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Parametric Quintic Spline for Time–Fractional Burger's and Coupled Burgers' Equations

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Abstract: In this paper, the solutions of time–fractional Burger's equation and time–fractional coupled Burgers' equations are obtained using the parametric quintic spline method with a local truncation error of order eight. Moreover, the stability analysis of the present method is demonstrated using Von–Neumann method. Also, to show the accuracy of this method, some examples with different cases for Burger's and coupled Burgers' equations are presented and compared with the previous methods.

Keywords: Parametric quintic spline method, Fractional Burger's equation, Fractional coupled Burger's equations, Von–Neumann stability analysis

1. Introduction

Consider the time–fractional Burger's equation (TFBE) [1– 17] defined as following:

$$D_t^{\alpha} u(x,t) + u(x,t)\, D_x u(x,t) - s\, D_{xx} u(x,t) = f(x,t), \;\; 0 < \alpha < 1 \tag{1.1}$$

Also, we consider the time–fractional coupled Burgers' equations (TFCBEs) are defined as [18– 25]:

$$D_t^{\alpha}\big(u(x,t)\big) = D_{xx}\big(u(x,t)\big) + 2u(x,t)\, D_x\big(u(x,t)\big) - D_x\big(u(x,t)\, v(x,t)\big), \tag{1.2}$$

$$D_t^{\beta}\big(v(x,t)\big) = D_{xx}\big(v(x,t)\big) + 2v(x,t)\, D_x\big(v(x,t)\big) - D_x\big(u(x,t)\, v(x,t)\big), \tag{1.3}$$

where s is a viscosity parameter and $D_t^{\alpha} u(x,t)$, $D_t^{\beta} v(x,t)$ are the Caputo fractional derivatives of order α and β [4, 7, 8, 9– 11, 15, 29, 31, 34, 35, 37] defined as following:

$$D_t^{\alpha} u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,\xi)}{\partial \xi} (t-\xi)^{-\alpha}\, d\xi, \;\; 0 < \alpha < 1, \tag{1.4}$$

by the same manner, we can define $D_t^{\beta} v(x,t)$.

Firstly, many numerical techniques are used to obtain the solutions of equations (1.1)– (1.3) such as adomian decomposition method (ADM) [1], variational iteration method (VIM) [2], cubic parametric spline (CPS) method [3], quadratic B–spline Galerkin method (QBSGM) [4], cubic trigonometric B–splines method (CTBSM) [7], Legendre–Galerkin spectral method (LGSM) [9], Crank–Nicolson approach (CNA) [14], finite difference method (FDM) [16], Chebyshev collocation method (CCM) [20], spectral collocation method (SCM) [22] … etc. For more details, Momani [1] investigated a non–perturbative analytical solutions of TFBE using ADM while Inc [2] solved it using VIM. El-Danaf and Hadhoud [3] used cubic parametric spline method to obtain the numerical solutions of equation (1.1). Moreover, the quadratic B–spline Galerkin method has been used to solve it by Esen and Tasbozan [4]. Also, Yokus and Kaya [5] applied the expansion method and the Cole–Hopf transformation to get its solutions. Furthermore, Hassani and Naraghirad [6] and Yaseen and Abbas [7] gave its solutions through an optimization method based on the generalized polynomials [6] and CTBSM [7]. Alsaedi et al. [8] gave smooth solutions of TFBE while Li et al. [9, 10] discussed its solution using LGSM [9] and the local discontinuous–Galerkin method (LDGM) [10]. A numerical technique based on an extended cubic B–spline function used to solve TFBE by Akram et al. [12] and Majeed et al. [13]. Also, Onal and Esen [14], Chen et al. [15], Yadav and Pandey [16] and Wang [17] approximated its solution using CNA [14], Fourier spectral method [15], FDM [16] and separation of variables method [17]. The backward substitution method (BSM) and FDM were used by Safari and Chen [18] to obtain the solutions of TFCBEs. Doha et al. [19] and Albuohimad and Adibi [20] established their solutions by Jacobi–Gauss–Lobatto collocation method (JGLCM) [19] and CCM [20]. Ahmed et. al [21, 22] solved them using Laplace–adomian decomposition method (LADM) [21], Laplace–variational iteration method (LVIM) [21] and SCM [22]. Also, the generalized differential transform method [23] and the meshfree spectral method [24] have been used to solve TFCBEs by Liu and Hou [23] and Hussain et al. [24]. Furthermore, Bekir and Guner [26] solved equations (1.1)– (1.3) using the (G'/G)–expansion method while Abdel-Salam and Hassan [27] solved equation (1.1) using the generalized exp–function method.

Secondly, non–polynomial splines are introduced to solve many fractional order partial differential equations such as fractional sub–diffusion problems [28–31], fractional diffusion–wave problems [32, 33], fractional Schrödinger equation [34, 35] and fractional differential equations [36, 37]. In this study, we used the parametric quintic spline method (PQSM) to get the solutions of TFBE and TFCBEs. Moreover, we demonstrated the stability and compared the results with other methods such as CPS [3], QBSGM [4] and BSM [18].

Finally, the structure of this paper is given as following: in Sections 2 and 3, the scheme of PQSM is presented in details, then we applied it to solve TFBE and TFCBEs. Sections 4 contains the stability analysis of the proposed method. The numerical results and the conclusion are given in sections 5 and 6.

2. Parametric Quintic Spline Method

Firstly, suppose that $x_i = x_0 + ih$ be the nodes of a uniform partition of the interval $[a, b]$ with n subintervals, where $h = \frac{x_n - x_0}{n}$, $x_0 = a$, $x_n = b$ and $i = 0, 1, 2, \dots, n$. Also, consider $t_j = jk$, where $j = 0, 1, 2, \dots$, $t \in [0, T]$ and $k = \Delta t = (t_{j+1} - t_j)$ is the time step. Secondly, let $Q_i(x, t_j)$ be the parametric quintic spline function defined as following:

$$Q_i(x, t_j) = a_i(t_j) + b_i(t_j)\,(x - x_i) + c_i(t_j)\,(x - x_i)^2 + d_i(t_j)\,(x - x_i)^3 + e_i(t_j)\,Sinh(\tau(x - x_i)) + f_i(t_j)\,Cosh(\tau(x - x_i)),$$
$$\text{for } i = 0, 1, 2, \dots, n-1 \text{ and } x \in [x_i, x_{i+1}], \tag{2.1}$$

where $a_i(t_j), b_i(t_j), c_i(t_j), d_i(t_j), e_i(t_j), f_i(t_j)$ are the spline coefficients and τ is a constant.

Suppose that $\Phi = h\,\tau$, $u_i^j = Q_i(x_i, t_j)$, $u_{i+1}^j = Q_i(x_{i+1}, t_j)$, $R_i^j = Q_i''(x_i, t_j)$, $R_{i+1}^j = Q_i''(x_{i+1}, t_j)$, $L_i^j = Q_i^{(4)}(x_i, t_j)$ and $L_{i+1}^j = Q_i^{(4)}(x_{i+1}, t_j)$. Hence, the spline coefficients can be determined as following:

$$a_i(t_j) = -\frac{L_i^j}{\tau^4} + u_i^j,$$

$$b_i(t_j) = \frac{1}{6\Phi\tau^3}\left[2(3 + \Phi^2)L_i^j + (-6 + \Phi^2)L_{i+1}^j - \Phi^2\tau^2(2R_i^j + R_{i+1}^j) + 6\tau^4(u_{i+1}^j - u_i^j)\right],$$

$$c_i(t_j) = \frac{1}{2}\left(-\frac{L_i^j}{\tau^2} + R_i^j\right),$$

$$d_i(t_j) = -\frac{1}{6\Phi\tau}\left(L_{i+1}^j - L_i^j + \tau^2(R_i^j - R_{i+1}^j)\right), \tag{2.2}$$

$$e_i(t_j) = \frac{1}{\tau^4}\left(-Coth(\Phi)\,L_i^j + Csch(\Phi)\,L_{i+1}^j\right)$$

and $f_i(t_j) = \frac{1}{\tau^4}(L_i^j)$.

Using the continuity of 1st and 3rd derivatives of the spline function at (x_i, t_j), we have $Q_{i-1}'(x_i, t_j) = Q_i'(x_i, t_j)$ and $Q_{i-1}^{(3)}(x_i, t_j) = Q_i^{(3)}(x_i, t_j)$, then for $i = 1, 2, 3, \dots, n-1$, we get

$$R_{i-1}^j + 4R_i^j + R_{i+1}^j = \frac{6}{h^2}(u_{i-1}^j - 2u_i^j + u_{i+1}^j) - 6h^2(\gamma_1\,L_{i-1}^j + 2\delta_1\,L_i^j + \gamma_1\,L_{i+1}^j), \tag{2.3}$$

$$R_{i-1}^{j} - 2R_{i}^{j} + R_{i+1}^{j} = h^{2}\left(\gamma\, L_{i-1}^{j} + 2\delta\, L_{i}^{j} + \gamma\, L_{i+1}^{j}\right), \tag{2.4}$$

where

$$\gamma = \frac{1}{\Phi^{2}}(\Phi\ \mathrm{Csch}(\Phi) - 1),\ \ \delta = \frac{1}{\Phi^{2}}\left(1 - \Phi\ \mathrm{Coth}(\Phi)\right),$$

$$\gamma_{1} = \frac{-1}{\Phi^{2}}\left(\frac{1}{6} + \gamma\right) \ \text{ and } \ \delta_{1} = \frac{-1}{\Phi^{2}}\left(\frac{1}{3} + \delta\right).$$

By multiplying equation (2.3) by (γ) and equation (2.4) by $(-6\gamma_{1})$, then subtracting these equations, we get

$$L_{i}^{j} = \frac{1}{12h^{2}(\gamma_{1}\delta - \gamma\,\delta_{1})}\Big[(\gamma + 6\gamma_{1})\, R_{i-1}^{j} + 4(\gamma - 3\gamma_{1})\, R_{i}^{j} + (\gamma + 6\gamma_{1})\, R_{i+1}^{j} - \frac{6\gamma}{h^{2}}\left(u_{i-1}^{j} - 2u_{i}^{j} + u_{i+1}^{j}\right)\Big], \tag{2.5}$$

similarly, we can get L_{i-1}^{j} and L_{i+1}^{j}.

By substituting L_{i}^{j} , L_{i-1}^{j} and L_{i+1}^{j} into equation (2.4), we get

$$(\gamma + 6\gamma_{1})\left(R_{i-2}^{j} + R_{i+2}^{j}\right) + 2(2\,\gamma + \delta - 6\,\gamma_{1} + 6\,\delta_{1})\left(R_{i-1}^{j} + R_{i+1}^{j}\right) + 2(\,\gamma + 4\delta\ + 6\,\gamma_{1} - 12\,\delta_{1})R_{i}^{j} = \frac{6}{h^{2}}\left[\gamma\left(u_{i-2}^{j} + u_{i+2}^{j}\right) + 2(\delta - \gamma)\left(u_{i-1}^{j} + u_{i+1}^{j}\right) + 2(\gamma - 2\delta)u_{i}^{j}\right], \text{ for } i = 2, 3, 4, \ldots, n-2, \tag{2.6}$$

The system of equations (2.6) gives $(n-3)$ equations in $(n-3)$ unknowns. Furthermore, to have a unique solution, we need two more equations which can be defined as following:

$$\sum_{l=0}^{3} a_{l}\, u_{l}^{j} + h^{2} \sum_{l=0}^{4} b_{l} R_{l}^{j} + t_{1} = 0, \text{ for } \ i = 1, \tag{2.7}$$

$$\sum_{l=0}^{3} a_{l}\, u_{n-l}^{j} + h^{2} \sum_{i=0}^{4} b_{l} R_{n-l}^{j} + t_{n-1} = 0, \text{ for } \ i = n-1, \tag{2.8}$$

By expanding equations (2.7) and (2.8) by Taylor's approximation about x_{1} and x_{n-1}, respectively, we get

$$(a_{0}, a_{1}, a_{2}, a_{3}) = (0, 1, -2, 1),$$

$$(b_{0}, b_{1}, b_{2}, b_{3}, b_{4}) = \left(\frac{1}{240}, -\frac{1}{10}, -\frac{97}{120}, -\frac{1}{10}, \frac{1}{240}\right),$$

$$t_{1} = \frac{31}{60480}(h^{8})\, u^{(8)}\left(x_{1}, t_{j}\right),$$

and $t_{n-1} = \frac{31}{60480}(h^{8})\, u^{(8)}\left(x_{n-1}, t_{j}\right)$.

The local truncation error (LTE) of PQSM can be established by expanding equation (2.6) by Taylor's approximation about x_i, then we have

$$LTE_i = \frac{1}{6}h^2(\gamma + \delta + 12\gamma_1 + 12\delta_1)u^{(4)}(x_i, t_j) + \frac{1}{180}h^4(19\gamma + 4\delta + 210\gamma_1 + 30\delta_1)u^{(6)}(x_i, t_j) + \frac{1}{30240}h^6(571\gamma + 25\delta + 5208\gamma_1 + 168\delta_1)u^{(8)}(x_i, t_j) + \frac{1}{907200}h^8(1439\gamma + 14\delta + 11430\gamma_1 + 90\delta_1)u^{(10)}(x_i, t_j) + \cdots, \tag{2.9}$$

By assuming $L_1 = \gamma + \delta + 12\gamma_1 + 12\delta_1$, $L_2 = 19\gamma + 4\delta + 210\gamma_1 + 30\delta_1$ and $L_3 = 571\gamma + 25\delta + 5208\gamma_1 + 168\delta_1$. Hence, for $L_1 = L_2 = L_3 = 0$, we get

$$\gamma = \frac{31}{95}\delta,\ \gamma_1 = -\frac{109}{2850}\delta,\ \delta_1 = -\frac{103}{1425}\delta, \tag{2.10}$$

and $LTE_i = \left(\frac{79}{1795500}\delta\right) h^8\, u^{(10)}(x_i, t_j) + O(h^{10})$,

hence, $LTE_i = O(h^8)$.

3. Applying PQSM on Fractional Burgers' Equations

To investigate the solutions of TFBE and TFCBEs using PQSM method, we consider the discretization of the fractional derivative equation (1.4) is defined as in [3] as following:

$$D_t^{\alpha}u(x,t) = \frac{k^{-\alpha}}{(1-\alpha)\,\Gamma(1-\alpha\,)} \sum_{m=0}^{j-1}\left[\left((j-m)^{1-\alpha} - (j-m-1)^{1-\alpha}\right)\left(u(x_i, t_{m+1}) - u(x_i, t_m)\right)\right]. \tag{3.1}$$

3.1 Time fractional Burger's equation

From equation (1.1), we have

$$R_i^j = (u_{xx})_i^j \ = \frac{1}{s}\left(D_t^{\alpha}U_i^j + U_i^j(U_x)_i^j - f(x_i, t_j)\right), \tag{3.2}$$

by using CNA, we can rewrite equation (3.2) as following:

$$R_i^j = \frac{1}{s}\left(D_t^{\alpha}U_i^j + \frac{1}{2}U_i^j\left[(U_x)_i^j + (U_x)_i^{j-1}\right] - f(x_i, t_j)\right), \tag{3.3}$$

by substituting $(D_t^{\alpha}u)$ from equation (3.1) into equation (3.3), we get

$$R_i^j = \frac{1}{s}\left[\left(\frac{k^{-\alpha}}{(1-\alpha)\,\Gamma(1-\alpha\,)}\right)\sum_{m=0}^{j-1}\left[\left((j-m)^{1-\alpha} - (j-m-1)^{1-\alpha}\right)\left(U_i^{m+1} - U_i^m\right)\right]\right] + \left(\frac{1}{2s}\right)U_i^j\left[(U_x)_i^j + (U_x)_i^{j-1}\right] - \left(\frac{1}{s}\right)f(x_i, t_j), \tag{3.4}$$

where $i = 0, 1, 2, \ldots, n$, $j = 1, 2, 3, \ldots$, and $(U_x)_i^j = \begin{cases} \frac{1}{h}\left(U_{i+1}^j - U_i^j\right), & i = 0, \\ \frac{1}{2h}(U_{i+1}^j - U_{i-1}^j), & 1 \le i \le n-1, \\ \frac{1}{h}\left(U_i^j - U_{i-1}^j\right), & i = n. \end{cases}$

By substituting R_i^j from equation (3.4) into equations (2.6)– (2.8) and solving this nonlinear system with the initial and boundary conditions using any numerical method such as Newton–Raphson method, we get the solution of time–fractional Burger's equation (1.1).

3.2 Time fractional Coupled Burgers' equations

By using CNA, we can rewrite equations (1.2) and (1.3) as following:

$$R_{1}{}_i^j = D_t^\alpha\, U_i^j - \left((U\, U_x)_i^j + (U\, U_x)_i^{j-1}\right) + \frac{1}{2}\left[\left((U\, V_x)_i^j + (U\, V_x)_i^{j-1}\right) + \left((V\, U_x)_i^j + (V\, U_x)_i^{j-1}\right)\right], \tag{3.5}$$

$$R_{2}{}_i^j = D_t^\beta\, V_i^j - \left((V\, V_x)_i^j + (V\, V_x)_i^{j-1}\right) + \frac{1}{2}\left[\left((U\, V_x)_i^j + (U\, V_x)_i^{j-1}\right) + \left((V\, U_x)_i^j + (V\, U_x)_i^{j-1}\right)\right], \tag{3.6}$$

where $R_{1}{}_i^j = (u_{xx})_i^j$ and $R_{2}{}_i^j = (v_{xx})_i^j$.

The nonlinear terms in equations (3.5) and (3.6) can be defined as in [38] as following:

$$(A\, B_x)_i^j = A_i^j\, B_{x}{}_i^j,$$
$$(A\, B_x)_i^{j-1} = A_i^j\, B_{x}{}_i^{j-1} + A_i^{j-1}\, B_{x}{}_i^j - A_i^j\, B_{x}{}_i^j, \tag{3.7}$$

where the symbols A and B in equation (3.7) can be used for U or V.

From equation (3.7) into equations (3.5) and (3.6) and by substituting $D_t^\alpha U_i^j$ and $D_t^\beta V_i^j$ from equation (3.1), we get

$$R_{1}{}_i^j = \left(\frac{k^{-\alpha}}{(1-\alpha)\,\Gamma(1-\alpha)}\right) \sum_{m=0}^{j-1}[((j-m)^{1-\alpha} - (j-m-1)^{1-\alpha})\,(U_i^{m+1} - U_i^m)] - \left[U_i^{j-1}\, U_{x}{}_i^j + U_i^j\, U_{x}{}_i^{j-1}\right] + \frac{1}{2}\left[U_i^{j-1}\, V_{x}{}_i^j + U_i^j\, V_{x}{}_i^{j-1} + V_i^{j-1}\, U_{x}{}_i^j + V_i^j\, U_{x}{}_i^{j-1}\right], \tag{3.8}$$

$$R_{2}{}_i^j = \left(\frac{k^{-\beta}}{(1-\beta)\,\Gamma(1-\beta)}\right) \sum_{m=0}^{j-1}[((j-m)^{1-\alpha} - (j-m-1)^{1-\alpha})\,(V_i^{m+1} - V_i^m)] - \left[V_i^{j-1}\, V_{x}{}_i^j + V_i^j\, V_{x}{}_i^{j-1}\right] + \frac{1}{2}\left[U_i^{j-1}\, V_{x}{}_i^j + U_i^j\, V_{x}{}_i^{j-1} + V_i^{j-1}\, U_{x}{}_i^j + V_i^j\, U_{x}{}_i^{j-1}\right], \tag{3.9}$$

where $i = 0, 1, 2, \dots, n,\ j = 1, 2, 3, \dots$

$$\text{and}\quad C_{x}{}_i^j = \begin{cases} \frac{1}{h}\left(C_{i+1}^j - C_i^j\right), & i = 0, \\ \frac{1}{2h}(C_{i+1}^j - C_{i-1}^j), & 1 \le i \le n-1, \\ \frac{1}{h}\left(C_i^j - C_{i-1}^j\right), & i = n. \end{cases} \tag{3.10}$$

where the symbol C in equation (3.10) can be used for U or V.

Replacing R_i^j in equations (2.6)– (2.8) by ${R_1}_i^j$ once and ${R_2}_i^j$ again, then substituting ${R_1}_i^j$ and ${R_2}_i^j$ from equations (3.8) and (3.9). Now, we can solve this nonlinear system with the initial and boundary conditions using Newton–Raphson method to obtain the solutions of time–fractional coupled Burgers' equations (1.2) and (1.3).

4. Von–Neumann Stability Analysis

In this section, we analyze the stability of PQSM using Von–Neumann method. For this purpose, firstly, we need to linearize the nonlinear terms in equation (3.4) as in [3] as following:

$$R_i^j = \sum_{m=0}^{j-1} \left[\frac{k^{-\alpha}}{s(1-\alpha)\,\Gamma(1-\alpha)}\left((j-m)^{1-\alpha} - (j-m-1)^{1-\alpha}\right)\left(U_i^{m+1} - U_i^m\right)\right] + \left(\frac{\varphi_1}{2s\,h}\right)\left[U_{i+1}^j - U_{i-1}^j + U_{i+1}^{j-1} - U_{i-1}^{j-1}\right], \text{ for } 1 \le i \le n-1 \text{ and } j \ge 1, \quad (4.1)$$

where $\varphi_1 = U_i^j$ is a local constant and for simplicity, we assume $f(x_i, t_j) = 0$.

Secondly, we consider U_i^j is defined as following:

$$U_i^j = \xi^j e^{l(kih)}, \quad (4.2)$$

where $l = \sqrt{-1}$ and ξ is the growth factor.

Substituting U_i^j from equation (4.2) into (4.1), we get

$$R_i^j = e^{l(kih)}\left[\sum_{m=0}^{j-1} \varphi_2\left[\Psi_{j,m}^{\alpha}\left(\xi^{m+1} - \xi^m\right)\right] + \left(\frac{\varphi_1}{2s\,h}\right)\left(\xi^j + \xi^{j-1}\right)\left(e^{l(kh)} - e^{-l(kh)}\right)\right], \text{ for } 0 \le i \le n \text{ and } j \ge 1, \quad (4.3)$$

where $\Psi_{j,m}^{\alpha} = (j-m)^{1-\alpha} - (j-m-1)^{1-\alpha}$ and $\varphi_2 = \frac{k^{-\alpha}}{s\,(1-\alpha)\,\Gamma(1-\alpha)}$.

Substituting U_i^j and R_i^j from equations (4.2) and (4.3) into equation (2.6), we get

$$\left[\sum_{m=0}^{j-1} \varphi_2\left[\Psi_{j,m}^{\alpha}\left(\xi^{m+1} - \xi^m\right)\right] + \left(\frac{\varphi_1}{2s\,h}\right)\left(\xi^j + \xi^{j-1}\right)\left(e^{l(kh)} - e^{-l(kh)}\right)\right]\left[\rho_1\left(e^{l(-2kh)} + e^{l(2kh)}\right) + \rho_2\left(e^{l(-kh)} + e^{l(kh)}\right) + \rho_3\right] = \frac{6}{h^2}\xi^j\left[\gamma\left(e^{l(-2kh)} + e^{l(2kh)}\right) + 2(\delta - \gamma)\left(e^{l(-kh)} + e^{l(kh)}\right) + 2(\gamma - 2\delta)\right], \text{ for } j = 1, 2, 3, \ldots, \quad (4.4)$$

where $\rho_1 = \gamma + 6\gamma_1$, $\rho_2 = 2(2\,\gamma + \delta - 6\,\gamma_1 + 6\,\delta_1)$

and $\rho_3 = 2(\gamma + 4\delta + 6\gamma_1 - 12\delta_1)$.

Since $\left(e^{l(-kh)} + e^{l(kh)}\right) = 2Cos(kh)$ and $\left(e^{l(kh)} - e^{-l(kh)}\right) = 2l\,Sin(kh)$, then we have

$$\left[\sum_{m=0}^{j-1} \varphi_2\left[\Psi_{j,m}^{\alpha}\left(\xi^{m+1} - \xi^{m}\right)\right] + l\left(\frac{\varphi_1}{s\,h}\right)\left(\xi^{j} + \xi^{j-1}\right) Sin(kh)\right]\left[2\rho_1 Cos(2kh) + 2\rho_2 Cos(kh) + \rho_3\right] = \frac{6}{h^2}\xi^{j}\left[2\gamma\, Cos(2kh) + 4(\delta - \gamma)\, Cos(kh) + 2(\gamma - 2\delta)\right],$$
for $j = 1, 2, 3, \ldots$, (4.5)

Putting $j = 1$ in equation (4.5), we have

$$\left[\left(\varphi_2\Psi_{1,0}^{\alpha} + l\left(\frac{\varphi_1}{s\,h}\right) Sin(kh)\right)\left(2\rho_1 Cos(2kh) + 2\rho_2 Cos(kh) + \rho_3\right) - \frac{6}{h^2}\left(2\gamma\, Cos(2kh) + 4(\delta - \gamma)\, Cos(kh) + 2(\gamma - 2\delta)\right)\right]\xi^1 = \left[\left(\varphi_2\Psi_{1,0}^{\alpha} - l\left(\frac{\varphi_1}{v\,h}\right) Sin(kh)\right)\left(2\rho_1 Cos(2kh) + 2\rho_2 Cos(kh) + \rho_3\right)\right]\xi^0, \quad (4.6)$$

hence,

$$\left|\frac{\xi^1}{\xi^0}\right| = \left|\frac{(\varphi_2 - l\omega_1)\varphi_3}{(\varphi_2 + l\omega_1)\varphi_3 - \varphi_4}\right|, \quad (4.7)$$

where $\Psi_{1,0}^{\alpha} = (1)^{1-\alpha} = 1$, $\omega_1 = \left(\frac{\varphi_1}{s\,h}\right) Sin(\theta)$, $\varphi_3 = (2\rho_1 Cos(2\theta) + 2\rho_2 Cos(\theta) + \rho_3)$, $\varphi_4 = \frac{6}{h^2}\left(2\gamma\, Cos(2\theta) + 4(\delta - \gamma)\, Cos(\theta) + 2(\gamma - 2\delta)\right)$ and $\theta = kh$.

Now, we can study the worth cases for equation (4.7), when $Cos(\theta) = \pm 1$ as following:

i. For $\theta = 2\pi n$, $n = 0, \pm 1, \pm 2, \ldots$, we get

$\omega_1 = 0$, $\varphi_3 = 12(\gamma + \delta)$ and $\varphi_4 = 0$,

hence, $\left|\frac{\xi^1}{\xi^0}\right| = 1$,

ii. For $\theta = \pi n$, $n = \pm 1, \pm 3, \pm 5, \ldots$, we get

$\omega_1 = 0$, $\varphi_3 = 4(-\gamma + \delta + 12\gamma_1 - 12\delta_1)$ and $\varphi_4 = -\frac{48}{h^2}(\delta - \gamma)$,

hence, $\left|\frac{\xi^1}{\xi^0}\right| = \frac{\varphi_2\varphi_3}{\varphi_2\varphi_3 + \frac{48}{h^2}(\delta - \gamma)} < 1$, for $\delta - \gamma > 0$,

therefore, $\left|\frac{\xi^1}{\xi^0}\right| \leq 1$ and $\delta > \gamma$. (4.8)

Similarly, putting $j = 2$ in equation (4.5), we get

$$\left[\varphi_2\left(\Psi_{2,0}^{\alpha}\,(\xi^1 - \xi^0)\right) + \varphi_2\left(\Psi_{2,1}^{\alpha}(\xi^2 - \xi^1)\right) + l\,\omega_1(\xi^2 + \xi^1)\right]\varphi_3 = (\varphi_4)\xi^2, \tag{4.9}$$

since $\Psi_{2,0}^{\alpha} = ((2)^{1-\alpha} - 1)$, $\Psi_{2,1}^{\alpha} = 1$ and from equation (4.8), we can take $\xi^1 = \xi^0$ and $\delta > \gamma$, then we have

$$[\varphi_2(\xi^2 - \xi^0) + l\omega_1(\xi^2 + \xi^0)]\varphi_3 = (\varphi_4)\xi^2, \tag{4.10}$$

hence,

$$\left|\frac{\xi^2}{\xi^0}\right| = \left|\frac{(\varphi_2 - l\omega_1)\varphi_3}{(\varphi_2 + l\omega_1)\varphi_3 - \varphi_4}\right| \le 1. \tag{4.11}$$

In general, for $j = 1, 2, 3, \ldots$, we have $\left|\frac{\xi^j}{\xi^0}\right| \le 1$ and $\delta > \gamma$. Thus, the proposed scheme is stable.

5. Numerical Results

In this section, the numerical solutions of TFBE and TFCBEs are obtained by using PQSM method as introduced in sections 2 and 3. In addition, we discuss their solutions in two cases according to the spline parameters given by equation (2.10) as following:

Case1: as in [29, 33, 35], if we take $\gamma + \delta = \frac{1}{2}$, we obtain

$$\gamma = \frac{31}{252}\,,\ \delta = \frac{95}{252},\ \ \gamma_1 = -\frac{109}{7560},\ \ \delta_1 = -\frac{103}{3780}\ \text{and}\ LTE = \frac{79}{4762800}h^8.$$

Case2: let $\delta = 0.001$, hence we have

$$\gamma = \frac{31}{95000},\ \ \gamma_1 = -\frac{109}{2850000},\ \ \delta_1 = -\frac{103}{1425000}\ \text{and}\ LTE = \frac{79}{1795500000}h^8.$$

To show the accuracy of the proposed method, we obtain the absolute errors or the error norms L_2 and L_∞ in each example. The error norms are demonstrated as following:

$$L_\infty = Max_{1\le i\le n-1}\left|(u_{Exact})_i^j - U_i^j\right|, \tag{5.1}$$

$$L_2 = \sqrt{h\sum_{i=1}^{n-1}\left((u_{Exact})_i^j - U_i^j\right)^2}. \tag{5.2}$$

Problem 5.1: Consider the time fractional Burger's equation (1.1) [4] with

$$f(x,t) = \frac{2}{\Gamma(3-\alpha)}(t^{2-\alpha}e^{x}) + t^{4}e^{2x} - s\,(t^{2}e^{x}), \tag{5.3}$$

the initial condition is:

$$u(x,0) = 0,\ \ 0 \le x \le 1, \tag{5.4}$$

and the boundary conditions are:

$$u(0,t) = t^{2} \text{ and } u(1,t) = e\,t^{2}, \tag{5.5}$$

The exact solution is: $u(x,t) = t^{2}e^{x}$. (5.6)

Table 1 contains the exact and approximated solutions of problem 5.1 by using PQSM with $k = 0.002$ and QBSGM [4] with $k = 0.00025$ at $t = 1$, $s = 1, \alpha = 0.5$ and $h = 0.05, 0.025$ and 0.0125. Also, table 2 illustrate the numerical results for $t = 1, s = 1, \alpha = 0.5, n = 80$ and different values of k . these results show that the L_∞ error norm at $n = 80$ for PQSM with $k = 0.002$ is $O(10^{-7})$ while it's $O(10^{-4})$ for QBSGM [4] with $k = 0.00025$, hence PQSM is better than QBSGM [4]. Moreover, the numerical results for $\alpha = 0.25$, $\alpha = 0.75, s = 1, h = 0.025$ and $t = 1$ is shown in table 3. Figure 1 shows the relation between exact solution and PQSM solution. Also, the absolute errors behavior for $\alpha = 0.5$ and 0.75 is given in figure 2. These computations are obtained according to case 1.

Table 1. the numerical solutions and error norms of problem 5.1 for $s = 1, \alpha = 0.5$ and $t = 1$

x	PQSM ($k = 0.002$)			QBSGM [4] ($k = 0.00025$)			Exact
	$n = 20$	$n = 40$	$n = 80$	$n = 20$	$n = 40$	$n = 80$	
0.1	1.105145	1.105166	1.105171	1.105287	1.105216	1.105197	1.105170918
0.2	1.221357	1.221394	1.221403	1.221644	1.221493	1.221455	1.221402758
0.3	1.349794	1.349846	1.349860	1.350217	1.349992	1.349935	1.349858808
0.4	1.491745	1.491809	1.491825	1.492287	1.491996	1.491922	1.491824698
0.5	1.648629	1.648703	1.648722	1.649270	1.648922	1.648838	1.648721271
0.6	1.822021	1.822099	1.822119	1.822727	1.822342	1.822247	1.822118800
0.7	2.013657	2.013733	2.013753	2.014378	2.013979	2.013882	2.013752707
0.8	2.225462	2.225523	2.225541	2.226118	2.225747	2.225661	2.225540928
0.9	2.459564	2.459593	2.459603	2.460020	2.459745	2.459680	2.459603111
L_2	3.11×10^{-4}	9.06×10^{-5}	3.90×10^{-6}	8.48×10^{-4}	1.62×10^{-4}	9.26×10^{-5}	
L_∞	9.83×10^{-5}	2.04×10^{-5}	7.16×10^{-7}	6.25×10^{-4}	2.27×10^{-4}	1.33×10^{-4}	

Table 2. the numerical solutions and error norms of problem 5.1 for $s = 1, \alpha = 0.5$, $n = 80$ and $t = 1$

$\boldsymbol{x}$	PQSM	QBSGM [4]		
	$\boldsymbol{k = 0.002}$	$\boldsymbol{k = 0.002}$	$\boldsymbol{k = 0.001}$	$\boldsymbol{k = 0.00025}$
$\mathbf{0.1}$	1.105171	1.105356	1.105276	1.105216
$\mathbf{0.2}$	1.221403	1.221768	1.221611	1.221493
$\mathbf{0.3}$	1.349860	1.350395	1.350164	1.349992
$\mathbf{0.4}$	1.491825	1.492516	1.492218	1.491996
$\mathbf{0.5}$	1.648722	1.649543	1.649188	1.648922
$\mathbf{0.6}$	1.822119	1.823031	1.822636	1.822342
$\mathbf{0.7}$	2.013753	2.014687	2.014282	2.013979
$\mathbf{0.8}$	2.225541	2.226387	2.226020	2.225747
$\mathbf{0.9}$	2.459603	2.460180	2.459931	2.459745
$\boldsymbol{L_2}$	3.90×10^{-6}	6.61×10^{-4}	3.75×10^{-4}	9.26×10^{-5}
$\boldsymbol{L_\infty}$	7.16×10^{-7}	9.37×10^{-4}	5.30×10^{-4}	1.33×10^{-4}

Table 3. the numerical solutions and error norms of problem 5.1 for $s = 1, n = 40$ and $t = 1$

$\boldsymbol{x}$	PQSM ($\boldsymbol{k = 0.002}$)		QBSGM [4] ($\boldsymbol{k = 0.00025}$)	
	$\alpha = \mathbf{0.25}$	$\alpha = \mathbf{0.75}$	$\alpha = \mathbf{0.25}$	$\alpha = \mathbf{0.75}$
$\mathbf{0.1}$	1.105164	1.105179	1.105217	1.105216
$\mathbf{0.2}$	1.221390	1.221418	1.221495	1.221493
$\mathbf{0.3}$	1.349841	1.349880	1.349995	1.349990
$\mathbf{0.4}$	1.491802	1.491850	1.492000	1.491993
$\mathbf{0.5}$	1.648696	1.648749	1.648928	1.648920
$\mathbf{0.6}$	1.822091	1.822147	1.822348	1.822339
$\mathbf{0.7}$	2.013725	2.013779	2.013984	2.013977
$\mathbf{0.8}$	2.225517	2.225563	2.225750	2.225744
$\mathbf{0.9}$	2.459589	2.459618	2.459747	2.459744
$\boldsymbol{L_2}$	1.26×10^{-4}	1.33×10^{-4}	1.65×10^{-4}	1.60×10^{-4}
$\boldsymbol{L_\infty}$	2.80×10^{-5}	2.85×10^{-5}	2.33×10^{-4}	2.25×10^{-4}

Problem 5.2: Consider the time fractional Burger's equation (1.1) [3] with

$f(x,t) = 0,$

the initial condition is:

$$u(x,0)=\frac{\mu+\gamma+(\gamma-\mu)\mathrm{Exp}\left[\frac{\mu}{s}(x-\xi)\right]}{1+\mathrm{Exp}\left[\frac{\mu}{s}(x-\xi)\right]},\ -3\le x\le 3, \tag{5.7}$$

and the boundary conditions are given as following:

i. For $\alpha=1$,

$$u(-3,t)=\frac{\mu+\gamma+(\gamma-\mu)\mathrm{Exp}\left[\frac{\mu}{s}(-3-\gamma\, t-\xi)\right]}{1+\mathrm{Exp}\left[\frac{\mu}{s}(-3-\gamma\, t-\xi)\right]}, \tag{5.8}$$

and $$u(3,t)=\frac{\mu+\gamma+(\gamma-\mu)\mathrm{Exp}\left[\frac{\mu}{s}(3-\gamma\, t-\xi)\right]}{1+\mathrm{Exp}\left[\frac{\mu}{s}(3-\gamma\, t-\xi)\right]}, \tag{5.9}$$

ii. For $0<\alpha<1$,

$$u(-3,t)\cong 0.699993+(1.07\times 10^{-5})\frac{t^{\alpha}}{\Gamma(1+\alpha)}-(9.67\times 10^{-6})\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}+(1.16\times 10^{-5})\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}, \tag{5.10}$$

and

$$u(3,t)\cong 0.100815+(1.3\times 10^{-3})\frac{t^{\alpha}}{\Gamma(1+\alpha)}+(1.17\times 10^{-3})\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}-(5.72\times 10^{-6})\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}, \tag{5.11}$$

The exact solution is: $u(x,t)=\frac{\mu+\gamma+(\gamma-\mu)\mathrm{Exp}\left[\frac{\mu}{s}(x-\gamma\, t-\xi)\right]}{1+\mathrm{Exp}\left[\frac{\mu}{s}(x-\gamma\, t-\xi)\right]}$, for $\alpha=1$. (5.12)

The approximated solutions of problem 5.2 using PQSM and CPS [3] are given in tables 4– 6 with the following conditions: $h=k=0.01$, $s=0.1$, $\mu=0.3$, $\gamma=0.4$, $\xi=0.8$ and $\alpha=0.2\,,0.8$ and 1. These results indicate that PQSM is the most accurate. Table 4 shows that the maximum absolute errors of PQSM is $O(h^{-4})$ while it's $O(h^{-3})$ for CPS [3]. The exact solution and the numerical solution using PQSM for different values of α are given in figures 4 and 5. These results are determined according to case 2.

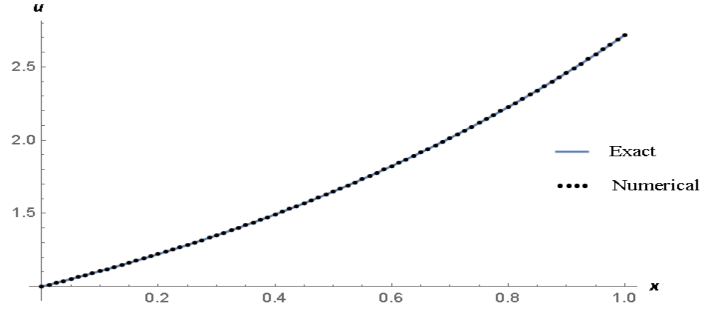

Fig. 1. The exact solution and the numerical solution using PQSM of problem 5.1 for $h=0.0125$, $k=0.002$, $\alpha=0.5$ and $t=1$.

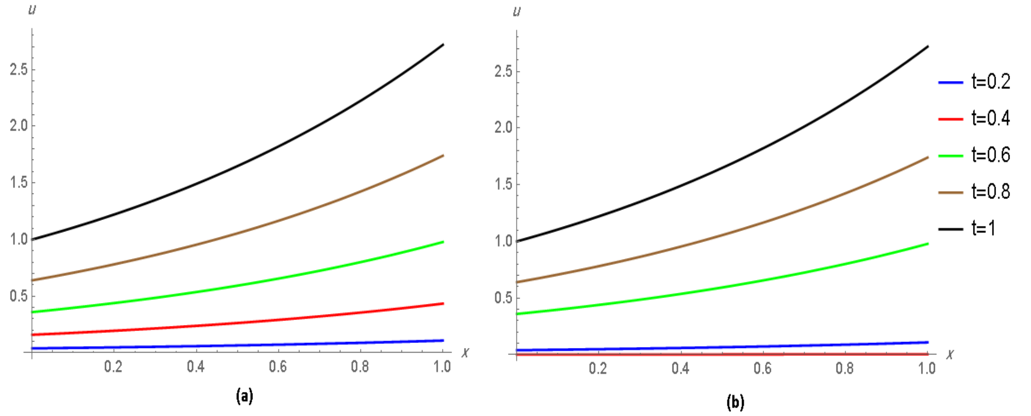

Fig. 2. The approximated solutions of problem 5.1 using PQSM for $h = 0.025$, $k = 0.002$, **(a)** $\alpha = 0.5$ and **(b)** $\alpha = 0.75$.

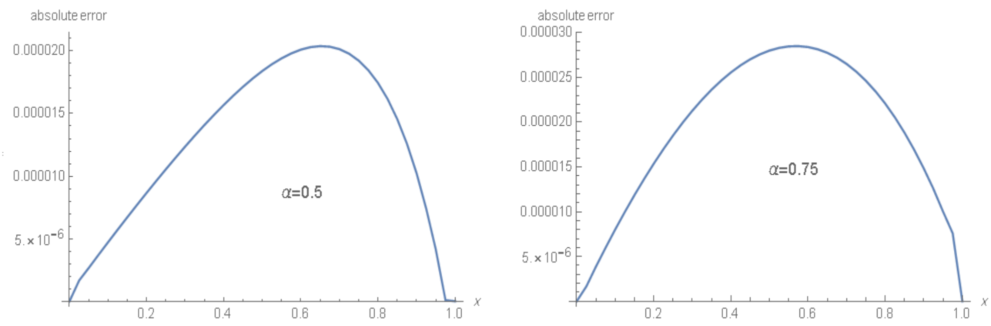

Fig. 3. The absolute errors distribution of problem 5.1 for $h = 0.025$, $k = 0.002$, and $t = 1$.

Table 4. the absolute errors of problem 5.2 for $\alpha = 1, s = 0.1,\ \mu = 0.3, \gamma = 0.4$ and $\xi = 0.8$

t	PQSM	CPS [3]
1.0	4.346×10^{-4}	4.632×10^{-3}
2.0	6.528×10^{-4}	5.267×10^{-3}
2.5	7.264×10^{-4}	5.569×10^{-3}
3.0	7.840×10^{-4}	5.857×10^{-3}

Table 5. the numerical solutions and the absolute errors of problem 5.2 for $t = 3$, $\alpha = 1, s = 0.1,\ \mu = 0.3, \gamma = 0.4$ and $\xi = 0.8$

x	Exact Solution	PQSM	CPS [3]	Absolute errors of	
				PQSM	CPS [3]
-1.80	0.6999932828	0.69999336	0.69999302	7.614×10^{-8}	2.608×10^{-7}
-1.50	0.6999834786	0.69998367	0.69998276	1.938×10^{-7}	7.193×10^{-7}
-0.96	0.6999165251	0.69991749	0.69991259	9.657×10^{-7}	3.931×10^{-6}
-0.48	0.6989641713	0.69897337	0.69963109	9.198×10^{-6}	6.669×10^{-4}
0.00	0.6985164261	0.69852846	0.69844644	1.203×10^{-5}	6.998×10^{-5}
0.48	0.6937877575	0.69381151	0.69349918	2.375×10^{-5}	2.886×10^{-4}
0.96	0.6746261369	0.67458239	0.67349066	4.375×10^{-5}	1.135×10^{-3}
1.50	0.5905446857	0.58997244	0.58651442	5.722×10^{-4}	4.030×10^{-3}
1.80	0.4873937837	0.48661686	0.48167270	7.769×10^{-4}	5.721×10^{-3}

Table 6. the numerical solutions of problem 5.2 for $t = 2$, $s = 0.1$, $\mu = 0.3$, $\gamma = 0.4$ and $\xi = 0.8$

x	$\alpha = 0.2$		$\alpha = 0.8$	
	PQSM	CPS [3]	PQSM	CPS [3]
-1.80	0.69990818	0.69990801	0.69995843	0.69995837
-1.50	0.69976764	0.69976704	0.69989353	0.69989305
-0.96	0.69881774	0.69881444	0.69945724	0.69945423
-0.48	0.68593695	0.69505832	0.69348579	0.69771999
0.00	0.68023291	0.68019612	0.69080108	0.69075808
0.48	0.62975137	0.62969696	0.66570607	0.66557967
0.96	0.50954256	0.50952078	0.59278878	0.59246071
1.50	0.32247745	0.32242313	0.42059669	0.41986080
1.80	0.23615545	0.23607121	0.30646604	0.30566973

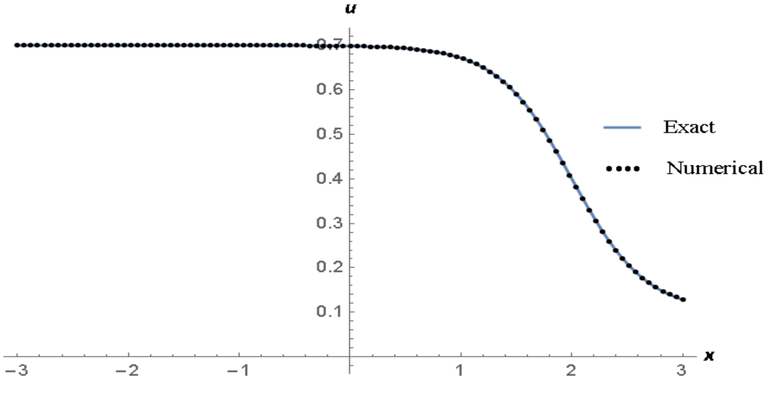

Fig. 4. The exact solution and the numerical solution using PQSM of problem 5.2 for $h = k = 0.01$, $\alpha = 1$ and $t = 3$.

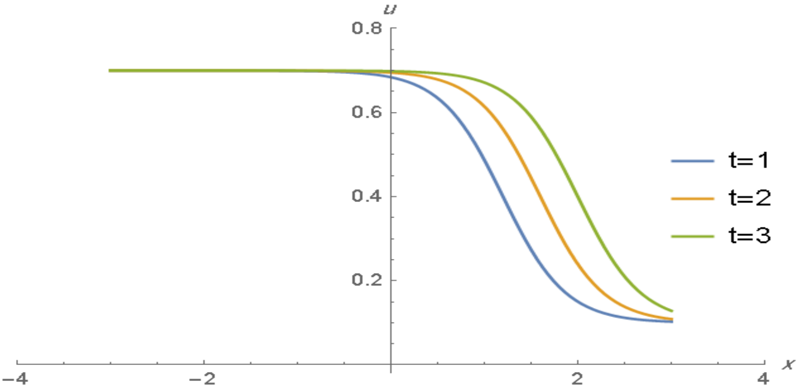

Fig. 5. The approximated solutions of problem 5.2 using PQSM for $\alpha = 1$, $h = k = 0.01$ and $t = 1, 2$ and 3.

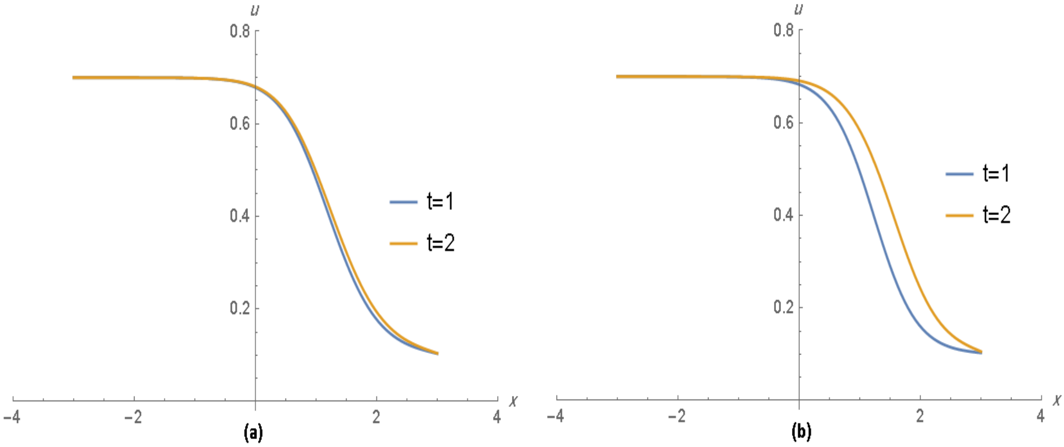

Fig. 6. The approximated solutions of problem 5.2 using PQSM for $h = k = 0.01$, **(a)** $\alpha = 0.2$ and **(b)** $\alpha = 0.8$

Problem 5.3: Consider the time fractional coupled Burgers' equations (1.2) and (1.3) [18] with the initial condition is:

$$u(x,0) = v(x,0) = Sin(x),\ 0 \le x \le 1, \tag{5.13}$$

and the boundary conditions are:

$$u(0,t) = v(0,t) = 0 \ \text{ and } \ u(1,t) = v(1,t) = Sin(1)\, e^{-t}, \tag{5.14}$$

The exact solution is:

$$u(x,t) = v(x,t) = Sin(x)\, e^{-t}. \tag{5.15}$$

Table 7 illustrate the exact solution and the numerical solution of problem 5.3 given by PQSM for $\alpha = \beta = 1$ and at time $t = 1$. A comparison between PQSM and BSM [18] is given in table 8 for $\alpha = \beta = 0.6, 0.9$ and 1 which conclude that PQSM gave better results than BSM [18]. Also, the approximated solution for $\alpha = \beta = 0.2, 0.6$ and 0.9 are shown in table 9. The solution of problem 5.3 and the absolute errors are shown in figures 7 and 8. These computations are investigated according to case 1 and 2.

Table 7. the numerical solution and the absolute errors of problem 5.3 for $t = 1$, $h = 0.1, k = 0.01$ and $\alpha = \beta = 1$.

x	Exact	PQSM	Absolute errors
0.1	0.036726662	0.0367541	2.749×10^{-5}
0.2	0.073086362	0.0731449	5.859×10^{-5}
0.3	0.108715808	0.1088009	8.510×10^{-5}
0.4	0.143259002	0.1433644	1.054×10^{-4}
0.5	0.176370799	0.1764883	1.175×10^{-4}
0.6	0.207720358	0.2078400	1.197×10^{-4}
0.7	0.236994443	0.2371047	1.103×10^{-4}
0.8	0.263900558	0.2639884	8.788×10^{-5}
0.9	0.288169866	0.2882213	5.141×10^{-5}

Table 8. the L_2 error norm of problem 5.3 for $k = 0.01$.

t	$\alpha = \beta = 1$		$\alpha = \beta = 0.9$		$\alpha = \beta = 0.6$	
	PQSM	**BSM [18]**	**PQSM**	**BSM [18]**	**PQSM**	**BSM [18]**
0.01	6.48×10^{-5}	4.16×10^{-2}	6.17×10^{-3}	2.68×10^{-2}	4.02×10^{-2}	2.49×10^{-2}
0.05	2.43×10^{-4}	3.99×10^{-2}	1.57×10^{-2}	3.01×10^{-2}	6.31×10^{-2}	2.58×10^{-2}
0.10	3.72×10^{-4}	3.79×10^{-2}	1.74×10^{-2}	2.88×10^{-2}	5.82×10^{-2}	2.31×10^{-2}
0.50	4.31×10^{-4}	2.55×10^{-2}	2.80×10^{-4}	1.64×10^{-2}	5.23×10^{-3}	5.46×10^{-3}
1.00	2.71×10^{-4}	1.54×10^{-2}	8.13×10^{-3}	7.87×10^{-3}	2.41×10^{-2}	4.84×10^{-3}

Table 9. the numerical solution of problem 5.3 for $t = 1, h = 0.1$ and $k = 0.01$.

x	$\alpha = \beta = 0.2$	$\alpha = \beta = 0.6$	$\alpha = \beta = 0.9$
0.1	0.0421081	0.0394294	0.0376431
0.2	0.0833382	0.0783387	0.0748691
0.3	0.1230931	0.1161984	0.1112536
0.4	0.1607006	0.1524908	0.1463852
0.5	0.1955216	0.1867052	0.1798625

0.6	0.2269384	0.2183411	0.2112986
0.7	0.2543721	0.2469104	0.2403234
0.8	0.2772507	0.2719389	0.2665878
0.9	0.2949838	0.2929689	0.2897663

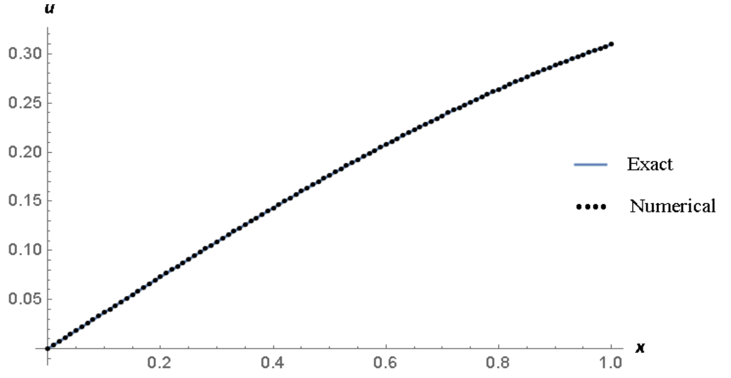

Fig. 7. The exact solution and the numerical solution using PQSM of problem 5.3 for $h = k = 0.01$, $\alpha = \beta = 1$ and $t = 1$.

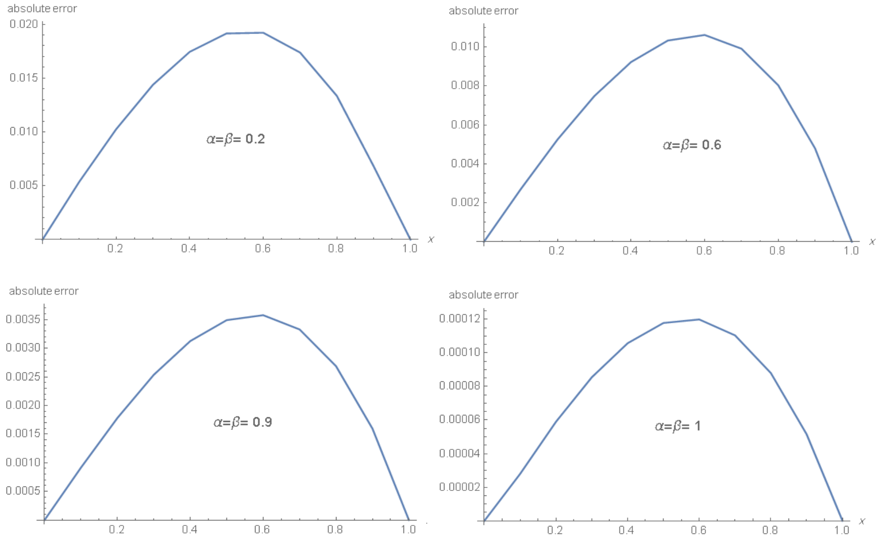

Fig. 8. The absolute errors distribution of problem 5.3 using PQSM for $h = 0.1$, $k = 0.01$, $t = 1$ and $\alpha = 0.2, 0.6, 0.9$ and 1.

6. Conclusion

In this work, the solutions of TFBE and TFCBEs have been investigated using PQSM which local truncation error is $O(h^8)$. we showed that the proposed method is stable. In addition, the given results are obtained for different values of the fractional order (α) and compared with the pervious methods which verified that the present method has a good accuracy and efficiency.

Declarations

Ethics approval and consent to participate

Not applicable.

Consent for publication

Not applicable.

Availability of data and materials

Not applicable.

Competing interests

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