Locality and Conservation Laws: How, in the presence of symmetry, locality restricts realizable unitaries

Iman Marvian (iman.marvian@duke.edu)
Duke University https://orcid.org/0000-0001-6719-061X

Article

Keywords: quantum computing, Locality and Conservation Laws

Posted Date: January 25th, 2021

DOI: https://doi.org/10.21203/rs.3.rs-149584/v1

Version of Record: A version of this preprint was published at Nature Physics on January 10th, 2022. See the published version at https://doi.org/10.1038/s41567-021-01464-0.

Locality and Conservation Laws: How, in the presence of symmetry, locality restricts realizable unitaries

Iman Marvian[1]

[1]*Departments of Physics & Electrical and Computer Engineering, Duke University, Durham, North Carolina 27708, USA*

According to an elementary result in quantum computing, any unitary transformation on a composite system can be generated using 2-local unitaries, i.e., those that act only on two subsystems. Beside its fundamental importance in quantum computing, this result can also be regarded as a statement about the dynamics of systems with local Hamiltonians: although locality puts various constraints on the short-term dynamics, it does not restrict the possible unitary evolutions that a composite system with a general local Hamiltonian can experience after a sufficiently long time. We ask if such universality remains valid in the presence of conservation laws and global symmetries. In particular, can k-local symmetric unitaries on a composite system generate *all* symmetric unitaries on that system? Surprisingly, it turns out that the answer is negative in the case of continuous symmetries, such as U(1) and SU(2): generic symmetric unitaries cannot be implemented, even approximately, using local symmetric unitaries. In the context of quantum thermodynamics this means that generic energy-conserving unitary transformations on a composite system cannot be implemented by applying local energy-conserving unitary transformations on the components. We also show how this no-go theorem can be circumvented via catalysis: any globally energy-conserving unitary can be implemented using a sequence of 2-local energy-conserving unitaries, provided that one can use a single ancillary qubit (catalyst).

Locality and symmetry are fundamental and ubiquitous properties of physical systems and their interplay leads to diverse emergent phenomena, such as spontaneous symmetry breaking. They also put various constraints on both equilibrium and dynamical properties of physical systems. For instance, symmetry implies conservation laws, as highlighted by the Noether's theorem [1, 2], and locality of interactions implies finite speed of propagation of information, as highlighted by the Lieb-Robinson bound [3]. Nevertheless, in spite of the restrictions imposed by locality on the short-term dynamics, it turns out that after a sufficiently long time and in the absence of symmetries, a composite system with a general local (time-dependent) Hamiltonian can experience any arbitrary unitary time evolution. This is related to a fundamental result in quantum computing: any unitary transformation on a composite system can be generated by a sequence of 2-local unitary transformations, i.e., those that couple, at most, two subsystems [4–6].

In this Letter, we study this phenomenon in the presence of conservation laws and global symmetries. In particular, we ask whether this universality remains valid in the presence of symmetries, or whether locality puts additional constraints on the possible unitary evolutions of a composite system. Clearly, if all the local unitaries obey a certain symmetry, then the overall unitary evolution also obeys the same symmetry. The question is if *all* symmetric unitaries on a composite system can be generated using *local* symmetric unitaries on the system. Surprisingly, it turns out that the answer is negative in the case of continuous symmetries, such as SU(2) and U(1). In fact, we show that generic symmetric unitaries cannot be implemented, even approximately, using local symmetric unitaries. Furthermore, the difference between the dimensions of the manifold of all symmetric unitaries and the submanifold of unitaries generated by k-local symmetric unitaries with a fixed k, constantly increases with the system size.

This result implies that, in the presence of locality, symmetries of Hamiltonian impose extra constraints on the time evolution of the system, which are not captured by the Noether's theorem. These additional constraints can have interesting implications, e.g., in the context of quantum chaos and thermalization of many-body systems.

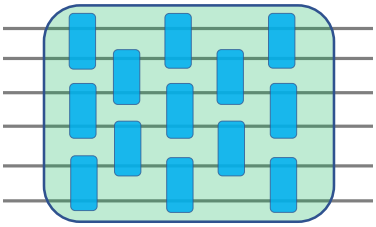

FIG. 1: **Local Symmetric Quantum Circuits.** A quantum circuit with 2-local unitaries on 6 subsystems (e.g., qubits). A Local Symmetric Quantum Circuit (LSQC) only contains local unitaries that respect a certain symmetry. For instance, they are all invariant under rotations around z axis. Such circuits can model the time evolution of systems with local symmetric Hamiltonians. Conversely, any LSQC corresponds to the time evolution generated by a local symmetric (time-dependent) Hamiltonian. Therefore, by studying LSQC, we can also characterize general features of time evolution under local symmetric Hamiltonians.

This surprising phenomenon also has important implications in the context of quantum thermodynamics and, specifically, the operational approach to thermodynamics, which is often called the *resource theory* of quantum thermodynamics [7–14]. A fundamental assumption in this framework is that all energy-conserving unitaries, i.e., those commuting with the intrinsic Hamiltonian of the system, are *free*, that is, they can be implemented with negligible thermodynamic costs. This is assumed even for composite systems with arbitrarily large number of subsystems. However, our result implies that general energy-conserving unitaries on a composite system cannot be implemented by applying *local* energy-conserving unitaries on the subsystems. In fact, even by com-

posing energy-conserving unitaries that act on $n-1$ subsystems, one still cannot generate all energy-conserving unitaries on n subsystems. Note that energy-conserving unitaries are those that are invariant under the time-translation symmetry $\{e^{-iH_0t} : t \in \mathbb{R}\}$ generated by the intrinsic Hamiltonian H_0; a *continuous* symmetry, which is isomorphic to the group U(1) in the case of periodic systems.

Therefore, this no-go theorem suggests that there might be some *hidden* thermodynamic costs for implementing general energy-conserving unitaries, using *local* energy-conserving unitaries and, in principle, this additional cost can increase with the system size. However, interestingly, we find that this no-go theorem can be circumvented using ancillary qubits, i.e., auxiliary systems initially prepared in a fixed state which return to their initial states at the end of the process. In the context of quantum thermodynamics, such ancillary systems can be interpreted as *catalysts*. For instance, we show that using 2-local Hamiltonian $XX+YY$ and local Pauli Z, which are both invariant under rotations around z, it is possible to implement all uniatiries that are invariant under this symmetry, provided that one can employ an ancillary qubit. Moreover, any energy-conserving unitary on a composite system can be implemented in a similar fashion, using a single ancillary qubit (See Theorem 1).

I. PRELIMENARIES

We start by a few definitions that are needed to formulate the problem. We present two slightly different formulations in terms of the notions of (i) local symmetric quantum circuits and (ii) local symmetric Hamiltonians, and argue that they are indeed equivalent. We also briefly explain the Lie-algebraic approach that will be used to study this problem.

A. Local Symmetric Quantum Circuits (LSQC)

Consider an arbitrary composite system formed from local subsystems or *sites* (e.g., qubits or spins). In this paper we focuse on systems with finite-dimensional Hilbert spaces. An operator is called k-local if it acts non-trivially on the Hilbert spaces of, at most, k sites. For systems with a given geometry, such as lattice systems, we also consider a stronger notion of locality, namely *geometric locality*, where the operator acts non-trivially only on a local neighborhood of the system, e.g., a pair of nearest-neighbor sites. Consider a symmetry described by a general group G. To simplify the following discussion, unless otherwise stated, we assume all sites in the system have identical Hilbert spaces and carry the same unitary representation of group G (In Appendix A we consider a more general case). In particular, on a system with n sites, assume each group element $g \in G$ is represented by the unitary $U(g) = u(g)^{\otimes n}$. An operator A acting on the total system is called G-invariant, or *symmetric*, if satisfies $U(g)AU^\dagger(g) = A$, for any group element $g \in G$. The set of symmetric unitaries itself forms a group, denoted by

$$\mathcal{V}^{\mathrm{G}} \equiv \{V : VV^\dagger = I, [V, U(g)] = 0, \forall g \in G\}\,, \qquad (1)$$

where I is the identity operator.

As an example, we consider a system with n qubits and the U(1) symmetry corresponding to global rotations around the z axis. Then, an operator A is symmetric if $(e^{-i\theta Z})^{\otimes n} A (e^{i\theta Z})^{\otimes n} = A$, for $\theta \in [0, 2\pi)$, or, equivalently, if it commutes with $\sum_{j=1}^n Z_j$, where X_j, Y_j, Z_j denote Pauli operators on qubit j tensor product with the identity operators on the rest of qubits. Depending on the context, this symmetry can have different physical interpretations. For instance, if each qubit has Hamiltonian $\frac{\Delta E}{2} Z$, then $\frac{\Delta E}{2} \sum_{j=1}^n Z_j$ is the total Hamiltonian of the system. Then, unitaries that satisfy this symmetry are the energy-conserving unitaries.

We define $\mathcal{V}_k^G$ to be the set of all unitary transformations that can be implemented with Local Symmetric Quantum Circuits (LSQC) with k-local unitaries (See Fig.1). More formally, $\mathcal{V}_k^G$ is the set of unitaries $V = \prod_{i=1}^m V_i$, generated by composing symmetric k-local unitaries $V_i : i = 1 \cdots m$, for a finite m. It can be easily seen that $\mathcal{V}_k^G$ is a subgroup of $\mathcal{V}^G = \mathcal{V}_n^G$, the group of all symmetric unitaries. More generally, for $k \le l \le n$, we have $\mathcal{V}_k^G \subseteq \mathcal{V}_l^G \subseteq \mathcal{V}^G$. We are interested in characterizing each subgroup $\mathcal{V}_k^G$ and, in particular, to determine if there exists $k < n$, such that k-local symmetric unitaries become *universal*, that is $\mathcal{V}_k^G = \mathcal{V}_n^G = \mathcal{V}^G$. As we discussed before, in the absence of symmetries, i.e., when G is the trivial group, this holds for $k = 2$.

It is worth noting that for composite systems with a given geometry, one can consider the stronger constraint of geometric locality in the above definitions: the k-local symmetric unitaries should act on local neighborhoods, e.g., only on k nearest-neighbor sites. However, provided that the sites lie on a connected graph, e.g., on a connected 1D chain, adding this additional constraint does not change the generated group $\mathcal{V}_k^G$. This is true because the swap unitary that exchanges the states of two nearest-neighbor sites is 2-local and respects the symmetry, for all symmetry groups. If the graph is connected, by combining these 2-local permutations on pairs of neighboring sites, we can generate all permutations and hence change the order of sites arbitrarily. Therefore, any k-local symmetric unitary can be realized by a sequence of k-local symmetric unitaries on k nearest-neighbor sites.

B. Time evolution under local symmetric Hamiltonians

Next, we consider a slightly different formulation of this problem in terms of the notion of local symmetric Hamiltonians. A generic *local* Hamiltonian $H(t)$ acts non-trivially on all subsystems in the system, but, it has a decomposition as $H(t) = \sum_j h_j(t)$, where each term $h_j(t)$ is k-local for a fixed k, which is often much smaller than the total number of subsystems in the system. The unitary evolution generated by

this Hamiltonian is determined by the Schrödinger equation

$$\frac{dV(t)}{dt} = -iH(t)V(t) = -i\left[\sum_j h_j(t)\right] V(t)\,, \qquad (2)$$

with the initial condition $V(0) = I$. Suppose, in addition to the above locality constraint, the Hamiltonian $H(t)$ also respects the symmetry described by the group G, such that $[U(g), H(t)] = 0$, for all $g \in G$, and all $t \geq 0$. Then, it can be shown that the family of unitaries $\{V(t) : t \geq 0\}$ generated by any such Hamiltonian belongs to $\mathcal{V}_k^G$, i.e., the group of symmetric unitaries that can be implemented by k-local symmetric unitaries (See Appendix A). Conversely, any unitary in this group is generated by a Hamiltonian $H(t)$ satisfying the above locality and symmetry constraints (Any quantum circuit can be thought of as the time evolution generated by a time-dependent local Hamiltonian). Therefore, by characterizing $\mathcal{V}_k^G$ and studying its relation with the group of all symmetric unitaries $\mathcal{V}^G$, we can also unveil possible constraints on the time evolution under local symmetric Hamiltonians, which are not captured by the standard conservation laws and Noether's theorem.

As an example, consider a closed spin chain with n qubits with Hamiltonian $H(t)$ equal to

$$\sum_{j=1}^{n} a_j(t)[X_jX_{j+1} + Y_jY_{j+1}] + b_j(t)\; Z_jZ_{j+1} + c_j(t)Z_j\,.$$

For any choices of the real functions $a_j(t), b_j(t)$, and $c_j(t)$, the Hamiltonian commutes with $\sum_j Z_j$, and clearly can be written as the sum of 2-local terms. Therefore, all unitaries generated by this family of Hamiltonians should be in $\mathcal{V}_2^{U(1)}$ for the U(1) symmetry corresponding to rotations around the z axis. We are interested to determine if this set includes all unitaries that respect the symmetry, or, the fact that the Hamiltonian is 2-local implies additional constraints on the time evolution of the system.

C. The Lie algebra generated by local symmetric Hamiltonians

To investigate these questions, we use the Lie algebraic methods of quantum control theory [15, 16], which have also been previously used to study the universality of 2-local gates in the absence of symmetries [4, 5, 17–22]. Here, we briefly discuss the main relevant result. The details and precise statements can be found in Appendix A.

Suppose one can implement the unitary time evolutions generated by Hamiltonians $\pm A$ and $\pm B$ for an arbitrary amount of time $t \geq 0$; that is one can turn on and off these Hamiltonians at will. Then, combining these time evolutions one can obtain unitaries

$$e^{-iB(c_2\delta t)}e^{-iA(c_1\delta t)} = e^{-i(c_1A+c_2B)\delta t} + \mathcal{O}(\delta t^2) \qquad (3a)$$

$$e^{-iA\delta t}e^{-iB\delta t}e^{iA\delta t}e^{iB\delta t} = e^{-[A,B]\delta t^2} + \mathcal{O}(\delta t^3)\,, \qquad (3b)$$

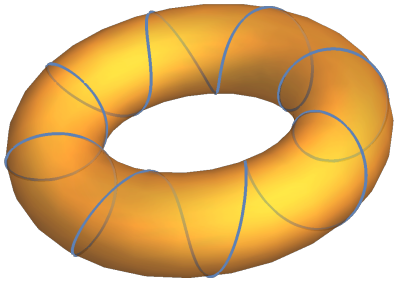

FIG. 2: The schematic relation between the group of all symmetric unitaries (the torus) and the subgroup generated by Local Symmetric Quantum Circuits (the blue curve). They are both closed connected Lie groups and hence closed manifolds. Unitary evolution under any local symmetric Hamiltonian is restricted to the submanifold corresponding to LSQC. In other words, adding a perturbation to the Hamiltonian can bring the evolution outside this submanifold, only if it is non-local or symmetry-breaking.

for arbitrary coefficients $c_1, c_2 \in \mathbb{R}$, and for sufficiently small δt. This means that using Hamiltonians $\pm A$ and $\pm B$, one can approximately simulate the time evolutions generated by any Hamiltonian in the linear span of A and B as well as the Hamiltonian $i[A, B]$. Furthermore, by repeating such combinations of unitaries, one can obtain a larger class of unitaries. In fact, it can be proven that using finite sequences of unitaries generated by Hamiltonians $\pm A$ and $\pm B$, one obtains all unitary transformations $\{e^{-iHt} : t \in \mathbb{R}\}$ generated by any Hermitian operator H if, and only if, H belongs to the real Lie algebra generated by A and B, i.e., it can be written as a linear combination of A, B, and their (nested) commutators, $i[A, B]$, $[[A, B], A]$, $[[A, B], B]$, ..., with real coefficients [15]. As we explain more in Appendix A, this result means that to characterize the group $\mathcal{V}_k^G$ generated by k-local symmetric unitaries, it suffices to characterize the Lie algebra generated by k-local symmetric Hermitian operators. In particular, the dimension of this Lie algebra, as a vector space over $\mathbb{R}$, is equal to $\dim(\mathcal{V}_k^G)$, the dimension of the manifold associated to $\mathcal{V}_k^G$, which is also equal to the number of real parameters needed to specify a general element of $\mathcal{V}_k^G$. Using this relation, we establish an upper bound on $\dim(\mathcal{V}_k^G)$, which is discussed next.

II. MAIN RESULTS

A. A no-go theorem: Non-universality of local unitaries in the presence of symmetries

We show that in the case of continuous symmetries such as U(1) and SU(2), most symmetric unitaries cannot be implemented, even approximately, using local symmetric unitaries: First, as we prove in Appendix A, for any group G, the set of symmetric unitaries $\mathcal{V}^G = \mathcal{V}_n^G$ and its subgroup $\mathcal{V}_k^G$ generated by k-local symmetric unitaries, are both connected compact Lie groups, and hence closed manifolds (See Fig.2). This means that if a unitary V is not in $\mathcal{V}_k^G$, then there is a neighborhood of symmetric unitaries around V, none of which can

be implemented using k-local symmetric unitaries. On the other hand, if V belongs to $\mathcal{V}_k^G$, then it can be implemented with a uniformly finite number of such unitaries, that is upper bounded by a fixed number independent of V [15].

Secondly, we prove that for any finite or compact Lie group G, the difference between the dimensions of the manifolds associated to all symmetric unitaries $\mathcal{V}^G = \mathcal{V}_n^G$ and its submanifold $\mathcal{V}_k^G$ is lower bounded by

$$\dim(\mathcal{V}^G) - \dim(\mathcal{V}_k^G) \geq |\text{Irreps}_G(n)| - |\text{Irreps}_G(k)| \,, \quad (4)$$

where for any integer l, $|\text{Irreps}_G(l)|$ is the number of inequivalent irreducible representations (irreps) of group G, appearing in the representation $\{u(g)^{\otimes l} : g \in G\}$, i.e., in the action of symmetry on l subsystems. We conclude that, unless $|\text{Irreps}_G(n)| = |\text{Irreps}_G(k)|$, there is a family of symmetric unitaris on n subsystems that cannot be implemented with k-local symmetric unitaries. In the case of continuous symmetries such as U(1) and SU(2), $|\text{Irreps}_G(n)|$ grows unboundedly with n. This means that there is no fixed integer k, such that k-local symmetric unitaries become universal for all system size n. This is in a sharp contrast with the universality of 2-local unitaries in the absence of symmetries.

Moreover, in the special case of connected compact Lie groups, such as U(1) and SU(2), we prove a more fine-grained version of Eq.(4): for any integer l in the interval $k \leq l \leq n$, it holds that

$$\dim(\mathcal{V}_l^G) - \dim(\mathcal{V}_k^G) \geq |\text{Irreps}_G(l)| - |\text{Irreps}_G(k)| \,. \quad (5)$$

This means that, if $|\text{Irreps}_G(l)| > |\text{Irreps}_G(k)|$, then there are symmetric unitaries that can be implemented with l-local symmetric unitaries, but not with k-local symmetric unitaries. In other words, as k increases from 1 to n, i.e., as the local unitaries become more *non-local*, the subgroup $\mathcal{V}_k^G$ generated by k-local symmetric unitaries gradually becomes larger.

The above results are obtained using a technique introduced in Sec. IV, called *charge vectors*. In Appendix A, we present the complete proofs and also show how the no-go theorem can be extended to the case where the subsystems can have different representations of the symmetry.

1. Example: U(1) symmetry for qubits

Recall the example of the U(1) symmetry corresponding to rotations around the z axis for a system with n qubits. In this case, the representation of symmetry on l sites is $\{(e^{i\theta Z})^{\otimes l} : \theta \in [0, 2\pi)\}$, which decomposes to $l+1$ inequivalent irreps of U(1). Hence, $|\text{Irreps}_{\text{U}(1)}(l)| = l + 1$. Therefore, Eq.(4) implies that for a system with n qubits the difference between the dimensions of the manifold of all symmetric unitaries and those generated by k-local symmetric unitaries is, at least, $n - k$. This means that under any restrictions on locality, i.e., if $k < n$, most symmetric unitaries cannot be implemented using local symmetric unitaries. As an example, we show that if $k < n$ then the family of unitaries $\{e^{i\theta Z^{\otimes n}} : \theta \in [0, 2\pi)\}$ generated by the Hamiltonian $Z^{\otimes n}$ cannot be implemented using k-local U(1)-invariant unitaries (except a finite set of points in this family).

More generally, consider diagonal Hamiltonians in the *computational* basis $\{|\mathbf{z}\rangle, \mathbf{z} \in \{0,1\}^n\}$ of n qubits, defined as the tensor products of $\{|0\rangle, |1\rangle\}$, the eigenbasis of Pauli Z operator. Any such Hamiltonian can be written as

$$H_{\text{diag}} = \sum_{\mathbf{z}\in\{0,1\}^n} h(\mathbf{z})\, |\mathbf{z}\rangle\langle\mathbf{z}| = \sum_{\mathbf{b}\in\{0,1\}^n} \tilde{h}(\mathbf{b})\, \mathbf{Z}^{\mathbf{b}} \,, \quad (6)$$

where for any bit string $\mathbf{b} = b_1 \cdots b_n \in \{0,1\}^n$, we have defined $\mathbf{Z}^{\mathbf{b}} = Z_1^{b_1} \cdots Z_n^{b_n}$ and $\tilde{h}(\mathbf{b}) = 2^{-n} \sum_{\mathbf{z}\in\{0,1\}^n} (-1)^{\mathbf{b}\cdot\mathbf{z}}\, h(\mathbf{z})$ is the Fourier transform of $h(\mathbf{z})$. We show that for any diagonal Hamiltonian H_{diag} the family of unitaries $\{e^{-itH_{\text{diag}}} : t \in \mathbb{R}\}$ belongs to $\mathcal{V}_k^{\text{U}(1)}$ and hence can be implemented using k-local U(1)-invariant unitaries if, and only if, the following condition holds: for all y in the interval $k+1 \leq y \leq n$,

$$\sum_{\mathbf{b}\in\{0,1\}^n : w(\mathbf{b})=y} \tilde{h}(\mathbf{b}) = 0 \,, \quad (7)$$

where the summation is over all bit strings with Hamming weight $w(\mathbf{b}) = \sum_{l=1}^n b_l$ equal to y. It is interesting to compare this condition with the stronger constraint $\tilde{h}(\mathbf{b}) = 0$ for all $\mathbf{b}$ with $w(\mathbf{b}) \geq k+1$, which is relevant if rather than all k-local U(1)-invariant unitaries, one is restricted to *diagonal* k-local unitaries. As we show in Appendix B, the necessity of condition in Eq.(7) follows immediately using the notion of charge vectors, introduced in Sec.IV. In the rest of this section, we sketch the argument that proves the sufficiency of this condition.

For any pair of distinct qubits $r, s \in \{1, \cdots, n\}$, consider the 2-local U(1)-invariant Hamiltonians

$$R_{rs} = \frac{X_r X_s + Y_r Y_s}{2} \quad , \quad T_{rs} = \frac{i}{2}[Z_r, R_{rs}] \,. \quad (8)$$

It can be shown that for any set of distinct v qubits $r_1, r_2, \cdots, r_v \in \{1, \cdots, n\}$ it holds that

$$\begin{aligned} &(Z_{r_1} - Z_{r_v})\, Z_{r_1} Z_{r_2} \ldots Z_{r_v} = \quad (9)\\ &\quad c_v \left[\left[...[R_{r_1 r_2}, R_{r_2 r_3}], ..., R_{r_{v-1}, r_v} \right], R_{r_v, r_1} \right] : \ v \text{ odd} \,, \\ &\quad ic_v \left[\left[...[R_{r_1 r_2}, R_{r_2 r_3}], ..., iR_{r_{v-1}, r_v} \right], T_{r_v, r_1} \right] : v \text{ even} \,, \end{aligned}$$

where $c_v = \pm 1$, depending on v. Then, the Lie algebraic result discussed under Eq.(3) implies that using Hamiltonians $\{R_{rs}\}$ and local Z on qubits, we can implement the unitary evolutions generated by all Hamiltonians in the form $(Z_{r_1} - Z_{r_v})\, Z_{r_1} Z_{r_2} \ldots Z_{r_v}$ and their linear combinations with real coefficients. It turns out that the linear combinations of these Hamiltonians and local Z on all qubits, is equal to the set of all diagonal Hamiltonians satisfying the constraint in Eq.(7) for all y in the interval $2 \leq y \leq n$. In other words, all such Hamiltonians can be implemented using interaction $XX + YY$ and local Z. Finally, combining this family of Hamiltonians with k-local diagonal Hamiltonians, we obtain

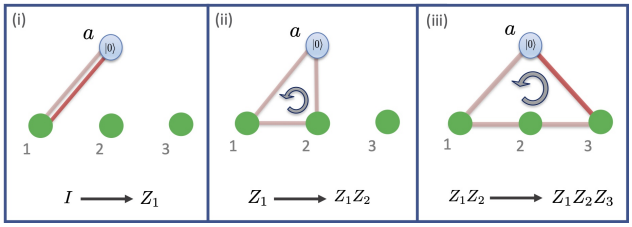

FIG. 3: **A protocol for implementing U(1)-invariant unitaries using interaction *XX+YY*.** We show that using 2-local interactions that are invariant under rotations around the z axis, the family of unitaries $\{e^{i\theta Z_1 Z_2 Z_3} : \theta \in [0, 2\pi)\}$ cannot be implemented, unless one uses ancillary systems. Using a single ancillary qubit a, we can implement this family of unitraies just using interactions $R_{rs} = (X_r X_s + Y_r Y_s)/2$ for $r, s \in \{1, 2, 3, a\}$ and local Z_a on the ancillary qubit, which are both invariant under rotations around z. The ancillary qubit a, highlighted by blue, is initially prepared in state $|0\rangle$ and returns to the same state at the end of the process. In part (i) we use Hamiltonians R_{a1} and Z_a to simulate Hamiltonian Z_1. To achieve this, we consider the commutator $i[R_{a1}, T_{1a}] = Z_1 - Z_a$, where $T_{1a} = \frac{i}{2}[Z_a, R_{1a}]$. Since a is initially in state $|0\rangle$, the effect of Hamiltonian $Z_1 - Z_a$ is equivalent to Hamiltonian $Z_3 - I$, where I is the identity operator. Therefore, by applying R_{a1} and Z_a in a proper order, we can implement unitaries generated by Z_1, up to a global phase. Part (ii) corresponds to the commutator $\Big[\big[[R_{a1}, R_{12}], R_{2a}\big]\Big] = Z_1(Z_2 - Z_a)$. The equality means that by applying R_{a1}, R_{12} and R_{2a} in a proper order, we can implement Hamiltonian $Z_1 Z_2 - Z_1$. Combining it with Z_1 obtained in the step (i) we obtain $Z_1 Z_2$. Part (iii) corresponds to the commutator $i\Big[\big[[R_{a1}, R_{12}], R_{23}\big], T_{3a}\Big] = Z_1 Z_2 (Z_3 - Z_a)$. Since qubit a is initially in state $|0\rangle$, the effect of this time evolution on qubits 1, 2, 3 is equivalent to the time evolution generated by Hamiltonian $Z_1 Z_2 Z_3 - Z_1 Z_2$ (See Eq.10). Combining this with Hamiltonian $Z_1 Z_2$ obtained from step (ii), we obtain $Z_1 Z_2 Z_3$.

diagonal Hamiltonians satisfying the constraints in Eq.(7) for y in the interval $k + 1 \leq y \leq n$. This proves the sufficiency of condition in Eq.(7) (See Appendix B for further details).

B. Circumventing the no-go theorem with ancillary systems

Interestingly, it turns out that in the case of U(1) symmetry the above constraints can be circumvented, provided that one is allowed to interact with an ancillary qubit. Here, we explain the main idea with an example. See Appendix B for the general result and further details.

Suppose the goal is to implement the family of unitaries $\{e^{i\theta Z_1 Z_2 Z_3} : \theta \in [0, 2\pi)\}$ on a system with $n = 3$ qubits, labeled as 1, 2, 3. The condition in Eq.(7) implies that this family cannot be generated by 2-local U(1)-invariant unitaries. Now suppose in addition to these 3 qubits, we can use an ancillary qubit labeled as a. As we explain in Fig.3, Eq.(9) implies that by applying the unitaries generated by $XX + YY$ and local Z Hamiltonians, which are both 2-local and invariant under rotations around z, one can simulate Hamiltonian $(Z_3 - Z_a)Z_1 Z_2$, where Z_a is Pauli Z on the ancillary qubit a tensor product the identity operators on the rest of qubits. Assuming the ancillary qubit is initially prepared in state $|0\rangle_a$ and qubits 1, 2, 3 are in an arbitrary state $|\psi\rangle$, under the time evolution generated by this Hamiltonian, the initial state $|\psi\rangle \otimes |0\rangle_a$ evolves to

$$e^{i\theta(Z_3 - Z_a)Z_1 Z_2}\,(|\psi\rangle \otimes |0\rangle_a) = \big(e^{i\theta(Z_1 Z_2 Z_3 - Z_1 Z_2)}|\psi\rangle\big) \otimes |0\rangle_a\,, \tag{10}$$

for $\theta \in [0, 2\pi)$. Therefore, if after applying this unitary we apply the 2-local symmetric unitary $e^{i\theta Z_1 Z_2}$ on qubits 1 and 2, the overall unitary evolution of qubits 1, 2, 3 will be the desired unitary $e^{i\theta Z_1 Z_2 Z_3}$. Note that because at the end of the process the ancillary qubit goes back to its initial state, we can use it again to implement other unitary transformations. As we discuss in Fig.3 and prove in Appendix B, the commutation relations in Eq.(9) imply that this technique can be generalized to implement all diagonal Hamiltonians, just using interactions $XX+YY$, and local Z on the ancillary qubit.

Finally, we show that combining diagonal unitaries with unitaries generated by 2-local interaction $XX + YY$, one obtains all U(1)-invariant unitaries. In summary, we prove that: *all unitaries that are invariant under rotations around z, i.e., those preserving $\sum_j Z_j$, can be implemented using a single ancillary qubit and via interactions $XX+YY$ and local Z on the ancillary qubit*. Furthermore, as we discuss in Appendix B, this result remains valid if there are further geometric constraints on the interactions between qubits. In particular, if the qubits in the system form a chain and only nearest-neighbor $XX+YY$ interactions between them are allowed, we can still implement general U(1)-invariant unitaries, provided that the ancillary qubit can interact with all the qubits via $XX + YY$ interaction. Alternatively, if in addition to $XX + YY$ interaction, one can also apply ZZ interactions to the nearest-neighbor qubits, then the ancillary qubit only needs to interact with one qubit in the chain, e.g., the qubit at one end of the chain.

C. Quantum thermodynamics with local interactions

Energy-conserving unitaries, i.e., those commuting with the intrinsic Hamiltonian of a system, play a fundamental role in quantum thermodynamics. In fact, the central assumption of the resource theory of quantum thermodynamics is that any such unitary can be implemented with negligible thermodynamic costs, regardless of the system size [7–13]. However, according to our no-go theorem, a generic energy-conserving unitary on a composite system cannot be implemented by applying local energy-conserving unitaries on the subsystems. Therefore, under the restriction to local interactions (which is inevitable due to fundamental and practical reasons) to implement a general energy-conserving unitary, one needs to apply non-energy-conserving transformations on the system. That is the subsystems need to exchange energy with an external system. Can we implement a general energy-conserving unitary by interacting with a finite-size external system? The following theorem addresses this question (See theorem 9 in Appendix C for a more precise statement).

Theorem 1. *Consider a finite set of closed systems with the property that for each system the gap between any consecutive pairs of energy levels is ΔE (See Fig.4). Then, any global*

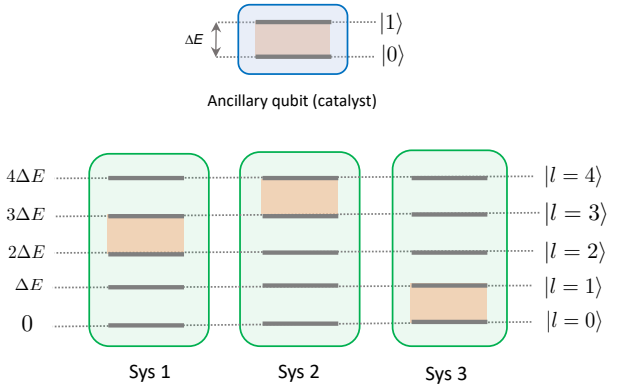

FIG. 4: **A scheme for implementing energy-conserving unitaries on composite systems.** In each system we consider a pair of energy eigenstates with energy difference ΔE, which can be interpreted as a single qubit. Then, we can apply the protocol discussed in the previous section for qubits with U(1) symmetry (See Fig.3). To implement this protocol, the qubits defined in different systems are sequentially coupled to each other and to an ancillary qubit, via 2-local energy-conserving interactions in Eq.(12) and Eq.(11). The ancilla is initially prepared in its ground state $|0\rangle$, whose energy gap with the excited state $|1\rangle$, is ΔE. This implies that the interaction in Eq.(11), which couples the ancilla to the systems, is energy-conserving. Using these 2-local energy-conserving interactions and following the protocol introduced in the previous section, we can implement energy-conserving unitaries on the selected energy pairs. Repeating these steps with other pairs of energy eigenstates with energy difference ΔE, we obtain all energy-conserving unitaries (See Appendix C).

energy-conserving unitary transformation on these systems can be implemented by a finite sequence of 2-local energy-conserving unitaries, provided that the systems can interact with a single ancillary qubit with energy gap ΔE between its two levels.

Therefore, the assumption of the resource theory of quantum thermodynamics [7–13] that all energy-conserving unitaries (and, hence all thermal operations) are *free*, is consistent with the locality of interactions, provided that one allows the use of ancillary systems. In the context of quantum thermodynamics such systems can be interpreted as catalysts [14, 23]. It is worth mentioning that the assumption in this theorem on the energy gap ΔE between consecutive levels can be relaxed, provided that one can use larger catalysts with more energy levels.

To prove this result, we first show how to implement general *diagonal* energy-conserving unitaries, i.e., those that conserve the energy of all individual subsystems. Then, we show that by combining diagonal unitaries with 2-local energy-conserving unitaries, we can implement all energy-conserving unitaries.

In the rest of this section, we briefly sketch a specific scheme for implementing energy-conserving unitaries (See Appendix C for further details). This is, in fact, a generalization of the protocol introduced in the previous section, for the special case of qubits with U(1) symmetry (Note that for the family of Hamiltonians considered in the above theorem, energy conservation is equivalent to a U(1) symmetry). In particular, this scheme uses an ancillary qubit with Hamiltonian $\Delta E|1\rangle\langle 1|_{\rm a}$, initially prepared in the ground state $|0\rangle_{\rm a}$. Suppose in each system we pick a pair of energy levels with energy difference ΔE, which can be interpreted as a qubit (See Fig.4). Then, following the protocol introduced in the previous section, we can implement all energy-conserving unitaries defined on these qubits using 2-local energy-conserving unitaries. For concreteness, suppose in system j we choose energy eigenstates $|l-1\rangle_j$ and $|l\rangle_j$ with energies $(l-1)\times\Delta E$ and $l\times\Delta E$. Consider the 2-local interaction

$$R^{(l)}_{j,{\rm a}} = |l-1\rangle\langle l|_j \otimes |1\rangle\langle 0|_{\rm a} + \text{H.C.}\,, \tag{11}$$

which couples the selected levels in system j to the ancillary qubit, where H.C. denotes the Hermitian conjugate of the first term. Similarly, suppose in system j', we pick levels $\{|l'-1\rangle_{j'}, |l'\rangle_{j'}\}$ with energy difference ΔE. Consider the 2-local interaction

$$R^{(l,l')}_{j,j'} = |l-1\rangle\langle l|_j \otimes |l'\rangle\langle l'-1|_{j'} + \text{H.C.}\,. \tag{12}$$

Note that both interactions $R^{(l)}_{j,{\rm a}}$ and $R^{(l,l')}_{j,j'}$ conserve the total energy of the systems and ancilla. Therefore, by turning on and off each of these interactions we can implement 2-local energy-conserving unitaries. In fact, if we interpret states $|l-1\rangle_j$ and $|l\rangle_j$ as states $|0\rangle$ and $|1\rangle$ of a qubit, then $R^{(l)}_{j,{\rm a}}$ will be equivalent to the interaction $\frac{1}{2}(XX+YY)$ between a pair of qubits. Similarly, $R^{(l,l')}_{j,j'}$ can be interpreted as interaction $\frac{1}{2}(XX+YY)$ between a pair of qubits, defined in systems j and j'. Recall that the protocol defined in the previous section can be implemented using interactions $XX+YY$ and local Z on the ancillary qubit. Therefore, interactions defined in Eq.(11) and Eq.(12) together with local $Z_{\rm a}$ on the ancillary qubit, allow us to apply this protocol and implement all energy-conserving unitaries defined on the the selected pairs of energy eigenstates. It turns out that combining these energy-conserving unitaries with those that can be implemented on other pairs of eigenstates with energy difference ΔE, we obtain all energy-conserving unitaries (See Appendix C).

Finally, it is worth mentioning that that the system-system interactions in Eq.(12) that are required in the above scheme, can also be engineered using the system-ancilla interactions in Eq.(11), provided that one can use a second ancillary qubit b. This follows from the fact that

$$R^{(l,l')}_{j,j'} Z_{\rm b} = \frac{1}{2}\left[R^{(l')}_{j',{\rm b}}\,, [R^{(l)}_{j,{\rm b}}\,, Z_{\rm b}]\right]\,. \tag{13}$$

Therefore, if qubit b is initially prepared in an eigenstate of $Z_{\rm b}$, then by composing the time evolutions generated by $R^{(l)}_{j,{\rm b}}$, $R^{(l')}_{j',{\rm b}}$, and $Z_{\rm b}$, we can implement the time evolution generated by $R^{(l,l')}_{j,j'}$ (This is similar to the argument in Eq.(10)). In conclusion, all energy-conserving unitaries can be implemented without any direct interactions between the systems,

just by coupling them to a pair of ancillary qubits, using 2-local energy-conserving interactions in Eq.(11).

III. DISCUSSION

We conclude that while for a general (time-dependent) Hamiltonian, the locality of interactions alone does not put any constraints on the long-term time evolution, in the presence of symmetries, locality can restrict possible time evolutions of the system. That is the unitary evolution of the system satisfies certain additional constraints, apart from the standard conservation laws. In particular, unlike the standard conservation laws, these extra constraints can be violated by adding *non-local symmetric terms* to the Hamiltonian (Note that a long-range interaction, such as the Coulomb interaction, still could be 2-local). Understanding the implications of these non-trivial constraints on thermalization, quantum chaos and scrambling of quantum information in many-body systems is an interesting open question.

This observation also raises other interesting questions about general properties of Local Symmetric Quantum Circuits. While this family of quantum circuits has been used in various areas of quantum physics, from classifying Symmetry-Protected Topological (SPT) phases [24, 25] to the study of quantum chaos with conserved charges [26], many important aspects of this family is not well-understood yet.

For instance, an interesting open question in this area is to understand how the notion of *circuit complexity* changes under the constraint of symmetry. Recall that the circuit complexity of a unitary transformation or a state is the minimum number of local gates needed to implement the unitary or to prepare the state from a fixed (product) state [27]. For a symmetric unitary or a symmetric state, we can define a modified notion of complexity, which can be called *Symmetry-Protected Complexity* (SPC) and is defined as the minimum number *symmetric* local unitaries needed to implement a symmetric unitary or to prepare a symmetric state. Certain aspects of this notion of complexity has been studied in the context of SPT phases. In particular, it is known that for certain family of states the SPC grow linearly with the number of subsystems, whereas the regular complexity remains constant. Given the conjectured roles of complexity in the context of holography and AdS/CFT correspondence [28–30], it is interesting to further study the notion of SPC and compare it with the regular complexity.

Finally, this work also raises interesting questions in the context of quantum resource theories and specifically motivates the study of *Local Quantum Resource Theories*: For a general resource theory, one can ask if the set of free operations can be generated by k-local free operations with a fixed k, (and, possibly, with the help of a catalyst). A negative answer to the this question suggests that the resource theory may not correspond to any operationally well-motivated restrictions (See [31, 32] for a discussion on physical justifications of quantum resource theories).

IV. METHODS

Here, we present an overview of the proofs of our no-go theorem in Sec.II A. See Appendix A for details.

A. Charge vectors

We start by introducing the idea of *charge vectors*, which is our main tool for deriving constraints on the unitary evolutions generated by local symmetric Hamiltonians.

Recall that $\text{Irreps}_G(n)$ denotes the set of inequivalent irreps of group G that appear in the representation $\{U(g) = u(g)^{\otimes n} : g \in G\}$ and $|\text{Irreps}_G(n)|$ is the number of these inequivalent irreps. Let Π_μ be the projector to the subspace corresponding to irrep $\mu \in \text{Irreps}_G(n)$, also known as the *isotypic* component of μ. For any operator A define the *charge vector* of A as

$$|\chi_A\rangle \equiv \sum_{\mu \in \text{Irreps}_G(n)} \text{Tr}(\Pi_\mu A)\, |\mu\rangle\,, \tag{14}$$

where $\{|\mu\rangle : \mu \in \text{Irreps}_G(n)\}$ is a set of orthonormal vectors in an abstract vector space with dimension $|\text{Irreps}_G(n)|$. A general G-invariant Hamiltonian can have any charge vector with real coefficients. In particular, for any set of real numbers $\{h_\mu \in \mathbb{R} : \mu \in \text{Irreps}_G(n)\}$, the Hermitian operator $\sum_{\mu \in \text{Irreps}_G(n)} \frac{h_\mu}{\text{Tr}(\Pi_\mu)} \Pi_\mu$ is G-invariant and has the charge vector $\sum_{\mu \in \text{Irreps}_G(n)} h_\mu\, |\mu\rangle$. In other words, under the linear map $A \to |\chi_A\rangle$, the image of the linear space of Hermitian G-invariant operators has dimension $|\text{Irreps}_G(n)|$.

On the other hand, it turns out that if the the unitary evolutions generated by Hamiltonian H can be simulated by k-local G-invariant unitaries, i.e., if $\forall t \in \mathbb{R} : e^{-iHt} \in \mathcal{V}_k^G$, then the charge of vector of H should satisfy certain constraints. Let $\mathcal{S}_k$ be the set of charge vectors for all such Hamiltonians, i.e.,

$$\mathcal{S}_k \equiv \{|\chi_H\rangle : e^{-iHt} \in \mathcal{V}_k^G\ ,\ \forall t \in \mathbb{R}\}\,. \tag{15}$$

We prove that $\mathcal{S}_k$ is a linear subspace (over the field $\mathbb{R}$) with dimension

$$\dim(\mathcal{S}_k) \leq |\text{Irreps}_G(k)|\,, \tag{16}$$

and the equality holds if G is a connected Lie group, such as U(1) and SU(2). Therefore, if $|\text{Irreps}_G(k)| < |\text{Irreps}_G(n)|$, then $\dim(\mathcal{S}_k) < |\text{Irreps}_G(n)|$, which means there are G-invariant Hamiltonians whose charge vectors do not belong to $\mathcal{S}_k$, which in turn implies they cannot be simulated using k-local symmetric unitaries. This proves our main no-go theorem. As we explain in Appendix A, Eq.(16) also implies bounds in Eq.(4) and Eq.(5) on $\dim(\mathcal{V}_k^G)$. Moreover, in the example of U(1) symmetry, the necessity of the constraints in Eq.(7) follows from the constraints on charge vectors (In fact, Eq.(7) is simply the statement that the charge vector of H_{diag} should be in $\mathcal{S}_k$). In the following, we discuss more about the subspace $\mathcal{S}_k$ and the bound in Eq.(16) on its dimension. In particular, in theorem 2 below, we present a simple method

for testing the condition $|\chi_H\rangle \in \mathcal{S}_k$, based on the Fourier transform of $|\chi_H\rangle$.

To determine $\mathcal{S}_k$ we use the fact that if $e^{-iHt} \in \mathcal{V}_k^G$ for all $t \in \mathbb{R}$, then H should be in the Lie algebra generated by k-local G-invariant operators, i.e., $H = \sum_j c_j A_j + \sum_{j_1,j_2} c_{j_1,j_2}\ i[A_{j_1}, A_{j_2}] + \cdots$, where A_j are Hermitian k-local G-invariant operators and coefficients $c_j, c_{j_1,j_2}, \cdots$ are real numbers. It can be shown that the commutators appearing in this expansion, do not contribute in the charge vector of H, i.e., $|\chi_H\rangle = \sum_j c_j |\chi_j\rangle$, where $|\chi_j\rangle \equiv \sum_{\mu \in \text{Irreps}_G(n)} \text{Tr}(\Pi_\mu A_j)\ |\mu\rangle$ is the charge vector of A_j. To see this note that for any irrep $\mu \in \text{Irreps}_G(n)$, $\text{Tr}([A_{j_1}, A_{j_2}]\Pi_\mu) = \text{Tr}(A_{j_1}[A_{j_2}, \Pi_\mu]) = 0$, where the first equality follows from the cyclic property of trace and the second equality follows from the assumption that A_{j_2} is G-invariant, and therefore commutes with Π_μ. It follows that the commutator $[A_{j_1}, A_{j_2}]$ and other nested commutators do not contribute in $|\chi_H\rangle$. This implies that $\mathcal{S}_k$ is spanned by the charge vectors of k-local G-invariant Hermitian operators, i.e., $\mathcal{S}_k$ is equal to

$$\text{Span}_{\mathbb{R}}\big\{|\chi_A\rangle : A = A^\dagger, A \text{ is } k\text{-local}, [A, U(g)] = 0 : \forall g \in G\big\}\ . \tag{17}$$

Next, note that for any k-local operator A, by applying a properly chosen permutation operator S which changes the order of sites, we can obtain an operator in the form $SAS^\dagger = \tilde{A} \otimes I_{\text{rest}}$ with the property that $\tilde{A}$ acts on a fixed set of k sites (e.g., the first k sites according to a certain ordering) and I_{rest} is the identity operator on the remaining $n-k$ sites. Since charge vectors remain invariant under permutations, operators A and $SAS^\dagger = \tilde{A} \otimes I_{\text{rest}}$ have the same charge vectors. It follows that the subspace in Eq.(17) is equal to the set of the charge vectors of G-invariant Hermitian operators that act non-trivially only on a fixed set of k sites (e.g., the first k sites). Therefore, as the number of total sites n increases, $\dim(\mathcal{S}_k)$ remains bounded by a number independent of n. In other words, even though using k-local G-invariant unitaries we can simulate Hamiltonians that are not k-local, they can only have charge vectors which are allowed for k-local G-invariant Hamiltonians. Next, we show how the bound on $\dim(\mathcal{S}_k)$ in Eq.(16) can be established.

Fourier transform of charge vectors

To characterize the subspace $\mathcal{S}_k$ and determine its dimension, it is useful to consider the Fourier transform of charge vectors. For any operator A, let χ_A be the complex function over group G, defined by $\chi_A(g) \equiv \text{Tr}(A\ U(g))$ for all $g \in G$. It turns out that for any G-invariant operator A, the charge vector $|\chi_A\rangle$ and function χ_A are related via a one-to-one linear map, namely the Fourier transform over group G (See Appendix A). In Appendix A, we prove

Theorem 2. *Consider a system with n identical sites, and let $\{U(g) = u(g)^{\otimes n} : g \in G\}$ be the unitary representation of an arbitrary finite or compact Lie group G on this system. Suppose the family of unitaries $\{e^{-iHt} : t \in \mathbb{R}\}$ generated by Hamiltonian H, belongs to $\mathcal{V}_k^G$, i.e., can be implemented by G-invariant k-local unitaries. Then, there exists a set of real numbers $c_\nu \in \mathbb{R}$, such that for all group elements $g \in G$, it holds that*

$$Tr\big(H\ u(g)^{\otimes n}\big) = [Tr(u(g))]^{n-k} \times \sum_{\nu \in Irreps_G(k)} c_\nu\ f_\nu(g)\ , \tag{18}$$

where the summation is over all inequivalent irreducible representations of G appearing in the representation $\{u(g)^{\otimes k} : g \in G\}$, and $f_\nu : G \to \mathbb{C}$ is the character of the irrep ν. Conversely, for any set of real numbers $c_\nu \in \mathbb{R}$, there exists a Hermitian operator H satisfying the above equality, such that all unitaries $\{e^{-iHt} : t \in \mathbb{R}\}$ generated by H belong to $\mathcal{V}_k^G$. Furthermore, for any unitary $V \in \mathcal{V}_k^G$, there exists a G-invariant Hermitian operator H, such that $V = e^{-iH}$, and H satisfies the above condition for a set of real numbers $c_\nu \in \mathbb{R}$.

The right-hand side of Eq.(18) defines a linear subspace in the space of complex functions over group G, namely the subspace $\text{Span}_{\mathbb{R}}\{r^{(n-k)}\ f_\nu : \nu \in \text{Irreps}_G(k)\}$, where $r(g) \equiv \text{Tr}(u(g))$, for all $g \in G$. The theorem implies that if Hamiltonian H can be simulated using k-local G-invariant unitaries, then function χ_H should be restricted to the subspace $\text{Span}_{\mathbb{R}}\{r^{(n-k)} f_\nu : \nu \in \text{Irreps}_G(k)\}$. This condition is equivalent to the condition in terms of the charge vectors, i.e.,

$$|\chi_H\rangle \in \mathcal{S}_k \iff \chi_H \in \text{Span}_{\mathbb{R}}\{r^{(n-k)} f_\nu : \nu \in \text{Irreps}_G(k)\}\ . \tag{19}$$

The subspace in the right-hand side has dimension less than or equal to $|\text{Irreps}_G(k)|$ (In fact, in the case of connected Lie groups this dimension is equal to $|\text{Irreps}_G(k)|$). Since χ_H and $|\chi_H\rangle$ are related via an invertible linear map, it follows that $\dim(\mathcal{S}_k) \leq |\text{Irreps}_G(k)|$. Note that for a general G-invariant Hamiltonian H, χ_H can be any function in the subspace $\text{Span}_{\mathbb{R}}\{f_\nu : \nu \in \text{Irreps}_G(n)\}$, which has dimension $|\text{Irreps}_G(n)|$ (This follows from the linear independence of the characters of different irreps).

Finally, we note that theorem 2 has other interesting implications, which will be studied in a future work. For instance, in the case of Abelian groups, it implies that if there exists a group element $g \in G$ such that $\text{Tr}(u(g)) = 0$, then k-local symmetric unitaries are not universal, unless $k = n$. In other words, in the case of Abelian groups if $\text{Tr}(u(g)) = 0$ for some group element $g \in G$, then any restrictions on locality implies non-universality (See Appendix A).

Acknowledgements: I am grateful to Austin Hulse, Hadi Salmasian, and Nicole Yunger-Halpern for reading the manuscript carefully and providing many useful comments.

[1] E. Noether, Invariante Variationsprobleme (1918).

[2] E. Noether, Transport Theory and Statistical Physics **1**, 186

(1971).
[3] E. H. Lieb and D. W. Robinson, in *Statistical mechanics* (Springer, 1972), pp. 425–431.
[4] S. Lloyd, Physical Review Letters **75**, 346 (1995).
[5] D. P. DiVincenzo, Physical Review A **51**, 1015 (1995).
[6] D. E. Deutsch, A. Barenco, and A. Ekert, Proceedings of the Royal Society of London. Series A: Mathematical and Physical Sciences **449**, 669 (1995).
[7] M. Horodecki and J. Oppenheim, Nat. Commun. **4**, 1 (2013).
[8] F. G. Brandao, M. Horodecki, J. Oppenheim, J. M. Renes, and R. W. Spekkens, Physical review letters **111**, 250404 (2013).
[9] D. Janzing, P. Wocjan, R. Zeier, R. Geiss, and T. Beth, Int. J. Theor. Phys. **39**, 2717 (2000).
[10] M. Lostaglio, K. Korzekwa, D. Jennings, and T. Rudolph, Physical Review X **5**, 021001 (2015).
[11] N. Y. Halpern, P. Faist, J. Oppenheim, and A. Winter, Nature communications **7**, 12051 (2016).
[12] N. Y. Halpern and J. M. Renes, Physical Review E **93**, 022126 (2016).
[13] Y. Guryanova, S. Popescu, A. J. Short, R. Silva, and P. Skrzypczyk, Nature communications **7**, ncomms12049 (2016).
[14] E. Chitambar and G. Gour, Reviews of Modern Physics **91**, 025001 (2019).
[15] D. d'Alessandro, *Introduction to quantum control and dynamics* (CRC press, 2007).
[16] V. Jurdjevic and H. J. Sussmann, Journal of Differential equations **12**, 313 (1972).
[17] J.-L. Brylinski and R. Brylinski, Mathematics of quantum computation **79** (2002).
[18] A. M. Childs, D. Leung, L. Mančinska, and M. Ozols, arXiv preprint arXiv:1004.1645 (2010).
[19] P. Zanardi and S. Lloyd, Physical Review A **69**, 022313 (2004).
[20] P. Giorda, P. Zanardi, and S. Lloyd, Physical Review A **68**, 062320 (2003).
[21] D. Bacon, J. Kempe, D. A. Lidar, and K. B. Whaley, Phys. Rev. Lett. **85**, 1758 (2000), URL `http://link.aps.org/doi/10.1103/PhysRevLett.85.1758`.
[22] D. A. Lidar, I. L. Chuang, and K. B. Whaley, Physical Review Letters **81**, 2594 (1998).
[23] D. Jonathan and M. B. Plenio, Physical Review Letters **83**, 3566 (1999).
[24] X. Chen, Z.-C. Gu, and X.-G. Wen, Physical review b **82**, 155138 (2010).
[25] X. Chen, Z.-C. Gu, and X.-G. Wen, Physical review b **83**, 035107 (2011).
[26] V. Khemani, A. Vishwanath, and D. A. Huse, Physical Review X **8**, 031057 (2018).
[27] S. Aaronson, arXiv preprint arXiv:1607.05256 (2016).
[28] L. Susskind, Fortschritte der Physik **64**, 24 (2016).
[29] A. R. Brown, D. A. Roberts, L. Susskind, B. Swingle, and Y. Zhao, Physical review letters **116**, 191301 (2016).
[30] D. Stanford and L. Susskind, Physical Review D **90**, 126007 (2014).
[31] I. Marvian and R. W. Spekkens, Physical Review A **94**, 052324 (2016).
[32] E. Chitambar and G. Gour, Physical review letters **117**, 030401 (2016).
[33] A. L. Onishchik and E. B. Vinberg, *Lie groups and algebraic groups* (Springer Science & Business Media, 2012).

Appendices

The Appendices contain the formal proofs and further discussions on the results presented in the paper: Appendix A contains the proofs of the results presented in Sec. II A, and in particular, the proof of our no-go theorem and Eq.(5), Appendix B contains the results on the U(1) symmetry and Eq.(7), and Appendix C contains the proof of results in Sec.(II C) on energy-conserving unitaries and theorem 1.

Appendix A: Characterizing the group generated by local symmetric quantum circuits

In this section we use a Lie algebraic approach to characterize the family of unitaries that can be implemented using local symmetric quantum circuits. We start with a general setting, where the systems are not necessarily identical.

Consider n systems, labeled as $j = 1, \cdots, n$. Assume the dimension of the Hilbert space of system j is $d_j < \infty$. Therefore, the total Hilbert space of the composite system is

$$\bigotimes_{j=1}^{n} \mathbb{C}^{d_j} \ . \tag{A1}$$

Suppose for each system we are given a unitary representation of a symmetry group G. In particular, let $u^{(j)}(g)$ be the action of group element $g \in G$ on the Hilbert space of system j. Then, the action of this group element on the joint system is described by the unitary

$$U(g) = \bigotimes_{j=1}^{n} u^{(j)}(g) \ : \ \ g \in G \, . \tag{A2}$$

1. Lie Algebraic formulation of the problem

Suppose one can implement unitary transformations $\{e^{-itK_j} \ : \ t \in \mathbb{R}, j = 1, \cdots\}$, generated by a set of Hamiltonians $\{\pm K_j\}_j$. Composing these unitaris one can implement the group of unitaries

$$\left\{ e^{-it_m K_{j_m}} \cdots e^{-it_2 K_{j_2}} e^{-it_1 K_{j_1}} \ : \ t_1, \cdots, t_m \in \mathbb{R}, m \in \mathbb{N} \right\} . \tag{A3}$$

Using the standard results in the theory of Lie groups and control theory [15, 16], it turns out that this group is a connected Lie group and is fully characterized by its corresponding Lie algebra. In particular, for any Hermitian operator H, the one-parameter family of unitaries $\{e^{-iHt} : t \in \mathbb{R}\}$ is in this group if, and only if, iH is in the real Lie algebra generated by skew-Hermitian operators $\{iK_j\}_j$, denoted by $\mathfrak{alg}\{iK_j\}_j$, such that

$$iH = \sum_j \alpha_j \, iK_j + \sum_{j_1, j_2} \beta_{j_1, j_2} \, [iK_{j_1}, iK_{j_2}] + \sum_{j_1, j_2, j_3} \gamma_{j_1, j_2, j_3} \, \big[[iK_{j_1}, iK_{j_2}], iK_{j_3} \big] + \cdots \ ,$$

for some real coefficients $\alpha_j, \beta_{j_1,j_2}, \gamma_{j_1,j_2,j_3} \cdots$.

We apply the above fact to study

$$\mathcal{V}_k^G \equiv \left\langle V : VV^\dagger = I, V \text{ is } k\text{-local}, [V, U(g)] = 0, \forall g \in G \right\rangle , \tag{A4}$$

i.e., the group generated by k-local symmetric unitaries, where I is the identity operator on the total Hilbert space in Eq.(A1) and $\langle \cdot \rangle$ denotes the generated group. Note that for any k-local symmetric Hamiltonian h, the family of unitaries generated by h i.e., $\{e^{-iht} : t \in \mathbb{R}\}$ are all k-local and symmetric. Conversely, any k-local symmetric unitary V can be obtained by applying a k-local symmetric Hamiltonian on the system for a finite time. Hence, we conclude that

Proposition 1. *Let*

$$\mathfrak{h}_k \equiv \mathfrak{alg}_{\mathbb{R}} \big\{ A : k\text{-local}, A + A^\dagger = 0 \ , \ [A, U(g)] = 0 : \forall g \in G \big\} \ , \tag{A5}$$

be the real Lie algebra generated by the k-local, skew-Hermitian, G-invariant operators. For any Hermitian operator H, the family of unitaries $\{e^{-iHt} : t \in \mathbb{R}\}$ can be implemented using k-local symmetric unitaries, if and only if, $iH \in \mathfrak{h}_k$, i.e.

$$\forall t : e^{-iHt} \in \mathcal{V}_k^G \iff iH \in \mathfrak{h}_k \,. \tag{A6}$$

Therefore, in the following, to characterize $\mathcal{V}_k^G$, we study the Lie algebra $\mathfrak{h}_k$. Note that $\mathfrak{h}_k$ corresponds to the tangent space (at the identity) of the manifold associated to $\mathcal{V}_k^G$. In particular, the dimension of $\mathcal{V}_k^G$ as a manifold, is equal to the dimension of $\mathfrak{h}_k$ as a vector space (over the field $\mathbb{R}$), i.e.

$$\dim(\mathcal{V}_k^G) = \dim(\mathfrak{h}_k) \,. \tag{A7}$$

It is also worth noting that for $l > k$, $\mathcal{V}_k^G$ is a subgroup of $\mathcal{V}_l^G$ and $\mathfrak{h}_k$ is a sub-algebra of $\mathfrak{h}_l$. If $\dim(\mathfrak{h}_l) > \dim(\mathfrak{h}_k)$, then there are unitaries that can be implemented with l-local symmetric unitaries but not with k-local symmetric unitaries.

2. Compactness of $\mathcal{V}_k^G$ and the impossibility of approximate implementation of symmetric unitaries

As we show next, the group $\mathcal{V}_k^G$ generated by k-local G-invariant unitaries is compact. To prove this, we use the following fact, which can be proven using the techniques of the theory of algebraic groups (See e.g., chapter 5 of [33] and [17]).

Fact 1: For any compact Lie group, the subgroup generated by a finite set of compact connected subgroups is itself a compact connected Lie group.

Recall that in this paper, we only consider systems with finite-dimensional Hilbert spaces, and therefore the unitary group defined on the Hilbert space of the systems is a compact connected Lie group. Then, using this fact we can show that

Proposition 2. *For any symmetry group G, the group of unitaries $\mathcal{V}_k^G$ generated by k-local G-invariant unitaries is a compact connected Lie group.*

Proof. First, it can be easily seen that the group of G-invariant unitaries $\mathcal{V}^G$ is a connected Lie group (i.e., there is a smooth path between the identity and any other group element V). Furthermore, defined as the commutant of a set of operators, the subgroup of symmetric unitaries $\mathcal{V}^G$ is closed. Therefore, we conclude that it is a compact connected subgroup of the unitary group. In fact, as we discuss later, $\mathcal{V}^G$ has a simple characterization, as the direct product of groups isomorphic to the unitary group $\mathrm{U}(m_\mu)$, for different integers m_μ.

Using similar arguments, we can also see that for any finite subset of sites, the group of G-invariant unitaries which act non-trivially only on those sites is also a connected compact Lie group. Finally, recall that $\mathcal{V}_k^G$ is generated by k-local G-invariant unitaries, i.e., G-invariant unitaries that act non-trivially on arbitrary subset of k sites. Assuming the system has a finite number of sites n, this means that $\mathcal{V}_k^G$ is generated by a finite set of connected compact Lie groups. Using Fact 1 about compact Lie groups, we conclude that $\mathcal{V}_k^G$ itself is a compact connected Lie group. □

The compactness of the group $\mathcal{V}_k^G$ has several useful implications. For instance, as we mentioned before, compactness implies that $\mathcal{V}_k^G$ is uniformly finitely generated by k-local symmetric unitaries [15]. Another useful implication of compactness follows from the following fact about Lie groups:

Fact 2: For compact connected Lie groups, the exponential map from the Lie algebra to the Lie group is surjective.

It follows that

Corollary 3. *A unitary V can be implemented using k-local symmetric unitaries, i.e., $V \in \mathcal{V}_k^G$ if, and only if, there exists $C \in \mathfrak{h}_k$ such that $V = e^C$. In other words, $\mathcal{V}_k^G = e^{\mathfrak{h}_k}$.*

Therefore, by characterizing the Lie algebra $\mathfrak{h}_k$ we also find a full and direct characterization of $\mathcal{V}_k^G$. Hence, in the following, we focus on the study of the Lie algebra $\mathfrak{h}_k$.

3. Unitary evolution generated by local symmetric Hamiltonians

Although we defined the group $\mathcal{V}_k^G$ in terms of local symmetric quantum circuits, it can also be equivalently defined in terms of the unitary evolutions generated by local symmetric Hamiltonians. Clearly, any quantum circuit can be interpreted as the unitary time evolution generated under a time-dependent Hamiltonian, whose symmetries and locality are determined by the symmetries

and locality of the quantum circuit. In particular, any unitary $W \in \mathcal{V}_k^G$ can be generated by a Hamiltonian $H(t) = \sum_j h_j(t)$, where each term $h_j(t)$ is k-local and G-invariant. In particular, we can choose k-local, G-invariant terms $h_j(t)$ such that the family of unitaries $V(t)$, generated by Hamiltonian $H(t) = \sum_j h_j(t)$ under the Schrödinger equation,

$$\frac{dV(t)}{dt} = -iH(t)V(t) = -i\,\big[\sum_j h_j(t)\big]\,V(t), \tag{A8}$$

satisfy $V(t=0) = I$ and $V(t=1) = W$, where I is the identity operator on the total Hilbert space.

In the following, we argue that the converse is also true, i.e., the time evolution generated by any local symmetric Hamiltonian can also be realized by a finite local symmetric quantum circuit.

Proposition 3. *Suppose for all time $t \geq 0$, the Hermitian operator $H(t)$ is G-invariant and can be written as the sum of k-local terms. Then, for all time $t \geq 0$, the unitary evolution $V(t)$ generated by Hamiltonian $H(t)$ according to the Schrödinger equation belongs to the Lie group $\mathcal{V}_k^G$, i.e., can be implemented by a quantum circuit with a finite number of k-local G-invariant gates. Conversely, any unitary in $\mathcal{V}_k^G$ can be generated using a G-invariant Hamiltonian $H(t)$ that can be written as the sum of k-local terms.*

Proof. As we explained above the second part of this proposition is trivial. To prove the first part, suppose $H(t) = \sum_j h_j(t)$ is G-invariant. This does not imply that the k-local terms $\{h_j(t)\}$ are also G-invariant. However, we can easily show that $H(t)$ can be written as the sum of k-local G-invariant terms, i.e. $H(t) = \sum_j \tilde{h}_j(t)$, where each $\tilde{h}_j(t)$ is both k-local and G-invariant. This can be shown, for instance, by twirling over group G, using the uniform (Haar) distribution over group G. E.g. for a compact Lie group G, suppose we define

$$\tilde{h}_j(t) \equiv \int dg\, U(g) h_j(t) U(g)^\dagger\,, \tag{A9}$$

where dg is the uniform measure over group G. It can be easily seen that $\tilde{h}_j(t)$ becomes G-invariant. Furthermore, because $U(g) = \bigotimes_{s=1}^n u^{(s)}(g)$, the operator $U(g)h_j(t)U^\dagger(g)$ acts trivially on all systems except on the k systems, where $h_j(t)$ acts non-trivially. It follows that $\tilde{h}_j(t)$ is also k-local. Finally, the assumption that $H(t) = \sum_j h_j(t)$ is G-invariant implies

$$H(t) = \int dg\, U(g) H(t) U(g)^\dagger = \int dg\, U(g)\big[\sum_j h_j(t)\big]U(g)^\dagger = \sum_j \tilde{h}_j(t)\,. \tag{A10}$$

Since all operators $\{\tilde{h}_j(t) : t \geq 0\}_j$ are k-local and G-invariant, the Lie algebra generated by operators $\{i\tilde{h}_j(t) : t \geq 0\}_j$ is a sub-algebra of $\mathfrak{h}_k$, the Lie algebra associated to the Lie group $\mathcal{V}_k^G$.

Finally, we use a standard result of quantum control theory [15, 16], which implies that the 1-parameter family of unitaries $\{V(t) : t \geq 0\}$ satisfying the Schrödinger equation in Eq.(A8) with the initial condition $V(t=0) = I$ are in the Lie group associated to the Lie algebra generated by the set of operators $\{ih_j(t) : t \geq 0\}$. Together with the above result this implies that the family of unitaries $\{V(t) : t \geq 0\}$ belongs to $\mathcal{V}_k^G$ for all $t \geq 0$.

□

4. Charge vectors of general symmetric Hamiltonians

Consider the decomposition of the unitary representation $\{U(g) : g \in G\}$ into the irreps of group G. Under this decomposition the Hilbert space decomposes as

$$\mathcal{H} \cong \bigoplus_{\mu \in \text{Irreps}(n)} \mathcal{H}_\mu\,, \tag{A11}$$

where the summation is over Irreps(n), the set of inequivalent irreducible representations (irreps) of G appearing in this representation and $\mathcal{H}_\mu$ is the subspace corresponding to irrep μ, also known as the *isotypic* component of μ. Using Schur's lemmas, we can show that any G-invariant operator is block-diagonal with respect to this decomposition and, in general, can have support in any arbitrary subset of these sectors. However, as we will see in the following, for Hamiltonians generated by a fixed set of G-invariant Hamiltonians $\{H_j\}_j$, the supports in different subspaces $\{\mathcal{H}_\mu\}$ satisfy particular constraints, dictated by Hamiltonians $\{H_j\}_j$.

For any operator A, consider the vector

$$|\chi_A\rangle \equiv \sum_{\mu \in \text{Irreps}(n)} \text{Tr}(A\Pi_\mu)\, |\mu\rangle \;, \tag{A12}$$

where Π_μ is the projector to the subspace $\mathcal{H}_\mu$ and $\{|\mu\rangle : \mu \in \text{Irreps}(n)\}$ is a set of orthonormal vectors in an abstract vector space (not the state space of the system). Vector $|\chi_A\rangle$, which will be called the *charge vector* of operator A, encodes information about the components of this operator in different sectors $\{\mathcal{H}_\mu\}$. A general G-invariant Hamiltonian can have any charge vector with real coefficients. In particular, for any set of real numbers $\{a_\mu \in \mathbb{R}\}$, the Hermitian operator $\sum_{\mu\in\text{Irreps}(n)} a_\mu \Pi_\mu/\text{Tr}(\Pi_\mu)$ is G-invariant and has the charge vector $\sum_{\mu\in\text{Irreps}(n)} a_\mu\, |\mu\rangle$. In other words, under the linear map $A \to |\chi_A\rangle$, the image of the Lie algebra of skew-Hermitian G-invariant operators is $\{i\sum_\mu a_\mu|\mu\rangle : a_\mu \in \mathbb{R}\}$, which is a vector space over field $\mathbb{R}$, with dimension equal to $|\text{Irreps}_G(n)|$.

Lemma 1. *Consider a set of G-invariant Hermitian operators $\{H_j\}_j$. For any operator A, if $iA \in \mathfrak{alg}\{iH_j\}_j$ then the charge vector of A is in the subspace spanned by the charge vectors of $\{H_j\}$, i.e., $|\chi_A\rangle \in \textit{Span}_\mathbb{R}\{|\chi_j\rangle\}_j$, where $|\chi_j\rangle = \sum_{\mu\in\textit{Irreps}(n)} \textit{Tr}(H_j\Pi_\mu)\, |\mu\rangle$, Irreps$(n)$ is the set of inequivalent irreps of G in the representation $\{U(g) : g \in G\}$ and Π_μ is the projector to the subspace corresponding to irrep μ. Furthermore,*

$$\textit{dim}(\mathfrak{h}_n) - \textit{dim}(\mathfrak{alg}_\mathbb{R}\{iH_j\}) \geq |\textit{Irreps}(n)| - \textit{dim}(\textit{Span}_\mathbb{R}\{|\chi_j\rangle\}_j) \;, \tag{A13}$$

where $\mathfrak{h}_n$ is the Lie algebra of all skew-Hermitian G-invariant operators.

Proof. Recall that the elements of the Lie algebra $\mathfrak{alg}_\mathbb{R}\{iH_j\}$ can be written as linear combination of terms $\{iH_j\}_j$, and their nested commutators $\{[iH_{j_1}, iH_{j_2}], [[iH_{j_1}, iH_{j_2}], iH_{j_3}], ..\}$. This implies that if $iA \in \mathfrak{alg}_\mathbb{R}\{iH_j\}_j$, then there exists a set of real coefficients $\{a_j \in \mathbb{R}\}$ and a Hermitian operator B, such that

$$A = \sum_j a_j H_j + i[B, H_j] \;. \tag{A14}$$

Let Π_μ be the projector to the subspace corresponding to irrep μ. Then, this equation implies

$$\begin{aligned}
\text{Tr}(A\Pi_\mu) &= \sum_j a_j \text{Tr}(\Pi_\mu H_j) + i\text{Tr}(\Pi_\mu [H_j, B]) && \text{(A15)}\\
&= \sum_j a_j \text{Tr}(\Pi_\mu H_j) + i\text{Tr}([\Pi_\mu, H_j] B]) && \text{(A16)}\\
&= \sum_j a_j \text{Tr}(\Pi_\mu H_j) \;, && \text{(A17)}
\end{aligned}$$

where the second line follows from the cyclic property of trace and the last line follows from the assumption that for each j, operator H_j is G-invariant, which implies $[\Pi_\mu, H_j] = 0$. Therefore, we conclude that

$$|\chi_A\rangle \equiv \sum_\mu \text{Tr}(A\Pi_\mu)|\mu\rangle = \sum_j a_j \sum_\mu \text{Tr}(\Pi_\mu H_j)|\mu\rangle = \sum_j a_j |\chi_j\rangle \;, \tag{A18}$$

i.e., $|\chi_A\rangle \in \text{span}_\mathbb{R}\{|\chi_j\rangle\}$, which proves the first part of lemma.

To prove the second part, we use the rank-nullity theorem for the linear map $C \to i|\chi_C\rangle$: Since $\mathfrak{alg}_\mathbb{R}\{iH_j\}$ is a subspace of $\mathfrak{h}_n$, using the rank-nullity theorem, we find that the difference between the dimensions of $\mathfrak{h}_n$ and its subspace $\mathfrak{alg}_\mathbb{R}\{iH_j\}$ is larger than, or equal to, the difference between the dimensions of their images under the linear map $C \to i|\chi_C\rangle$. As we have seen before, the dimension of these images are respectively, $|\text{Irreps}(n)|$ and $\dim(\text{Span}_\mathbb{R}\{|\chi_j\rangle\}_j)$. This immediately implies Eq.(A13). □

5. Charge vector and its Fourier Transform

Condition in lemma 1 can be stated more generally without referring to the irreducible decomposition of the symmetry. For any operator A, consider the function $\chi_A : G \to \mathbb{C}$ defined by equation

$$\chi_A(g) \equiv \text{Tr}(AU(g)) \;:\; g \in G \;. \tag{A19}$$

Then, using an argument similar to the argument used in the proof of lemma 1, we can easily prove

Lemma 2. *Assume $iA \in \mathfrak{alg}_{\mathbb{R}}\{iH_j\}$, where $\{H_j\}_j$ are G-invariant Hermitian operators. Then, $\chi_A \in Span_{\mathbb{R}}\{\chi_j\}_j$, where $\chi_j(g) = Tr(H_j U(g)) : \forall g \in G$. Furthermore,*

$$dim(\mathfrak{h}_n) - dim(\mathfrak{alg}_{\mathbb{R}}\{iH_j\}) \geq |Irreps(n)| - dim(Span_{\mathbb{R}}\{\chi_j\}_j) \,. \tag{A20}$$

For any G-invariant operator A, the function χ_A and the charge vector $|\chi_A\rangle$ are related via Fourier transform. To understand the connection better, consider the decomposition of the representation $\{U(g) : g \in G\}$ to the irreducible representations of G. If G is a finite or compact Lie group, then every finite-dimensional representation is completely reducible, i.e., there exists a unitary W such that

$$WU(g)W^\dagger = \bigoplus_{\mu \in \text{Irreps}(n)} u^{(\mu)}(g) \otimes I_{m_\mu} \;, \quad \forall g \in G \tag{A21}$$

and the Hilbert space $\mathcal{H}$ decomposes as

$$\mathcal{H} \cong \bigoplus_{\mu \in \text{Irreps}(n)} \mathcal{H}_\mu \;\cong\; \bigoplus_{\mu \in \text{Irreps}(n)} \mathbb{C}^{d_\mu} \otimes \mathbb{C}^{m_\mu} \,, \tag{A22}$$

where Irreps(n) is the set of inequivalent irreps of G appearing in the representation $\{U(g) : g \in G\}$, $\{u^{(\mu)}(g) : g \in G\}$ is the irreducible representation which acts irreducibly on $\mathbb{C}^{d_\mu}$, d_μ is the dimension of irrep μ and m_μ is its multiplicity, and I_{m_μ} is the identity operator on $\mathbb{C}^{m_\mu}$. Using Schur's lemmas, one can show that in this basis any G-invariant operator A can be written as

$$WAW^\dagger = \bigoplus_{\mu \in \text{Irreps}(n)} I_{d_\mu} \otimes A^{(\mu)} \,, \tag{A23}$$

where I_{d_μ} is the identity operator on $\mathbb{C}^{d_\mu}$, and $A^{(\mu)}$ is an operator acting on $\mathbb{C}^{m_\mu}$. Using this decomposition, the charge vector of operator A is

$$|\chi_A\rangle = \sum_{\mu \in \text{Irreps}(n)} \text{Tr}(\Pi_\mu A)\, |\mu\rangle = \sum_{\mu \in \text{Irreps}(n)} d_\mu \times \text{Tr}(A^{(\mu)})\, |\mu\rangle = \sum_{\mu \in \text{Irreps}(n)} d_\mu \times a_\mu\, |\mu\rangle \,, \tag{A24}$$

where $a_\mu = \text{Tr}(A^{(\mu)})$. On the other hand,

$$\chi_A(g) = \text{Tr}(AU(g)) = \text{Tr}\big(\bigoplus_{\mu \in \text{Irreps}(n)} u^{(\mu)}(g) \otimes A^{(\mu)}\big) = \sum_{\mu \in \text{Irreps}(n)} \text{Tr}(u^{(\mu)}(g)) \times \text{Tr}(A^{(\mu)}) = \sum_{\mu \in \text{Irreps}(n)} a_\mu\, f_\mu(g) \,, \tag{A25}$$

where $f_\mu(g) = \text{Tr}(u^{(\mu)}(g))$ is the character of irrep μ. Using the orthogonality relations for the characters, one can obtain $|\chi_A\rangle$ from $\chi_A(g)$ and vice versa, using Fourier transforms. In partciular, for compact Lie groups

$$|\chi_A\rangle = \int_G dg\; \chi_A(g) \sum_{\mu \in \text{Irreps}(n)} d_\mu\; f^*_\mu(g) |\mu\rangle \,, \tag{A26}$$

where the integral is over Haar (uniform) measure, d_μ is the dimension of irrep μ, and f^*_μ is the complex conjugate of the character f_μ.

Finally, it is worth noting that Eq.(A23) implies that the group of symmetric unitaries $\mathcal{V}^G$ is isomorphic to the direct product of unitary groups U(m_μ), for all $\mu \in$ Irreps(n), where m_μ is the multiplicity of irrep μ in the representation $\{U(g) : g \in G\}$.

6. Charge vectors of the Lie algebra generated by k-local symmetric Hamiltonians: The case of identical subsystems

In this section, we focus on the special case where all subsystems $j = 1, \cdots n$, have identical Hilbert spaces isomorphic to $\mathbb{C}^d$ for a finite d, and identical unitary representation of symmetry G. In particular, we assume the action of group element $g \in G$ on the composite system is represented by $U(g) = u(g)^{\otimes n}$.

Suppose the family of unitaries $\{e^{-iHt} : t \in \mathbb{R}\}$ can be implemented using k-local G-invariant unitarie, i.e., suppose $iH \in \mathfrak{h}_k$. Then, lemma 1 implies that the charge vector of H should be in the subspace spanned by the charge vector of k-local G-invariant

Hamiltonians, denoted by

$$\mathcal{S}_k \equiv \{|\chi_H\rangle : e^{-iHt} \in \mathcal{V}_k^G, \forall t \in \mathbb{R}\} = \{|\chi_A\rangle : iA \in \mathfrak{h}_k\} = \{|\chi_A\rangle : A = A^\dagger, A \text{ is } k\text{-local}, \ [A, U(g)] = 0 \ : \forall g \in G\} \ , \quad \text{(A27)}$$

where the last equality follows from lemma 1. Using this result, we can immediately see that the dimension of $\mathcal{S}_k$ does not grow with n, the total number of sites, and only depends of k: First, note that the charge vectors are invariant under permutations of sites, i.e., for any permutation S, the charge vector of A and $SAS^\dagger$ are equal. This follows from the assumption that all sites have identical representation of symmetry and any such permutation leaves the total charge in the system invariant, i.e., $[S, \Pi_m u] = 0$ for all $\mu \in \text{Irreps}(n)$.

Furthermore, by applying a proper permutation S, any k-local operator A can be converted to an operator $SAS^\dagger$, which acts non-trivially only on a fixed set of k sites. Therefore, the dimension of $\mathcal{S}_k$ is equal to the dimension of the subspace spanned by the charge vectors of G-invariant operators which act nontrivially only on a fixed k sites. This immediately implies the dimension of $\mathcal{S}_k$ does not grow with n. In fact, as we show next, it can be easily seen that $\dim(\mathcal{S}_k) \leq |\text{Irreps}_G(k)|$, where $|\text{Irreps}_G(k)|$ is the number of distinct irreps of G appearing in representation $\{u(g)^{\otimes k} : g \in G\}$. This follows from the following lemma, which also gives a simple criterion for testing whether the charge vector of A is in $\mathcal{S}_k$, or not.

Lemma 3. *Suppose $iB \in \mathfrak{h}_k \equiv \mathfrak{alg}\{A : k\text{-local}, A + A^\dagger = 0, [A, u(g)^{\otimes n}] = 0 : \forall g \in G\}$. Then, there exists a set of real coefficients $\{b_\mu \in \mathbb{R}\}$, such that*

$$\forall g \in G : \qquad \textit{Tr}(u(g)^{\otimes n} B) = [\textit{Tr}(u(g))]^{n-k} \times \sum_{\mu \in \textit{Irreps}_G(k)} b_\mu \ f_\mu(g) \ , \quad \text{(A28)}$$

where the summation is over $\textit{Irreps}_G(k)$, the set of distinct irreps of G appearing in representation $\{u(g)^{\otimes k} : g \in G\}$ and f_μ is the character of irrep μ. Furthermore, for any set of real numbers $\{b_\mu\}$, there exists a Hermitian G-invariant operator B, satisfying this equation.

Proof. For any k-local Hermitian operator B, there exists a permutation operator S such that $SBS^\dagger = \tilde{B} \otimes I_d^{\otimes(n-k)}$, where $\tilde{B}$ is a Hermitian operator defined on the first k systems and I_d is the identity operator on the Hilbert space of each system. It follows that

$$\text{Tr}(B \ u(g)^{\otimes n}) = Tr([SBS^\dagger] u(g)^{\otimes n}) = [\text{Tr}(u(g))]^{n-k} \times \text{Tr}(\tilde{B} \ u(g)^{\otimes k}) \ . \quad \text{(A29)}$$

Furthermore, if $[B, u(g)^{\otimes n}] = 0$, then $[\tilde{B}, u(g)^{\otimes k}] = 0$, for all $g \in G$. Then, using Eq.(A25) we find

$$\text{Tr}(\tilde{B} \ u(g)^{\otimes k}) = \sum_{\mu \in \text{Irreps}(k)} b_\mu \ f_\mu(g) \ , \quad \text{(A30)}$$

where $\{b_\mu\}$ is a set of real coefficients. This proves the first part of the lemma. To see the second part, we note that if in the left-hand side of Eq.(A30) we choose $\tilde{B}$ to be the projector to the subspace corresponding to irrep $\mu \in \text{Irreps}(k)$, then $\text{Tr}(\tilde{B} \ u(g)^{\otimes k}) = c_\mu f_\mu(g), \forall g \in G$, for a constant $c_\mu > 0$. More generally, considering the linear combinations of projectors corresponding to all $\text{Irreps}_G(k)$, we conclude that for any set of real coefficients $\{b_\mu : \mu \in \text{Irreps}_G(k)\}$, there is a Hermitian G-invariant operator $\tilde{B}$ acting on $(\mathbb{C}^d)^{\otimes n}$, such that $\sum_{\mu \in \text{Irreps}_G(k)} b_\mu \ f_\mu(g)$. Then, operator $B = \tilde{B} \otimes I_d^{\otimes n-k}$, is a k-local G-invariant Hermitian operator and satisfies Eq.(A30). This proves the the second part of the lemma.

□

Note that theorem 2 in the main paper follows from this lemma together with proposition 1 and corollary 3. An interesting corollary of this result is

Corollary 4. *Suppose G is an Abelian group. If there exists a group element $g \in G$ such that $\textit{Tr}(u(g)) = 0$, then for any $k < n$, $\textit{dim}(\mathfrak{h}_k) < \textit{dim}(\mathfrak{h}_n)$.*

Proof. We prove a slightly more general result: Suppose the representation $\{u(g)^{\otimes n} : g \in G\}$ contains a 1D irrep of group G, and let Π be the projector to the subspace corresponding to this 1D irrep. In the case of Abelian groups all irreps are 1D and therefore this assumption is always satisfied. Now we show that if there exists $g \in G$ such that $\text{Tr}(u(g)) = 0$ and if $k < n$, then $i\Pi \notin \mathfrak{h}_k$, whereas clearly, $i\Pi \in \mathfrak{h}_n$. This implies $\dim(\mathfrak{h}_k) < \dim(\mathfrak{h}_n)$. To prove this claim, we assume it is not true, i.e., $i\Pi \in \mathfrak{h}_k$ and derive a contradiction. If $i\Pi \in \mathfrak{h}_k$, then Eq.(A28) should be satisfied for $B = \Pi$ and certain coefficients $b_\mu \in \mathbb{R}$. However, if there exists a group elements $g \in G$ such that $\text{Tr}(u(g)) = 0$ and $k < n$, then the right-hand side of this equation is zero for this group element. On the other hand, the left-hand side, i.e., $\text{Tr}(\Pi u(g)^{\otimes n})$ is non-zero: because Π is a projector to a 1D irrep, $|\text{Tr}(\Pi u(g)^{\otimes n})| = \text{Tr}(\Pi) > 0$. Therefore, the assumption that $i\Pi \in \mathfrak{h}_k$ is in contradiction with the assumptions that $k < n$ and $\text{Tr}(u(g)) = 0$, for some $g \in G$. This completes the proof. □

Another useful corollary of lemma 3 is the following result.

Corollary 5. *Recall the definition* $\mathcal{S}_k \equiv \{|\chi_H\rangle : e^{-iHt} \in \mathcal{V}_k^G, \forall t \in \mathbb{R}\} = \{|\chi_A\rangle : iA \in \mathfrak{h}_k\}$. *For any group G,* $dim(\mathcal{S}_k) \leq \big|Irreps_G(k)\big|$. *Furthermore, for any connected Lie group G, this holds as as equality, i.e.,* $dim(\mathcal{S}_k) = \big|Irreps_G(k)\big|$.

Proof. Using Eq.(A27), we have

$$\dim(\mathcal{S}_k) = \dim\big(\{|\chi_A\rangle : A = A^\dagger, A \text{ is } k\text{-local}, \ [A, U(g)] = 0 \ : \forall g \in G\}\big) \tag{A31}$$

$$= \dim\big(\{\chi_A : A = A^\dagger, A \text{ is } k\text{-local}, \ [A, U(g)] = 0 \ : \forall g \in G\}\big) \ , \tag{A32}$$

where $\chi_A : G \to \mathbb{C}$ is defined by $\chi_A(g) = \text{Tr}(AU(g)) = \text{Tr}(Au(g)^{\otimes n})$, and to get the second line we have used the fact that there is an invertible linear map between the charge vector $|\chi_A\rangle$ and function χ_A, namely the Fourier transform in Eq.(A26). Define

$$v(g) \equiv \text{Tr}(u(g)) : \quad g \in G \ . \tag{A33}$$

Then, using Eq.(A28)

$$\Big\{\chi_A : A = A^\dagger, A \text{ is } k\text{-local}, \ [A, U(g)] = 0 \ : \forall g \in G\Big\} = \Big\{v^{n-k} \times \sum_{\mu \in \text{Irreps}_G(k)} a_\mu \ f_\mu \ : a_\mu \in \mathbb{R}\Big\} \ . \tag{A34}$$

Together with Eq.(A32) this immediately implies that

$$\dim(\mathcal{S}_k) = \dim\big(\{\chi_A : A = A^\dagger, A \text{ is } k\text{-local}, \ [A, U(g)] = 0 \ : \forall g \in G\}\big) \tag{A35}$$

$$= \dim\Big(\Big\{v^{n-k} \times \sum_{\mu \in \text{Irreps}_G(k)} a_\mu \ f_\mu \ : a_\mu \in \mathbb{R}\Big\}\Big) \ , \tag{A36}$$

$$\leq |\text{Irreps}_G(k)| \ . \tag{A37}$$

This bound holds for any group G. In the special case of connected Lie group, functions $\{v^{n-k} \times f_\mu : \mu \in \text{Irreps}_G(k)\}$ are linearly independent. To prove this we assume the contrary holds, i.e., there exists a set of real coefficients a_μ, such that

$$v^{n-k} \sum_{\mu \in \text{Irreps}_G(k)} a_\mu \times f_\mu = 0 \ . \tag{A38}$$

Assuming G is a connected Lie group, then there is a finite neighborhood around the group identity, such that for all group elements g in the neighborhood, $v(g) = \text{Tr}(u(g)) \neq 0$ (Recall that the representation is finite-dimensional). Therefore, if Eq.(A38) holds then for all group elements g in this neighborhood $\sum_{\mu \in \text{Irreps}_G(k)} a_\mu f_\mu(g) = 0$. But, because the characters are analytic functions, if $\sum_{\mu \in \text{Irreps}_G(k)} a_\mu f_\mu(g) = 0$ is zero in a finite neighborhood, then it should vanish everywhere. Finally, using the fact that characters are linearly independent, we find that Eq.(A38) holds if and only if all $a_\mu = 0$ for all $\mu \in \text{Irreps}_G(k)$. We conclude that the set of functions $\{v^{n-k} \times f_\mu : \mu \in \text{Irreps}_G(k)\}$ are linearly independent. Therefore, if G is a connected Lie group, Eq.(A36) implies that $\dim(\mathcal{S}_k) = |\text{Irreps}_G(k)|$. □

7. Non-universality of LSQC and proof of the bound in Eq.(5): The case of identical subsystems

Recall the following definitions

$\mathcal{V}_k^G$: The Lie group generated by k-local G-invariant unitaries

$\mathfrak{h}_k$: The Lie algebra corresponding to $\mathcal{V}_k^G$

$\mathcal{S}_k$: The corresponding charge vectors

Note that

$$\dim(\mathcal{V}_k^G) = \dim(\mathfrak{h}_k) \ , \tag{A39}$$

where the left-hand side is the dimension of $\mathcal{V}_k^G$ as a manifold and the right-hand side is the dimension of $\mathfrak{h}_k$ as a vector space.

Theorem 6. *For a general group G, and integer $k \leq n$*

$$\textit{dim}(\mathfrak{h}_n) - \textit{dim}(\mathfrak{h}_k) \geq \textit{dim}(\mathcal{S}_n) - \textit{dim}(\mathcal{S}_k) \geq \textit{Irreps}_G(n) - \textit{Irreps}_G(k) \ . \tag{A40}$$

Furthermore, if G is connected Lie group, then for any integers k and l, satisfying $1 \leq k \leq l \leq n$, it holds that

$$\textit{dim}(\mathfrak{h}_l) - \textit{dim}(\mathfrak{h}_k) \geq \textit{dim}(\mathcal{S}_l) - \textit{dim}(\mathcal{S}_k) = \textit{Irreps}_G(l) - \textit{Irreps}_G(k) \ . \tag{A41}$$

Proof. This theorem follows immediately by applying the rank-nullity theorem for the linear map $A \to |\chi_A\rangle$, together with corollary 5. In particular, note that for $k \leq l$, the Lie algebra $\mathfrak{h}_k$ is a subspace of $\mathfrak{h}_l$. Since $\mathcal{S}_k$ and $\mathcal{S}_l$ are their images under a linear map (up to the imaginary i), using the rank-nullity theorem, we have

$$\dim(\mathfrak{h}_l) - \dim(\mathfrak{h}_k) \geq \dim(\mathcal{S}_l) - \dim(\mathcal{S}_k) \ . \tag{A42}$$

Combining this with corollary 5 together with the fact that $\dim(\mathcal{S}_n) = \text{Irreps}_G(n)$, proves the theorem. □

8. The general case of non-identical subsystems

In the previous section we focused on the special case where all the subsystems are identical and, in particular, they carry the same representation of the group G. However, note that the general argument about charge vectors and, in particular, lemma 1 and 2 are valid in the case of non-identical subsystems. Using these lemmas it can be easily seen that the argument that proves the non-universality of local symmetric unitaries can be generalized to the more general case where the subsystems are not identical. Here, we sketch the main idea.

Assume there are a finite number of *types* of subsystems, where each type carries a particular representation of group G. More precisely, suppose each subsystem has one of T possible representations $\{v^{(1)}, \cdots, v^{(T)}\}$, where for each $t \in \{1, \cdots, T\}$, $\{v^{(t)}(g) : g \in G\}$ is a finite-dimensional unitary representation of group G.

Then, our previous argument can be easily generalized to show that $\mathcal{S}_k$, the set of charge vectors for k-local G-invariant Hamiltonians, is a finite-dimensional subspace, whose dimension is bounded by a number which is independent of n, the total number of sites. In fact, the dimension of $\mathcal{S}_k$ is upper bounded by the total number of inequivalent irreps of G, which appear in all tensor product representations

$$\bigotimes_{i=1}^{k} v^{(t_i)} \ : \ t_1, \cdots, t_k \in \{1, \cdots, T\} \ . \tag{A43}$$

This follows from the fact that any k-local operator can act non-trivially on at most k sites, and the representation of group G on those k sites is equivalent to one of the representations listed above. Clearly, the total number of inequivalent irreps appearing in the above representations, is independent of n, the total number of sites.

On the other hand, let $\bigotimes_{i=1}^{n} v^{(t_i)}$ be the representation of group G on the total system, where $v^{(t_i)}$ is the representation of group G on site i and $t_i \in \{1, \cdots, T\}$. For a compact connected Lie group G, such as U(1) and SU(2), as the number of sites carrying a non-trivial representation of G grows, the number of distinct irreps which appear in this representation also increases unboundedly, and for sufficiently large n, this will be larger than the dimension of $\mathcal{S}_k$. Therefore, by lemma 1 we conclude that for sufficiently large n, there are G-invariant unitaries which cannot be implemented using k-local G-invariant unitaries.

Appendix B: U(1) symmetry with qubits

In this section we study a system of qubits with U(1) symmetry corresponding to rotations around the z axis. Specifically, we consider n qubits, labeled as $j = 1, \cdots, n$, and the group of unitaries

$$U(e^{i\theta}) = (e^{i\theta Z})^{\otimes n} = e^{i\theta \sum_{j=1}^{n} Z_j} = \sum_{q=-n}^{n} e^{i\theta q}\, \Pi_q \quad : \theta \in [0, 2\pi)\ . \tag{B1}$$

where Π_q is the projector to the subspace corresponding to eigenvalue q of operator $\sum_j Z_j$. We study the group $\mathcal{V}_k^{U(1)}$ generated by k-local unitaries that are invariant under this symmetry. It turns out that the constraints in lemma 1 on charge vectors are the only constraints on diagonal Hamiltonians that can be simulated using these unitaries. In particular, we show that

Theorem 7. *Consider an arbitrary diagonal Hamiltonian*

$$H = \sum_{\mathbf{z}\in\{0,1\}^n} h(\mathbf{z})\, |\mathbf{z}\rangle\langle\mathbf{z}| = \sum_{\mathbf{b}\in\{0,1\}^n} \tilde{h}(\mathbf{b})\, \mathbf{Z}^{\mathbf{b}}\ , \tag{B2}$$

where $\tilde{h}(\mathbf{b}) = 2^{-n} \sum_{\mathbf{z}\in\{0,1\}^n} (-1)^{\mathbf{b}\cdot\mathbf{z}}\, h(\mathbf{z})$ *is the Fourier transform of* $h(\mathbf{z})$*. The family of diagonal unitaries* $\{e^{-itH} : t \in \mathbb{R}\}$ *can be generated using* k*-local U(1)-invariant Hamiltonians if, and only if*

$$\sum_{\mathbf{b}\in\{0,1\}^n : w(\mathbf{b})=v} \tilde{h}(\mathbf{b}) = 0\ , \qquad \textit{for}\ \ \forall v : k < v \le n\ , \tag{B3}$$

where the summation is over all bit strings with Hamming weight v*.*

The necessity of constraints in Eq.(B3) follows immediately from constraints on charge vectors in lemma 1.

Furthermore, in Sec.B 2 we show that using diagonal Hamiltonians together with Hamiltonians $XX+YY$, we can implement all unitaries which are invariant under this U(1) symmetry (See theorem 8). In Appendix C, we present a more general (and more formal) version of this result in the case of energy-conserving unitaries.

Charge vectors of diagonal operators in the Lie algebra generated by local U(1)-invariant Hamiltonians

(Necessity of conditions in Eq.(B3) of theorem 7)

For any operator A acting on $(\mathbb{C}^2)^{\otimes n}$, the charge vector is

$$|\chi_A\rangle \equiv \sum_{q=-n}^{n} \mathrm{Tr}(\Pi_q A)\, |q\rangle\ , \tag{B4}$$

and its Fourier transform is the function

$$\chi_A(e^{i\theta}) = \mathrm{Tr}(AU(e^{i\theta})) = \mathrm{Tr}(A(e^{i\theta Z})^{\otimes n}) = \sum_{q=-n}^{n} e^{iq\theta}\, \mathrm{Tr}(\Pi_q A)\ . \tag{B5}$$

Let $\mathfrak{h}_k$ be the Lie algebra generated by $k-$local U(1)-invariant Hamiltonians, and $\tilde{\mathcal{S}}_k$ be the subspace

$$\tilde{\mathcal{S}}_k \equiv \{\chi_A :\ iA \in \mathfrak{h}_k\ \}\ . \tag{B6}$$

First, using lemma 2, we find

$$\tilde{\mathcal{S}}_k \equiv \{\chi_A :\ iA \in \mathfrak{h}_k\ \} = \mathrm{Span}_{\mathbb{R}}\Big\{\chi_A :\ A = A^\dagger,\ [A, \sum_r Z_r] = 0, A \text{ is } k\text{-local}\ \Big\}\ , \tag{B7}$$

i.e., $\tilde{\mathcal{S}}_k$ is the span of functions χ_A for all $k-$local Hermitian, U(1)-invariant operators. We claim that this subspace is equal to

$$\tilde{\mathcal{S}}_k = \mathrm{Span}_{\mathbb{R}}\{\xi_v : 0 \le v \le k\}\ ,\ \ \text{where}\ \ \xi_v(e^{i\theta}) = (\cos\theta)^{n-v}\, (i\sin\theta)^v\ . \tag{B8}$$

To prove Eq.(B8) first note that

$$\chi_A(e^{i\theta}) = \mathrm{Tr}(\mathcal{D}(A)U(e^{i\theta}))\ , \tag{B9}$$

where

$$\mathcal{D}(A) \equiv \sum_{\mathbf{b}\in\{0,1\}^n} |\mathbf{b}\rangle\langle\mathbf{b}|A|\mathbf{b}\rangle\langle\mathbf{b}| = \sum_{b_1,\cdots,b_n\in\{0,1\}} (|b_1\rangle\langle b_1|\otimes\cdots\otimes|b_n\rangle\langle b_n|)\, A\, (|b_1\rangle\langle b_1|\otimes\cdots\otimes|b_n\rangle\langle b_n|)\,, \tag{B10}$$

is the diagonal part of A. This definition immediately implies that if A is k-local, then $\mathcal{D}(A)$ is also k-local, i.e., acts non-trivially on, at most, k sites. Furthermore, any diagonal operator can be written as a linear combination of operators $\{\mathbf{Z}^{\mathbf{b}} : \mathbf{b}\in\{0,1\}^n\}$. We conclude that

$$\tilde{\mathcal{S}}_k = \text{Span}_{\mathbb{R}}\Big\{\chi_A : A = \sum_{\mathbf{b}} a_{\mathbf{b}}\mathbf{Z}^{\mathbf{b}}\,, a_{\mathbf{b}}\in\mathbb{R},\ A \text{ is } k\text{-local}\Big\}\,. \tag{B11}$$

Next, using the fact that operators $\{\mathbf{Z}^{\mathbf{b}} : \mathbf{b}\in\{0,1\}^n\}$ are linearly independent and pairwise orthogonal relative to the Hilbert-Schmidt inner product, we find that any k-local operator $A = \sum_{\mathbf{b}} a_{\mathbf{b}}\mathbf{Z}^{\mathbf{b}}$ can be written as

$$A = \sum_{\mathbf{b}:w(\mathbf{b})\le k} a_{\mathbf{b}}\,\mathbf{Z}^{\mathbf{b}}\,, \tag{B12}$$

where $w(\mathbf{b}) = \sum_{l=1}^{N} b_l$ is the Hamming weight of the bit string $\mathbf{b} = b_1\cdots b_n$, and the summation is over all bit strings with Hamming weight $w(\mathbf{b}) \le k$. In other words, A can be written as a linear combination of those operators in $\{\mathbf{Z}^{\mathbf{b}}\}$, which are themselves k-local.

This immediately implies

$$\tilde{\mathcal{S}}_k = \text{Span}_{\mathbb{R}}\Big\{\chi_{\mathbf{Z}^{\mathbf{b}}} :\ w(\mathbf{b})\le k\Big\}\ , \tag{B13}$$

where

$$\chi_{\mathbf{Z}^{\mathbf{b}}}(e^{i\theta}) \equiv \text{Tr}\big(\mathbf{Z}^{\mathbf{b}}(e^{i\theta Z})^{\otimes n}\big) = 2^n(\cos\theta)^{n-w(\mathbf{b})}(i\sin\theta)^{w(\mathbf{b})}\,, \tag{B14}$$

where the equality follows from the fact that $\text{Tr}(e^{i\theta Z}) = 2\cos\theta$ and $\text{Tr}(Ze^{i\theta Z}) = 2i\sin\theta$. This proves our claim in Eq.(B8).

Similarly, using Eq.(B14), for any operator $A = \sum_{\mathbf{b}} a_{\mathbf{b}}\mathbf{Z}^{\mathbf{b}}$, we have

$$\chi_A(e^{i\theta}) = \text{Tr}(A(e^{i\theta Z})^{\otimes n}) = \sum_{\mathbf{b}} a_{\mathbf{b}}\text{Tr}(\mathbf{Z}^{\mathbf{b}}(e^{i\theta Z})^{\otimes n}) = 2^n\sum_{v=0}^{n}\xi_v(e^{i\theta})\times\sum_{\mathbf{b}\in\{0,1\}^n:w(\mathbf{b})=v} a_{\mathbf{b}}\,. \tag{B15a}$$

By lemma 2, an operator $iA = i\sum_{\mathbf{b}} a_{\mathbf{b}}\mathbf{Z}^{\mathbf{b}}$ with real coefficients $\{a_{\mathbf{b}}\}$ is in $\mathfrak{h}_k$ only if χ_A is in the subspace $\tilde{\mathcal{S}}_k$. Since functions $\{\xi_v : v = 0,\cdots,n\}$ are linearly independent, and subspace $\tilde{\mathcal{S}}_k$ is the linear span of $\{\xi_v : 0\le v\le k\}$, χ_A is in the subspace $\tilde{\mathcal{S}}_k$, only if the coefficients of ξ_v for $v > k$ is zero, i.e.

$$\sum_{\mathbf{b}:w(\mathbf{b})=v} a_{\mathbf{b}} = 0\ :\quad \forall v > k\,, \tag{B16}$$

where the summation is over all bit strings with Hamming weight v, and the condition should hold for all $v > k$. This proves the necessity of condition in Eq.(B3) in theorem 7. Next, we prove the sufficiency of this condition.

Diagonal operators in the Lie algebra generated by local U(1)-invariant Hamiltonians
(Sufficiency of condition in Eq.(B3) of theorem 7)

Recall the definitions

$$R_{rs} = \frac{X_r X_s + Y_r Y_s}{2} \quad , \quad T_{rs} = \frac{i}{2}[Z_r, R_{rs}] \, . \tag{B17}$$

We prove the following lemma

Lemma 4. *For $n \leq 2$ qubits labeled as $1, \cdots, n$, it holds that*

$$\Big\{ i \sum_{\boldsymbol{b}} a_{\boldsymbol{b}} \, \mathbf{Z}^{\boldsymbol{b}} : \; a_{\boldsymbol{b}} \in \mathbb{R}, \sum_{\boldsymbol{b}: w(\boldsymbol{b})=v} a_{\boldsymbol{b}} = 0 \; , \; \forall v : 1 \leq v \leq n \Big\} \subset \mathfrak{alg}_{\mathbb{R}} \Big\{ i R_{rs}, i T_{rs} : r \neq s \in \{1, \cdots, n\} \Big\} \, , \tag{B18}$$

where $\sum_{\boldsymbol{b}: w(\boldsymbol{b})=v}$ is the summation over all bit strings with Hamming weight v, and the right-hand side is the real Lie algebra generated by 2-local U(1)-invariant operators $\{iR_{rs}, iT_{rs} : r \neq s \in \{1, \cdots, n\}\}$.

The lemma implies that the subspace in Eq.(B18) is a subspace of $\mathfrak{h}_k$ for $k \geq 2$, where $\mathfrak{h}_k$ is the Lie algebra generated by k-local U(1)-invariant skew-Hermitian operators. By definition, in addition to this subspace, $\mathfrak{h}_k$ also includes arbitrary linear combinations of operators $\{i\mathbf{Z}^{\mathbf{b}} : w(\mathbf{b}) \leq k\}$. Linear combinations of these operators with the set of operators in Eq.(B18), yield all diagonal Hamiltonians satisfying condition in Eq.(B16). This proves the sufficiency of this condition and completes the proof of theorem 7.

In the rest of this section, we prove lemma 4. To prove this lemma we use the fact that, for any subset $t \leq n$ distinct qubits $l_1, l_2, \cdots, l_t \in \{1, \cdots, n\}$, we have

$$\begin{aligned} ic_t \times (Z_{l_1} ... Z_{l_{t-1}} - Z_{l_2} ... Z_{l_t}) = & \big[[..[[iR_{l_1 l_2}, iR_{l_2 l_3}], iR_{l_3 l_4}]..., iR_{l_{t-1}, l_t}], iR_{l_t, l_1} \big] \quad : t \text{ odd}, \\ & \big[[..[[iR_{l_1 l_2}, iR_{l_2 l_3}], iR_{l_3 l_4}]..., iR_{l_{t-1}, l_t}], iT_{l_t, l_1} \big] \quad : t \text{ even} \, , \end{aligned} \tag{B19}$$

where $c_t = \pm 1$, depending on t. Because of the usefulness of this commutation relation, for completeness we repeat it again, using a slightly different notation:

$$R_{rs} \equiv \frac{X_r X_s + Y_r Y_s}{2} \quad : r \neq s$$

$$\forall m \geq 2 : \qquad (Z_1 - Z_m) \, Z_1 \cdots Z_m = \begin{cases} c_m \Big[R_{1,m} \, , \, \big[R_{m,m-1} \, , \, \cdots [R_{4,3} \, , \, [R_{3,2}, R_{2,1}]] \cdots \big] \Big] & : \quad m \text{ odd} \\ \\ c_m \Big[R_{1,m} \, , \, \big[R_{m,m-1} \, , \cdots \, [R_{3,2} \, , \, [R_{2,1}, \frac{Z_1}{2}]] \cdots \big] \Big] & : \quad m \text{ even} \end{cases}$$

$$c_m = \pm 1$$

Proof. (lemma 4) For any pair of bit strings $\mathbf{b}_1, \mathbf{b}_2 \in \{0,1\}^n$, let $d(\mathbf{b}_1, \mathbf{b}_2)$ be their Hamming distance, i.e., the number of bits that should be flipped to transform one bit string to another. Using Eq.(B19), for any pair of bit strings $\mathbf{b}_1, \mathbf{b}_2 \in \{0,1\}^n$, the operator $i(\mathbf{Z}^{\mathbf{b}_1} - \mathbf{Z}^{\mathbf{b}_2})$ can be obtained from these commutators, provided that $\mathbf{b}_1$ and $\mathbf{b}_2$ have equal Hamming weights, i.e. $w(\mathbf{b}_1) = w(\mathbf{b}_2) = t$, and their Hamming distance $d(\mathbf{b}_1, \mathbf{b}_2) = 2$. This means that the linear span of operators in Eq.(B19) for a fixed t in the interval $1 \leq t \leq n$ contains all operators

$$\text{Span}_{\mathbb{R}} \Big\{ i(\mathbf{Z}^{\mathbf{b}_1} - \mathbf{Z}^{\mathbf{b}_2}) : w(\mathbf{b}_1) = w(\mathbf{b}_2) = t - 1, d(\mathbf{b}_1, \mathbf{b}_2) = 2 \Big\} \, . \tag{B20}$$

Next, we prove that

$$\text{Span}_{\mathbb{R}} \Big\{ i(\mathbf{Z}^{\mathbf{b}_1} - \mathbf{Z}^{\mathbf{b}_2}) : w(\mathbf{b}_1) = w(\mathbf{b}_2) = t - 1, d(\mathbf{b}_1, \mathbf{b}_2) = 2 \Big\} = \text{Span}_{\mathbb{R}} \Big\{ i(\mathbf{Z}^{\mathbf{b}_1} - \mathbf{Z}^{\mathbf{b}_2}) : \; w(\mathbf{b}_1) = w(\mathbf{b}_2) = t - 1 \Big\} \, , \tag{B21}$$

i.e. the restriction $d(\mathbf{b}_1, \mathbf{b}_2) = 2$ in the left-hand side can be removed. To prove this we use the fact that any pair of bit strings strings $\mathbf{c}_1, \mathbf{c}_2 \in \{0,1\}^n$ with equal Hamming weights $w(\mathbf{c}_1) = w(\mathbf{c}_2) = t-1$ are related to each other by a permutation of bits. Furthermore, any permutation can be realized by a sequence *transpositions*, i.e., 2-bit permutations, which only exchange the value of two-bits. It follows that for any pair of bit strings $\mathbf{c}_1, \mathbf{c}_2 \in \{0,1\}^n$ with equal Hamming weights $w(\mathbf{c}_1) = w(\mathbf{c}_2) = t-1$, there is a path in the space of bit strings with Hamming weight $t-1$ from $\mathbf{c}_1$ to $\mathbf{c}_2$, i.e.

$$\mathbf{f}_1, \cdots, \mathbf{f}_L \in \{0,1\}^n : \quad w(\mathbf{f}_k) = v, \quad \mathbf{f}_1 = \mathbf{c}_1,\ \mathbf{f}_L = \mathbf{c}_2 \ , \tag{B22}$$

and

$$\mathbf{c}_1 = \mathbf{f}_1 \longrightarrow \mathbf{f}_2 \longrightarrow \cdots \longrightarrow \mathbf{f}_L = \mathbf{c}_2 \ , \tag{B23}$$

where each consecutive pair of bit strings have Hamming distance 2, i.e.

$$d(\mathbf{f}_r, \mathbf{f}_{r+1}) = 2 : \quad 1 \leq r \leq L-1 \ . \tag{B24}$$

Therefore, $i(\mathbf{Z}^{\mathbf{c}_1} - \mathbf{Z}^{\mathbf{c}_2})$ can be obtained using the linear combination

$$i(\mathbf{Z}^{\mathbf{c}_1} - \mathbf{Z}^{\mathbf{c}_2}) = i(\mathbf{Z}^{\mathbf{f}_1} - \mathbf{Z}^{\mathbf{f}_L}) = i(\mathbf{Z}^{\mathbf{f}_1} - \mathbf{Z}^{\mathbf{f}_2}) + i(\mathbf{Z}^{\mathbf{f}_2} - \mathbf{Z}^{\mathbf{f}_3}) + \cdots + i(\mathbf{Z}^{\mathbf{f}_{L-1}} - \mathbf{Z}^{\mathbf{f}_L}) \ . \tag{B25}$$

This proves Eq.(B21). Next, it can be easily seen that

$$\text{Span}_{\mathbb{R}}\Big\{ i(\mathbf{Z}^{\mathbf{b}_1} - \mathbf{Z}^{\mathbf{b}_2}) : \ w(\mathbf{b}_1) = w(\mathbf{b}_2) = t-1 \Big\} = \Big\{ i \sum_{\mathbf{b}:w(\mathbf{b})=v} a_{\mathbf{b}}\, \mathbf{Z}^{\mathbf{b}} : \ \ a_{\mathbf{b}} \in \mathbb{R} \ , \sum_{\mathbf{b}:w(\mathbf{b})=t-1} a_{\mathbf{b}} = 0 \Big\} \ , \tag{B26}$$

where the right-hand side is the subspace of all linear combinations $i \sum_{\mathbf{b}:w(\mathbf{b})=t-1} a_{\mathbf{b}}\, \mathbf{Z}^{\mathbf{b}}$ for bit strings with Hamming weight $t-1$, which satisfy the linear constraint $\sum_{\mathbf{b}:w(\mathbf{b})=t-1} a_{\mathbf{b}} = 0$. Finally, considering the linear combinations of subspaces with different Hamming weights $v < n$, we obtain the subspace

$$\{ i \sum_{\mathbf{b}} a_{\mathbf{b}}\, \mathbf{Z}^{\mathbf{b}} : \ a_{\mathbf{b}} \in \mathbb{R}, \forall t \leq n : \sum_{\mathbf{b}:w(\mathbf{b})=t} a_{\mathbf{b}} = 0 \} \ . \tag{B27}$$

This completes the proof of lemma 4. □

1. All diagonal unitaries with a single ancillary qubit

Here, we show how we can circumvent the no-go theorem, and implement arbitrary diagonal unitary, using a single ancillary qubit. The argument is a straightforward generalization of the idea discussed in Eq.(10). In Appendix C, where we study the case of energy-conserving unitaries, a more general and formal version of this result is presented (See lemma 5 and theorem 9).

Consider a pair of bit strings $\mathbf{b}, \mathbf{b}' \in \{0,1\}^n$, such that $\mathbf{b}'$ can be obtained from $\mathbf{b}$ by flipping a bit with value 1, i.e., its Hamming weight is $w(\mathbf{b}') = w(\mathbf{b}) - 1$ and its Hamming distance with $\mathbf{b}$ is $d(\mathbf{b}', \mathbf{b}) = 1$. Let Z_a and I_a be, respectively, the Pauli z and the identity operator on the ancillary qubit. Then, Eq.(B19) implies that the Hamiltonian

$$\mathbf{Z}^{\mathbf{b}} \otimes I_a - \mathbf{Z}^{\mathbf{b}'} \otimes Z_a$$

can be generated using Hamiltonians $R_{rs} : \ r, s \in \{1, \cdots, n\} \cup \{a\}$ together with Pauli z on the ancillary qubit. Assuming this qubit is initially in state $|0\rangle_a$, under this Hamiltonian any initial state $|\psi\rangle$ of n qubits evolves to

$$e^{i\theta[\mathbf{Z}^{\mathbf{b}} \otimes I_a - \mathbf{Z}^{\mathbf{b}'} \otimes Z_a]} \ (|\psi\rangle \otimes |0\rangle_a) = (e^{i\theta[\mathbf{Z}^{\mathbf{b}} - \mathbf{Z}^{\mathbf{b}'}]} |\psi\rangle) \otimes |0\rangle_a \ . \tag{B28}$$

Note that at the end of the process, the ancillary qubit goes back to its initial state. Therefore, repeating this, we can implement all Hamiltonians

$$\Big\{ \mathbf{Z}^{\mathbf{b}} - \mathbf{Z}^{\mathbf{b}'} : \mathbf{b}, \mathbf{b}' \in \{0,1\}^n, w(\mathbf{b}') = w(\mathbf{b}) - 1, d(\mathbf{b}', \mathbf{b}) = 1 \Big\} = \ . \tag{B29}$$

Furthermore, by applying Pauli z Hamiltonian on the ancillary qubit, i.e., Hamiltonian $I^{\otimes n} \otimes Z_a$ on the total system, we can also implement the constant Hamiltonian $I^{\otimes n}$ on n qubits. Linear combinations of this Hamiltonian with Hamiltonians in Eq.(B29),

give all diagonal Hamiltonians (Note that a similar argument works if rather than state $|0\rangle_a$, the ancillary qubit is prepared in state $|1\rangle_a$). As we show next combining diagonal Hamiltonians with Hamiltonians $R_{rs} : r, s \in \{1, \cdots n\}$, one can generate all $U(1)$-invariant Hamiltonians.

2. From diagonal Hamiltonians to all symmetric Hamiltonians

In this section we prove that if one can implement all diagonal Hamiltonians as well as 2-local Hamiltonians $\{R_{j,j+1} = (X_jX_{j+1} + Y_jY_{j+1})/2 : j = 1, \cdots, n\}$, then one can implement all symmetric unitaries, i.e., those commuting with $\sum_j Z_j$. To show this, we prove the following theorem:

Theorem 8. *The real Lie algebra generated by the set of diagonal skew-Hermitian operators and operators $\{iR_{j,j+1} = i(X_jX_{j+1} + Y_jY_{j+1})/2 : j = 1, \cdots, n-1\}$ is equal to the set of all skew-Hermitian $U(1)$-invariant operators, i.e., those commuting with $\sum_j Z_j$. In other words,*

$$\mathfrak{h}_n \equiv \Big\{A \in \mathcal{L}((\mathbb{C}^2)^{\otimes n}) : A + A^\dagger = 0\ ,\ [A, \sum_{r=1}^n Z_r] = 0\Big\} = \mathfrak{alg}_{\mathbb{R}}\Big\{iR_{j,j+1}, i\boldsymbol{Z^b} : \boldsymbol{b} \in \{0,1\}^n, j \in \{1, \cdots, n-1\}\Big\}\ . \quad \text{(B30)}$$

Proof. It is clear that the Lie algebra $\mathfrak{alg}_{\mathbb{R}}\Big\{iR_{j,j+1}, i\mathbf{Z}^{\mathbf{b}} : \mathbf{b} \in \{0,1\}^n, j \in \{1, \cdots, n-1\}\Big\}$ is contained in $\mathfrak{h}_n$. Here, we prove the converse, i.e., we show

$$\mathfrak{h}_n \subseteq \mathfrak{alg}_{\mathbb{R}}\Big\{iR_{j,j+1}, i\mathbf{Z}^{\mathbf{b}} : \mathbf{b} \in \{0,1\}^n, j \in \{1, \cdots, n-1\}\Big\}\ . \quad \text{(B31)}$$

Any arbitrary operator $A \in \mathcal{L}((\mathbb{C}^2)^{\otimes n})$ can be written as

$$A = \sum_{\mathbf{b},\mathbf{b}' \in \{0,1\}^n} a_{\mathbf{b},\mathbf{b}'}\ |\mathbf{b}\rangle\langle\mathbf{b}'|\ . \quad \text{(B32)}$$

Using the fact that

$$\big(\sum_{r=1}^n Z_r\big)|\mathbf{b}\rangle = [n - 2w(\mathbf{b})]\ |\mathbf{b}\rangle\ , \quad \text{(B33)}$$

we find that

$$[A, \sum_{r=1}^n Z_r] = 2\sum_{\mathbf{b},\mathbf{b}'} a_{\mathbf{b},\mathbf{b}'}[w(\mathbf{b}') - w(\mathbf{b})]\ |\mathbf{b}\rangle\langle\mathbf{b}'|\ . \quad \text{(B34)}$$

This implies that if $[A, \sum_{r=1}^n Z_r] = 0$, then

$$a_{\mathbf{b},\mathbf{b}'} = 0 \quad \text{for}\ \ w(\mathbf{b}) \neq w(\mathbf{b}')\ . \quad \text{(B35)}$$

In other words, the off-diagonal terms for bit strings with different Hamming weights vanish.

Therefore, the space of $U(1)$-invariant operators is spanned by

$$\{|\mathbf{b}\rangle\langle\mathbf{b}'| : w(\mathbf{b}) = w(\mathbf{b}');\ \mathbf{b}, \mathbf{b}' \in \{0,1\}^n\}\ . \quad \text{(B36)}$$

This implies that $\mathfrak{h}_n$, the space of anti-Hermitian $U(1)$-invariant operators is spanned by

$$\mathfrak{h}_n = \text{Span}_{\mathbb{R}}\Big\{i\big(|\mathbf{b}\rangle\langle\mathbf{b}'| + |\mathbf{b}'\rangle\langle\mathbf{b}|\big), |\mathbf{b}\rangle\langle\mathbf{b}'| - |\mathbf{b}'\rangle\langle\mathbf{b}| : w(\mathbf{b}) = w(\mathbf{b}')\ ;\ \mathbf{b}, \mathbf{b}' \in \{0,1\}^n\Big\}\ . \quad \text{(B37)}$$

Using the fact that for any pair of bit strings $\mathbf{b}, \mathbf{b}' \in \{0,1\}^n$,

$$\Big[i|\mathbf{b}\rangle\langle\mathbf{b}|\ ,\ (|\mathbf{b}\rangle\langle\mathbf{b}'| - |\mathbf{b}'\rangle\langle\mathbf{b}|)\Big] = i\big(|\mathbf{b}\rangle\langle\mathbf{b}'| + |\mathbf{b}'\rangle\langle\mathbf{b}|\big)\ , \quad \text{(B38)}$$

we find that the Lie algebra $\mathfrak{h}_n$ is generated by

$$\mathfrak{h}_n = \mathfrak{alg}\Big\{ i|\mathbf{b}\rangle\langle\mathbf{b}| \ , \ |\mathbf{b}\rangle\langle\mathbf{b}'| - |\mathbf{b}'\rangle\langle\mathbf{b}| : w(\mathbf{b}) = w(\mathbf{b}') \ ; \ \mathbf{b}, \mathbf{b}' \in \{0,1\}^n \Big\} \ . \tag{B39}$$

Next, we prove that this algebra is generated by the following set of operators

$$\big\{ i|\mathbf{b}\rangle\langle\mathbf{b}| \big\} \cup \big\{ iR_{j,j+1} = i(X_jX_{j+1} + Y_jY_{j+1})/2 : j = 1, \cdots, n-1 \big\} \ , \tag{B40}$$

i.e., we prove that

$$\mathfrak{alg}\Big\{ i|\mathbf{b}\rangle\langle\mathbf{b}|, iR_{j,j+1} = i(X_jX_{j+1} + Y_jY_{j+1})/2 : j = 1, \cdots, n-1, \mathbf{b} \in \{0,1\}^n \Big\} \tag{B41a}$$

$$= \mathfrak{alg}\Big\{ i|\mathbf{b}\rangle\langle\mathbf{b}| \ , \ |\mathbf{b}\rangle\langle\mathbf{b}'| - |\mathbf{b}'\rangle\langle\mathbf{b}| : w(\mathbf{b}) = w(\mathbf{b}') \ ; \ \mathbf{b}, \mathbf{b}' \in \{0,1\}^n \Big\} = \mathfrak{h}_n \ . \tag{B41b}$$

To prove this claim, first note that for any bit string $\mathbf{b} \in \{0,1\}^n$, and any pair of distinct qubits $l, r \in \{1, \cdots, n\}$, it holds that

$$\big[i|\mathbf{b}\rangle\langle\mathbf{b}|, iR_{lr} \big] = |\mathbf{b}'\rangle\langle\mathbf{b}| - |\mathbf{b}\rangle\langle\mathbf{b}'| \equiv F(\mathbf{b}', \mathbf{b}) \ , \tag{B42}$$

where $\mathbf{b}'$ is the bit string obtained by exchanging bits l and r of bit string $\mathbf{b}$, and for any pair of bit strings $\mathbf{d}$ and $\mathbf{e}$, we have defined the notation

$$F(\mathbf{d}, \mathbf{e}) \equiv |\mathbf{d}\rangle\langle\mathbf{e}| - |\mathbf{e}\rangle\langle\mathbf{d}| \ . \tag{B43}$$

Next, note that for any three distinct bit strings $\mathbf{b}, \mathbf{b}', \mathbf{b}'' \in \{0,1\}^n$, it holds that

$$F(\mathbf{b}, \mathbf{b}'') = \Big[F(\mathbf{b}, \mathbf{b}'), F(\mathbf{b}', \mathbf{b}'') \Big] \ . \tag{B44}$$

By combining these two steps, we can obtain $F(\mathbf{c}_1, \mathbf{c}_2) = |\mathbf{c}_1\rangle\langle\mathbf{c}_2| - |\mathbf{c}_2\rangle\langle\mathbf{c}_1|$, for any pair of bit strings $\mathbf{c}_1, \mathbf{c}_2 \in \{0,1\}^n$ with equal Hamming weights: Recall that any pair of bit strings with equal Hamming weights are related via a permutation and any such permutation can be realized by combining *transpositions*, i.e., 2-bit permutations. Therefore, there exists a sequence

$$\mathbf{c_1} = \mathbf{b_1} \longrightarrow \mathbf{b_2} \ \cdots, \longrightarrow \mathbf{b_L} = \mathbf{c_2} \ , \tag{B45}$$

where $\mathbf{b_1} = \mathbf{c_1}$, $\mathbf{b_L} = \mathbf{c_2}$, $d(\mathbf{b}_p, \mathbf{b}_{p+1}) = 2$ for $1 \leq p \leq L-1$. In fact, because any permutation on n bits can be generated by transpositions on nearest-neighbor pairs of bits j and $j+1$, for $j = 1, \cdots, n-1$, in the chain in Eq.(B45), we can assume any two consecutive bit strings $\mathbf{b_p}$ and $\mathbf{b_{p+1}}$ are identical for all bits, except a pair of nearest-neighbor bits j and $j+1$.

Then, using Eq.(B44) we have

$$F(\mathbf{c_1}, \mathbf{c_2}) = F(\mathbf{b_1}, \mathbf{b_L}) = \Big[\cdots \big[\big[\big[[F(\mathbf{b_1}, \mathbf{b_2}), F(\mathbf{b_2}, \mathbf{b_3})], F(\mathbf{b_3}, \mathbf{b_4}) \big], F(\mathbf{b_4}, \mathbf{b_5}) \big] \cdots \big], F(\mathbf{b_{L-1}}, \mathbf{b_L}) \Big] \ . \tag{B46}$$

This proves Eq.(B41), i.e., the Lie algebra $\mathfrak{h}_n$ is generated by operators $\{i|\mathbf{b}\rangle\langle\mathbf{b}|, iR_{j,j+1}\}$, and completes proof of theorem 8. □

3. Implementing U(1)-invariant unitaries with geometrically local interactions

So far, in our discussion we have not assumed any particular geometry for the system and the labels $1, \cdots, n$ of n qubits was arbitrary. Next, we assume the qubits lie on an open chain and their labeling corresponds to their order in the chain. For instance, the qubits are ordered from left to right, and the leftmost qubit is labeled as qubit 1.

Suppose one can turn on and off $XX + YY$ interactions between nearest-neighbor qubits. Furthermore, suppose an ancillary qubits a can interact with all qubits with $XX + YY$ interaction. Furthermore, in addition to $XX + YY$ interactions between the qubits, suppose one can also apply local Pauli Z on the ancillary qubit. Then, the overall, Hamiltonian is in the form

$$H(t) = \sum_{j=1}^{n-1} c_j(t) \ (X_jX_{j+1} + Y_jY_{j+1}) + d_j(t) \ (X_jX_a + Y_jY_a) + z(t)Z_a \ , \tag{B47}$$

where $c_j(t)$, $d_l(t)$ and $z(t)$ are arbitrary real functions. As before, we assume the ancilllary qubit is initially in state $|0\rangle$.

This Hamiltonian does not allow direct interactions between arbitrary pairs of qubits. Nevertheless, it turns out that using this family of Hamiltonians we can implement all U(1)-invariant unitary transformations on qubits $1, \cdots, n$. To see this first not that for any connected subset of qubits, corresponding to a gapless sequence of integers $j, j+1, \cdots, j'-1, j'$, we can implement the corresponding Hamiltonian $Z_j \cdots Z_{j'}$. This follows immediately from Eq.(B19) together with the argument in Eq.(B28). In particular, we can simulate interaction $Z_j Z_{j+1}$ for any neighboring qubits j and $j+1$. Now the key observation is that by combining interactions $Z_j Z_{j+1}$ and $X_j X_{j+1} + Y_j Y_{j+1}$ one can implement the swap unitary on qubits j and $j+1$, i.e., the unitary that exchanges the states of these qubits. In particular,

$$e^{i\frac{\pi}{4}(X_j X_{j+1} + Y_j Y_{j+1} + Z_j Z_{j+1})} = e^{i\frac{\pi}{4}} S_{j,j+1} \ , \tag{B48}$$

where $S_{j,j+1}$ is the swap operator that exchanges the state of qubits j and $j+1$. By combining swaps on nearest-neighbor qubits, we obtain all permutations on n qubits. Therefore, we can change the order of qubits, arbitrarily. Combining this with the above technique we can implement arbitrary diagonal unitary transformations. For instance, to implement the unitary $e^{i\theta Z_l Z_m}$ between any two arbitrary qubits l and m with $l < m$, we first apply the permutation operator $S_{l+1,m}$ that exchanges the states of qubits m and $l+1$ and leave the other qubits unchanged. Then we apply the unitary $e^{i\theta Z_l Z_{l+1}}$ and finally exchange the state of qubits $l+1$ and m again. In this way, we obtain

$$S_{l+1,m} e^{i\theta Z_l Z_{l+1}} S_{l+1,m} = e^{i\theta Z_l Z_m} \ . \tag{B49}$$

Finally, recall that according to theorem 8, by combining diagonal unitaries with unitaries generated by Hamiltonians $\{R_{j,j+1} = \frac{1}{2}(X_j X_{j+1} + Y_j Y_{j+1}) : j = 1, \cdots, n-1\}$, we can implement all U(1)-invariant unitaries. This proves the claim that using a single ancillary qubit in initial state $|0\rangle$ and by properly choosing functions c_j, d_j and z, we can implement a general energy-conserving unitary using Hamiltonian $H(t)$ in Eq.(B47).

In the above scheme, the ancillary qubit a needs to interact with all qubits in the system. We can relax this requirement if in addition to interactions $XX + YY$, we have access to interactions ZZ. More precisely, consider the family of Hamiltonians

$$H'(t) = r(r)(X_1 X_a + Y_1 Y_a) + s(t) Z_1 Z_a + z(t) Z_a + \sum_{j=1}^{n-1} c_j(t)\ (X_j X_{j+1} + Y_j Y_{j+1}) + b_j(t)\ Z_j Z_{j+1} \ , \tag{B50}$$

where c_j, b_j, r, s and z are arbitrary real functions. Using an argument similar to the above argument, we can easily see that universality can also be achieved using this family of Hamiltonians. Again, the key point is that by combining $XX + YY$ and ZZ interactions on nearest neighbor qubits, we can swap their orders, and therefore, we can permute the order of all qubits arbitrarily. Hence, the restriction to nearest-neighbor interactions becomes irrelevant.

Appendix C: A scheme for implementing energy-conserving unitaries on composite systems (Proof of Theorem 1)

In this section, we prove theorem 1, i.e., we show that: *all energy-conserving unitary transformations on a composite system can be implemented using 2-local energy-conserving unitaries and a single ancillary qubit*. We also show that if one can use a second ancillary qubit, then a general energy-conserving unitary can be implemented without any direct interactions between the systems, i.e., just using system-ancilla interactions. Theorem 9 and corollary 10 contain the precise statements of these results.

The proposed scheme for implementing general energy-conserving unitaries is a generalization of the protocol used in the qubit case with U(1) symmetry. In particular, similar to that case, there are two main steps in the argument: First, we show how a general *diagonal* energy-conserving unitary can be implemented, and then we show that by combining diagonal unitaries with 2-local energy-conserving unitaries, we obtain all energy-conserving unitaries.

To simplify the notation and analysis, we focus on the case of systems with identical Hilbert spaces and Hamiltonians. We also assume the Hamiltonians are non-degenerate (These assumptions are non-essential in the argument and can be relaxed). In particular, we consider $n \geq 1$ systems each with a d-dimensional Hilbert space and with the intrinsic Hamiltonian $\Delta E \sum_{r=0}^{d-1} |r\rangle\langle r|$. The systems are labeled by $j = 1, \cdots, n$. We assume before and after applying the energy-conserving unitary, the systems are non-interacting, i.e., their total intrinsic Hamiltonian is

$$H_{\text{intrinsic}} = \sum_{j=1}^{n} H_j \ , \tag{C1}$$

where

$$H_j = \Delta E \sum_{r=0}^{d-1} |r\rangle\langle r|_j \ , \tag{C2}$$

is the Hamiltonian of system j tensor product with the identity operators on the rest of systems.

1. A Scheme with a Single Ancillary Qubit

In this scheme we use an ancillary qubit a, with Hamiltonian $\Delta E \frac{Z+I}{2} = \Delta E |1\rangle\langle 1|$, initially prepared in the ground state $|0\rangle$. The required interactions for implementing this scheme are

$$R^{(l)}_{\text{a},j} \equiv R^{(l)}_{j,\text{a}} \equiv |l-1\rangle\langle l|_j \otimes |1\rangle\langle 0|_\text{a} + |l\rangle\langle l-1|_j \otimes |0\rangle\langle 1|_\text{a} \qquad : l = 1, \cdots, d-1 \ ; \ \ j = 1, \cdots, n \tag{C3a}$$

$$R^{(l,l')}_{j,j+1} \equiv R^{(l',l)}_{j+1,j} \equiv |l-1\rangle\langle l|_j \otimes |l'\rangle\langle l'-1|_{j+1} + |l\rangle\langle l-1|_j \otimes |l'-1\rangle\langle l'|_{j+1} \qquad : l, l' = 1, \cdots, d-1 \ ; \ \ j = 1, \cdots, n-1 \ . \tag{C3b}$$

Note that the above operators are defined on the Hilbert space $(\mathbb{C}^d)^{\otimes n} \otimes \mathbb{C}^2$, corresponding to n system and the ancillary qubit. Operator $R^{(l)}_{j,\text{a}}$ acts non-trivially on system j and ancillary qubit a and $R^{(l',l)}_{j+1,j}$ acts non-trivially on systems j and $j+1$. Therefore, these interactions are 2-local. Furthermore, if the systems are placed in the order corresponding to their labels $j = 1, \cdots, n$, then the terms $R^{(l,l')}_{j,j+1}$ are interactions between 2 nearest-neighbor systems.

Also, note that the above interactions are all energy-conserving, i.e., commute with the total intrinsic Hamiltonian of n systems and the ancillary qubits

$$H_{\text{tot}} \equiv \sum_{j=1}^{n} H_j + \Delta E |1\rangle\langle 1|_\text{a} \ . \tag{C4}$$

We show that

Lemma 5. *Consider a Hermitian operator $H_{\mathbf{n}}$ acting on the Hilbert space $(\mathbb{C}^d)^{\otimes n}$ that commutes with $H_{intrinsic} = \sum_{j=1}^{n} H_j$. Then, there exists an operator $\tilde{H}_{\mathbf{n},a}$ acting on $(\mathbb{C}^d)^{\otimes n} \otimes \mathbb{C}^2$, such that $i\tilde{H}_{\mathbf{n},a}$ is in the real Lie algebra, generated by iZ_a , $iR^{(l)}_{j,a}, iR^{(l,l')}_{j,j+1}$, i.e.*

$$i\tilde{H}_{\mathbf{n},a} \in \mathfrak{alg}_{\mathbb{R}}\Big\{ iZ_a \ , iR^{(l)}_{j,a}, iR^{(l,l')}_{j,j+1} : \ l, l' = 1, \cdots, d-1 \ ; \ \ j = 1, \cdots, n \Big\} \ , \tag{C5}$$

and

$$|\psi\rangle \in (\mathbb{C}^d)^{\otimes n}: \qquad \tilde{H}_{\mathbf{n},a}(|\psi\rangle \otimes |0\rangle_a) = H_{\mathbf{n}}|\psi\rangle \otimes |0\rangle_a \,. \tag{C6}$$

Any energy-conserving unitary $V_{\mathbf{n}}$ on the n systems can be written as $e^{i\tilde{H}_{\mathbf{n}}}$, where $\tilde{H}_{\mathbf{n}}$ commutes with the intrinsic Hamiltonian $H_{\text{intrinsic}} = \sum_{j=1}^n H_j$. Therefore, the above lemma implies that there exists an operator $i\tilde{H}_{\mathbf{n},\text{a}}$ in the real Lie algebra, generated by iZ_a , $iR_{j,\text{a}}^{(l)}, iR_{j,j+1}^{(l,l')}$, such that

$$|\psi\rangle \in (\mathbb{C}^d)^{\otimes n}: \qquad e^{i\tilde{H}_{\mathbf{n},\text{a}}}(|\psi\rangle \otimes |0\rangle_\text{a}) = e^{iH_{\mathbf{n}}}|\psi\rangle \otimes |0\rangle_\text{a} = V_{\mathbf{n}}|\psi\rangle \otimes |0\rangle_\text{a} \,. \tag{C7}$$

Furthermore, the fact that $i\tilde{H}_{\mathbf{n},\text{a}} \in \mathfrak{alg}_{\mathbb{R}}\big\{iZ_\text{a}\ , iR_{j,\text{a}}^{(l)}, iR_{j,j+1}^{(l,l')} :\ l,l' = 1,\cdots,d-1\ ;\ \ j = 1,\cdots,n\big\}$ implies that the unitary $e^{i\tilde{H}_{\mathbf{n},\text{a}}}$ is in the Lie group generated by unitaries

$$\exp(i\theta Z_\text{a})\ ,\ \exp(i\theta R_{j,\text{a}}^{(l)})\ ,\ \exp(i\theta R_{j,\text{a}}^{(l)})\ ,\ \exp(i\theta R_{j,j+1}^{(l,l')}) \quad : \theta \in [0,2\pi);\ l,l' = 1,\cdots,d-1\ ;\ \ j = 1,\cdots,n\ , \tag{C8}$$

which are all 2-local and energy-conserving. Moreover, using Fact 1 in Appendix A, the group generated by these unitaries is compact and therefore using the result of [15], any unitary in this group is uniformly finitely generated by the generating set in Eq.(C8). This result, which is summarized in theorem 9, proves the statement of theorem 1 in the main text.

This result means that there is a map from energy-conserving unitaries on $(\mathbb{C}^d)^{\otimes n}$ to energy-conserving unitaries on $(\mathbb{C}^d)^{\otimes n} \otimes \mathbb{C}^2$, namely

$$V_{\mathbf{n}} \longrightarrow\ \tilde{V}_{\mathbf{n},\text{a}} = V_{\mathbf{n}} \otimes |0\rangle\langle 0|_\text{a} + W_{\mathbf{n}} \otimes |1\rangle\langle 1|_\text{a}\ , \tag{C9}$$

where $W_{\mathbf{n}}$ is also an energy-conserving unitary on $(\mathbb{C}^d)^{\otimes n}$, and $\tilde{V}_{\mathbf{n},\text{a}}$ can be generated by the family of unitaries in Eq.(C8).

Hamiltonian Picture:

Alternatively, we can understand this result in the Hamiltonian picture. Suppose to implement an energy-conserving unitary, we perturb the intrinsic Hamiltonian of systems and ancilla $H_{\text{tot}} = \sum_{j=1}^n H_j + \Delta E|1\rangle\langle 1|_\text{a}$. In particular, suppose we add 2-local energy-conserving interactions in Eq.(C3) to H_{tot}, and obtain the family of Hamiltonians

$$H_{\mathbf{n},\text{a}}(t) = H_{\text{tot}} + g_\text{a}(t)\ Z_\text{a} + \sum_{j=1}^{n}\sum_{l=1}^{d-1} g_{j,\text{a}}^{(l)}(t)\ R_{j,\text{a}}^{(l)} + \sum_{j=1}^{n-1}\sum_{l,l'=1}^{d-1} g_{j,j+1}^{(l,l')}(t)\ R_{j,j+1}^{(l,l')}\ , \tag{C10}$$

where g_a, $g_{j,\text{a}}^{(l)}$ and $g_{j,j'}^{(l,l')}$ are real functions of time t, which vanish for $t<0$, and for sufficiently large t. We are interested in the unitary transformations generated by this family of Hamiltonians for different choices of these functions, i.e., unitaries satisfying

$$\frac{d}{dt}V_{\mathbf{n},\text{a}}(t) = -iH_{\mathbf{n},\text{a}}(t)V_{\mathbf{n},\text{a}}(t)\ ,\ \ t \geq 0 \tag{C11}$$

where $V_{\mathbf{n},\text{a}}(0)$ is the identity operator on $(\mathbb{C}^d)^{\otimes n} \otimes \mathbb{C}^2$. Clearly,

$$\forall t \geq 0: \quad \Big[H_{\mathbf{n},\text{a}}(t), H_{\text{tot}}\Big] = 0\ , \tag{C12}$$

which means the family of unitaries generated by these Hamiltonians are energy-conserving. Note that Hamiltonian $H_{\mathbf{n},\text{a}}(t)$ in Eq.(C10) contains a time-independent term H_{tot}, which corresponds to the intrinsic Hamiltonians of the n systems and the ancilla. It turns out that the existence of this constant term does not restrict the family of unitaries generated by this family of Hamiltonians. In particular, this family contains the family of unitaries generated by Hamiltonians

$$g_\text{a}(t)\ Z_\text{a} + \sum_{j=1}^{n}\sum_{l=1}^{d-1} g_{j,\text{a}}^{(l)}(t)\ R_{j,\text{a}}^{(l)} + \sum_{j=1}^{n-1}\sum_{l,l'=1}^{d-1} g_{j,j+1}^{(l,l')}(t)\ R_{j,j+1}^{(l,l')}\ , \tag{C13}$$

where we have dropped the term H_{tot} in Eq.(C10). This follows from the fact that H_{tot} commutes with all other terms in the Hamiltonian $H_{\mathbf{n},\text{a}}(t)$ and, furthermore, it generates a periodic time evolution, with period $2\pi/\Delta E$. Therefore, if the total time of implementing the desired unitary is an integer multiple of $2\pi/\Delta E$, the presence of H_{tot} does not have any effect on the implemented unitary. This can always be achieved by adding a time delay less than $2\pi/\Delta E$, during which the other terms are

turned off.

As we have seen before, the standard results of quantum control theory [15, 16] imply that, using the family of Hamiltonians in Eq.(C13) and assuming $g_{\rm a}$, $g^{(l)}_{j,{\rm a}}$, and $g^{(l,l')}_{j,j+1}$ are arbitrary real functions, we can generate any unitary e^{iG}, with iG in the Lie algebra defined in Eq.(C5). Combining this with the argument in Eq.(C7), we conclude that

Theorem 9. *Consider an arbitrary energy-conserving unitary $V_{\mathbf{n}}$ acting on $(\mathbb{C}^d)^{\otimes n}$ (i.e., a unitary satisfying $[V_{\mathbf{n}}\,, H_{intrinsic}] = 0$). Then, there exists a unitary $\tilde{V}_{\mathbf{n},a}$ acting on $(\mathbb{C}^d)^{\otimes n}\otimes\mathbb{C}^2$, such that*

$$\forall|\psi\rangle\in(\mathbb{C}^d)^{\otimes n}:\quad \tilde{V}_{\mathbf{n},a}(|\psi\rangle|0\rangle_a)=V_{\mathbf{n}}|\psi\rangle\otimes|0\rangle_a\,, \tag{C14}$$

and $\tilde{V}_{\mathbf{n},a}$ can be generated by a finite sequence of 2-local energy-conserving unitaries in Eq.(C8). Equivalently, $\tilde{V}_{\mathbf{n},a}$ can be implemented with the family of energy-conserving Hamiltonians $H_{\mathbf{n},a}(t)$ defined in Eq.(C10).

To complete the proof of this result we need to prove lemma 5. But, first we discuss a modified version of this scheme.

2. A Modified Scheme with Two Ancillary Qubits

In the above scheme we need system-system interactions $R^{(l,l')}_{j,j'}$. It turns out that this interaction can be easily engineered using system-ancilla interactions, provided that we can use a second ancillary qubit, labeled as qubit b, with Hamiltonian $\Delta E|1\rangle\langle 1|_{\rm b}$. This follows from the fact that

$$R^{(l,l')}_{j,j'}Z_{\rm b}=\frac{1}{2}\big[R^{(l)}_{j,{\rm b}}\,,[R^{(l')}_{j',{\rm b}}\,,\,Z_{\rm b}]\big]\,. \tag{C15}$$

Therefore, if qubit b is initially in state $|0\rangle_{\rm b}$, then

$$\exp\big(i\theta\frac{1}{2}\big[R^{(l)}_{j,{\rm b}}\,,[R^{(l')}_{j',{\rm b}}\,,\,Z_{\rm b}]\big]\big)|\phi\rangle|0\rangle_{\rm b}=\exp\big(i\theta R^{(l,l')}_{j,j'}\big)|\phi\rangle\otimes|0\rangle_{\rm b}\,, \tag{C16}$$

where $|\phi\rangle$ is an arbitrary state of the rest of systems.

Based on this observation, we consider interactions between systems $j=1,\cdots n$ and ancillary qibits a and b:

$$R^{(l)}_{{\rm a},j}\equiv R^{(l)}_{j,{\rm a}}\equiv|l-1\rangle\langle l|_j\otimes|1\rangle\langle 0|_{\rm a}+|l\rangle\langle l-1|_j\otimes|0\rangle\langle 1|_{\rm a}\qquad:l=1,\cdots,d-1\,,\;\;j=1,\cdots,n \tag{C17}$$

$$R^{(l)}_{{\rm b},j}\equiv R^{(l)}_{j,{\rm b}}\equiv|l-1\rangle\langle l|_j\otimes|1\rangle\langle 0|_{\rm b}+|l\rangle\langle l-1|_j\otimes|0\rangle\langle 1|_{\rm b}\qquad:l=1,\cdots,d-1\,,\;\;j=1,\cdots,n\,. \tag{C18}$$

Then, in this modified scheme instead of 2-local energy-conserving unitaries in Eq.(C8), we consider unitaries

$$\exp(i\theta Z_{\rm a})\,,\;\exp(i\theta R^{(l)}_{j,{\rm a}})\,,\;\exp(i\theta R^{(l)}_{j,{\rm a}})\,,\;\exp(i\theta R^{(l)}_{j,{\rm b}})\quad:\theta\in[0,2\pi);\;l,l'=1,\cdots,d-1\,;\;\;j=1,\cdots,n\,. \tag{C19}$$

Similarly, in the Hamiltonian picture, instead of Hamiltonians in Eq.(C10), we consider the family of Hamiltonians

$$H_{\mathbf{n},{\rm a},{\rm b}}(t)=\Big(\sum_{j=1}^n H_j+\Delta E|1\rangle\langle 1|_{\rm a}+\Delta E|1\rangle\langle 1|_{\rm b}\Big)+g_{\rm a}(t)\;Z_{\rm a}+g_{\rm b}(t)\;Z_{\rm b}+\sum_{j=1}^n\sum_{l=1}^{d-1}g^{(l)}_{j,{\rm a}}(t)\;R^{(l)}_{j,{\rm a}}+\sum_{j=1}^n\sum_{l=1}^{d-1}g^{(l)}_{j,{\rm b}}(t)\;R^{(l)}_{j,{\rm b}}\,, \tag{C20}$$

where $g_{\rm a}$, $g_{\rm b}$, $g^{(l)}_{j,{\rm a}}$ and $g^{(l)}_{j,{\rm b}}$ are arbitrary real functions. Note that

$$\forall t\geq 0:\quad\Big[H_{\mathbf{n},{\rm a},{\rm b}}(t),\Big(\sum_{j=1}^n H_j+\Delta E|1\rangle\langle 1|_{\rm a}+\Delta E|1\rangle\langle 1|_{\rm b}\Big)\Big]=0\,, \tag{C21}$$

and therefore the family of unitaries generated by these Hamiltonians are energy-conserving.

Then, combining the observation in Eq.(C16) with theorem 9, we conclude that

Corollary 10. *Consider an arbitrary energy-conserving unitary $V_{\mathbf{n}}$ acting on $(\mathbb{C}^d)^{\otimes n}$ (i.e., a unitary satisfying $[V_{\mathbf{n}}\,, H_{intrinsic}] = 0$). There exists a unitary $\tilde{V}_{\mathbf{n},a,b}$ acting on $(\mathbb{C}^d)^{\otimes n}\otimes\mathbb{C}^2\otimes\mathbb{C}^2$, such that*

$$\forall|\psi\rangle\in(\mathbb{C}^d)^{\otimes n}:\quad\tilde{V}_{\mathbf{n},a,b}(|\psi\rangle|0\rangle_a|0\rangle_b)=V_{\mathbf{n}}|\psi\rangle\otimes|0\rangle_a|0\rangle_b\,, \tag{C22}$$

and $\tilde{V}_{\mathbf{n},a,b}$ can be generated by a finite sequence of 2-local energy-conserving unitaries in Eq.(C19). Equivalently, $\tilde{V}_{\mathbf{n},a,b}$ can be implemented with the family of energy-conserving Hamiltonians $H_{\mathbf{n},a,b}(t)$ defined in Eq.(C20).

3. Implementing diagonal unitaries with 2-local energy-conserving interactions (Proof of lemma 5 for the special case of diagonal Hamiltonians)

In this section we focus on diagonal unitaries, i.e., those that commute with the Hamiltonians of all systems $j = 1, \cdots, n$. In Sec.(II C) we briefly explained how a general diagonal unitary can be implemented with 2-local energy-conserving interactions and a single ancillary qubit (See Fig.4). Here, we explain the idea in more details.

Any diagonal unitary can be written as $e^{iH_{\text{diag}}}$, where H_{diag} is a diagonal Hermitian operator, i.e., can be written as

$$H_{\text{diag}} = \sum_{s_1,\cdots,s_n=0}^{d-1} h_{s_1,\cdots,s_n} \bigotimes_{j=1}^{n} |s_j\rangle\langle s_j| \,, \tag{C23}$$

where $h_{s_1,\cdots,s_n} \in \mathbb{R}$.

We start with the case of $n = 1$, i.e., a single system with the Hilbert space $\mathbb{C}^d$. Consider the pair of energy eigenstates $|l-1\rangle$ and $|l\rangle$ with energies $(l-1) \times \Delta E$ and $l \times \Delta E$, respectively. Define

$$Z^{(l)} \equiv |l-1\rangle\langle l-1| - |l\rangle\langle l| \qquad : l = 1, \cdots, d-1 \,. \tag{C24}$$

Consider the set of operators

$$\Big\{ I_d, Z^{(l)} \equiv |l-1\rangle\langle l-1| - |l\rangle\langle l| : l = 1, \cdots, d-1 \Big\} \,, \tag{C25}$$

where I_d is the identity operator on $\mathbb{C}^d$. It can be easily seen that the above d operators form a basis for diagonal operators. In particular,

$$\text{Span}_{\mathbb{R}}\big\{ |s\rangle\langle s| \ : 0 \le s \le d-1 \big\} = \text{Span}_{\mathbb{R}}\big\{ I_d, Z^{(l)} \equiv |l-1\rangle\langle l-1| - |l\rangle\langle l| : l = 1, \cdots, d-1 \big\} \,. \tag{C26}$$

Next, consider n systems labeled as $j = 1, \cdots, n$. The set of operators $\big\{ \bigotimes_{j=1}^{n} |s_j\rangle\langle s_j| \ : 0 \le s_j \le d-1 \big\}$ spans the space of diagonal operators on $(\mathbb{C}^d)^{\otimes n}$. Clearly, this set can be obtained as the $n-$fold tensor product of the set $\big\{ |s\rangle\langle s| \ : 0 \le s \le d-1 \big\}$, i.e.

$$\Big\{ \bigotimes_{j=1}^{n} |s_j\rangle\langle s_j| : s_j = 0, \cdots, d-1 \Big\} = \Big\{ |s\rangle\langle s| \ : 0 \le s \le d-1 \Big\}^{\otimes n} \,. \tag{C27}$$

Next, we consider the n-fold tensor product of operators $\big\{ I_d, Z^{(l)} \equiv |l-1\rangle\langle l-1| - |l\rangle\langle l| : l = 1, \cdots, d-1 \big\}$, which appear in the right-hand side of Eq.(C26). For each system j consider the pair of energy eigenstates $|l-1\rangle_j$ and $|l\rangle_j$ with energies $(l-1) \times \Delta E$ and $l \times \Delta E$, respectively. Define

$$Z_j^{(l)} \equiv |l-1\rangle\langle l-1|_j - |l\rangle\langle l|_j \qquad : l = 1, \cdots, d-1 \,, \ \ j = 1, \cdots, n \,. \tag{C28}$$

Then, the n-fold tensor product of $\big\{ I_d, Z^{(l)} \equiv |l-1\rangle\langle l-1| - |l\rangle\langle l| : l = 1, \cdots, d-1 \big\}$ gives the set of operators

$$\Big\{ I_d, Z^{(l)} : l = 1, \cdots, d-1 \Big\}^{\otimes n} = \Big\{ I, \prod_{r=1}^{t} Z_{j_r}^{(l_r)} : \ 1 \le t \le n \,, \ 0 < j_1 < j_2 < \cdots < j_t < n+1 \,, \ 1 \le l_r \le d-1 \Big\} \,, \tag{C29}$$

where $I = I_d^{\otimes n}$ is the identity operator on $(\mathbb{C}^d)^{\otimes n}$. Combining this with Eq.(C26) and Eq.(C27), we find

$$\text{Span}_{\mathbb{R}}\big\{ \bigotimes_{j=1}^{n} |s_j\rangle\langle s_j| \ : 0 \le s_j \le d-1 \big\} = \text{Span}_{\mathbb{R}}\Big\{ I, \prod_{r=1}^{t} Z_{j_r}^{(l_r)} : \ 1 \le t \le n \,, \ 0 < j_1 < j_2 < \cdots < j_t < n+1 \,, \ 1 \le l_r \le d-1 \Big\} \,. \tag{C30}$$

Given that all operators $\prod_{r=1}^{t} Z_{j_r}^{(l_r)}$ are traceless, we conclude that

Lemma 6. *Any Hermitian operator on $(\mathbb{C}^d)^{\otimes n}$ that is diagonal in the basis $\{\bigotimes_{j=1}^n |r_j\rangle : r_j = 0, \cdots, d-1\}$, can be written as*

$$H_{diag} = \sum_{r_1,\cdots,r_n=0}^{d-1} h_{r_1,\cdots,r_n} \bigotimes_{j=1}^{n} |r_j\rangle\langle r_j| \tag{C31}$$

$$= \frac{Tr(H_{diag})}{Tr(I)} I + \sum_{t=1}^{n} \sum_{0<j_1<j_2<\cdots<j_t<n+1}^{j_1,\cdots j_t} \sum_{l_1,\cdots,l_t=1}^{d-1} c_{j_1,\cdots,j_t}^{(l_1,\cdots,l_t)} Z_{j_1}^{(l_1)} Z_{j_2}^{(l_2)} \cdots Z_{j_t}^{(l_t)} , \tag{C32}$$

for a set of real coefficients $h_{r_1,\cdots,r_n}$, and $c_{j_1,\cdots,j_t}^{(l_1,\cdots,l_t)}$, where $Z_j^{(l)} \equiv |l-1\rangle\langle l-1|_j - |l\rangle\langle l|_j \ : l = 1,\cdots,d-1; j = 1,\cdots,n$.

Therefore, to generate a general diagonal unitary evolution, up to a global phase, it suffices to implement all Hamiltonians

$$Z_{j_1}^{(l_1)} Z_{j_2}^{(l_2)} \cdots Z_{j_t}^{(l_t)} \ : \ t = 1,\cdots,n \, ; \ 0 < j_1 < j_2 < \cdots < j_t < n+1 \, ; l_1,\cdots,l_j = 0,\cdots,d-1 \, . \tag{C33}$$

Next, note that each pair of states $\{|l-1\rangle_j, |l\rangle_j\}$ can be interpreted as a separate qubit and $Z_j^{(l)}$ can be interpreted as the Pauli Z operator associated to this qubit. Hence, we can mimic the argument in Appendix B in the case of qubits. Following this analogy, we define

$$T_{\mathrm{a},j}^{(l)} \equiv \frac{i}{2}[Z_{\mathrm{a}}, R_{\mathrm{a},j}^{(l)}] = i\Big(|l\rangle\langle l-1|_j \otimes |0\rangle\langle 1|_{\mathrm{a}} - |l-1\rangle\langle l|_j \otimes |1\rangle\langle 0|_{\mathrm{a}}\Big) \qquad : l = 1,\cdots,d-1 \, , \ \ j = 1,\cdots,n \, . \tag{C34}$$

With this definition we can easily see that

$$D_j^{(l)} \equiv Z_j^{(l)} - Z_{\mathrm{a}} = i\big[iR_{j,\mathrm{a}}^{(l)}, iT_{\mathrm{a},\,j}^{(l)}\big] = \frac{i}{2}\big[iR_{j,\mathrm{a}}^{(l)}, [iZ_{\mathrm{a}}, iR_{\mathrm{a},j}^{(l)}]\big] \, . \tag{C35}$$

Furthermore, rewriting Eq.(B19), we find that for any distinct $t \geq 2$ systems labeled by $j_1 < j_2 < \cdots < j_t$, it holds that

$$D_{j_1,\cdots,j_t}^{(l_1,\cdots,l_t)} \equiv (Z_{j_1}^{(l_1)} - Z_{\mathrm{a}}) \, Z_{j_2}^{(l_2)} ... Z_{j_t}^{(l_t)} = \begin{cases} c_t i \, \big[[\cdots[[iR_{j_1,j_2}^{(l_1,l_2)}, iR_{j_2j_3}^{(l_2,l_3)}], iR_{j_3j_4}^{(l_3,l_4)}]\cdots, iR_{j_t,\mathrm{a}}^{(l_t)}], iR_{\mathrm{a},j_1}^{(l_1)}\big] & : t \text{ even} \\ c_t i \, \big[[\cdots[[iR_{j_1,j_2}^{(l_1,l_2)}, iR_{j_2j_3}^{(l_2,l_3)}], iR_{j_3j_4}^{(l_3,l_4)}]\cdots, iR_{j_t,\mathrm{a}}^{(l_t)}], iT_{\mathrm{a},j_1}^{(l_1)}\big] & : t \text{ odd} \, , \end{cases} \tag{C36}$$

where $c_t = \pm 1$, depending on t.

Next, note that for any $|\psi\rangle \in (\mathbb{C}^d)^{\otimes n}$, it holds that

$$Z_{\mathrm{a}}(|\psi\rangle|0\rangle_{\mathrm{a}}) = |\psi\rangle|0\rangle_{\mathrm{a}} \, , \tag{C37}$$

$$D_j^{(l)}(|\psi\rangle|0\rangle_{\mathrm{a}}) = (Z_j^{(l)} - I)|\psi\rangle \otimes |0\rangle_{\mathrm{a}} \, , \tag{C38}$$

$$D_{j_1,\cdots,j_t}^{(l_1,\cdots,l_t)}(|\psi\rangle|0\rangle_{\mathrm{a}}) = ([Z_{j_1}^{(l_1)} \cdots Z_{j_t}^{(l_t)} - Z_{j_2}^{(l_2)} \cdots Z_{j_t}^{(l_t)}]|\psi\rangle) \otimes |0\rangle_{\mathrm{a}} \, , \qquad : t \geq 2 \, . \tag{C39}$$

Considering the linear combinations of the above terms, we find $\forall |\psi\rangle \in (\mathbb{C}^d)^{\otimes n}$,

$$\big[D_j^{(l)} + Z_{\mathrm{a}}\big](|\psi\rangle|0\rangle_{\mathrm{a}}) = Z_j^{(l)}|\psi\rangle \otimes |0\rangle_{\mathrm{a}} \, , \tag{C40}$$

$$\big[\sum_{r=1}^{t-1} D_{j_r,\cdots,j_t}^{(l_r,\cdots,l_t)} + D_{j_t}^{(l_t)} + Z_{\mathrm{a}}\big](|\psi\rangle|0\rangle_{\mathrm{a}}) = (Z_{j_1}^{(l_1)} \cdots Z_{j_t}^{(l_t)}|\psi\rangle) \otimes |0\rangle_{\mathrm{a}} \, , \quad : t \geq 2 \, . \tag{C41}$$

Combining this with lemma 6, we find that for any Hermitian operator H_{diag} on $(\mathbb{C}^d)^{\otimes n}$ that is diagonal in the basis $\{\bigotimes_{j=1}^n |r_j\rangle \ : \ r_j = 0,\cdots,d-1\}$, there exists an operator $\tilde{H}_{\text{diag}}$ on $(\mathbb{C}^d)^{\otimes n} \otimes \mathbb{C}^2$, such that $i\tilde{H}_{\text{diag}}$ is in the real Lie algebra $\mathfrak{alg}_{\mathbb{R}}\Big\{iZ_{\mathrm{a}} \, , iR_{j,\mathrm{a}}^{(l)}, iR_{j,j'}^{(l,l')} : \ l,l' = 1,\cdots,d-1 \, ; \ \ j \neq j' = 1,\cdots,n\Big\}$ and

$$\forall |\psi\rangle \in (\mathbb{C}^d)^{\otimes n} : \qquad \tilde{H}_{\text{diag}}(|\psi\rangle \otimes |0\rangle_{\mathrm{a}}) = H_{\text{diag}}|\psi\rangle \otimes |0\rangle_{\mathrm{a}} \, . \tag{C42}$$

Finally, it can be easily shown that the same result remains valid if instead of all Hamiltonians $R_{j,j'}^{(l,l')} : \ l,l' = 1,\cdots,d-1 \, ; \ \ j \neq$

$j' = 1, \cdots, n$, we are restricted to only nearest-neighbor interactions $R^{(l,l')}_{s,s+1}$, $l, l' = 1, \cdots, d-1\,; s = 1, \cdots, n-1$. The argument is similar to the argument in Sec.B 3) for qubits: Combining interactions $R^{(l,l')}_{s,s+1}$ and $Z^{(l)}_s Z^{(l')}_{s+1}$, we can swap the state of qubits defined by $\{|l-1\rangle_s, |l\rangle_s\}$ in system j and $\{|l'-1\rangle_{s+1}, |l'\rangle_{s+1}\}$ in system j'. Furthermore, by combining permutations on nearest-neighbor sites, we can change the order of qubits arbitrarily. Therefore, the additional restriction to nearest-neighbor interactions, does not restrict the of Hamiltonians that can be simulated.

We conclude that

Lemma 7. *For any Hermitian operator H_{diag} on $(\mathbb{C}^d)^{\otimes n}$ that is diagonal in the basis $\{\bigotimes_{j=1}^n |r_j\rangle \,:\, r_j = 0, \cdots, d-1\}$, there exists an operator $\tilde{H}_{diag}$ on $(\mathbb{C}^d)^{\otimes n} \otimes \mathbb{C}^2$, such that $i\tilde{H}_{diag}$ is in the real Lie algebra $\mathfrak{alg}_{\mathbb{R}}\Big\{iZ_a\,, iR^{(l)}_{j,a}, iR^{(l,l')}_{s,s+1} \,:\, l, l' = 1, \cdots, d-1\,;\;\; j = 1, \cdots, n; s = 1, \cdots, n-1\Big\}$ and*

$$\forall |\psi\rangle \in (\mathbb{C}^d)^{\otimes n}: \qquad \tilde{H}_{diag}(|\psi\rangle \otimes |0\rangle_a) = H_{diag}|\psi\rangle \otimes |0\rangle_a\,. \tag{C43}$$

This lemma is a special case of lemma 5, for the case of diagonal Hamiltonians. Next, we prove lemma 5 in the general case.

4. All energy-conserving unitaries from diagonal unitaries and 2-local energy-conserving unitaries

In this section, we show how a general energy-conserving unitary can be implemented by combining diagonal energy-conserving unitaries and 2-local energy-conserving unitaries. In particular, we study the Lie algebra of energy-conserving Hamiltonians and show

Lemma 8. *Let $\mathfrak{h}$ be the Lie algebra of energy-conserving skew-Hermitian operators, i.e., those commuting with $H_{intrinsic} = \sum_{j=1}^n H_j$. Then, $\mathfrak{h}$ is generated by the set of skew-Hermitian diagonal operators together with operators $iR^{(l,l')}_{j,j+1} \,:\, l, l' = 1, \cdots, d-1\,;\;\; j = 1, \cdots, n-1$, i.e.*

$$\mathfrak{h} \equiv \Big\{A : A + A^\dagger = 0, [A, \sum_{j=1}^n H_j] = 0\Big\} \tag{C44}$$

$$= \mathfrak{alg}_{\mathbb{R}}\Big(\Big\{i\bigotimes_{j=1}^n |r_j\rangle\langle r_j| : r_j = 0, \cdots, d-1\Big\} \cup \Big\{iR^{(l,l')}_{j,j+1} :\; l, l' = 1, \cdots, d-1\,;\;\; j = 1, \cdots, n-1\Big\}\Big)\,. \tag{C45}$$

This lemma is a generalization of lemma 8 for the qubit case with U(1) symmetry and can be proven in a similar way.

In the following, we use the notation

$$|\mathbf{r}\rangle \equiv |r_1\rangle \cdots |r_n\rangle = \bigotimes_{j=1}^n |r_j\rangle\,, \qquad \mathbf{r} \in \{0, \cdots, d-1\}^n\,, \tag{C46}$$

where $\mathbf{r} \equiv r_1 \cdots r_n$ and $r_j \in \{0, \cdots, d-1\}$. Then,

$$H_{\text{intrinsic}} = \sum_{j=1}^n H_j = \sum_{j=1}^n \sum_{r_j=0}^{d-1} (r_j \Delta E)|r_j\rangle\langle r_j|_j = \Delta E \sum_{\mathbf{r}\in\{0,\cdots,d-1\}^n} N(\mathbf{r})\; |\mathbf{r}\rangle\langle\mathbf{r}|\,, \tag{C47}$$

where

$$N(\mathbf{r}) \equiv \sum_{j=1}^n r_j\,. \tag{C48}$$

Proof. Any operator A acting on $(\mathbb{C}^d)^{\otimes n}$ that commutes with $H_{\text{intrinsic}}$ can be written as

$$A = \sum_{\mathbf{r},\mathbf{r}': N(\mathbf{r}) = N(\mathbf{r}')} a_{\mathbf{r},\mathbf{r}'}\; |\mathbf{r}\rangle\langle\mathbf{r}'|\,, \tag{C49}$$

where the summation is over all $\mathbf{r}, \mathbf{r}' \in \{0, \cdots, d-1\}^n$ satisfying the condition $N(\mathbf{r}) = N(\mathbf{r}')$. It follows that $\mathfrak{h}$, the Lie algebra

of skew-Hermitian energy-conserving operators, can be written as a linear combinations of 3 sets of operators, namely,

$$\mathcal{D} \equiv \Big\{ i|\mathbf{r}\rangle\langle\mathbf{r}| : \mathbf{r} \in \{0,\cdots,d-1\}^n \Big\} \tag{C50}$$

$$\mathcal{R} \equiv \Big\{ |\mathbf{r}\rangle\langle\mathbf{r}'| - |\mathbf{r}'\rangle\langle\mathbf{r}| \ : \mathbf{r},\mathbf{r}' \in \{0,\cdots,d-1\}^n \, , N(\mathbf{r}) = N(\mathbf{r}') \Big\} \tag{C51}$$

$$\mathcal{I} \equiv \Big\{ i(|\mathbf{r}\rangle\langle\mathbf{r}'| + |\mathbf{r}'\rangle\langle\mathbf{r}|) \ : \mathbf{r},\mathbf{r}' \in \{0,\cdots,d-1\}^n, N(\mathbf{r}) = N(\mathbf{r}') \Big\} \, , \tag{C52}$$

where the constraint $N(\mathbf{r}) = N(\mathbf{r}')$ means that states $|\mathbf{r}\rangle$ and $|\mathbf{r}'\rangle$ have the same energy. In other words,

$$\mathfrak{h} \equiv \Big\{ A : A + A^\dagger = 0, [A, H_{\text{intrinsic}}] = 0 \Big\} = \text{span}_{\mathbb{R}}(\mathcal{I} \cup \mathcal{D} \cup \mathcal{R}) \, . \tag{C53}$$

For any distinct pair $\mathbf{r}_1, \mathbf{r}_2 \in \{0,\cdots,d-1\}^n$, the following commutation relations hold:

$$\Big[(|\mathbf{r}_2\rangle\langle\mathbf{r}_1| \mp |\mathbf{r}_1\rangle\langle\mathbf{r}_2|) \, , |\mathbf{r}_1\rangle\langle\mathbf{r}_1| \Big] = |\mathbf{r}_2\rangle\langle\mathbf{r}_1| \pm |\mathbf{r}_1\rangle\langle\mathbf{r}_2| \, . \tag{C54}$$

This implies that the Lie algebra $\mathfrak{h}$ is generated by $\mathcal{D}$ and $\mathcal{R}$, i.e.

$$\mathfrak{h} \equiv \Big\{ A : A + A^\dagger = 0, [A, H_{\text{tot}}] = 0 \Big\} = \text{span}_{\mathbb{R}}(\mathcal{D} \cup \mathcal{I} \cup \mathcal{R}) = \mathfrak{alg}_{\mathbb{R}}(\mathcal{D} \cup \mathcal{R}) \, . \tag{C55}$$

Next, we consider the subset of $\mathcal{R}$ defined by

$$\mathcal{R}' = \Big\{ |\mathbf{r}\rangle\langle\mathbf{r}'| - |\mathbf{r}'\rangle\langle\mathbf{r}| \ : \mathbf{r},\mathbf{r}' \in \{0,\cdots,d-1\}^n, N(\mathbf{r}) = N(\mathbf{r}'), \text{dist}(\mathbf{r},\mathbf{r}') = 2 \Big\} \, , \tag{C56}$$

where for any pair $\mathbf{r} = r_1 \cdots r_n$ and $\mathbf{r}' = r'_1 \cdots r'_n$, we have defined the distance

$$\text{dist}(\mathbf{r},\mathbf{r}') \equiv \sum_{j=1}^{n} |r_j - r'_j| \, . \tag{C57}$$

Note that the two conditions

$$N(\mathbf{r}) = N(\mathbf{r}') \qquad \text{and} \qquad \text{dist}(\mathbf{r},\mathbf{r}') = 2 \, , \tag{C58}$$

together imply that $r_j = r'_j$ for all systems $j = 1,\cdots,n$ except two distinct systems v and w, i.e.,

$$r_j = r'_j \quad : j \neq v, w \, ; \quad r'_v = r_v + 1 \quad \text{and} \ \ r'_w = r_w - 1 \, . \tag{C59}$$

Next, we show that $\mathcal{R}$ can be generated by $\mathcal{R}'$, i.e.,

$$\mathcal{R} \subset \mathfrak{alg}_{\mathbb{R}}(\mathcal{R}'), \tag{C60}$$

and therefore

$$\mathfrak{h} = \mathfrak{alg}_{\mathbb{R}}(\mathcal{D} \cup \mathcal{R}) = \mathfrak{alg}_{\mathbb{R}}(\mathcal{D} \cup \mathcal{R}') \, . \tag{C61}$$

To see this first note that for any three distinct $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3 \in \{0,\cdots,d-1\}^n$, the following commutation relations hold:

$$\Big[|\mathbf{r}_3\rangle\langle\mathbf{r}_2| - |\mathbf{r}_2\rangle\langle\mathbf{r}_3|, |\mathbf{r}_2\rangle\langle\mathbf{r}_1| - |\mathbf{r}_1\rangle\langle\mathbf{r}_2| \Big] = |\mathbf{r}_3\rangle\langle\mathbf{r}_1| - |\mathbf{r}_1\rangle\langle\mathbf{r}_3| \, , \tag{C62}$$

or, equivalently,

$$[F(\mathbf{r}_3,\mathbf{r}_2), F(\mathbf{r}_2,\mathbf{r}_1)] = F(\mathbf{r}_3,\mathbf{r}_1) \, , \tag{C63}$$

where we have defined

$$F(\mathbf{r}',\mathbf{r}) \equiv |\mathbf{r}'\rangle\langle\mathbf{r}| - |\mathbf{r}\rangle\langle\mathbf{r}'| \, . \tag{C64}$$

Then, consider a pair of $\mathbf{r}_{\text{in}}, \mathbf{r}_{\text{fin}} \in \{0, 1, \cdots, d-1\}^n$ satisfying $N(\mathbf{r}_{\text{in}}) = N(\mathbf{r}_{\text{fin}})$, which means $|\mathbf{r}_{\text{in}}\rangle$ and $|\mathbf{r}_{\text{out}}\rangle$ have the same energy. It can be easily seen that any such pair can be converted to each other through a sequence of transitions

$$\mathbf{r}_{\text{in}} = \mathbf{r}_1 \longrightarrow \mathbf{r}_2 \longrightarrow \cdots\cdots \longrightarrow \mathbf{r}_m = \mathbf{r}_{\text{fin}} \,, \tag{C65}$$

where any consecutive pairs $\mathbf{r}_t$ and $\mathbf{r}_{t+1}$, satisfy

$$N(\mathbf{r}_t) = N(\mathbf{r}_{t+1}) \quad \text{and} \quad \text{dist}(\mathbf{r}_t, \mathbf{r}_{t+1}) = 2 \quad , 1 \leq t \leq m \,. \tag{C66}$$

This means that at each step in Eq.(C65), energy ΔE is transferred from one system to another. Combining this with Eq.(C63), we find

$$F(\mathbf{r}_{\text{out}}, \mathbf{r}_{\text{in}}) = [F(\mathbf{r}_m, \mathbf{r}_{m-1}), [\cdots [F(\mathbf{r}_3, \mathbf{r}_2), [F(\mathbf{r}_3, \mathbf{r}_2), [F(\mathbf{r}_3, \mathbf{r}_2), F(\mathbf{r}_2, \mathbf{r}_1)]]] \cdots]] \,. \tag{C67}$$

Furthermore, because at each step $N(\mathbf{r}_t) = N(\mathbf{r}_{t+1})$ and $\text{dist}(\mathbf{r}_t, \mathbf{r}_{t+1}) = 2$, then $F(\mathbf{r}_{t+1}, \mathbf{r}_t) \in \mathcal{R}'$. This proves that $\mathcal{R} \subset \mathfrak{alg}_{\mathbb{R}}(\mathcal{R}')$, and therefore implies Eq.(C61).

Next, we prove that

$$\mathcal{R}' \subset \mathfrak{alg}_{\mathbb{R}}\Big(\mathcal{D} \cup \big\{ iR^{(l,l')}_{j,j'} : \, l, l' = 1, \cdots, d-1 \,; \;\; j \neq j' = 1, \cdots, n \big\}\Big) \,. \tag{C68}$$

That is we show that for any $\mathbf{r}, \mathbf{r}' \in \{0, \cdots, d-1\}^n$, satisfying $N(\mathbf{r}) = N(\mathbf{r}')$ and $\text{dist}(\mathbf{r}, \mathbf{r}') = 2$, it holds that

$$|\mathbf{r}\rangle\langle\mathbf{r}'| - |\mathbf{r}'\rangle\langle\mathbf{r}| \in \mathfrak{alg}_{\mathbb{R}}\Big(\mathcal{D} \cup \big\{ iR^{(l,l')}_{j,j'} : \, l, l' = 1, \cdots, d-1 \,; \;\; j \neq j' = 1, \cdots, n \big\}\Big) \,. \tag{C69}$$

To see this note that, as we have seen in Eq.(C59), for any pair $\mathbf{r} = r_1 \cdots r_n$ and $\mathbf{r}' = r'_1 \cdots r'_n$, the two conditions $N(\mathbf{r}) = N(\mathbf{r}')$ and $\text{dist}(\mathbf{r}, \mathbf{r}') = 2$ together imply

$$r_j = r'_j \quad : j \neq v, w \,; \quad r'_v = r_v + 1 \quad \text{and} \;\; r'_w = r_w - 1 \,. \tag{C70}$$

It follows that

$$|\mathbf{r}'\rangle = R^{(r_v+1, r_w)}_{v,w}|\mathbf{r}\rangle = \frac{1}{2}\Big(|r_v+1\rangle\langle r_v|_v \otimes |r_w - 1\rangle\langle r_w|_w + |r_v\rangle\langle r_v + 1|_v \otimes |r_w\rangle\langle r_w - 1|_w\Big)|\mathbf{r}\rangle \,. \tag{C71}$$

This means that

$$F(\mathbf{r}', \mathbf{r}) = |\mathbf{r}'\rangle\langle\mathbf{r}| - |\mathbf{r}\rangle\langle\mathbf{r}'| = \big[i|\mathbf{r}\rangle\langle\mathbf{r}| \,, \; iR^{(r_v+1, r_w)}_{v,w} \big] \,. \tag{C72}$$

Since $\mathcal{R}' \equiv \big\{ |\mathbf{r}\rangle\langle\mathbf{r}'| - |\mathbf{r}'\rangle\langle\mathbf{r}| \;\; : \mathbf{r}, \mathbf{r}' \in \{0, \cdots, d-1\}^n, N(\mathbf{r}) = N(\mathbf{r}'), \text{dist}(\mathbf{r}, \mathbf{r}') = 2 \big\}$, we conclude that $\mathcal{R}' \subset \mathfrak{alg}_{\mathbb{R}}\big(\mathcal{D} \cup \big\{ iR^{(l,l')}_{j,j'} : \, l, l' = 1, \cdots, d-1 \,; \;\; j \neq j' = 1, \cdots, n \big\}\big)$. Combining this with Eq.(C61), we find

$$\begin{aligned}
\mathfrak{h} &\equiv \Big\{ A : A + A^\dagger = 0, [A, H_{\text{intrinsic}}] = 0 \Big\} && \text{(C73a)} \\
&= \text{span}_{\mathbb{R}}(\mathcal{D} \cup \mathcal{I} \cup \mathcal{R}) && \text{(C73b)} \\
&= \mathfrak{alg}_{\mathbb{R}}(\mathcal{D} \cup \mathcal{R}) && \text{(C73c)} \\
&= \mathfrak{alg}_{\mathbb{R}}(\mathcal{D} \cup \mathcal{R}') && \text{(C73d)} \\
&= \mathfrak{alg}_{\mathbb{R}}\big(\mathcal{D} \cup \big\{ iR^{(l,l')}_{j,j'} : \, l, l' = 1, \cdots, d-1 \,; \;\; j \neq j' = 1, \cdots, n \big\}\big) && \text{(C73e)} \\
&= \mathfrak{alg}_{\mathbb{R}}\Big(\Big\{ i\bigotimes_{j=1}^{n} |r_j\rangle\langle r_j| : r_j = 0, \cdots, d-1 \Big\} \cup \big\{ iR^{(l,l')}_{j,j'} : \, l, l' = 1, \cdots, d-1 \,; \;\; j \neq j' = 1, \cdots, n \big\}\Big) \,. && \text{(C73f)}
\end{aligned}$$

This means that all energy-conserving unitaries can be implemented using diagonal Hamiltonians together with interactions $\big\{ iR^{(l,l')}_{j,j'} : \, l, l' = 1, \cdots, d-1 \,; \;\; j \neq j' = 1, \cdots, n \big\}$.

Finally, we can easily see that the above conclusion remains valid if instead of all pairwise interactions $\big\{ R^{(l,l')}_{j,j'} : \, l, l' = 1, \cdots, d-1 \,; \;\; j \neq j' = 1, \cdots, n \big\}$, one only considers interactions $\big\{ R^{(l,l')}_{j,j+1} : \, l, l' = 1, \cdots, d-1 \,; \;\; j = 1, \cdots, n-1 \big\}$ on nearest-neighbor systems. Here, we sketch the argument: Using diagonal Hamiltonians together with interactions $\big\{ iR^{(l,l')}_{j,j+1} :$

$l, l' = 1, \cdots, d-1\,;\;\; j = 1, \cdots, n-1\Big\}$, one can implement swap unitaries on nearest-neighbor systems j and $j+1$, i.e., the unitaries that exchange the state of the two nearest-neighbor systems. To see this first note that for any pair of neighbor systems j and $j+1$ the swap unitary $S_{j,j+1}$ that exchanges the state of systems j and $j+1$, can be implemented using Hamiltonians

$$|r\rangle\langle r|_j \otimes |r'\rangle\langle r'|_{j+1} : \quad r, r' = 0, 1, \cdots, d-1 \quad \text{and} \tag{C74}$$

$$R^{(l,l')}_{j,j+1} = |l-1\rangle\langle l|_j \otimes |l'\rangle\langle l'-1|_{j+1} + |l\rangle\langle l-1|_j \otimes |l'-1\rangle\langle l'|_{j+1} : \quad l, l' = 1, \cdots, d-1\,. \tag{C75}$$

In other words, $S_{j,j+1}$ is in the Lie group associated to the Lie algebra

$$\mathfrak{alg}_{\mathbb{R}}\Big\{ i|r\rangle\langle r|_j \otimes |r'\rangle\langle r'|_{j+1} : r, r' = 0, 1, \cdots, d-1,\;\; ; iR^{(l,l')}_{j,j+1} : l, l' = 1, \cdots, d-1 \Big\}\,. \tag{C76}$$

This follows, for instance, using the above result in Eq.(C73) in the special case of $n = 2$, and the fact that $S_{j,j+1}$ is an energy-conserving unitary. Since this holds for all $j = 1, \cdots, n-1$, and since swaps of nearest-neighbor systems generate all possible permutations of the systems, we conclude that all permutations are in the Lie group associated to the Lie algebra

$$\mathfrak{alg}_{\mathbb{R}}\big(\mathcal{D} \cup \big\{ iR^{(l,l')}_{j,j+1} :\; l, l' = 1, \cdots, d-1\,;\;\; j = 1, \cdots, n-1 \big\}\big)\,. \tag{C77}$$

But, since the Lie algebra is closed under the adjoint action of the Lie group, it follows that the above Lie algebra is closed under all permutations of n systems. This, in particular, implies that for any pair of distinct systems j and j', operator $iR^{(l,l')}_{j,j'}$ is in the Lie algebra in Eq.(C77), for all $j \neq j' = 1, \cdots, n$ and $l, l' = 1, \cdots, n-1$. We conclude that

$$\mathfrak{alg}_{\mathbb{R}}\big(\mathcal{D} \cup \big\{ iR^{(l,l')}_{j,j+1} :\; l, l' = 1, \cdots, d-1\,;\;\; j = 1, \cdots, n-1 \big\}\big) \tag{C78}$$

$$= \mathfrak{alg}_{\mathbb{R}}\big(\mathcal{D} \cup \big\{ iR^{(l,l')}_{j,j'} :\; l, l' = 1, \cdots, d-1\,;\;\; j \neq j' = 1, \cdots, n \big\}\big) = \mathfrak{h}\,. \tag{C79}$$

This implies lemma 8, and therefore completes the proof of lemma 5.

□

Figures

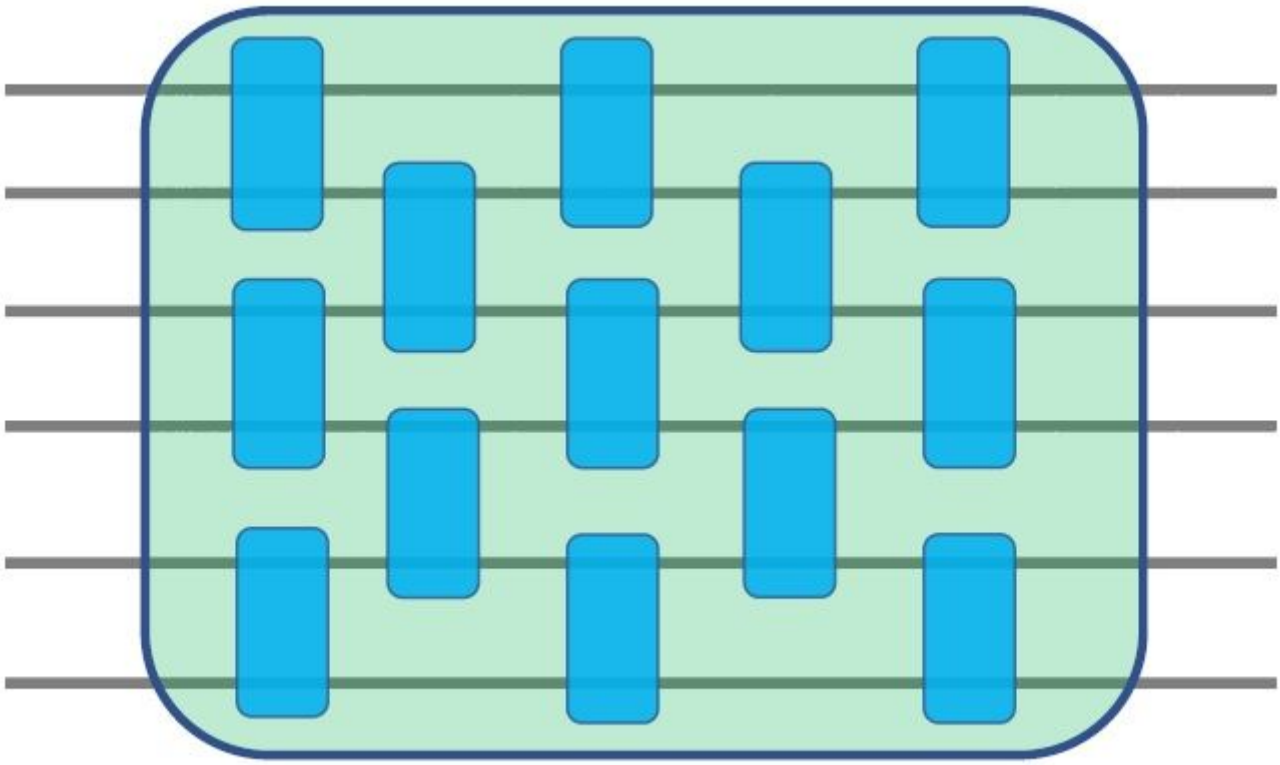

Figure 1

Local Symmetric Quantum Circuits. A quantum circuit with 2-local unitaries on 6 subsystems (e.g., qubits). A Local Symmetric Quantum Circuit (LSQC) only contains local unitaries that respect a certain symmetry. For instance, they are all invariant under rotations around z axis. Such circuits can model the time evolution of systems with local symmetric Hamiltonians. Conversely, any LSQC corresponds to the time evolution generated by a local symmetric (time-dependent) Hamiltonian. Therefore, by studying LSQC, we can also characterize general features of time evolution under local symmetric Hamiltonians.

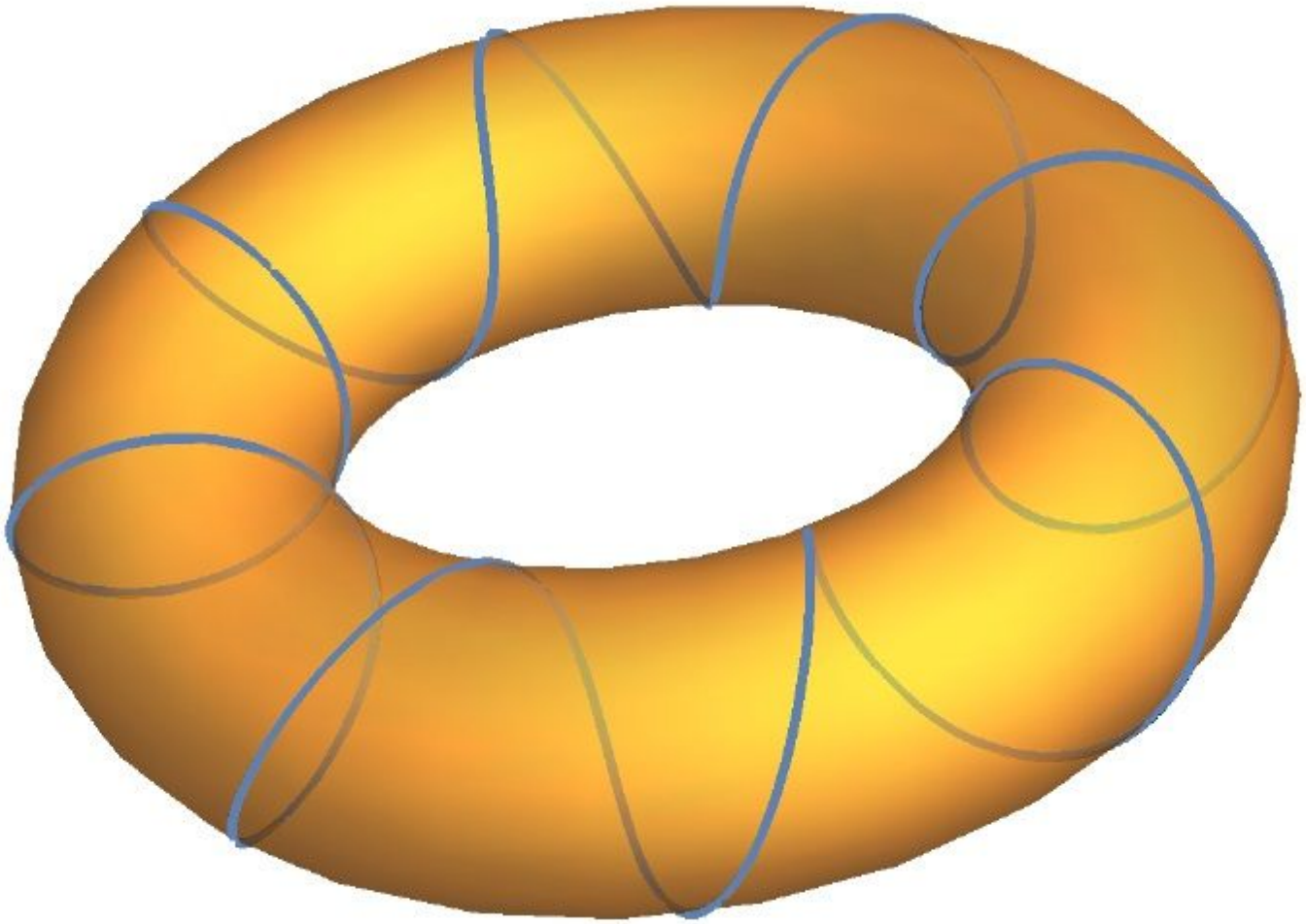

Figure 2

The schematic relation between the group of all symmetric unitaries (the torus) and the subgroup generated by Local Symmetric Quantum Circuits (the blue curve). They are both closed connected Lie groups and hence closed manifolds. Unitary evolution under any local symmetric Hamiltonian is restricted to the submanifold corresponding to LSQC. In other words, adding a perturbation to the Hamiltonian can bring the evolution outside this submanifold, only if it is non-local or symmetry-breaking.

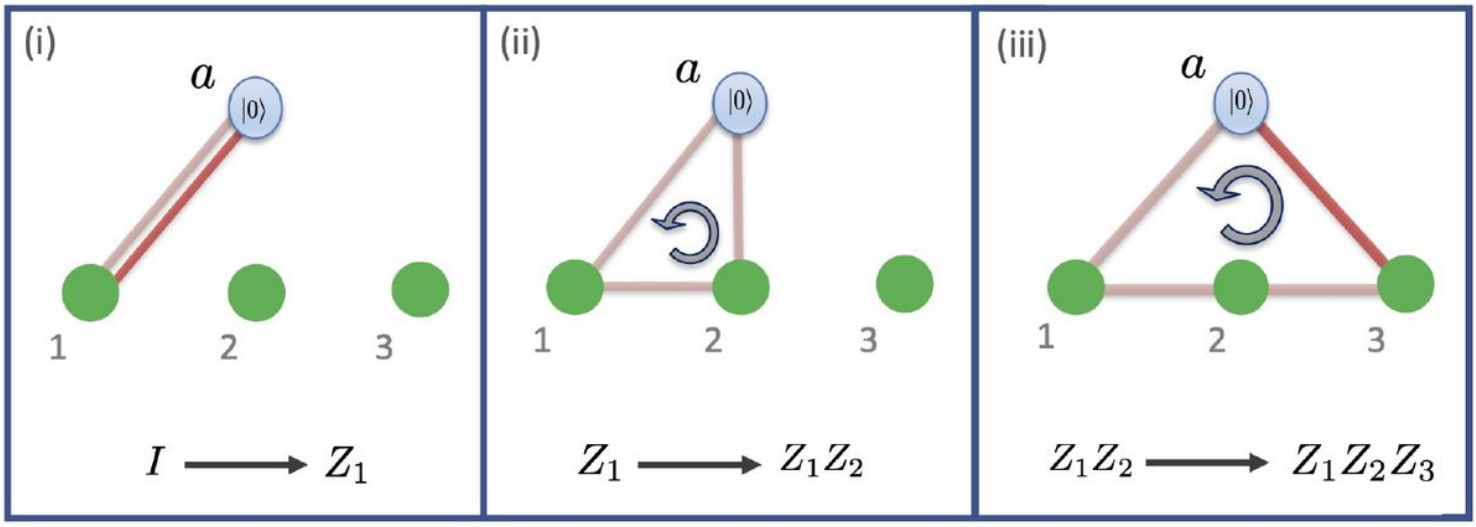

Figure 3

A protocol for implementing U(1)-invariant unitaries using interaction XX+YY.

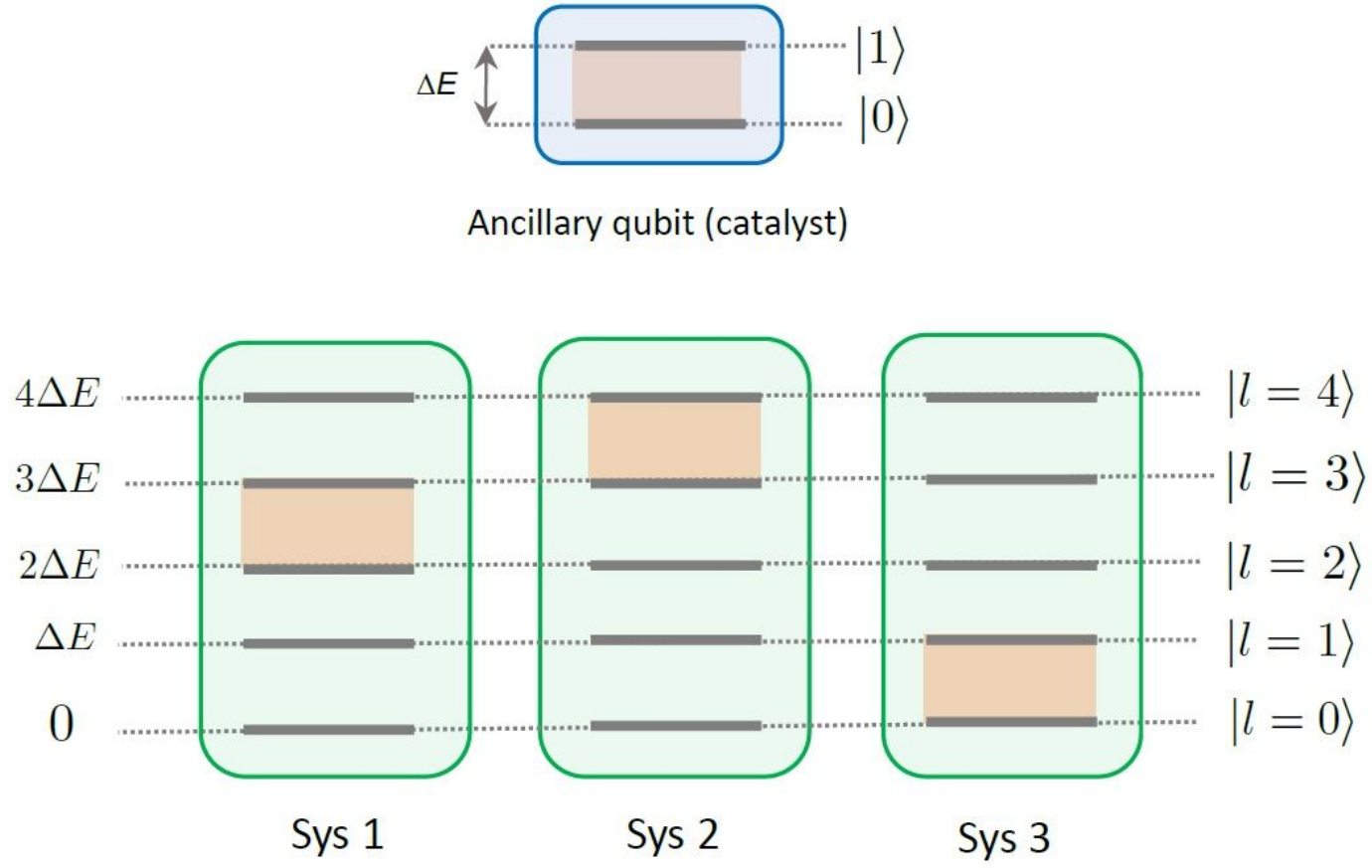

Figure 4

A scheme for implementing energy-conserving unitaries on composite systems.