Some Curvature Properties in Almost Parahermitian Manifolds

NECİP ERDOGAN (necerd@hotmail.com)

Research Article

Keywords: almost para contact manifolds, curvature, LC connection, para-Hermitian manifolds

Posted Date: April 8th, 2022

DOI: https://doi.org/10.21203/rs.3.rs-1538048/v1

[1]

Department of Mathematics, Eskişehir Technical University, 26470 Eskişehir, Turkey

[1]E.mail: mail adresini yaz

curvature properties in almost parahermitian manifolds

Necip Erdogan

Abstract

In this study, the curvature properties are investigated in almost parahermitian manifolds.

Keywords: curvature,Levi-civita covariant derivative, almost parahermitian manifolds.

1 Introduction

The almost paracontact manifolds and almost para-Hermitian manifolds play a crucial role

in differential geometry.In the first case it is easily seen that the vector field

ξ

is a Killing vector field and the integral curve is a geodesic.

In this study i have proved that the vector field $(\xi, 0)$ is a Killing vector field

in the product manifold $M * R$ and the integral curve is a geodesic.

2 Almost paracontact metric manifolds and almost parahermitian manifolds

A $(2n+1)$-dimensional smooth manifold M is called an almost paracontact metric manifod if the structure group of its tangent bundle reduces to $\mathbb{U}^{\pi}(n) \times \{1\}$, where $U^{\pi}(n)$ is paraunitary group [3]. Equivalently, if a smooth manifold M has a tensor field φ of type $(1,1)$, a vector field ξ, a 1-form η and a pseudo-riemannian metric g satisfying the following conditions

1. $\varphi^2(X) = X - \eta(X)\xi$, $\eta(\xi) = 1$,

2. There exists a distribution $\mathbb{D} : M \to T_pM$, $\mathbb{D}_p \subset T_pM$ such that $\mathbb{D}_p = Ker\eta = \{X \in T_pM : \eta(X) = 0\}$. This distribution is called paracontact distribution generated by η,

3. $g(\varphi(X), \varphi(Y)) = -g(X,Y) - \eta(X)\eta(Y)$,

then the manifold M is called an almost paracontact metric manifold. Note that signature of g is $(n+1, n)$. The fundamental 2-form on an almost paracontact metric manifold M is defined as

$$\Phi(X,Y) := g(\varphi(X),Y).$$

This 2-form is non-degenerate on the horizontal distribution $\mathbb{D}$ and $\eta\wedge\Phi^n \neq 0$. The covariant derivative of Φ with respect to the Levi-Civita connection ∇ of g is

$$\beta(X,Y,Z) = (\nabla_X)(Y,Z) = g\left((\nabla_X\varphi)(Y), Z\right).$$

The tensor β has the following properties:

$$\beta(X,Y,Z) = -\beta(X,Z,Y),$$

$$\beta(X,\varphi(Y),\varphi(Z)) = \beta(X,Y,Z) + \eta(Y)\beta(X,Z,\xi) - \eta(Z)\beta(X,Y,\xi).$$

The following 1-forms associated with β are defined as [3]:

$$\theta(X) := g^{ij}\beta(e_i,e_j,X) + \beta(\xi,\xi,X), \qquad \theta^*(X) := g^{ij}\beta(e_i,\varphi(e_j),X),$$

where $\{e_1, e_2, \cdots, e_{2n}, \xi\}$ is an orthonormal basis of TM and (g^{ij}) is the inverse matrix of (g_{ij}).

If a smooth manifold N has a tensor field J (almost product structure) and a pseudo-Riemannian metric h satisfying the following conditions

- $J^2(X) = X$,
- $h(J(X), J(Y)) = -h(X,Y)$

for all vector fields X, Y on N, then the manifold N is called an almost para-hermitian manifold [4]. An almost parahermitian manifold has even dimension ($\dim N = 2n$)

The fundamental 2-form F on N is defined by

$$F(X,Y) = h(J(X),Y)$$

for all X, Y vector fields on N. The covariant derivative of F with respect to the Levi-civita connection of h is

$$\alpha(X,Y,Z) = (\nabla_X F)(Y,Z) = g\left((\nabla_X J)(Y), Z\right).$$

The tensor α has the following properties:

$$\alpha(X,Y,Z) = -\alpha(X,Z,Y),$$

$$\alpha\left(X, J(Y), J(Z)\right) = \alpha(X,Y,Z).$$

3 Almost paraHermitian manifolds from almost paracontact manifold

In this section, first we define the almost parahermitian structure on the product of an almost paracontact manifold with $\mathbb{R}$. Then we give the relations between covariant derivatives.

Let $(M, \varphi, \xi, \eta, g)$ a $(2n+1)$-dimensional almost paracontact metric manifold and consider the product manifold $M \times \mathbb{R}$. A vector field on the manifold $M \times \mathbb{R}$ is the form $\left(X, a\frac{d}{dt}\right)$ where t is the coordinate of $\mathbb{R}$ and a is a smooth function on $M \times \mathbb{R}$. The almost para-complex structure (or almost product structure) J on $M \times \mathbb{R}$ is defined by

$$J\left(X, a\frac{d}{dt}\right) = \left(\varphi(X) + a\xi, \eta(X)\frac{d}{dt}\right) \tag{1}$$

and we define a pseudo-Riemannian metric on $M \times \mathbb{R}$ with signature $(n+1, n+1)$ by

$$h\left(\left(X, a\frac{d}{dt}\right), \left(Y, b\frac{d}{dt}\right)\right) := g(X,Y) - ab. \tag{2}$$

One can easily see that

$$h\left(J\left(X, a\frac{d}{dt}\right), J\left(Y, b\frac{d}{dt}\right)\right) = -h\left(\left(X, a\frac{d}{dt}\right), \left(Y, b\frac{d}{dt}\right)\right). \tag{3}$$

The para-Kaehler form F of the almost parahermitian manifold $(M \times \mathbb{R}, J, h)$ is given by

$$F\left(\left(X, a\frac{d}{dt}\right), \left(Y, b\frac{d}{dt}\right)\right) := h\left(J\left(X, a\frac{d}{dt}\right), \left(Y, b\frac{d}{dt}\right)\right). \tag{4}$$

Hence, one can express the form F in terms of Φ, η as

$$F\left(\left(X, a\frac{d}{dt}\right), \left(Y, b\frac{d}{dt}\right)\right) = \Phi(X, Y) + a\eta(Y) - b\eta(X). \tag{5}$$

Let ∇ be the pseudo-Riemannian connection of $(M, \varphi, \xi, \eta, g)$. Levi-Civita covariant derivative of the metric h on $M \times \mathbb{R}$ is obtained using the Kozsul formula as

$$\nabla_{\left(X, a\frac{d}{dt}\right)}\left(Y, b\frac{d}{dt}\right) = \left(\nabla_X Y, \left(X[b] + a\frac{db}{dt}\right)\frac{d}{dt}\right). \tag{6}$$

Note that the covariant derivative on the product manifold $M \times \mathbb{R}$ will also be denoted with the same symbol ∇. Also, covariant derivative of the 2-form F is calculated as

$$\left(\nabla_{\left(X, a\frac{d}{dt}\right)}F\right)\left(\left(X, a\frac{d}{dt}\right), \left(Y, b\frac{d}{dt}\right)\right) = \beta(X, Y, Z) - a\left(\nabla_X \eta\right)(Y) + b\left(\nabla_X \eta\right)(Z),$$

for any vector fields $\left(X, a\frac{d}{dt}\right)$, $\left(Y, b\frac{d}{dt}\right)$ and $\left(Z, c\frac{d}{dt}\right)$ on $M \times \mathbb{R}$. Differential of the fundamental 2-form F can be evaluated as

$$\begin{aligned} dF\left(\left(X, a\frac{d}{dt}\right), \left(Y, b\frac{d}{dt}\right), \left(Z, c\frac{d}{dt}\right)\right) &= d\Phi(X, Y, Z) - \frac{2a}{3}d\eta(Y, Z) \\ &\quad + \frac{2b}{3}d\eta(X, Z) - \frac{2c}{3}d\eta(X, Y). \end{aligned}$$

Let $\{e_1, \cdots, e_n, \varphi(e_1), \cdots, \varphi(e_n), \xi\}$ be a local pseudo-orthonormal $\varphi-$frame field on M. Then one can obtain an orthonormal frame field on $M \times \mathbb{R}$ as follows:

$$\left\{(e_1, 0), \cdots, (e_n, 0), (\varphi(e_1), 0), \cdots, (\varphi(e_n), 0), (\xi, 0), \left(0, \frac{d}{dt}\right)\right\}$$

Using this frame, the coderivative of F is calculated as

$$\delta F(X, \frac{d}{dt}) = -\theta(X) - a\theta^*(\xi). \tag{7}$$

Let (X,adt) and (Y,bdt) be two elements in the product manifold.

Then the Lie bracket is defined as:

$[(X, adt), (Y, bdt)] = ([X, Y], (X(b) - Y(a))dt)$

and the covariant derivative is:

$(\nabla_{(X,adt)}F)((Y, bdt), (Z, cdt)) = g((\nabla_X \varphi)Y, Z) + bg(\nabla_X \xi, Z) - c(\nabla_X \eta)Y$

using covariant derivative we can find out the curvature in the product manifold $M * R$

$R((X, adt), (Y, bdt))(\xi, 0dt) = \nabla_{(X,adt)}\nabla_{(Y,bdt)}(\xi, 0dt) - \nabla_{(Y,bdt)}\nabla_{(X,adt)}(\xi, 0dt) - \nabla_{[(X,adt),(Y,bdt)]}(\xi, 0dt)$

$R((X, adt), (Y, bdt))(\xi, 0dt) = \nabla_{(X,adt)}(\nabla_Y \xi, 0dt) - \nabla_{(Y,bdt)}(\nabla_X \xi, 0dt) - \nabla_{[(X,adt),(Y,bdt)]}\xi, 0dt)$

$R((X, adt), (Y, bdt))(\xi, 0dt) = (\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}\xi, 0dt) = (R(X, Y)\xi, 0dt)$

Theorem 3.1 $(\xi, 0)$ *is a Killing vector field in the product manifold* $M * R$

Proof. Let (X,adt),(Y,bdt) be two different vector fields.

if $(\xi, 0)$ is a Killing vector field then $(L_{(\xi,0)}h)((X, adt), (Y, bdt) = 0$

using the definition of the Lie derivative

$(L_{(\xi,0)}h)((X, adt), (Y, bdt) = (\xi, 0)((X, adt), (Y, bdt)) - h([(\xi, 0), (X, adt)], (Y, bdt)) - h((X, adt), [(\xi, 0), (Y, bdt)])$

the metric in the product manifold is defined by the metric in the almost para-contact manifold since

$h((X, adt), (Y, bdt)) = g(X, Y) - ab$

after algebraic calculations ; for all $(X, adt), (Y, bdt)$

$(L_{(\xi,0)}h)((X, adt), (Y, bdt) = 0$

which means that the Reeb vector field is a Killing vector field in the product manifold. ■

Theorem 3.2 *let* $(X, adt), (Y, bdt)$*be two vector fields and* $(\xi, 0)$ *be the Reeb vector field.*

then $R((X, adt), (Y, bdt))J(\xi, 0dt) = J(R((X, adt), (Y, bdt)(\xi, 0dt)))$

iff

$\varphi(R(X, Y)\xi) = 0, \eta(R(X, Y)\xi) = 0$

Proof.

let $(X, adt), (Y, bdt)$be two vector fields and $(\xi, 0)$ be the Reeb vector field.

$R((X, adt), (Y, bdt))J(\xi, 0) = R((X, adt), (Y, bdt))(0, dt)$

from the definition of the curvature operator

$R((X, adt), (Y, bdt))(0, dt) = \nabla_{(X,adt)}\nabla_{(Y,bdt)}(0, dt) - \nabla_{(Y,bdt)}\nabla_{(X,adt)}(0, dt) - \nabla_{[(X,adt),(Y,bdt)]}(0, dt)$

$R((X, adt), (Y, bdt))(\xi, 0dt) = \nabla_{(X,adt)}(0, 0dt) - \nabla_{(Y,bdt)}(0, 0dt) - \nabla_{[(X,adt),(Y,bdt)]}(0, 0dt) = (0, 0dt)$

and

$J(R((X,adt),(Y,bdt))(\xi,0)) = J(R(X,Y)\xi,0dt) = (\varphi(R(X,Y)\xi),\eta(R(X,Y)\xi)dt)$

so the equality holds iff $\varphi(R(X,Y)\xi) = 0, \eta(R(X,Y)\xi) = 0$

■

Definition 3.3 *Let* $(M*R),J,h)$ *be an almost Hermitian structure and* $(X,adt),(Y,bdt) \in \chi(M*R)$

if

$(L_{(X,adt)}J)(Y,bdt) = 0$*then* (X,adt) *is analytic vector field(infinitesimal automorphism)*

Theorem 3.4 *The Reeb vector field is an infinitesimal vector field*

i.e $((L_{(\xi,0dt)}J)(Y,bdt) = 0)$*iff* $Y = \xi$

Proof.

Suppose that the reeb vector field is an infinitesimal vector field then

$(L_{(\xi,0dt)}J)(Y,bdt) = 0$

$L_{(\xi,0dt)}J(Y,bdt) = J(L_{(\xi,0dt)}(Y,bdt))$

$[(\xi,0dt),J(Y,bdt)] = J([(\xi,0dt),(Y,bdt)])$

$[(\xi,0dt),(\varphi Y+b\xi,\eta(Y)dt)] = J([\xi,Y],(\xi(b))dt)$

$([\xi,\varphi Y+b\xi],(\xi(\eta(Y)))dt = (\varphi[\xi,Y]+(\xi(b))\xi,\eta([\xi,Y]dt)$

so

$[\xi,\varphi Y+b\xi] = \varphi[\xi,Y]+(\xi(b))\xi$

and

$\xi(\eta(Y)) = \eta([\xi,Y]$

from the first and second equality it can easily be seen that $Y = \xi$

On the contrary let us suppose that $Y = \xi$

using algebraic operations and the properties of the covariant derivative

we can say that the Reeb vector field is an infinitesimal vector field. ■

Theorem 3.5 *The vector field* $(\xi, 0)$ *is a geodesic in the product manifold.*

Proof. from the definition of the covariant derivative in the product manifold

it can easily seen that

$(\nabla_{\xi}\xi, 0dt) = (0, 0)$

the equality holds since the vector field ξ is a geodesic

in the almost contact manifold. ■

Theorem 3.6 *Let* $(M * R, J, h)$ *be the almost complex structure.*

$(X, adt), (\xi, 0dt) \in \chi(M * R)$

Let a be the constant function.Then (X, adt) *satisfies (the Jacobi equation)*

$\nabla_{(\xi,0)}\nabla_{(\xi,0)}(X, adt) + R((X, a), (\xi, 0))(\xi, 0) = 0$

if $\nabla_{\xi}[X, \xi] + \nabla_{[X,\xi]}\xi = 0$

Proof.

Let us suppose that (X, adt) be a vector field where a is a

constant function.

In the product manifold the covariant derivative is defined as

$\nabla_{(X,adt)}(Y, bdt) = (\nabla_X Y, (X[b] + a.\frac{db}{dt})dt)$

then

$\nabla_{(\xi,0)}(X, adt) = (\nabla_\xi X, \xi[a]dt)$

since a is a constant function then $\xi[a] = 0$ so

$\nabla_{(\xi,0)}(X, adt) = (\nabla_\xi X, 0dt)$

then

$\nabla_{(\xi,0)}(\nabla_\xi X, 0dt) + R((X, a), (\xi, 0))(\xi, 0) = 0$

or

$\nabla_\xi [X, \xi] + \nabla_{[X,\xi]}\xi = 0$

■

Theorem 3.7 *Let $(M, \xi, \eta, \varphi, g)$ be the almost contact manifold and let $(M * R, J, h)$ be the corresponding product manifold. X, Y be two vector fields in $\chi(M)$ and $(X, adt), (Y, bdt)$ be two vector fields in $\chi(M * R)$ such that a and b are constant functions. then*

$S((X, adt), (Y, bdt)) = (S(X, Y), 0dt)$

Proof.

Let $\{e_1, ..., e_n, \varphi e_1, ..., \varphi e_n, \xi\}$

be a basis in the almost contact structure

and $\{(e_1, 0), ..., (e_n, 0), (\varphi e_1, 0), ..., (\varphi e_n, 0), (\xi, 0), (0, dt)\}$

be a basis in the almost complex structure.

we know that

$\nabla_{(X,adt)}(Y, bdt) = (\nabla_X Y, (Xb - Ya)dt)$

since a and b are constant functions then $Xb = Ya = 0$

so;

$\nabla_{(X,adt)}(Y, bdt) = (\nabla_X Y, 0dt)$

Using the above equation and taking the summation over the basis elements then it can easily seen that

$S((X, adt), (Y, bdt)) = (S(X, Y), 0dt)$

■

Theorem 3.8 *Let $(M * R, J, h)$ be the almost complex manifold with the Hermitian metric.*

$(X, adt), (Y, bdt)$ be two vector fields in $\chi(M * R)$

if the complex manifold is an Einstein manifold then $M * R$

is a Ricci soliton and $(\xi, 0)$is a potential vector field.

Proof. Let us suppose that the product manifold is Einstein. Then

$S((X, adt), (Y, bdt)) = \lambda.h((X, adt), (Y, bdt))$ holds

and λ is a constant function.

then the Ricci Soliton equation holds

$(L_{(\xi,0)}h)(X, adt), (Y, bdt)+2S((X, adt), (Y, bdt))+(2k)h((X, adt), (Y, bdt)) = 0$

for $k = -\lambda$ since $(\xi, 0)$ is a Killing vector field.

■

Theorem 3.9 *(A new bracket operation on $M * R$)*

Let $(M, \xi, \eta, \varphi, g)$be an almost contact structure *and* $(M * R, J, h)$ *be the corresponding almost complex manifold.*

the operation defined on $\chi(M * R)$

$[(X, adt), (Y, bdt)]' = (\eta(Y)\varphi(X) + \eta(Y)a\xi - \eta(X)\varphi(Y)$

$-\eta(X)b\xi + [X, Y], 0dt)$

is linear and anti-symmetric in the almost complex structure.

Proof.

the linearity is trivial since $\xi, \eta and \varphi$ are linear.

$[(X, adt), (Y, bdt)]' = (\eta(Y)\varphi(X) + \eta(Y)a\xi - \eta(X)\varphi(Y)$

$-\eta(X)b\xi + [X, Y], 0dt)$

and

$[(Y, bdt), (X, adt)]' = (\eta(X)\varphi(Y) + \eta(X)b\xi - \eta(Y)\varphi(X)$

$-\eta(Y)a\xi + [Y, X], 0dt)$

from differential geometry we know that $[X, Y] = -[Y, X]$

so the anti-symmetry of the bracket is obvious. ■

References

[1] Chinea, D., Gonzalez, C., A Classification of Almost Contact Metric Manifolds, Annali di Matematica pura ed applicata, (IV), Vol. CLVI, 15-36, (1990).

[2] A. Gray, L. Hervella, *The Sixteen Classes of Almost Hermitian Manifolds and Their Linear Invariants* Ann. Mat. Pura, Appl. **123**, 35-58 (1980).

[3] S. Zamkovoy, G. Nakova, *The decomposition of almost paracontact metric manifolds in eleven classes revised*, J. Geom. **109:18**, (2018).

[4] P. M. Gadea, J. M. Masque, *Classification of almost parahermitian manifolds*, Rendiconti di Matematica, Serie VII **11**, 924 (1963).

[5] Bejan, C., A Classification of the Almost Parahermitian Manifolds, Differential Geometry and Applications Proceedings of the Conference, June 26-July 3, Dubrovnik, Yugoslavia, 1988.

[6] M. Manev, M. Ivanova, *A classification of the torsion tensors on almost contact manifolds with B-metric*, Cent. Eur. J. Math. **12(10)**, 1416-1432 (2014).

[7] Obina, Jose A., A Classification for Almost Contact Structures, preprint, (1980).

[8] Ivanov, S., Vassiliev, S., Zamkovoy, S., Conformal paracontact curvature and the local flatness Theorem, Geom. Dedicata 144, 79-100, 2010.

[9] Zamkovoy, S., On Para-Kenmotsu Manifolds, Filomat 32(14), 4971-4980, (2018).