Families of exact solutions of a Generalized (2+1)-dimensional Boussinesq type equation

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Families of exact solutions of a Generalized (2+1)-dimensional Boussinesq type equation

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Abstract In this paper, we study a Generalized (2+1)-dimensional Boussinesq type equation. Through the Hirota bilinear method, we give the N-order bright soliton solutions and dark soliton solutions. For the one-soliton solution, the bright soliton solution and the dark soliton solution have the same limit line and different extreme values. Based on soliton solutions, we give higher-order bright and dark breather solutions and mixed solutions. The dynamic behavior is characterized by images. Through the long-wave limit method, we obtain the bright and dark lump solution. It is worth noting that they have the same extreme points and different extreme values. In addition, we also get two semi-rational solutions as lump-soliton and lump-breather. It is found that the collision between lump and soliton is elastic.

Keywords Generalized (2+1)-dimensional Boussinesq type equation · Hirota bilinear method · Soliton solutions · Breather solutions · Semi-rational solutions

1 Introduction

As is known to us all, the study of local waves has always been the focus of research in the field of nonlinear science. Solitons, respiration, lumps and rogue waves are all important local waves in nonlinear physical systems. Solitary wave is a special local wave with uniform velocity and constant shape. Due to the cancellation of nonlinearity and dispersion effects, this waveform maintains its original characteristics after pairwise collision. Although this characteristic

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can be observed through numerical simulation[1], it is found that it does not completely return to the original shape after collision after one-step analysis of the numerical results[2]. Therefore, the study of collision is inseparable from the exact solution of the nonlinear evolution equation. When the dispersion and nonlinearity in the system reach a balance, the so-called soliton solution will be produced. Driven by computer technology, soliton theory, and its nonlinear physics have rapidly become a research hotspot. In recent years, many outstanding scientists in the fields of engineering and science have constructed various powerful methods [1,3–11] to find exact solutions to nonlinear evolution equations [12–25].

The Boussinesq equation was proposed by Boussinesq in 1871, which is used to describe the propagation of long waves in shallow water [26]. Today, many types of Boussinesq equations have been widely studied [27–32]. Among the (1+1)-dimensional equations, Boussinesq equation is one of the most studied equations due to its appearance in many physical systems [33,34]. Ma [35] gave a rational solution Wronskian formula for the (1+1)-dimensional Boussinesq equation. Clarkson [36] gave the rational solutions of the (1+1)-dimensional Boussinesq equation and the application of rogue waves. Singh [37] studied the higher-order bright dark strange wave solutions of a new (2+1)-dimensional integrable Boussinesq model. Cao [13] gave families of exact solutions for a new extended (2+1)-dimensional Boussinesq equation. Yan [38] studied the breath strange wave solution and its dynamic behavior of the generalized (3+1)-dimensional Boussinesq equation. Recently, Zhou [39] introduced (2+1)-dimensional Boussinesq type equation and gave the solutions of block and linear rogue waves.

In order to study the dynamics of wave propagation on the ocean surface, we will consider a new (2+1)-dimensional Boussinesq model. First, we consider the following new (3+1)-dimensional Boussinesq model [40]

$$u_{tt} - \gamma(u^2)_{xx} - u_{xx} - \delta u_{xxxx} + \frac{\nu^2}{4} u_{yy} + \nu u_{yt} + \beta u_{xz} = 0. \tag{1}$$

Here, the $u = u(x,\ y,\ z,\ t)$ is a differentiable function, and γ, δ, ν, β are real constants. This model describes how gravity waves travel on water. If $z = 0$ or $z = x$ or $z = y$, the Eq.(1) is reduced to (2+1)-dimension. Now, we mainly consider the case when $z = y$ and $\nu = 0$. To better discuss the influence of coefficients on waves and study some more general situations, this paper mainly studies the following Generalized (2+1)-dimensional Boussinesq type(GB-type) equation:

$$u_{tt} + \gamma(u^2)_{xx} - \alpha u_{xx} - \delta u_{xxxx} - \beta u_{xy} = 0. \tag{2}$$

Here, the $u = u(x,\ y,\ t)$ is a differentiable function, and γ, α, δ, β are real constants. When $\alpha \neq 0,\ \beta = 0$, the Eq.(2) degenerates into a (1+1)-dimensional Boussinesq equation [41]. The equation $\alpha = \pm 1$ is considered a bad and good Boussinesq equation in the literature [42]. The "good" Boussinesq equation depicts the two-dimensional non-rotating flow of ideal fluid in a uniform rectangular channel, while the "bad" Boussinesq equation is used to depict the plane flow of wavelet amplitude shallow-water waves. When $\alpha = 0,\ \beta = 0$, the Eq.(2) degenerates into a classic Benjamin-Ono equation [43]. This equation describes the propagation of small amplitude long waves in shallow water. When $\gamma = 3,\ \delta = 1$, the Eq.(2) degenerates into the (2+1)-dimensional Boussinesq type equation [39]. Although many people have done a lot of research on the Boussinesq equation in recent years. But as far as we know, the Generalized (2+1)-dimensional Boussinesq-type equation has not been studied.

The outline of this paper is as follows. In Sec.2, we give the bilinear form, N-soliton solution, breather solution, and the mixed solution of the GB-type equation by the Hirota bilinear method. In Sec.3, the lump solution of the GB-type equation is given by the long-wave limit method. The semi-rational solution of the GB-type equation is given in Sec.4. And we will give the research results of this paper in Sec.5.

2 *N*-soliton and breather solutions of the GB-type equation

2.1 *N*-soliton solutions

With the following logarithmic transformation

$$u = -\frac{12\delta}{\gamma}(\ln f)_{xx}, \tag{3}$$

the bilinear form of Eq.(2) is

$$(D_t^2 - \beta D_x D_y - \alpha D_x^2 - \delta D_x^4) f \cdot f = 0. \tag{4}$$

If we introduce polynomials

$$P(t,\ x,\ y) = t^2 - \delta x^4 - \alpha x^2 - \beta xy, \tag{5}$$

then Eq.(4) is equivalent to

$$P(D_t,\ D_x,\ D_y) f \cdot f = 0. \tag{6}$$

That is

$$ff_{tt} - f_t^2 - \beta ff_{xy} + \beta f_x f_y - \alpha ff_{xx} + \alpha f_x^2 - \delta ff_{xxxx} - 3\delta f_{xx}^2 + 4\delta f_{xxx} f_x = 0, \tag{7}$$

where $f = f(x, y, t)$. And D is the Hirota bilinear differential operator, defining as follows

$$D_x^n a \cdot b \equiv (\frac{\partial}{\partial x} - \frac{\partial}{\partial y})^n a(x) b(y) \mid_{y=x},$$

$$D_t^m D_x^n a \cdot b \equiv \frac{\partial^m}{\partial s^m} \frac{\partial^n}{\partial y^n} a(t+s,\ x+y) b(t-s,\ x-y) \mid_{s=0,\ y=0},$$

where m and n are nonnegative integers. By using the Hirota bilinear method, the N-soliton solution of Eq.(4) can be generated

$$f = \sum \exp\Big[\sum_{j=1}^{N} \mu_j \eta_j + \sum_{j<k}^{(N)} A_{jk} \mu_j \mu_k\Big]. \tag{8}$$

Where μ=0, 1, η_j and A_{jk} are as follows

$$\begin{aligned}
\eta_j &= p_j x + q_j y + \omega_j t + \eta_j^0, \\
\omega_j &= \sqrt{\delta p_j^4 + \alpha p_j^2 + \beta p_j q_j}, \\
A_{jk} &= \frac{B_{jk} + p_j p_k (4\delta p_j^2 - 6\delta p_j p_k) + C_{jk}}{B_{jk} + p_j p_k (4\delta p_j^2 + 6\delta p_j p_k) + C_{jk}}, \\
B_{jk} &= -2\sqrt{p_j(\delta p_j^3 + \alpha p_j + \beta q_j)}\sqrt{p_k(\delta p_k^3 + \alpha p_k + \beta q_k)}, \\
C_{jk} &= p_j(4\delta p_k^3 + 2\alpha p_k + \beta q_k) + \beta p_k q_j,
\end{aligned} \tag{9}$$

p_j, q_j, η_j^0 are arbitrary complex constants. $\delta p_j^4 + \alpha p_j^2 + \beta p_j q_j \geq 0$. Combining with transformation Eq.(3), N-soliton solution of the GB-type equation can be obtained.

When $N = 1$, with (3), (8) and (9), the one-soliton solution u_{1s} are get as follows

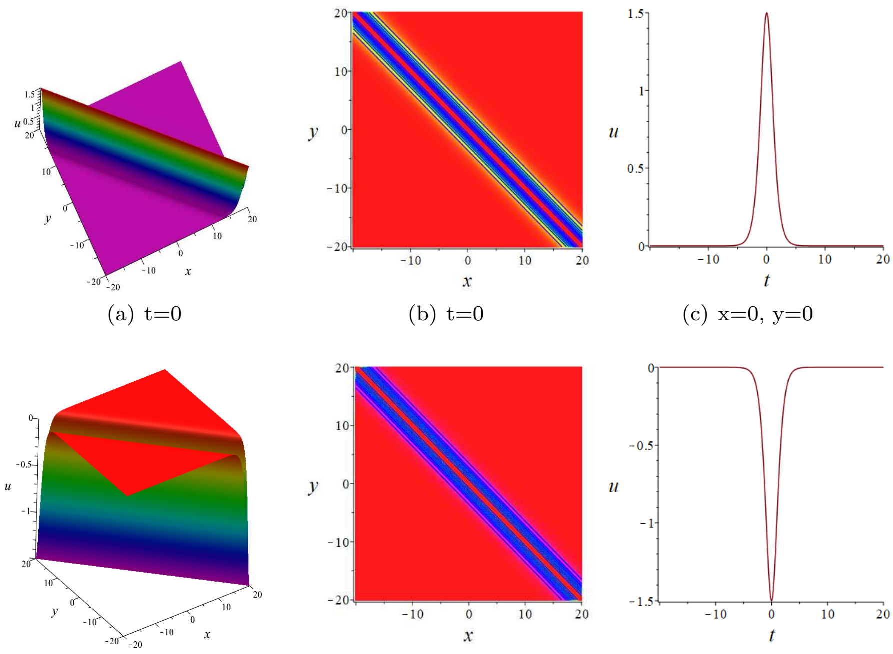
$$u_{1s} = -\frac{6\delta p_1^2}{\gamma[1 + \cosh(p_1 x + q_1 y + w_1 t + \eta_1^0)]}. \tag{10}$$

When $\gamma\delta < 0$, u_{1s} is a bright soliton as shown in Fig.1(a). Under this case, we can obtain the maximum value $\mid \frac{3\delta p_1^2}{\gamma} \mid$ when $p_1 x + q_1 y + w_1 t + \eta_1^0 = 0$. When $\gamma\delta > 0$, u_{1s} is a dark soliton as shown in Fig.1(d). Under this case, we can obtain the minimum value $- \mid \frac{3\delta p_1^2}{\gamma} \mid$ when $p_1 x + q_1 y + w_1 t + \eta_1^0 = 0$. Through the above analysis, it is not difficult to find that bright and dark solitons have the same limit line. At the same time, when the values of all parameters $\eta_j^0 (j = 1 \cdots 4)$ are set to 0, the bright and dark soliton images of two-soliton, three-soliton and four-soliton solutions are also given, as shown in Fig.2 respectively.

2.2 Breather solutions

According to previous studies, the breather solution can be obtained by selecting appropriate parameters on the basis of the soliton solution. Thus, the nth-order breather solutions of the (2+1)-dimensional GB-type equation can be obtained with

$$N = 2n,\ p_j^* = p_{j+1},\ q_j^* = q_{j+1},\ \alpha = 2,\ \beta = -1. \tag{11}$$



(a) t=0 (b) t=0 (c) x=0, y=0

(d) t=0 (e) t=0 (f) x=0, y=0

Fig. 1 One-soliton solution at $t = 0$, $p_1 = q_1 = 1$. (a), (b) and (c) Bright soliton: $\gamma = -2,\ \delta = 1$; (d), (e) and (f) Dark soliton: $\gamma = 2,\ \delta = 1$.

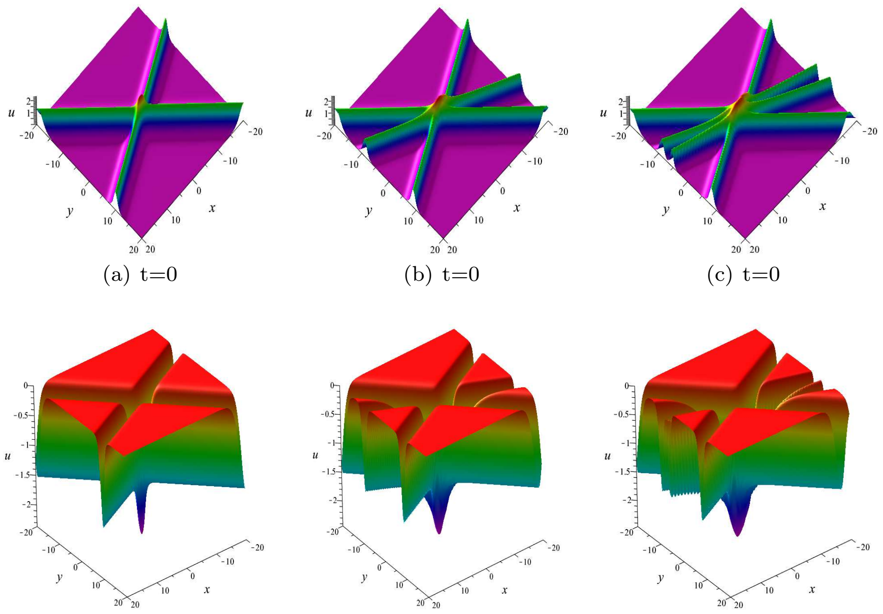
Case 1. $N = 2,\ \gamma = -2,\ \delta = 1,\ p_2 = p_1^* = \frac{i}{2},\ q_2 = q_1^* = 1 + \frac{i}{2},\ \eta_1^0 = \eta_2^0 = 0,$ the first-order bright breather solution u_{B1b} can be obtained as shown in Fig.3(a)

$$u_{B1b} = -\frac{3e^y\left(32C_1^0 + 32C_{1*}^0 + (41+3\sqrt{73})(C_2^0 + C_{2*}^0 + 4e^y C_2^0 C_{2*}^0)\right)}{\left((-3+\sqrt{73})(C_3^0 + C_{3*}^0) + (3+\sqrt{73})C_3^0 C_{3*}^0 - 3 + \sqrt{73}\right)^2}, \tag{12}$$

where

$$\begin{aligned} C_1^0 &= e^{\frac{t\sqrt{-3-8i}}{4} + \frac{t\sqrt{-3+8i}}{2} + (2-\frac{i}{2})y - \frac{ix}{2}}, \\ C_{1*}^0 &= e^{\frac{t\sqrt{-3+8i}}{4} + \frac{t\sqrt{-3-8i}}{2} + (2+\frac{i}{2})y + \frac{ix}{2}}, \\ C_2^0 &= e^{\frac{t\sqrt{-3-8i}}{4} + \frac{ix}{2} + \frac{iy}{2}}, \\ C_{2*}^0 &= e^{\frac{t\sqrt{-3+8i}}{4} - \frac{ix}{2} - \frac{iy}{2}}, \\ C_3^0 &= e^{\frac{t\sqrt{-3-8i}}{4} + (1+\frac{i}{2})y + \frac{ix}{2}}, \\ C_{3*}^0 &= e^{\frac{t\sqrt{-3+8i}}{4} + (1-\frac{i}{2})y - \frac{ix}{2}}. \end{aligned} \tag{13}$$

Case 2. Without changing other conditions, setting $\gamma = 2$ and $\delta = 1$, the first-order dark breather solution u_{D1b} of the GB-type equation can be obtained as



(a) t=0 (b) t=0 (c) t=0

(d) t=0 (e) t=0 (f) t=0

Fig. 2 Bright soliton at $t=0$, $\gamma=-2$, $\delta=1$: (a), (b) and (c). (a) $N=2$, $p_1=q_1=-1$, $p_2=1$, $q_2=-2$. (b)$N=3$, $p_1=q_1=-1$, $p_2=1$, $q_2=-2$, $p_3=1$, $q_3=2$. (c)$N=4$, $p_1=q_1=-1$, $p_2=1$, $q_2=-2$, $p_3=1$, $q_3=2$, $p_4=1$, $q_4=3$. Dark soliton at $t=0$, $\gamma=2$, $\delta=1$: (d), (e) and (f). (d) $N=2$, $p_1=q_1=-1$, $p_2=1$, $q_2=-2$. (e)$N=3$, $p_1=q_1=-1$, $p_2=1$, $q_2=-2$, $p_3=1$, $q_3=2$. (f)$N=4$, $p_1=q_1=-1$, $p_2=1$, $q_2=-2$, $p_3=1$, $q_3=2$, $p_4=1$, $q_4=3$.

shown in Fig.3(c)

$$u_{D1b}=\frac{3e^{y}\left(32C_1^0+32C_{1*}^0+(41+3\sqrt{73})(C_2^0+C_{2*}^0+4e^{y}C_2^0C_{2*}^0)\right)}{\left((-3+\sqrt{73})(C_3^0+C_{3*}^0)+(3+\sqrt{73})C_3^0C_{3*}^0-3+\sqrt{73}\right)^2}, \quad (14)$$

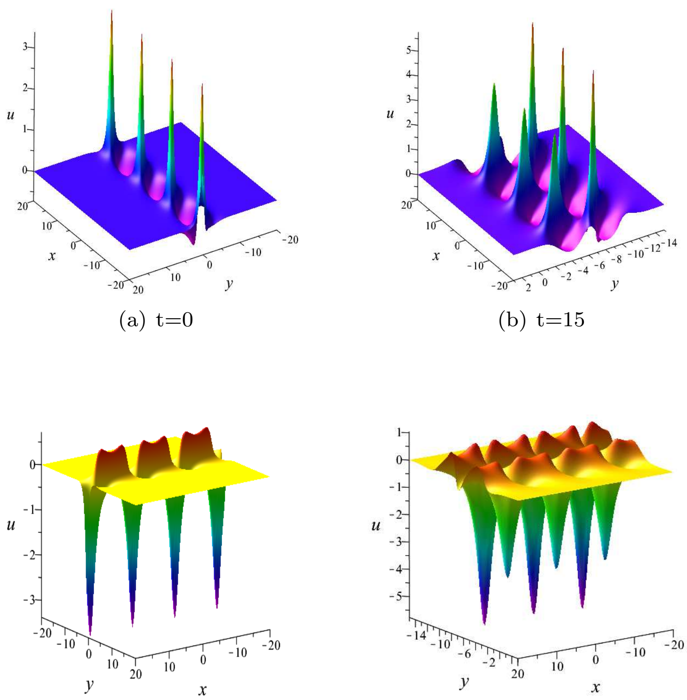
where C_1^0, C_{1*}^0, C_2^0, C_{2*}^0, C_3^0, C_{3*}^0 are the same as in Eq.(13).

Case 3. If $N=4, \gamma=-2,\ \delta=1,\ p_2=p_1^*=\frac{i}{2},\ q_2=q_1^*=1+\frac{i}{2},\ p_4=p_3^*=\frac{i}{2},\ q_4=q_3^*=1+i,\ \eta_1^0=\eta_2^0=\eta_3^0=\eta_4^0=0$. The second-order bright breather solution can be obtained, as shown in Fig.3(b).

Case 4. If $N=4, \gamma=2,\ \delta=1,\ p_2=p_1^*=\frac{i}{2},\ q_2=q_1^*=1+\frac{i}{2},\ p_4=p_3^*=\frac{i}{2},\ q_4=q_3^*=1+i,\ \eta_1^0=\eta_2^0=\eta_3^0=\eta_4^0=0$. The second-order dark breather solution can be obtained, as shown in Fig.3(d).

2.3 Mixed solutions

When $N>2$, the mixed solution of soliton and breather can be obtained by selecting appropriate parameters.



(a) t=0 (b) t=15

(c) t=0 (d) t=15

Fig. 3 Breather soliton. (a) First-order bright breather. (b) Second-order bright breather. (c) First-order dark breather. (d) Second-order dark breather.

Case 1. With $N = 3,\ \alpha = 2,\ \beta = -1,\ p_2 = p_1^* = \frac{i}{2},\ q_2 = q_1^* = 1 + \frac{i}{2},\ p_3 = 1,\ q_3 = -1,\ \gamma = -2,\ \delta = 1$, the mixed solution of first-order bright breather and bright one soliton can be obtained, as shown in Fig.4(a).

Case 2. With $N = 3,\ \alpha = 2,\ \beta = -1,\ p_2 = p_1^* = \frac{i}{2},\ q_2 = q_1^* = 1 + \frac{i}{2},\ p_3 = 1,\ q_3 = -1,\ \gamma = 2,\ \delta = 1$, the mixed solution of first-order dark breather and dark one soliton can be obtained, as shown in Fig.4(c). For higher order N, we also give the mixed solution of breather and soliton.

Case 3. With $N = 4,\ \alpha = 2,\ \beta = -1,\ p_2 = p_1^* = \frac{i}{2},\ q_2 = q_1^* = 1 + \frac{i}{2},\ p_3 = 1,\ q_3 = -1,\ p_4 = 1,\ q_4 = -2,\ \gamma = -2,\ \delta = 1$, the mixed solution of first-order bright breather and bright two soliton can be obtained, as shown in Fig.4(b).

Case 4. With $N = 4,\ \alpha = 2,\ \beta = -1,\ p_2 = p_1^* = \frac{i}{2},\ q_2 = q_1^* = 1 + \frac{i}{2},\ p_3 = 1,\ q_3 = -1,\ p_4 = 1,\ q_4 = -2,\ \gamma = 2,\ \delta = 1$, the mixed solution of first-order dark breather and dark two soliton can be obtained, as shown in Fig.4(d).

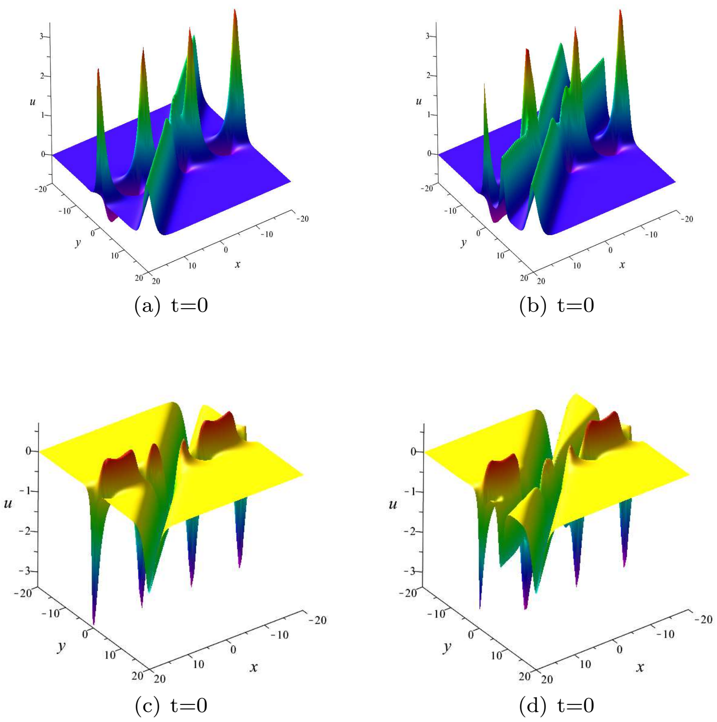


Fig. 4 (a) Mixed solution of first-order bright breather and bright one soliton at $t = 0$. (b) Mixed solution of first-order bright breather and bright two soliton at $t = 0$. (c) Mixed solution of first-order dark breather and dark one soliton at $t = 0$. (d) Mixed solution of first-order dark breather and dark two soliton at $t = 0$.

3 Lump solutions

In order to obtain the lump solution, when N=2,

$$q_1 = \lambda_1 p_1,\ q_2 = \lambda_2 p_2,\ \eta_1^0 = \eta_2^0 = i\pi. \tag{15}$$

Then letting $p_1,\ p_2 \to 0$, the function f can be written as follows

$$f = \theta_1\theta_2 + a_{12}, \tag{16}$$

where

$$\begin{aligned} \theta_j &= B_j t + x + y\lambda_j\ (j = 1,\ 2), \\ a_{12} &= -\frac{12\delta}{-2B_1B_2 + (\lambda_1 + \lambda_2)\beta + 2\alpha}, \\ B_j &= \gamma_j\sqrt{\beta\lambda_j + \alpha}\ (j = 1,\ 2). \end{aligned} \tag{17}$$

If $\gamma = -2,\ \delta = 1,\ \alpha = 2,\ \beta = -1,\ \gamma_1 = \gamma_2 = 1,\ \lambda_2 = \lambda_1^* = -i$ and combining Eq.(3), Eq.(16) and Eq.(17), the first-order bright lump solution

$u_{b1-lump}$ can be obtained

$$\begin{aligned}
u_{b1-lump} &= -12(\sqrt{5}-2)\frac{H_1\sqrt{2-i}+H_1^*\sqrt{2+i}+H_1^0}{(H_2\sqrt{2-i}+H_2^*\sqrt{2+i}+H_2^0)^2},\\
H_1 &= ((1-4i)y+ix)t,\\
H_2 &= ((-1+4i)y+ix)t,\\
H_1^0 &= (2t^2+x^2-y^2)\sqrt{5}-4t^2-2x^2+2y^2-6,\\
H_2^0 &= (-2t^2+x^2-y^2)\sqrt{5}+5t^2-2x^2-2y^2+6.
\end{aligned} \tag{18}$$

It is shown in Fig.5(a). In addition, we calculate the critical point of lump solution when $t=0$ as follows:

$$A_1=(0,\ 0),\ A_2=(\frac{3\sqrt{2\sqrt{5}-4}}{\sqrt{5}-2},\ 0),\ A_3=(-\frac{3\sqrt{2\sqrt{5}-4}}{\sqrt{5}-2},\ 0). \tag{19}$$

From Fig.5(a), we can see that the first-order bright lump solution has one maximum and two minimum. The maximum is $2\sqrt{5}-4$ at point A_1 and minimum is $\frac{2-\sqrt{5}}{4}$ at points A_2 and A_3.

If $\gamma=2,\ \delta=1,\ \alpha=2,\ \beta=-1,\ \gamma_1=\gamma_2=1,\ \lambda_2=\lambda_1^*=-i$ and combining Eq.(3), Eq.(16) and Eq.(17), the first-order dark lump solution $u_{d1-lump}$ can be obtained

$$u_{d1-lump}=12(\sqrt{5}-2)\frac{H_1\sqrt{2-i}+H_1^*\sqrt{2+i}+H_1^0}{(H_2\sqrt{2-i}+H_2^*\sqrt{2+i}+H_2^0)^2}, \tag{20}$$

here $H_1,\ H_2,\ H_1^0,\ H_2^0$ is the same as in Eq.(18). It is shown in Fig.5(c). In addition, we calculate the critical point of lump solution when $t=0$, same as in Eq.(19). From Fig.5(c), we can see that the first-order dark lump solution has one minimum and two maximum. Maximum is $\frac{\sqrt{5}-2}{4}$ at points A_2 and A_3, minimum is $4-2\sqrt{5}$ at point A_1.

4 Semi-rational solutions

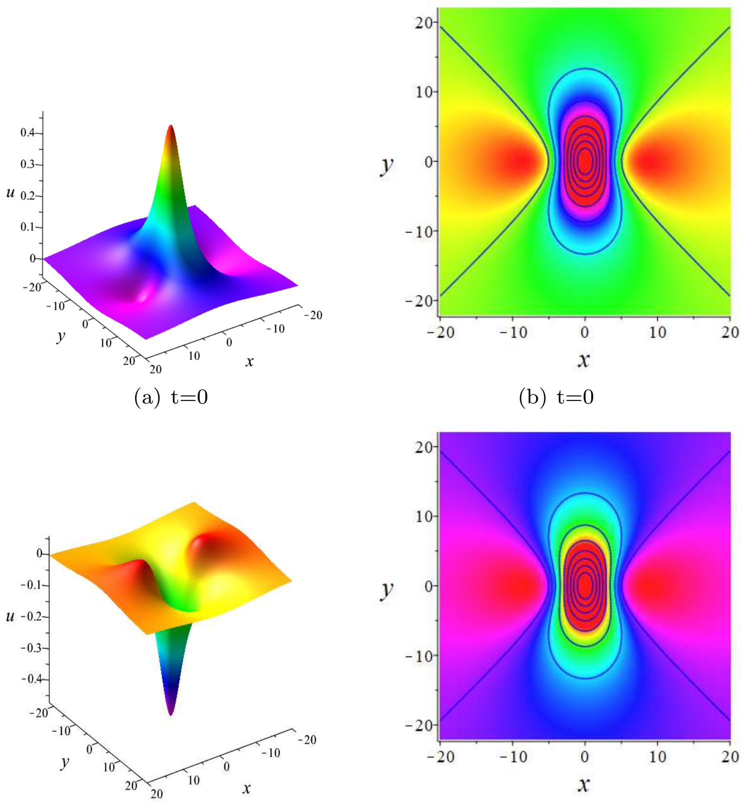
Next, for $N=3$ and $N=4$, semi-rational solution can be obtained by selecting appropriate parameters based on the soliton solution. This paper mainly discusses the two kinds of semi-rational solution: lump-soliton and lump-breather.

When $N=3$, with

$$q_1=\lambda_1 p_1,\ q_2=\lambda_2 p_2,\ \eta_1^0=\eta_2^0=i\pi, \tag{21}$$

and letting $p_1,\ p_2\to 0$. Then f in Eq.(8) can be expressed as follows

$$f=(a_{13}a_{23}+a_{13}\theta_2+a_{23}\theta_1+\theta_1\theta_2+a_{12})e^{\eta_3}+\theta_1\theta_2+a_{12}. \tag{22}$$



(a) t=0 (b) t=0

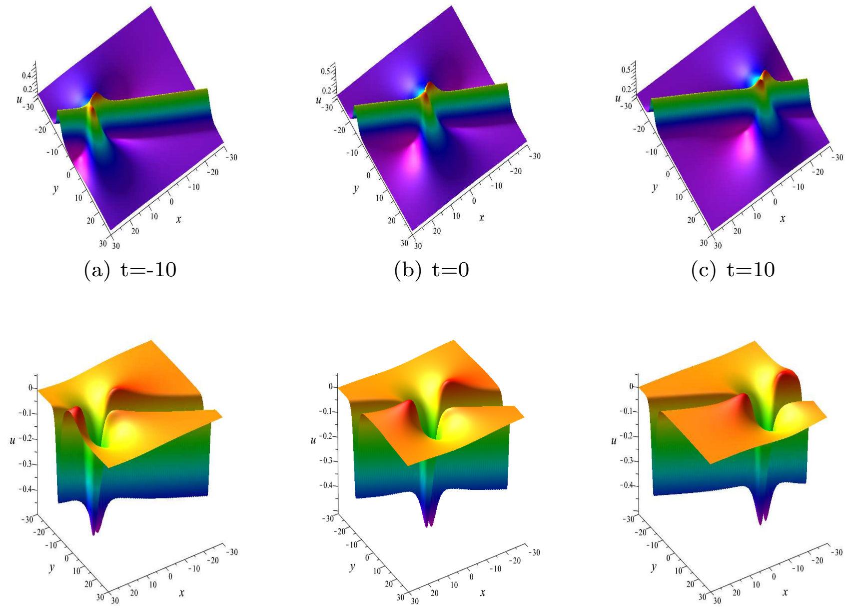
(c) t=0 (d) t=0

Fig. 5 (a), (b) First-order bright lump solution at $t = 0$. (c), (d) First-order dark lump solution at $t = 0$.

Here a_{12} and $\theta_j (j = 1,\ 2)$ in Eq.(17), and $a_{j3} (j = 1,\ 2)$ are as follows

$$
\begin{aligned}
a_{j3} &= -\frac{12\delta p_3^2}{-2B_j\sqrt{p_3(p_3^3\delta + p_3\alpha + \beta q_3)} + D_j^3},\ (j = 1,\ 2),\\
D_j^3 &= 4p_3^3\delta + 2p_3\alpha + \beta q_3 + \beta p_3\lambda_j,\ (j = 1,\ 2),
\end{aligned}
\tag{23}
$$

here B_j same as in Eq.(17). Combining Eq.(22), (23) and Eq.(3), we can obtain a semi-rational solution composed of one-soliton and first-order lump. Taking $\alpha = 2,\ \beta = -1,\ \delta = 1,\ \gamma = -2,\ \gamma_1 = \gamma_2 = 1,\ \lambda_2 = \lambda_1^* = -i,\ p_3 = \frac{1}{2},\ q_3 = 1,\ \eta_3^0 = 0$, bright lump-soliton solution can be obtained, as shown in Fig.6(a), (b), (c). Taking $\alpha = 2,\ \beta = -1,\ \delta = 1,\ \gamma = 2,\ \gamma_1 = \gamma_2 = 1,\ \lambda_2 = \lambda_1^* = -i,\ p_3 = \frac{1}{2},\ q_3 = 1,\ \eta_3^0 = 0$, dark lump-soliton solution can be obtained, as shown in Fig.6(d), (e), (f). It can be seen from the figure that the lump moves along the peak amplitude of the soliton wave. And the collision between soliton and lump is elastic.



(a) t=-10 (b) t=0 (c) t=10

(d) t=-10 (e) t=0 (f) t=10

Fig. 6 Time evolution in the $(x,\ y)$ plane of the two distinct patterns of lump-soliton. (a), (b), (c) Bright lump-soliton. (d), (e), (f) Dark lump-soliton.

When $N = 4$ taking

$$q_1 = \lambda_1 p_1,\ q_2 = \lambda_2 p_2,\ \eta_1^0 = \eta_2^0 = i\pi, \tag{24}$$

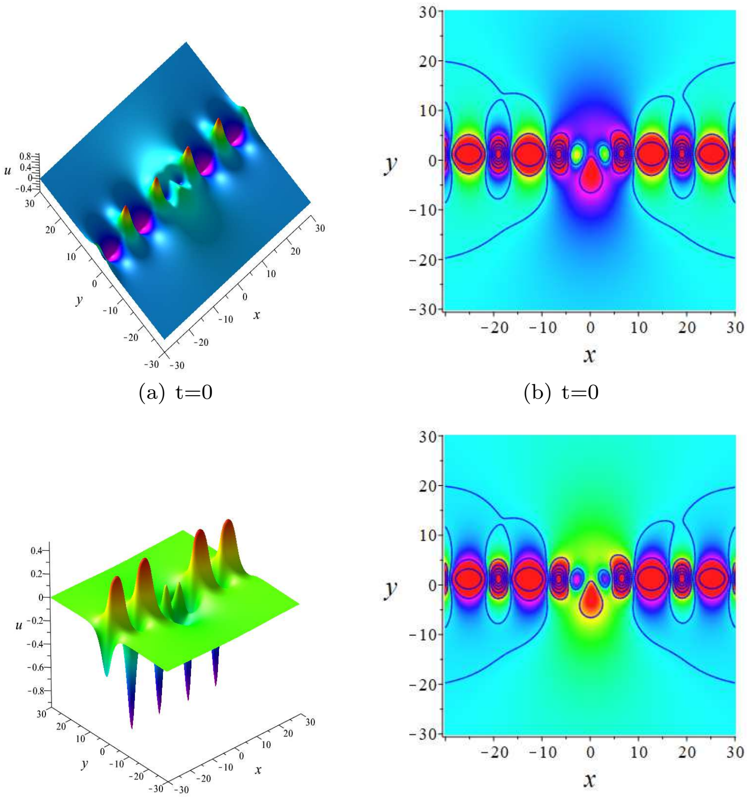
and letting $p_1,\ p_2 \to 0$. Then f in Eq.(8) can be expressed as follows

$$\begin{aligned} f = & e^{A_{34}}(a_{13}a_{23} + a_{13}a_{24} + a_{13}\theta_2 + a_{14}a_{23} + a_{14}a_{24} \\ & + a_{14}\theta_2 + a_{23}\theta_1 + a_{24}\theta_1 + \theta_1\theta_2 + a_{12})e^{\eta_3+\eta_4} \\ & + (a_{13}a_{23} + a_{13}\theta_2 + a_{23}\theta_1 + \theta_1\theta_2 + a_{12})e^{\eta_3} \\ & + (a_{14}a_{24} + a_{14}\theta_2 + a_{24}\theta_1 + \theta_1\theta_2 + a_{12})e^{\eta_4} \\ & + \theta_1\theta_2 + a_{12}. \end{aligned} \tag{25}$$

Where a_{12} and $\theta_j(j = 1,\ 2)$ in Eq.(18), and $a_{js}(j = 1,\ 2;\ s = 3,\ 4)$ are as follows

$$\begin{aligned} a_{js} &= -\frac{12\delta p_s^2}{-2B_j\sqrt{p_s(p_s^3\delta + p_s\alpha + \beta q_s)} + D_j^s}\ , \\ D_j^s &= 4p_s^3\delta + 2p_s\alpha + \beta q_s + \beta p_s\lambda_j\ , \end{aligned} \tag{26}$$

here B_j same as in Eq.(17). Combining Eq.(25), (26) and Eq.(3), we can obtain a semi-rational solution composed of first-order breather and first-order lump.



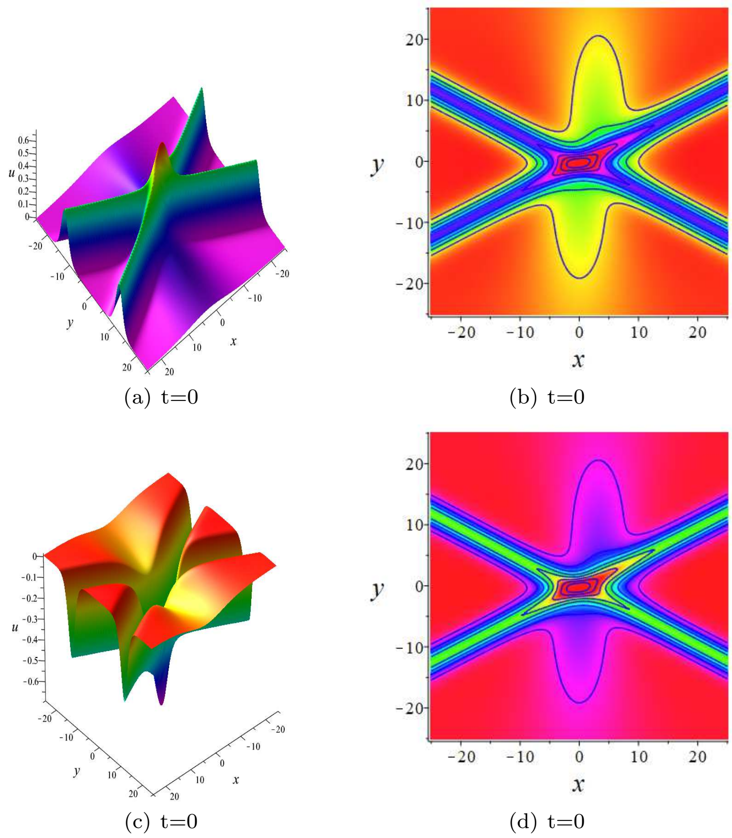
(a) t=0 (b) t=0

(c) t=0 (d) t=0

Fig. 7 Lump-breather solution in the $(x,\ y)$ plane at $t = 0$. (a), (b) Bright lump-breather. (c), (d) Dark lump-breather.

If $\alpha = 2,\ \beta = -1,\ \delta = 1,\ \gamma = -2,\ \gamma_1 = \gamma_2 = 1,\ \lambda_2 = \lambda_1^* = -i,\ p_4 = p_3^* = -\frac{i}{2},\ q_4 = q_3^* = \frac{1}{2},\ \eta_3^0 = \eta_4^0 = -\frac{\pi}{2}$, bright lump-breather solution can be obtained, as shown in Fig.7(a), (b). If $\alpha = 2,\ \beta = -1,\ \delta = 1,\ \gamma = 2,\ \gamma_1 = \gamma_2 = 1,\ \lambda_2 = \lambda_1^* = -i,\ p_4 = p_3^* = -\frac{i}{2},\ q_4 = q_3^* = \frac{1}{2},\ \eta_3^0 = \eta_4^0 = -\frac{\pi}{2}$, dark lump-breather solution can be obtained, as shown in Fig.7(c), (d).

In addition, the semi-rational solution composed of first-order lump and two-soliton can be obtained by selecting appropriate parameters when $N = 4$. Taking $\alpha = 2,\ \beta = -1,\ \delta = 1,\ \gamma = -2,\ \gamma_1 = \gamma_2 = 1,\ \lambda_2 = \lambda_1^* = -\frac{i}{2},\ p_3 = \frac{1}{2},\ p_4 = -\frac{1}{2},\ q_3 = q_4 = 1,\ \eta_3^0 = \eta_4^0 = 0$, bright lump-soliton solution can be obtained, as shown in Fig.8(a), (b). If $\alpha = 2,\ \beta = -1,\ \delta = 1,\ \gamma = 2,\ \gamma_1 = \gamma_2 = 1,\ \lambda_2 = \lambda_1^* = -\frac{i}{2},\ p_3 = \frac{1}{2},\ p_4 = -\frac{1}{2},\ q_3 = q_4 = 1,\ \eta_3^0 = \eta_4^0 = 0$, dark lump-soliton solution can be obtained, as shown in Fig.8(c), (d).



(a) t=0 (b) t=0

(c) t=0 (d) t=0

Fig. 8 The semi-rational solution composed of first-order lump and two-soliton in the $(x,\ y)$ plane at $t = 0$. (a), (b) Bright. (c), (d) Dark.

5 Conclusion

In this paper, a Generalized (2+1)-dimensional Boussinesq type equation is studied. On the basis of the Hirota bilinear method, N-soliton solution, high-order breather solution, mixed solution, lump solution, and semi-rational solution are obtained. Fig.(1) and Fig.(2) show the bright and dark soliton solutions of one-soliton to four-soliton. In addition, what's interesting is that the bright and dark soliton solutions have the same limit line and different extreme values. Based on the double soliton solution, the first-order bright and dark breather solution are generated by selecting appropriate parameters. Its dynamic behavior is shown in Fig.(3). When $N > 2$, we get the mixed solution of bright soliton and bright breather and the mixed solution of dark soliton and dark breather under suitable parameters, as shown in Fig.(4).

Moreover, by using the long-wave limit method, the lump solution and the semi-rational solution of GB-type equation are giving as shown in Fig.(5) to

(8). This article mainly considers the two types of lump-soliton and lump-breather, which are semi-rational. It is not difficult to find that they have different extremum, although they have the same extreme points. What is interesting is that for the lump-soliton solution in this article, the lump is moving along the peak amplitude of the soliton wave. Therefore, compared with previous reports on Boussinesq type equation, the results reported in this paper are more abundant. Another advantage is that the Boussinesq type equation studied is more general. These results may help explain some physical phenomena that exist in nature, and may also enrich the dynamic behavior of (2+1)-dimensional nonlinear evolution equations.

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Compliance with ethical standards

Ethical Statement Authors declare that they comply with ethical standards.

Conflict of interest Authors declare that they have no conflict of interest.

Data Availability Statement The data that support the findings of this article are available from the corresponding author, upon reasonable request.

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