Ai}$, so according to the formula (2-11), there is:

$$\sigma_i^2 = \sigma_{\Delta_B}^2 + \sigma_{\Delta_A}^2$$

For any two observation errors Δx_j and Δx_k , according to the formula (2-15), there is:

$$\begin{aligned}\sigma_{jk} &= E(\Delta x_j \Delta x_k) \\ &= E[(\Delta_B + \Delta_{Aj})(\Delta_B + \Delta_{Ak})] \\ &= E(\Delta_B^2) = \sigma_{\Delta_B}^2\end{aligned}$$

Therefore, according to the formula (3-11), there is:

$$\begin{bmatrix} \sigma_{\Delta_B}^2 + \sigma_{\Delta_A}^2 & \sigma_{\Delta_B}^2 & \cdots & \sigma_{\Delta_B}^2 & 1 \\ \sigma_{\Delta_B}^2 & \sigma_{\Delta_B}^2 + \sigma_{\Delta_A}^2 & \cdots & \sigma_{\Delta_B}^2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \sigma_{\Delta_B}^2 & \sigma_{\Delta_B}^2 & \cdots & \sigma_{\Delta_B}^2 + \sigma_{\Delta_A}^2 & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \\ \lambda_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

By solving the equation, we get the following results:

$$p_1 = p_2 = \cdots = p_n = \frac{1}{n} \quad \lambda_0 = -\sigma_{\Delta_B}^2 - \frac{\sigma_{\Delta_A}^2}{n}$$

Finally, according to (3-15) and (3-18), there are:

$$y_0 = \frac{x_1 + x_2 + \cdots + x_n}{n}, \quad \sigma^2(\Delta y_0) = \sigma_{\Delta_B}^2 + \frac{\sigma_{\Delta_A}^2}{n}.$$

It can be seen that in this case, the root cause of $p_1 = p_2 = \cdots = p_n = \frac{1}{n}$ is that each error Δx_i has the same variance $\sigma_i^2 = \sigma_{\Delta_B}^2 + \sigma_{\Delta_A}^2$ and any two errors Δx_i and Δx_j have the same covariance $\sigma_{jk} = \sigma_{\Delta_B}^2$. Moreover, it can be seen that the traditional measurement theory classifies Δ_B as systematic error, which is actually to negate the correlation between Δx_i and Δx_j .

Table 3. Observed values

i	1	2	3	4
a_i	5	10	15	20
l_i (mm)	2.87	5.72	8.60	11.37

Example 2, as shown in Fig 4, in order to obtain the accurate diameter value of the copper wire, the observed values of 5 times, 10 times, 15 times and 20 times of the diameter are measured with a caliper, and 4 observed values are shown in Table 3. Please give the best measured value of diameter and the evaluation of uncertainty.

The four measured values are as follows:

$$\begin{aligned} x_1 &= \frac{l_1}{a_1} = 0.574 \\ x_2 &= \frac{l_2}{a_2} = 0.572 \\ x_3 &= \frac{l_3}{a_3} = 0.5733 \\ x_4 &= \frac{l_4}{a_4} = 0.5685 \end{aligned}$$

The error Δl_i of the observed value l_i consists of three components: fixed error K , proportional error R and non-uniform dividing error c_i . That is:

$$\Delta l_i = K + l_i R + c_i$$

According to the definition (2-16) of variance, there is

Fig 4. Measuring the diameter of copper wire

$$\mathbf{D}(\Delta \mathbf{L}) = E \begin{bmatrix} \Delta l_1 \\ \Delta l_2 \\ \Delta l_3 \\ \Delta l_4 \end{bmatrix} \begin{bmatrix} \Delta l_1 & \Delta l_2 & \Delta l_3 & \Delta l_4 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_K^2 + l_1^2 \sigma_R^2 + \sigma_c^2 & \sigma_K^2 + l_2 l_1 \sigma_R^2 & \sigma_K^2 + l_3 l_1 \sigma_R^2 & \sigma_K^2 + l_4 l_1 \sigma_R^2 \\ \sigma_K^2 + l_1 l_2 \sigma_R^2 & \sigma_K^2 + l_2^2 \sigma_R^2 + \sigma_c^2 & \sigma_K^2 + l_3 l_2 \sigma_R^2 & \sigma_K^2 + l_4 l_2 \sigma_R^2 \\ \sigma_K^2 + l_1 l_3 \sigma_R^2 & \sigma_K^2 + l_2 l_3 \sigma_R^2 & \sigma_K^2 + l_3^2 \sigma_R^2 + \sigma_c^2 & \sigma_K^2 + l_4 l_3 \sigma_R^2 \\ \sigma_K^2 + l_1 l_4 \sigma_R^2 & \sigma_K^2 + l_2 l_4 \sigma_R^2 & \sigma_K^2 + l_3 l_4 \sigma_R^2 & \sigma_K^2 + l_4^2 \sigma_R^2 + \sigma_c^2 \end{bmatrix}$$

Because $x_i = l_i/a_i$ and $\Delta x_i = \Delta l_i/a_i$, there is

$$\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \\ \Delta x_4 \end{bmatrix} = \begin{bmatrix} a_1^{-1} & 0 & 0 & 0 \\ 0 & a_2^{-1} & 0 & 0 \\ 0 & 0 & a_3^{-1} & 0 \\ 0 & 0 & 0 & a_4^{-1} \end{bmatrix} \begin{bmatrix} \Delta l_1 \\ \Delta l_2 \\ \Delta l_3 \\ \Delta l_4 \end{bmatrix}$$

That is

$$\Delta \mathbf{X} = \mathbf{A} \Delta \mathbf{L}$$

According to the formula (2-17), there is:

$$\mathbf{D}(\Delta \mathbf{X}) = \mathbf{A} \mathbf{D}(\Delta \mathbf{L}) \mathbf{A}^\top$$

$$= \begin{bmatrix} a_1^{-2}(\sigma_K^2 + l_1^2 \sigma_R^2 + \sigma_c^2) & a_1^{-1} a_2^{-1}(\sigma_K^2 + l_2 l_1 \sigma_R^2) & a_1^{-1} a_3^{-1}(\sigma_K^2 + l_3 l_1 \sigma_R^2) & a_1^{-1} a_4^{-1}(\sigma_K^2 + l_4 l_1 \sigma_R^2) \\ a_2^{-1} a_1^{-1}(\sigma_K^2 + l_1 l_2 \sigma_R^2) & a_2^{-2}(\sigma_K^2 + l_2^2 \sigma_R^2 + \sigma_c^2) & a_2^{-1} a_3^{-1}(\sigma_K^2 + l_3 l_2 \sigma_R^2) & a_2^{-1} a_4^{-1}(\sigma_K^2 + l_4 l_2 \sigma_R^2) \\ a_3^{-1} a_1^{-1}(\sigma_K^2 + l_1 l_3 \sigma_R^2) & a_3^{-1} a_2^{-1}(\sigma_K^2 + l_2 l_3 \sigma_R^2) & a_3^{-2}(\sigma_K^2 + l_3^2 \sigma_R^2 + \sigma_c^2) & a_3^{-1} a_4^{-1}(\sigma_K^2 + l_4 l_3 \sigma_R^2) \\ a_4^{-1} a_1^{-1}(\sigma_K^2 + l_1 l_4 \sigma_R^2) & a_4^{-1} a_2^{-1}(\sigma_K^2 + l_2 l_4 \sigma_R^2) & a_4^{-1} a_3^{-1}(\sigma_K^2 + l_3 l_4 \sigma_R^2) & a_4^{-2}(\sigma_K^2 + l_4^2 \sigma_R^2 + \sigma_c^2) \end{bmatrix}$$

Assuming $\sigma_K = \pm 0.01 \text{ mm}$, $\sigma_R = \pm 1 \times 10^{-5}$ and $\sigma_c = \pm 0.02 \text{ mm}$, there is:

$$\mathbf{D}(\Delta \mathbf{X}) = \begin{bmatrix} 2 \times 10^{-5} & 2 \times 10^{-6} & 1.33333 \times 10^{-6} & 1 \times 10^{-6} \\ 2 \times 10^{-6} & 5 \times 10^{-6} & 6.66667 \times 10^{-7} & 5 \times 10^{-7} \\ 1.33333 \times 10^{-6} & 6.66667 \times 10^{-7} & 2.22222 \times 10^{-6} & 3.33334 \times 10^{-7} \\ 1 \times 10^{-6} & 5 \times 10^{-7} & 3.33334 \times 10^{-7} & 1.25 \times 10^{-6} \end{bmatrix}$$

According to the formula (3-14), there is:

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ \lambda_0 \end{bmatrix} = \begin{bmatrix} -0.014285714 \\ 0.085714286 \\ 0.299999996 \\ 0.628571432 \\ -9.14286 \times 10^{-7} \end{bmatrix}$$

According to the formula (3-15), there is:

$$y_0 = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0.5702 (\text{mm})$$

According to the formula (3-18), there is:

$$\sigma^2(\Delta y_0) = -\lambda_0$$

$$= 9.14286 \times 10^{-7} (\text{mm}^2)$$

Therefore, the uncertainty is:

$$\sigma(\Delta y_0) \approx \pm 0.00096(mm)$$

As you can see, p_1 is a negative value, which is beyond our inherent concept. In the past, we used to follow the conceptual logic of traditional theory to determine weights according to precision, ignoring the correlation between errors, so that $\mathbf{D}(\Delta \mathbf{X})$ is a diagonal matrix, and p_i is always positive. Now, variance is the dispersion of all possible values of error, and any error has variance and covariance, so $\mathbf{D}(\Delta \mathbf{X})$ is no longer a diagonal matrix, and p_i can be negative.

However, this negative weight is also a positive contribution to the final measured value. In this case, a_1 is the smallest, and the fixed error K is too harmful to the observed value x_1 . Therefore, Lagrange algorithm can only choose negative weights to correct this harm, which is exactly the scientific point of this algorithm.

3.2. Indirect measurement model for single measurand

The observation error equation is as follows.

$$\begin{bmatrix} P_1^{1/2} V_1 \\ P_2^{1/2} V_2 \\ \vdots \\ P_n^{1/2} V_n \end{bmatrix} = \begin{bmatrix} P_1^{1/2} x_1 \\ P_2^{1/2} x_2 \\ \vdots \\ P_n^{1/2} x_n \end{bmatrix} - \begin{bmatrix} P_1^{1/2} & 0 & \cdots & 0 \\ 0 & P_2^{1/2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P_n^{1/2} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} [Y] \quad (3-21)$$

According to the least square method, there is:

$$y = \frac{\sum P_i a_i x_i}{\sum P_i a_i^2} \quad (3-22)$$

Similar to the above direct measurement model, in order to realize the minimum value of $\sigma^2(\Delta y)$, it is assumed that $\sum P_i a_i^2 = 1$, and the formula (3-22) becomes:

$$\begin{aligned} y &= P_1 a_1 x_1 + P_2 a_2 x_2 + \cdots + P_n a_n x_n \\ &= \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} P_1 & 0 & \cdots & 0 \\ 0 & P_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \end{aligned} \quad (3-23)$$

That is

$$y = \mathbf{A}^T \mathbf{P} \mathbf{X} \quad (3-24)$$

Among them, $\mathbf{A} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$, $\mathbf{P} = \begin{bmatrix} P_1 & 0 & \cdots & 0 \\ 0 & P_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P_n \end{bmatrix}$, $\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$.

The error propagation equation is obtained by taking the total differential of formula (3-24), that is:

$$\Delta y = \mathbf{A}^T \mathbf{P} \Delta \mathbf{X} \quad (3-25)$$

Applying to the formula (2-17) to equation (3-25), there is:

$$\begin{aligned} \sigma^2(\Delta y) &= \mathbf{A}^T \mathbf{P} \mathbf{D}(\Delta \mathbf{X}) \mathbf{P} \mathbf{A} \\ &= P_1^2 a_1^2 \sigma_1^2 + P_2^2 a_2^2 \sigma_2^2 + \cdots + P_n^2 a_n^2 \sigma_n^2 \\ &\quad + 2P_1 P_2 a_1 a_2 \sigma_{12} + 2P_1 P_3 a_1 a_3 \sigma_{13} + \cdots \end{aligned} \quad (3-26)$$

Similarly, according to Lagrange multiplier method and omitting the derivation process, it is concluded that:

$$\begin{bmatrix} a_1^2 \sigma_1^2 & a_1 a_2 \sigma_{12} & \cdots & a_1 a_n \sigma_{1n} & a_1^2 \\ a_2 a_1 \sigma_{21} & a_2^2 \sigma_2^2 & \cdots & a_2 a_n \sigma_{2n} & a_2^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_n a_1 \sigma_{n1} & a_n a_2 \sigma_{n2} & \cdots & a_n^2 \sigma_n^2 & a_n^2 \\ a_1^2 & a_2^2 & \cdots & a_n^2 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \\ \lambda_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad (3-27)$$

Making $\mathbf{A}' = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_n \end{bmatrix}$, the formula (3-27) becomes:

$$\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \\ \lambda_0 \end{bmatrix} = \begin{bmatrix} & & & a_1^2 \\ & & & a_2^2 \\ & & & \vdots \\ & & & a_n^2 \\ a_1^2 & a_2^2 & \cdots & a_n^2 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad (3-28)$$

It can be seen that the best weights are actually determined by $\mathbf{D}(\Delta \mathbf{X})$ and $\mathbf{A}'$ together.

In this way, according to formula (3-24), the optimal measured value is

$$\begin{aligned} y_0 &= y|_{P_i=p_i} \\ &= \mathbf{A}^T \mathbf{P} \mathbf{X}|_{P_i=p_i} \end{aligned} \quad (3-29)$$

And according to (3-28), there is:

$$\mathbf{A}' \mathbf{D}(\Delta \mathbf{X}) \mathbf{A}' \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} = \begin{bmatrix} a_1^2 \\ a_2^2 \\ \vdots \\ a_n^2 \end{bmatrix} \begin{bmatrix} -\lambda_0 \end{bmatrix} \quad (3-30)$$

So there is

$$\begin{aligned} \begin{bmatrix} p_1 & p_2 & \cdots & p_n \end{bmatrix} \mathbf{A}' \mathbf{D}(\Delta \mathbf{X}) \mathbf{A}' \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} &= \begin{bmatrix} p_1 & p_2 & \cdots & p_n \end{bmatrix} \begin{bmatrix} a_1^2 \\ a_2^2 \\ \vdots \\ a_n^2 \end{bmatrix} \begin{bmatrix} -\lambda_0 \end{bmatrix} \\ &= -\lambda_0 \end{aligned} \quad (3-31)$$

And because there is:

$$\begin{aligned} \mathbf{A}^T \mathbf{P} \mathbf{D}(\Delta \mathbf{X}) \mathbf{P} \mathbf{A} \Big|_{P_i=p_i} &= \begin{bmatrix} p_1 & p_2 & \cdots & p_n \end{bmatrix} \mathbf{A}' \mathbf{D}(\Delta \mathbf{X}) \mathbf{A}' \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} \\ &= -\lambda_0 \end{aligned} \quad (3-32)$$

So according to formula (3-26), there is:

$$\begin{aligned} \sigma^2(\Delta y_0) &= \sigma^2(\Delta y) \Big|_{P_i=p_i} \\ &= \mathbf{A}^T \mathbf{P} \mathbf{D}(\Delta \mathbf{X}) \mathbf{P} \mathbf{A} \Big|_{P_i=p_i} \\ &= -\lambda_0 \end{aligned} \quad (3-33)$$

Combining formulas (3-29) and (3-32), we can get

$$\mathbf{D}(\Delta \mathbf{X}) \begin{bmatrix} a_1 p_1 \\ a_2 p_2 \\ \vdots \\ a_n p_n \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \left[\sigma^2(\Delta y_0) \right] \quad (3-34)$$

It can be seen that formula (3-19) is a special form of formula (3-34) when $a_1 = a_2 = \cdots = a_n = 1$.

Example 3, the data of example 2 is still used, and the solution is carried out according to the (3-21) model.

The observation error equation is as follows.

$$\begin{aligned} \begin{bmatrix} P_1^{1/2} V_1 \\ P_2^{1/2} V_2 \\ P_3^{1/2} V_3 \\ P_4^{1/2} V_4 \end{bmatrix} &= \begin{bmatrix} P_1^{1/2} l_1 \\ P_2^{1/2} l_2 \\ P_3^{1/2} l_3 \\ P_4^{1/2} l_4 \end{bmatrix} - \begin{bmatrix} P_1^{1/2} & 0 & 0 & 0 \\ 0 & P_2^{1/2} & 0 & 0 \\ 0 & 0 & P_3^{1/2} & 0 \\ 0 & 0 & 0 & P_4^{1/2} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} [Y] \\ \text{There are } \mathbf{L} = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \\ l_4 \end{bmatrix} &= \begin{bmatrix} 2.87 \\ 5.72 \\ 8.6 \\ 11.37 \end{bmatrix}, \mathbf{A} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 15 \\ 20 \end{bmatrix} \text{ and } \mathbf{A}' = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 15 & 0 \\ 0 & 0 & 0 & 20 \end{bmatrix}. \end{aligned}$$

Because

$$\Delta l_i = K + l_i R + c_i$$

According to the definition (2-16) of variance, there is

$$\mathbf{D}(\Delta \mathbf{L}) = E \begin{bmatrix} \Delta l_1 \\ \Delta l_2 \\ \Delta l_3 \\ \Delta l_4 \end{bmatrix} \begin{bmatrix} \Delta l_1 & \Delta l_2 & \Delta l_3 & \Delta l_4 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_K^2 + l_1^2 \sigma_R^2 + \sigma_c^2 & \sigma_K^2 + l_2 l_1 \sigma_R^2 & \sigma_K^2 + l_3 l_1 \sigma_R^2 & \sigma_K^2 + l_4 l_1 \sigma_R^2 \\ \sigma_K^2 + l_1 l_2 \sigma_R^2 & \sigma_K^2 + l_2^2 \sigma_R^2 + \sigma_c^2 & \sigma_K^2 + l_3 l_2 \sigma_R^2 & \sigma_K^2 + l_4 l_2 \sigma_R^2 \\ \sigma_K^2 + l_1 l_3 \sigma_R^2 & \sigma_K^2 + l_2 l_3 \sigma_R^2 & \sigma_K^2 + l_3^2 \sigma_R^2 + \sigma_c^2 & \sigma_K^2 + l_4 l_3 \sigma_R^2 \\ \sigma_K^2 + l_1 l_4 \sigma_R^2 & \sigma_K^2 + l_2 l_4 \sigma_R^2 & \sigma_K^2 + l_3 l_4 \sigma_R^2 & \sigma_K^2 + l_4^2 \sigma_R^2 + \sigma_c^2 \end{bmatrix}$$

Substituting $\sigma_K = \pm 0.01mm$, $\sigma_R = \pm 1 \times 10^{-5}$ and $\sigma_c = \pm 0.02mm$, and according to the formula (3-28), there is:

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ \lambda_0 \end{bmatrix} = \begin{bmatrix} & & & & a_1^2 \\ & & & & a_2^2 \\ & & & & a_3^2 \\ & & & & a_4^2 \\ a_1^2 & a_2^2 & a_3^2 & a_4^2 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.000571429 \\ 0.000857143 \\ 0.001333333 \\ 0.001571429 \\ -9.14286 \times 10^{-7} \end{bmatrix}$$

According to formula (3-29), the best measured value is as follows:

$$y_0 = \mathbf{A}^T \mathbf{P} \mathbf{X} \Big|_{p_i = p_i} = 0.5702(mm)$$

According to the formula (3-33), there is:

$$\begin{aligned} \sigma^2(\Delta y_0) &= -\lambda_0 \\ &= 9.14286 \times 10^{-7}(mm^2) \end{aligned}$$

Therefore, the uncertainty is $\sigma(\Delta y_0) = \pm 0.00096(mm)$

It can be seen that the final results of example 3 and example 2 are the same.

3.3. Indirect measurement model for multiple measurands

The observation error equation is as follows.

$$\begin{bmatrix} P_1^{1/2} V_1 \\ P_2^{1/2} V_2 \\ \vdots \\ P_n^{1/2} V_n \end{bmatrix} = \begin{bmatrix} P_1^{1/2} x_1 \\ P_2^{1/2} x_2 \\ \vdots \\ P_n^{1/2} x_n \end{bmatrix} - \begin{bmatrix} P_1^{1/2} & 0 & \cdots & 0 \\ 0 & P_2^{1/2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & P_n^{1/2} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1t} \\ a_{21} & a_{22} & \cdots & a_{2t} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nt} \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_t \end{bmatrix} \quad (3-35)$$

According to the least square method, there is:

$$\mathbf{Y} = [\mathbf{A}^T \mathbf{P} \mathbf{A}]^{-1} \mathbf{A}^T \mathbf{P} \mathbf{X} \quad (3-36)$$

$$\mathbf{P} = \begin{bmatrix} P_1 & 0 & \cdots & 0 \\ 0 & P_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & P_n \end{bmatrix}, \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1t} \\ a_{21} & a_{22} & \cdots & a_{2t} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nt} \end{bmatrix}, \mathbf{Y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_t \end{bmatrix}, \mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

The error propagation equation is obtained by taking the total differential of formula (3-36), that is:

$$\Delta \mathbf{Y} = [\mathbf{A}^T \mathbf{P} \mathbf{A}]^{-1} \mathbf{A}^T \mathbf{P} \Delta \mathbf{X} \quad (3-37)$$

Applying the law of covariance propagation to formula (3-37), we get:

$$\mathbf{D}(\Delta \mathbf{Y}) = [\mathbf{A}^T \mathbf{P} \mathbf{A}]^{-1} \mathbf{A}^T \mathbf{P} \mathbf{D}(\Delta \mathbf{X}) \mathbf{P} \mathbf{A} [\mathbf{A}^T \mathbf{P} \mathbf{A}]^{-1} \quad (3-38)$$

However, in the error propagation equation (3-37), the propagation law of each error Δy_j is

different from each other, so it is difficult to ensure that the variance of all errors can be minimized at the same time.

Therefore, at this time, we have to give up discussing the influence of each coefficient a_{ij} on the weights, and can only consider the influence of the variance $\mathbf{D}(\Delta\mathbf{X})$ on the weights alone. That is, the weights are obtained according to formula (3-14).

4. Conclusion

Following the pure concept of probability theory, both the observed value and the measured value are numerical values and belong to constant, while the error and true value are unknown and belong to random variable; the variance of error is the dispersion of all possible values of the error, and is the evaluation value of the probability interval where the error exists; any error is a deviation and has its variance, so it can't be classified as systematic error or random error. Based on these concepts, and analysing the covariance propagation relationship between the observation error and the final error of measured value, it is a pure mathematical problem to give the weight assignment with the minimum variance, and the dilemma of traditional measurement theory has been solved.

In an adjustment model, the weight allocation of observations is not only determined by the covariance array of observation error, but also related to the parameters in the adjustment model, which is particularly important for a single measurand model. For the adjustment model with multiple measurands, because the errors of multiple measured values come from different propagation laws, the variances of all errors cannot be minimized at the same time, which is a question that needs further research.

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