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Rigorous Method of Weights Calculation in Adjustment of Measurement Data with Different Qualities

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Abstract: In the traditional measurement theory, precision is defined as the dispersion of measured value, and is used as the basis of weights calculation in the adjustment of measurement data with different qualities, which leads to the trouble that trueness is completely ignored in the weight allocation. In this paper, following the concepts of pure probability theory, the measured value (observed value) is regarded as a constant, the error as a random variable, and the variance is interpreted as the dispersion of all possible values of the error. Thus, a rigorous formula for weights calculation and variance propagation is derived, which solves the theoretical problem of determining the weight values in the adjustment of multi-channel observation data with different qualities.

Keywords: measurement theory, random variable, uncertainty, survey adjustment, weight.

1. Introduction

The traditional measurement error theory is interpreted according to the logic of error classification, in which the measured value (observed value) and random error are regarded as random variables, and the variance is the dispersion of the measured value or random error; while the systematic error is regarded as constant without variance^[1,2,3,4,5,6]. In this way, when multiple observations with different qualities are obtained from a single measurand, according to the conceptual logic of existing measurement theory, the weight of each observed value is calculated by its precision, and finally the weighted average value of each observed value is taken as the final measured value.

However, the biggest trouble faced by this method is that the systematic error of each observation is different from each other. Some observations with large systematic error but small random error get larger weights, which can make the final measured value have better precision, but its trueness will be worse. Although we can explain the existing theory based on the assumption that the systematic error does not exist or can be ignored, this assumption is usually not tenable in actual measurement.

For example, in the field of geodesy, the multiplicative constant error of a geodimeter is considered as systematic error, its measured value given by metrology field is usually used to correct the observed value, and people think that the residual systematic error after correction can be ignored. However, the multiplicative constant error of geodimeter is originally the residual error after correction in the field of instrument manufacturing, which comes from the residual error of quartz crystal frequency after temperature correction, and still changes with temperature. In the field of instrument, the limited range of this error is also given. Therefore, it is actually meaningless to use a test sample given by metrology field to correct the observed value.

Therefore, this theory with obvious defects makes the measurement practice troubled.

For example, in 2020, the National Bureau of surveying, mapping and geographic information of China announced that the elevation of Mount Everest is 8848.86m, which is actually an

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adjustment value obtained from the trigonometric leveling data of geodimeter and GNSS satellite survey data according to a certain weight proportion. However, at present, we have not seen the public information about weight calculation, nor the official data about error evaluation.

In recent years, literature [7,8,9,10,11,12,13] proposed an interpretation method of measurement theory based new concepts, which clearly pointed out that the traditional measurement theory's understanding of constant and random variable concepts is inconsistent with probability theory. Because the constant in probability theory refers to a numerical value, and the random variable refers to a quantity whose numerical value is unknown, this theory regards the observed value (measured value) as a constant, the error and the true value as random variables, and the variance of error is the dispersion of all possible values of the error. In this conceptual system, errors cannot be classified as systematic error and random error.

Although these documents have been published formally, they do not deal with the problem of weight calculation in multi-channel data adjustment, so this paper will complete the mathematical deduction of this calculation principle. This deduction will completely overcome the confusion of error categories in traditional measurement theory, and completely solve the probability evaluation of error of adjustment value.

2. Basic concepts

In references[7,8,9,10,11,12,13], Fig 1 is used to interpret its basic measurement concepts, in which an observed value x_0 is used as the final measured value.

Fig 1. Conceptual sketch of new concept theory

Because the measured value x_0 is a numerical value and belongs to a constant, according to the concept of probability theory, there are:

$$\begin{cases} E(x_0) = x_0 \\ \sigma^2(x_0) = 0 \end{cases} \quad (2-1)$$

For the unknown deviation $\Delta_A = x_0 - E(X)$, because it has the same sample space $\{x_k - E(X)\}$ as the random variable $\Delta X = X - E(X)$, we can regard the deviation Δ_A as $\Delta X = X - E(X)$, that is, $\Delta_A = X - E(X)$. Therefore, there are

$$\begin{cases} E(\Delta_A) = E[X - E(X)] = 0 \\ \sigma^2(\Delta_A) = E(\Delta_A^2) = E[X - E(X)]^2 \end{cases} \quad (2-2)$$

Its concept meaning is that variance $\sigma^2(\Delta_A)$ is the dispersion of all possible values

$\{x_k - E(X)\}$ of deviation Δ_A , and is the evaluation value of the probability interval of deviation Δ_A .

For the unknown deviation $\Delta_B = E(X) - X_T$, because it comes from the previous measurement, its formation principle is similar to that of deviation Δ_A , and it also has all its possible values, so there are

$$\begin{cases} E(\Delta_B) = 0 \\ \sigma^2(\Delta_B) = E(\Delta_B^2) \end{cases} \quad (2-3)$$

That is to say, both Δ_A and Δ_B are unknown deviations, have their own variances, and cannot be classified as random error and systematic error respectively. That is, the concepts of precision and trueness were abolished.

For example, the multiplicative constant error of a geodimeter, which is generally considered as systematic error without variance in the field of geodesy, is actually the output error in the field of instrument manufacturing, and the instrument manufacturer also uses its error samples under all possible measurement conditions to count its probability interval (usually expressed by the maximum permissible error MPE).

In this way, there are

$$\Delta = \Delta_A + \Delta_B \quad (2-4)$$

$$\begin{aligned} E(\Delta) &= E(\Delta_A) + E(\Delta_B) \\ &= 0 \end{aligned} \quad (2-5)$$

$$\begin{aligned} \sigma^2(\Delta) &= E[\Delta - E(\Delta)]^2 \\ &= E(\Delta^2) \\ &= E(\Delta_A + \Delta_B)^2 \\ &= \sigma^2(\Delta_A) + \sigma^2(\Delta_B) \end{aligned} \quad (2-6)$$

The variance $\sigma^2(\Delta)$ is the evaluation value of the probability range of the total error Δ and the dispersion of all possible values of the total error Δ , which is called the uncertainty of the error Δ . Please note that this uncertainty is not the dispersion of the measured value, which is different from the traditional measurement theory.

Therefore, for the true value X_T , there are:

$$X_T = x_0 - \Delta \quad (2-7)$$

$$\begin{aligned} E(X_T) &= E(x_0) - E(\Delta) \\ &= x_0 \end{aligned} \quad (2-8)$$

$$\begin{aligned} \sigma^2(X_T) &= E[X_T - E(X_T)]^2 \\ &= E(x_0 - \Delta - x_0)^2 \\ &= \sigma^2(\Delta) \end{aligned} \quad (2-9)$$

In conclusion, the probability expressions of measured value, error and true value are shown in Table 1.

Table 1. The probability expressions of measured value, error and true value

Measured value x_0	Error Δ	True value X_T
$\begin{cases} E(x_0) = x_0 \\ \sigma^2(x_0) = 0 \end{cases}$	$\begin{cases} E(\Delta) = 0 \\ \sigma^2(\Delta) = E(\Delta^2) \end{cases}$	$\begin{cases} E(X_T) = x_0 \\ \sigma^2(X_T) = \sigma^2(\Delta) \end{cases}$

Where $E(\Delta) = 0$ means that the mean value of all possible values of error Δ after measurement is 0, because once $E(\Delta) = C \neq 0$, this C must be corrected to the measured value x_0 .

The above conclusion is derived by using an observed value x_0 as the final measured value. When multiple observations x_i are used to obtain the best measured value, the analysis process is shown in example 1 in the next section.

Any error has variance to evaluate its probability range, and the regular error is of course the same, because the variance of error is the dispersion of error values under all possible measurement conditions. Fig 2 is a conceptual diagram describing the regularity and randomness of the periodic error of the geodimeter. When observing the corresponding relationship between the error value and the range, we can see the regularity. When observing the density distribution of all possible values of the error, we can see the randomness.

Fig 2. Regularity and randomness of sine error

Because errors vary with various measurement conditions, the variation of measurement conditions in repeated measurement determines the contribution form of errors to repeated observations (deviation, divergence, outlier or both). It is precisely because errors can lead to the dispersion of repeated observations, so the variance of any error can be obtained through experimental statistics, and these data exist in the instrument specifications or instructions.

Because any error has variance, the concept of variance can be extended to any two errors, which is called covariance. It comes from the common component of the two errors.

$$\sigma(\Delta x_j \Delta x_k) = E(\Delta x_j \Delta x_k) \quad (2-10)$$

Thus, for an error array $\Delta \mathbf{X} = (\Delta x_1, \Delta x_2, \dots, \Delta x_n)^\top$, the definition of variance is as follows:

$$\begin{aligned} \mathbf{D}(\Delta \mathbf{X}) &= E(\Delta \mathbf{X} \Delta \mathbf{X}^\top) \\ &= \begin{bmatrix} \sigma_{11}^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22}^2 & \cdots & \sigma_{2n} \\ \vdots & \vdots & & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn}^2 \end{bmatrix} \end{aligned} \quad (2-11)$$

Therefore, for the error propagation equation $\Delta \mathbf{Z} = \mathbf{K} \cdot \Delta \mathbf{X}$, according to the definition of variance (2-11), we can get:

$$\mathbf{D}(\Delta \mathbf{Z}) = \mathbf{K} \cdot \mathbf{D}(\Delta \mathbf{X}) \cdot \mathbf{K}^\top \quad (2-12)$$

This is the law of covariance propagation, which is the propagation relation of probability

interval between errors, or the propagation relation of the dispersion of all possible values of errors, rather than the propagation relation of the dispersion of measured value explained by traditional theory.

After the data processing, the difference between the measured value and the true value is an unknown deviation. Therefore, the task of measurement is to adjust the measured value to minimize the variance of the deviation and submit its value.

3. The solution of weight values

The smaller the error is, the better the quality of the measured value is. However, because the error value is unknown, and we can only use the variance of the error to describe its probability interval, so we can only think that the quality of the measured value is the best when the variance of the error is the smallest. Therefore, the mathematical problem we are faced with is to minimize the variance of the error of the final measured value by assigning the weights of each observed value.

a) Direct measurement model for single measurand

When n different observed values x_i are obtained by repeated observation of a measurand, assuming that the best measured value is y , the error equation is as follows:

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} [y] \quad (3-1)$$

According to the formula in Table 1, there are

$$E(V) = 0 \quad (3-2)$$

$$\begin{aligned} \sigma^2(V) &= E(V^2) \\ &\approx \frac{\sum v_i^2}{n} \\ &= \frac{1}{n} (x_1^2 + x_2^2 + \cdots + x_n^2 - 2x_1y - 2x_2y - \cdots - 2x_ny + ny^2) \end{aligned} \quad (3-3)$$

The best measured value is the value when $\sigma^2(V)$ or $\sum v_i^2$ takes the minimum value, so

making $\frac{d(\sigma^2(V))}{dy} = 0$, the best measured value is obtained as follows:

$$y = \frac{\sum_{i=1}^n x_i}{n} \quad (3-3)$$

This is the least square method. Of course, formula (3-3) is actually an equal weight processing method, that is, the weight of each observation x_i is $\frac{1}{n}$. However, if n observation errors Δx_i have different variances and covariances, that is, the quality of each observation x_i is different from each other, then the equal weight processing is not scientific. At this time, different weights must be

given to each observed value to ensure that the quality of the final measured value is the best. Therefore, in order to study each weight p_i when $\sum p_i v_i^2$ takes the minimum value, the equation (3-1) is modified as:

$$\begin{bmatrix} p_1^{1/2} v_1 \\ p_2^{1/2} v_2 \\ \vdots \\ p_n^{1/2} v_n \end{bmatrix} = \begin{bmatrix} p_1^{1/2} x_1 \\ p_2^{1/2} x_2 \\ \vdots \\ p_n^{1/2} x_n \end{bmatrix} - \begin{bmatrix} p_1^{1/2} & 0 & \cdots & 0 \\ 0 & p_2^{1/2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_n^{1/2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} [y] \quad (3-4)$$

By using the least square method, there is

$$y = (p_1 x_1 + p_2 x_2 + \cdots + p_n x_n) / \sum_{i=1}^n p_i \quad (3-5)$$

For the convenience of calculation, making $\sum_{i=1}^n p_i = 1$, there is

$$y = p_1 x_1 + p_2 x_2 + \cdots + p_n x_n \quad (3-6)$$

Now, using the quality of each x_i , we begin to solve the general calculation method of each weight p_i .

Taking the total differential of equation (3-6), the error propagation equation is obtained as follows:

$$\Delta y = p_1 \Delta x_1 + p_2 \Delta x_2 + \cdots + p_n \Delta x_n \quad (3-7)$$

Applying the law of covariance propagation (2-12) to equation (3-7), there is:

$$\begin{aligned} \sigma^2(\Delta y) &= \begin{bmatrix} p_1 & p_2 & \cdots & p_n \end{bmatrix} \mathbf{D}(\Delta \mathbf{X}) \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} \\ &= p_1^2 \sigma_1^2 + p_2^2 \sigma_2^2 + \cdots + p_n^2 \sigma_n^2 + 2p_1 p_2 \sigma_{12} + 2p_1 p_3 \sigma_{13} + \cdots \end{aligned} \quad (3-8)$$

Please note that Δx_i here refers to the deviation between the observed value x_i and its true value, and the variance σ_i^2 is the dispersion $\sigma^2(\Delta x_i)$ of all possible values of the error Δx_i , which expresses the evaluation of the probability interval of the error Δx_i . This concept of variance is consistent with probability theory, but it is different from the concept of dispersion of observations in traditional measurement theory.

When the value of the function $\sigma^2(\Delta y)$ is the minimum, the quality of the measured value y is the best, which is a problem of finding the minimum value of the function $\sigma^2(\Delta y)$ under the

restriction condition $\sum_{i=1}^n p_i = 1$. Therefore, according to the Lagrange multiplier method, the

following functions are formed:

$$\begin{aligned}
Y &= \sigma^2(\Delta y) + 2\lambda(p_1 + p_2 + \dots + p_n - 1) \\
&= p_1^2 \sigma_1^2 + p_2^2 \sigma_2^2 + \dots + p_n^2 \sigma_n^2 + 2p_1 p_2 \sigma_{12} + 2p_1 p_3 \sigma_{13} + \dots \\
&\quad + 2\lambda(p_1 + p_2 + \dots + p_n - 1)
\end{aligned} \tag{3-9}$$

By making the partial differential equation of each unknown p_i equal to 0, the Lagrange equation is obtained as follows:

$$\left. \begin{aligned}
p_1 \sigma_1^2 + p_2 \sigma_{12} + p_3 \sigma_{13} + \dots + p_n \sigma_{1n} + \lambda &= 0 \\
p_1 \sigma_{21} + p_2 \sigma_2^2 + p_3 \sigma_{23} + \dots + p_n \sigma_{2n} + \lambda &= 0 \\
&\vdots \\
p_1 \sigma_{n1} + p_2 \sigma_{n2} + p_3 \sigma_{n3} + \dots + p_n \sigma_n^2 + \lambda &= 0 \\
p_1 + p_2 + \dots + p_n - 1 &= 0
\end{aligned} \right\} \tag{3-10}$$

That is:

$$\begin{bmatrix} \sigma_1^2 & \sigma_{12} & \dots & \sigma_{1n} & 1 \\ \sigma_{21} & \sigma_2^2 & \dots & \sigma_{2n} & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \dots & \sigma_n^2 & 1 \\ 1 & 1 & \dots & 1 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{3-11}$$

Because the covariance array of error array $\Delta \mathbf{X} = [\Delta x_1, \Delta x_2, \dots, \Delta x_n]^T$ is:

$$\begin{aligned}
\mathbf{D}(\Delta \mathbf{X}) &= E \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \vdots \\ \Delta x_n \end{bmatrix} \begin{bmatrix} \Delta x_1 & \Delta x_2 & \dots & \Delta x_n \end{bmatrix} \\
&= \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \dots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \dots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \dots & \sigma_n^2 \end{bmatrix}
\end{aligned} \tag{3-12}$$

So the equation (3-11) can be written as follows:

$$\begin{bmatrix} & & & 1 \\ & & & 1 \\ & & & \vdots \\ & & & 1 \\ 1 & 1 & \dots & 1 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{3-13}$$

Therefore

$$\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \\ \lambda \end{bmatrix} = \begin{bmatrix} & & & 1 \\ & & & 1 \\ & \mathbf{D}(\Delta \mathbf{X}) & & \vdots \\ & & & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad (3-14)$$

And according to formula (3-13), there is

$$\mathbf{D}(\Delta \mathbf{X}) \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} [\lambda] = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (3-15)$$

That is

$$\mathbf{D}(\Delta \mathbf{X}) \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} = \begin{bmatrix} -\lambda \\ -\lambda \\ \vdots \\ -\lambda \end{bmatrix} \quad (3-16)$$

By substituting equation (3-16) into equation (3-8), we can get:

$$\begin{aligned} \sigma^2(\Delta y) &= \begin{bmatrix} p_1 & p_2 & \cdots & p_n \end{bmatrix} \mathbf{D}(\Delta \mathbf{X}) \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} \\ &= \begin{bmatrix} p_1 & p_2 & \cdots & p_n \end{bmatrix} \begin{bmatrix} -\lambda \\ -\lambda \\ \vdots \\ -\lambda \end{bmatrix} \\ &= -\lambda \end{aligned} \quad (3-17)$$

Finally, by substituting formula (3-14) into formulas (3-6) and (3-17), the optimal measured value y and its error evaluation $\sigma^2(\Delta y)$ can be obtained.

In formula (3-11), if each $\sigma_{ij} = 0$, we can get the following results:

$$p_1 : p_2 : \cdots : p_n = \frac{1}{\sigma_1^2} : \frac{1}{\sigma_2^2} : \cdots : \frac{1}{\sigma_n^2} \quad (3-18)$$

This is the reason why we used to use a single $\frac{1}{\sigma_i^2}$ to determine the weight ratio. Obviously,

the formula (3-18) does not hold when the hypothesis $\sigma_{ij} = 0$ does not exist.

Next, let's see the actual cases.

Example 1, as shown in Figure 1, when n observations x_i are obtained by using the same sensor, please calculate the final measured value according to formula (3-6).

Because $\Delta x_i = \Delta_B + \Delta_{Ai}$, so according to the formula (2-6), there is:

$$\sigma_i^2 = \sigma_{\Delta_B}^2 + \sigma_{\Delta_A}^2$$

According to the formula (2-10), there is:

$$\begin{aligned}\sigma_{jk} &= E(\Delta x_j \Delta x_k) \\ &= E[(\Delta_B + \Delta_{Aj})(\Delta_B + \Delta_{Ak})] \\ &= E(\Delta_B^2) = \sigma_{\Delta_B}^2\end{aligned}$$

Therefore, according to the formula (3-11), there is:

$$\begin{bmatrix} \sigma_{\Delta_B}^2 + \sigma_{\Delta_A}^2 & \sigma_{\Delta_B}^2 & \cdots & \sigma_{\Delta_B}^2 & 1 \\ \sigma_{\Delta_B}^2 & \sigma_{\Delta_B}^2 + \sigma_{\Delta_A}^2 & \cdots & \sigma_{\Delta_B}^2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \sigma_{\Delta_B}^2 & \sigma_{\Delta_B}^2 & \cdots & \sigma_{\Delta_B}^2 + \sigma_{\Delta_A}^2 & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

By solving the equation, we get the following results:

$$p_1 = p_2 = \cdots = p_n = \frac{1}{n} \quad \lambda = -\sigma_{\Delta_B}^2 - \frac{\sigma_{\Delta_A}^2}{n}$$

Finally, according to (3-6) and (3-17), there are:

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n}, \quad \sigma^2(\Delta \bar{x}) = \sigma_{\Delta_B}^2 + \frac{\sigma_{\Delta_A}^2}{n}$$

It can be seen that in this case, the root cause of $p_1 = p_2 = \cdots = p_n = \frac{1}{n}$ is that each

observation error Δx_i has the same variance $\sigma_i^2 = \sigma_{\Delta_B}^2 + \sigma_{\Delta_A}^2$ and any two observation errors

have the same covariance $\sigma_{jk} = \sigma_{\Delta_B}^2$.

Example 2, as shown in Fig 3, in order to obtain the accurate diameter value of the copper wire, the observed values of 5 times, 10 times, 15 times and 20 times of the diameter are measured with a caliper, and 4 observed values are shown in Table 2. Please give the best measured value of

diameter and the evaluation of uncertainty.

Table 2. Observed values

i	1	2	3	4
a_i	5	10	15	20
l_i (mm)	2.87	5.72	8.60	11.37

The four measured values are as follows:

$$\begin{aligned} x_1 &= \frac{l_1}{a_1} = 0.574 \\ x_2 &= \frac{l_2}{a_2} = 0.572 \\ x_3 &= \frac{l_3}{a_3} = 0.5733 \\ x_4 &= \frac{l_4}{a_4} = 0.5685 \end{aligned}$$

The error Δl_i of the observed value l_i consists of three parts: fixed error K , proportional error R and non-uniform dividing error c_i . That is:

$$\Delta l_i = K + l_i R + c_i$$

According to the definition (2-11) of variance, there is

$$\begin{aligned} \mathbf{D}(\Delta \mathbf{L}) &= E \begin{bmatrix} \Delta l_1 & \Delta l_2 & \Delta l_3 & \Delta l_4 \end{bmatrix} \begin{bmatrix} \Delta l_1 \\ \Delta l_2 \\ \Delta l_3 \\ \Delta l_4 \end{bmatrix} \\ &= \begin{bmatrix} \sigma_K^2 + l_1^2 \sigma_R^2 + \sigma_c^2 & \sigma_K^2 + l_2 l_1 \sigma_R^2 & \sigma_K^2 + l_3 l_1 \sigma_R^2 & \sigma_K^2 + l_4 l_1 \sigma_R^2 \\ \sigma_K^2 + l_1 l_2 \sigma_R^2 & \sigma_K^2 + l_2^2 \sigma_R^2 + \sigma_c^2 & \sigma_K^2 + l_3 l_2 \sigma_R^2 & \sigma_K^2 + l_4 l_2 \sigma_R^2 \\ \sigma_K^2 + l_1 l_3 \sigma_R^2 & \sigma_K^2 + l_2 l_3 \sigma_R^2 & \sigma_K^2 + l_3^2 \sigma_R^2 + \sigma_c^2 & \sigma_K^2 + l_4 l_3 \sigma_R^2 \\ \sigma_K^2 + l_1 l_4 \sigma_R^2 & \sigma_K^2 + l_2 l_4 \sigma_R^2 & \sigma_K^2 + l_3 l_4 \sigma_R^2 & \sigma_K^2 + l_4^2 \sigma_R^2 + \sigma_c^2 \end{bmatrix} \end{aligned}$$

Because $x_i = l_i / a_i$ and $\Delta x_i = \Delta l_i / a_i$, there is

$$\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \\ \Delta x_4 \end{bmatrix} = \begin{bmatrix} a_1^{-1} & 0 & 0 & 0 \\ 0 & a_2^{-1} & 0 & 0 \\ 0 & 0 & a_3^{-1} & 0 \\ 0 & 0 & 0 & a_4^{-1} \end{bmatrix} \begin{bmatrix} \Delta l_1 \\ \Delta l_2 \\ \Delta l_3 \\ \Delta l_4 \end{bmatrix}$$

That is

$$\Delta \mathbf{X} = \mathbf{A} \Delta \mathbf{L}$$

According to the law of covariance propagation (2-12), there is:

$$\mathbf{D}(\Delta \mathbf{X}) = \mathbf{A} \mathbf{D}(\Delta \mathbf{L}) \mathbf{A}^T$$

$$= \begin{bmatrix} a_1^{-2}(\sigma_K^2 + l_1^2 \sigma_R^2 + \sigma_c^2) & a_1^{-1} a_2^{-1}(\sigma_K^2 + l_2 l_1 \sigma_R^2) & a_1^{-1} a_3^{-1}(\sigma_K^2 + l_3 l_1 \sigma_R^2) & a_1^{-1} a_4^{-1}(\sigma_K^2 + l_4 l_1 \sigma_R^2) \\ a_2^{-1} a_1^{-1}(\sigma_K^2 + l_1 l_2 \sigma_R^2) & a_2^{-2}(\sigma_K^2 + l_2^2 \sigma_R^2 + \sigma_c^2) & a_2^{-1} a_3^{-1}(\sigma_K^2 + l_3 l_2 \sigma_R^2) & a_2^{-1} a_4^{-1}(\sigma_K^2 + l_4 l_2 \sigma_R^2) \\ a_3^{-1} a_1^{-1}(\sigma_K^2 + l_1 l_3 \sigma_R^2) & a_3^{-1} a_2^{-1}(\sigma_K^2 + l_2 l_3 \sigma_R^2) & a_3^{-2}(\sigma_K^2 + l_3^2 \sigma_R^2 + \sigma_c^2) & a_3^{-1} a_4^{-1}(\sigma_K^2 + l_4 l_3 \sigma_R^2) \\ a_4^{-1} a_1^{-1}(\sigma_K^2 + l_1 l_4 \sigma_R^2) & a_4^{-1} a_2^{-1}(\sigma_K^2 + l_2 l_4 \sigma_R^2) & a_4^{-1} a_3^{-1}(\sigma_K^2 + l_3 l_4 \sigma_R^2) & a_4^{-2}(\sigma_K^2 + l_4^2 \sigma_R^2 + \sigma_c^2) \end{bmatrix}$$

Fig 3. Measuring the diameter of copper wire

Assuming $\sigma_K = \pm 0.01mm$, $\sigma_R = \pm 1 \times 10^{-5}$ and $\sigma_c = \pm 0.02mm$, there is:

$$\mathbf{D}(\Delta \mathbf{X}) = \begin{bmatrix} 2 \times 10^{-5} & 2 \times 10^{-6} & 1.33333 \times 10^{-6} & 1 \times 10^{-6} \\ 2 \times 10^{-6} & 5 \times 10^{-6} & 6.66667 \times 10^{-7} & 5 \times 10^{-7} \\ 1.33333 \times 10^{-6} & 6.66667 \times 10^{-7} & 2.22222 \times 10^{-6} & 3.33334 \times 10^{-7} \\ 1 \times 10^{-6} & 5 \times 10^{-7} & 3.33334 \times 10^{-7} & 1.25 \times 10^{-6} \end{bmatrix}$$

According to the formula (3-14), there is:

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ \lambda \end{bmatrix} = \begin{bmatrix} -0.014285714 \\ 0.085714286 \\ 0.2999999996 \\ 0.628571432 \\ -9.14286 \times 10^{-7} \end{bmatrix}$$

According to the formula (3-6), there is:

$$y = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0.5702(mm)$$

According to the formula (3-17), there is:

$$\begin{aligned} \sigma^2(\Delta y) &= -\lambda \\ &= 9.14286 \times 10^{-7}(mm^2) \end{aligned}$$

Therefore, the uncertainty is:

$$\sigma(\Delta y) \approx \pm 0.00096(mm)$$

As you can see, p_1 is a negative value, which is beyond our inherent concept. In the past, we used to follow the conceptual logic of traditional theory to determine weights according to precision, ignoring the correlation between errors, so that $\mathbf{D}(\Delta \mathbf{X})$ is a diagonal matrix, and p_i is always positive. Now, variance is the dispersion of all possible values of error, and any error has variance and covariance, so $\mathbf{D}(\Delta \mathbf{X})$ is no longer a diagonal matrix, and p_i can be negative.

However, this negative weight is also a positive contribution to the final measured value. In this case, a_1 is the smallest, and the fixed error K is too harmful to the observed value x_1 . Therefore, Lagrange algorithm can only choose negative weights to correct this harm, which is exactly the scientific point of this algorithm.

b) Indirect measurement model for single measurand

The observation error equation is as follows.

$$\begin{bmatrix} p_1^{1/2} v_1 \\ p_2^{1/2} v_2 \\ \vdots \\ p_n^{1/2} v_n \end{bmatrix} = \begin{bmatrix} p_1^{1/2} x_1 \\ p_2^{1/2} x_2 \\ \vdots \\ p_n^{1/2} x_n \end{bmatrix} - \begin{bmatrix} p_1^{1/2} & 0 & \cdots & 0 \\ 0 & p_2^{1/2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_n^{1/2} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} [y] \quad (3-19)$$

According to the least square method, there is:

$$y = \frac{\sum p_i a_i x_i}{\sum p_i a_i^2} \quad (3-20)$$

Similar to the above direct measurement model, in order to realize the minimum value of $\sigma^2(\Delta y)$, it is assumed that $\sum p_i a_i^2 = 1$, and the formula (3-20) becomes:

$$\begin{aligned} y &= p_1 a_1 x_1 + p_2 a_2 x_2 + \cdots + p_n a_n x_n \\ &= \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} p_1 & 0 & \cdots & 0 \\ 0 & p_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & p_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \end{aligned} \quad (3-21)$$

That is $y = \mathbf{A}^T \mathbf{P} \mathbf{X}$ (3-22)

Among them, $\mathbf{A} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \mathbf{P} = \begin{bmatrix} p_1 & 0 & \cdots & 0 \\ 0 & p_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & p_n \end{bmatrix}, \mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

Taking the total differential of formula (3-22), the error propagation equation is obtained as follows:

$$\Delta y = \mathbf{A}^T \mathbf{P} \Delta \mathbf{X} \quad (3-23)$$

According to the law of covariance propagation (2-12), there is:

$$\begin{aligned} \sigma^2(\Delta y) &= \mathbf{A}^T \mathbf{P} \mathbf{D}(\Delta \mathbf{X}) \mathbf{P} \mathbf{A} \\ &= p_1^2 a_1^2 \sigma_1^2 + p_2^2 a_2^2 \sigma_2^2 + \cdots + p_n^2 a_n^2 \sigma_n^2 \\ &\quad + 2p_1 p_2 a_1 a_2 \sigma_{12} + 2p_1 p_3 a_1 a_3 \sigma_{13} + \cdots \end{aligned} \quad (3-24)$$

Similarly, according to Lagrange multiplier method and omitting the derivation process, it is concluded that:

$$\begin{bmatrix} a_1^2 \sigma_1^2 & a_1 a_2 \sigma_{12} & \cdots & a_1 a_n \sigma_{1n} & a_1^2 \\ a_2 a_1 \sigma_{21} & a_2^2 \sigma_2^2 & \cdots & a_2 a_n \sigma_{2n} & a_2^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_n a_1 \sigma_{n1} & a_n a_2 \sigma_{n2} & \cdots & a_n^2 \sigma_n^2 & a_n^2 \\ a_1^2 & a_2^2 & \cdots & a_n^2 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad (3-25)$$

Making $\mathbf{A}' = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_n \end{bmatrix}$, the formula (3-25) becomes:

$$\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \\ \lambda \end{bmatrix} = \begin{bmatrix} & & & a_1^2 \\ & & & a_2^2 \\ & \mathbf{A}'\mathbf{D}(\Delta\mathbf{X})\mathbf{A}' & & \vdots \\ & & & a_n^2 \\ a_1^2 & a_2^2 & \cdots & a_n^2 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad (3-26)$$

It can be seen that the best weights are actually determined by $\mathbf{D}(\Delta\mathbf{X})$ and $\mathbf{A}'$ together.

And because according to (3-26), there is:

$$\mathbf{A}'\mathbf{D}(\Delta\mathbf{X})\mathbf{A}' \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} = \begin{bmatrix} a_1^2 \\ a_2^2 \\ \vdots \\ a_n^2 \end{bmatrix} \begin{bmatrix} -\lambda \end{bmatrix} \quad (3-27)$$

So there is

$$\begin{aligned} [p_1 \ p_2 \ \cdots \ p_n] \mathbf{A}'\mathbf{D}(\Delta\mathbf{X})\mathbf{A}' \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} &= [p_1 \ p_2 \ \cdots \ p_n] \begin{bmatrix} a_1^2 \\ a_2^2 \\ \vdots \\ a_n^2 \end{bmatrix} \begin{bmatrix} -\lambda \end{bmatrix} \\ &= -\lambda \end{aligned} \quad (3-28)$$

And because there is:

$$\mathbf{A}^\top \mathbf{P} \mathbf{D}(\Delta\mathbf{X}) \mathbf{P} \mathbf{A} = [p_1 \ p_2 \ \cdots \ p_n] \mathbf{A}'\mathbf{D}(\Delta\mathbf{X})\mathbf{A}' \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} \quad (3-29)$$

So formula (3-24) becomes:

$$\begin{aligned} \sigma^2(\Delta y) &= \mathbf{A}^\top \mathbf{P} \mathbf{D}(\Delta\mathbf{X}) \mathbf{P} \mathbf{A} \\ &= -\lambda \end{aligned} \quad (3-30)$$

Example 3, the data of example 2 is still used, and the solution is carried out according to the (3-19) model.

The observation error equation is as follows.

$$\begin{bmatrix} p_1^{1/2} v_1 \\ p_2^{1/2} v_2 \\ p_3^{1/2} v_3 \\ p_4^{1/2} v_4 \end{bmatrix} = \begin{bmatrix} p_1^{1/2} l_1 \\ p_2^{1/2} l_2 \\ p_3^{1/2} l_3 \\ p_4^{1/2} l_4 \end{bmatrix} - \begin{bmatrix} p_1^{1/2} & 0 & 0 & 0 \\ 0 & p_2^{1/2} & 0 & 0 \\ 0 & 0 & p_3^{1/2} & 0 \\ 0 & 0 & 0 & p_4^{1/2} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} [y]$$

There are $\mathbf{L} = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \\ l_4 \end{bmatrix} = \begin{bmatrix} 2.87 \\ 5.72 \\ 8.6 \\ 11.37 \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 15 \\ 20 \end{bmatrix}$ and $\mathbf{A}' = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 15 & 0 \\ 0 & 0 & 0 & 20 \end{bmatrix}$.

Because

$$\Delta l_i = K + l_i R + c_i$$

According to the definition (2-11) of variance, there is

$$\begin{aligned} \mathbf{D}(\Delta \mathbf{L}) &= E \begin{bmatrix} \Delta l_1 & \Delta l_2 & \Delta l_3 & \Delta l_4 \end{bmatrix} \begin{bmatrix} \Delta l_1 \\ \Delta l_2 \\ \Delta l_3 \\ \Delta l_4 \end{bmatrix} \\ &= \begin{bmatrix} \sigma_K^2 + l_1^2 \sigma_R^2 + \sigma_c^2 & \sigma_K^2 + l_2 l_1 \sigma_R^2 & \sigma_K^2 + l_3 l_1 \sigma_R^2 & \sigma_K^2 + l_4 l_1 \sigma_R^2 \\ \sigma_K^2 + l_1 l_2 \sigma_R^2 & \sigma_K^2 + l_2^2 \sigma_R^2 + \sigma_c^2 & \sigma_K^2 + l_3 l_2 \sigma_R^2 & \sigma_K^2 + l_4 l_2 \sigma_R^2 \\ \sigma_K^2 + l_1 l_3 \sigma_R^2 & \sigma_K^2 + l_2 l_3 \sigma_R^2 & \sigma_K^2 + l_3^2 \sigma_R^2 + \sigma_c^2 & \sigma_K^2 + l_4 l_3 \sigma_R^2 \\ \sigma_K^2 + l_1 l_4 \sigma_R^2 & \sigma_K^2 + l_2 l_4 \sigma_R^2 & \sigma_K^2 + l_3 l_4 \sigma_R^2 & \sigma_K^2 + l_4^2 \sigma_R^2 + \sigma_c^2 \end{bmatrix} \end{aligned}$$

Substituting $\sigma_K = \pm 0.01mm$, $\sigma_R = \pm 1 \times 10^{-5}$ and $\sigma_c = \pm 0.02mm$, and according to the formula (3-26), there is:

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ \lambda \end{bmatrix} = \begin{bmatrix} & & & & a_1^2 \\ & & & & a_2^2 \\ & & & & a_3^2 \\ & & & & a_4^2 \\ a_1^2 & a_2^2 & a_3^2 & a_4^2 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.000571429 \\ 0.000857143 \\ 0.001333333 \\ 0.001571429 \\ -9.14286 \times 10^{-7} \end{bmatrix}$$

According to formula (3-22), the best measured value is as follows:

$$y = \mathbf{A}^T \mathbf{P} \mathbf{X} = 0.5702(mm)$$

According to the formula (3-30), there is:

$$\begin{aligned} \sigma^2(\Delta y) &= -\lambda \\ &= 9.14286 \times 10^{-7}(mm^2) \end{aligned}$$

Therefore, the uncertainty is $\sigma(\Delta y) = \pm 0.00096(mm)$

It can be seen that the final results of example 3 and example 2 are the same.

c) Indirect measurement model for multiple measurand

The observation error equation is as follows.

$$\begin{bmatrix} p_1^{1/2} v_1 \\ p_2^{1/2} v_2 \\ \vdots \\ p_n^{1/2} v_n \end{bmatrix} = \begin{bmatrix} p_1^{1/2} x_1 \\ p_2^{1/2} x_2 \\ \vdots \\ p_n^{1/2} x_n \end{bmatrix} - \begin{bmatrix} p_1^{1/2} & 0 & \cdots & 0 \\ 0 & p_2^{1/2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_n^{1/2} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1t} \\ a_{21} & a_{22} & \cdots & a_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nt} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_t \end{bmatrix} \quad (3-31)$$

According to the least square method, there is:

$$\mathbf{Y} = [\mathbf{A}^T \mathbf{P} \mathbf{A}]^{-1} \mathbf{A}^T \mathbf{P} \mathbf{X} \quad (3-32)$$

$$\mathbf{P} = \begin{bmatrix} p_1 & 0 & \cdots & 0 \\ 0 & p_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_n \end{bmatrix}, \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1t} \\ a_{21} & a_{22} & \cdots & a_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nt} \end{bmatrix}, \mathbf{Y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_t \end{bmatrix}, \mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Taking the total differential of formula (3-32), the error propagation equation is obtained as follows:

$$\Delta \mathbf{Y} = [\mathbf{A}^T \mathbf{P} \mathbf{A}]^{-1} \mathbf{A}^T \mathbf{P} \Delta \mathbf{X} \quad (3-33)$$

However, in the error propagation equation (3-33), the propagation law of each error Δy_j is different from each other, so it is difficult to ensure that the variance of all errors can be minimized at the same time. At this time, it is suggested to use formula (3-14) to obtain a relatively fair weight ratio.

4. The main mistakes of traditional measurement theory in weight calculation method

The traditional theory explains its measurement concept according to Fig 4.

Fig 4. Schematic diagram in traditional measurement theory

Because the observed value and the measured value are regarded as random variables, which directly leads to the philosophy of error classification. Therefore, it is believed that only the random error has variance and the systematic error has no variance. Finally, the weight calculation can only be based on the precision.

Compared with the concepts and principles in sections 2 and 3, we can see that the traditional measurement theory has the following conceptual logic mistakes:

- 1、 Any observed value x_i (or measured value) is a numerical value and belongs to a constant,

and there must be $\begin{cases} E(x_i) = x_i \\ \sigma^2(x_i) = 0 \end{cases}$. The traditional theory regards x_i as a random variable,

which violates the basic concept of probability theory. For example, in case 2, according to the concept of pure probability theory, there must be:

$$\begin{cases} E(x_1) = E(0.574) = 0.574 \\ \sigma^2(x_1) = \sigma^2(0.574) = 0 \\ E(x_2) = E(0.572) = 0.572 \\ \sigma^2(x_2) = \sigma^2(0.572) = 0 \\ E(x_3) = E(0.5733) = 0.5733 \\ \sigma^2(x_3) = \sigma^2(0.5733) = 0 \\ E(x_4) = E(0.5685) = 0.5685 \\ \sigma^2(x_4) = \sigma^2(0.5685) = 0 \end{cases}$$

It can be seen that using $\sigma^2(x_i)$ as the basis to determine the weight p_i is a kind of conceptual logic confusion.

- 2、Because x_i is a constant, the error $x_i - E(X)$ is a deviation, which is not different from the deviation $E(x) - x_T$. Therefore, the theory of error classification is not correct.
- 3、The precision $\sigma^2(x_i)$ in the traditional theory is actually the evaluation of the probability range of the deviation $x_i - E(x)$, rather than the dispersion of the observed value x_i . Therefore, the variance of deviation $E(x) - x_T$ is ignored when the weight is determined only by precision.

5. Conclusion

According to the concept of pure probability theory, both the observed value and the measured value are numerical values and belong to constant, while the error and true value are unknown and belong to random variable. The variance of error is the dispersion of all possible values of the error, and is the evaluation value of the probability interval where the error exists. Any error is a deviation and has its variance, so it can't be classified as systematic error or random error. Through the covariance propagation relationship between the observation error and the final error, it is a pure mathematical problem to give the weight assignment with the minimum variance, and the dilemma of traditional measurement theory has been solved.

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Figures

Figure 1

Conceptual sketch of new concept theory

Figure 2

Regularity and randomness of sine error

Figure 3

Measuring the diameter of copper wire

Figure 4

Schematic diagram in traditional measurement theory