The white noise paradox in an experimental design

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Research Article

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Abstract

The study aims to investigate approaches that determine the observation regions, ensuring minimum estimation errors for small samples burdened by noise. The investigation is performed under the condition of *complete reflection of noise effects* on an optimal solution to a design problem. A design is constructed under *the Tikhonov regularization paradigm*. The case study focuses on the functional features of classical exponential regressions. Classical model functions are chosen because their features are well known. However, new inferences are defined. The main factors that govern the estimation errors of the exponential regressions are determined. The peculiarities of their optimal designs are revealed. It is proven that *noise generates a nonidentifiable observation case*, and starting with a specific threshold of noise, the design problem offers no solution. The widely used design approaches do not determine the optimal solution features revealed in the framework of the Tikhonov regularization paradigm. *The estimation error distribution*, *factors of noise effect reduction* and *peculiarities of the optimal design* are defined to satisfy this requirement. The results indicate the effectiveness of an alternative viewpoint on traditional solutions to design problems.

1 Introduction

For experimental data processing, the classical observational model assumes the effect of a random component whose stochastic properties are known. The most common model is *the white noise model*. In this case, the measurement error distribution is assumed to follow a normal law and there is no bias with respect to the mean.

For measurements of stable quantities, the conditions for asserting the suitability of the white noise model can be satisfied in practice with commonly known effort. The situation becomes much more complicated when the observed quantity changes during observations. It is practically impossible to make a very large number of measurements or, mathematically, an infinite number of measurements.

Introducing additional noise properties into the observational model can improve estimates, but from a mathematical viewpoint, the original *approximation of the observed sample* continues to be replaced by its *interpolation*. Theoretically, this means that experimental data can be processed with *arbitrary levels of noise*.

This results in the white noise paradox: *very detailed information about the stochastic properties of the noise with respect to the observed sample is postulated, but, however, the final estimate turns out to be independent of the level of noise.*

To avoid this paradoxical result, new observation models must be introduced. In this paper, *the model of worst observation errors* is proposed. Its practical application is demonstrated by the problem of designing observations of signals whose dynamics of change can be described by known phenomenological laws.

The paper is organized as follows. In Section 3, the abstract design problem is formulated in the framework of the regularization paradigm. The observation model is proposed to account for the small sample features. In Section 4, the essential features of the design problem regularization are specified. Section 5 gives a general description of the regularized design algorithm. In Section 6, the mathematical models traditionally studied in design theory are considered. New properties of the classical optimal design are revealed. Commonly known design inferences are essentially clarified. Section 7 underlines the peculiar properties of the design solution under unimprovable noise. Section 8 summarizes the results obtained.

2 An Overview Of Problems

The issue of observational design optimization is a logical continuation of the information channel capacity problem [42] in which, after the maximum amount of information has been resolved, it is necessary to define conditions for its determination. The main challenge here is a mapping from the desired parameter space to the observation space.

For the case of a linear mapping, the Fisher information matrix (FIM) [18] expresses the necessary and sufficient conditions for obtaining the maximum information in a received signal [49]. For nonlinear mapping, determining the best observation conditions within the FIM approach is based on the linearization of the original model [17]. Here, robust design construction is based on the idea of the optimality φ-criterion, where φ is some function of the information matrix [51, 12]. This approach requires setting a specific range for the desired quantities. The locally optimal design should be sought based on the best guesses regarding the desired quantities. The next step is the sequential analysis [10], which facilitates a practical application of the locally optimal design. A disadvantage of locally optimal designs is that the desired solution depends on *a priori* information of the parameters and may be very sensitive to these values [26]. The FIM defines a set of approaches, criteria, methods, algorithms, and software that make up *the modern design paradigm* [19]. This paradigm is actively employed in the design of inverse problems [1, 4, 5, 29, 46, 50]. Contemporary computational problems of the paradigm are studied in [21]. The Tikhonov regularization of a numerical solution under the FIM paradigm is considered in [30].

Recent advances in experimental design approaches are based on the idea of bounded noise [41]. A relevant approach is developed in [6, 22, 34, 48]. Its application reduces the requirements for postulating the properties of operating noise. However, the design paradigm remains unchanged and involves FIM norm minimization.

A popular statistical idea for the design, the Bayesian methodology, was proposed in [8] to eliminate the disadvantage mentioned above. This approach is based on the specifications of the prior distribution law to obtain the desired quantities. A similar postulation is effective in cases where prior information is used to choose an experiment, explanation, model, and many others. The Bayesian approach provides a mechanism for incorporating formally prior information into the design process.

All known approaches based on FIM and statistical paradigms are actively used to investigate a wide range of nonlinear mathematical models. However, the current design theory has severe limitations in studying noisy data. It is still unclear what is occurring with optimal designs when the noise is varied. In particular, the FIM and statistical paradigms cannot answer the question '*Does an optimal design exist for any noisy observations*?'. Moreover, a design problem is formulated so that the input data contain all the necessary noise information [1, 4, 5], but the known optimal designs are noise independent. Such a formulation leads to many more open questions. Similar solutions are considered in the following.

One of the central problems of numerical simulation with discrete data is the proper accounting for the noise effect [28]. The introduced notion of regularization [44] for approximately known initial data indicates that a design problem should be provided for *the constraints of a feasible solution domain and matched with the noise*. After this, the effect of noise on the existence and precision of the solution can be studied. As a result, the conditions for reducing the estimation errors are determined, and the strict noise model requirements, such as a zero mean value, can be excluded from the design problem formulation.

Optimal sensor placements for noisy data under the classical approximation were studied in [31]. Design error determination based on FIM and using noisy data of zero mean was proposed in [4]. In the theory of ill-posed problems, determining reconstruction accuracy in the case of fixed and unimprovable observational errors was developed in [3, 14, 25, 27, 47]. The so-called *a posteriori* error of solving an ill-posed problem was introduced, and an approach to control the reconstruction accuracy was proposed. Note that from the viewpoint of design theory, it is essential that the developed approach requires a sufficiently large sample with a low noise level. Additionally, this approach uses an explicit operator representation of the sought quantity mapping to the observations. Because of this, at present, the *a posteriori* reconstruction error of the is obtained for a single unknown quantity. Small samples, the most informative regions of an observation space, and arbitrary noise effects are not studied. Experimental conditions that reduce the noise effect cannot be identified. All of these issues are key questions for design theory.

The modern trend toward increasing the information content of data processing requires selecting a mathematical model that accounts for the phenomenological properties of the received signals and samples. The observed data can be approximated by direct problem solutions with physically interpretable terms of *a mathematical model*. From a mathematical viewpoint, the involvement of phenomenological models (for example, differential equations) in data processing contributes to the compression of the interpretive basis. This basis determines an approximation of the observed data in the framework of which the interpretation effectiveness at the simulation and numerical levels can be elevated. For such formulations, it is typical to reconstruct *a few unknown model parameters simultaneously* and to consider *the poor sensitivity* to variations in the sought parameters. With this strategy, large samples are not required to increase the informativeness of observed data processing, and numerical implementation of the design should be performed under solution stability control.

Thus, further development of the theory requires introducing an observation model for *small-size samples and design regularization to reconstruct a broad set of quantities*. From a practical viewpoint, the study is motivated by *whether changing the modern design paradigm to the regularization paradigm of ill-posed problems is effective*.

3 A Model Of The Worst Observation Errors

Let us describe the design problem formulation based on the regularization paradigm of ill-posed problems [44]. Consider an abstract signal $y(x,t) \in$ Y that, in a spatial region Q(x) and an interval of time (0, T), satisfies the equation

L_θ $y = f$, in Q(x) · (0, T), (2.1)

where $\theta = \{\theta_k\}_{k=\overline{1,p}} \in$ A is the vector of unknown quantities to be estimated, $f \in$F is a known external excitation (abstract 'driving force'), and L_θ: Y → F denotes a continuous operator acting in a Banach space.

The existence of a unique element $y \in$Y for the given $\theta \in$A and $f \in$F is assumed. There are no restrictions on the form of the operator L_θ. Its choice is defined by the phenomenological models that adequately describe the signal. The signal is thereby parameterized and studied in physically interpretable terms. Such parameterization extends the informativeness of signal processing. However, it has significant mathematical peculiarities.

The operator L_θ, generally, has not a continuous inverse operator L_θ^{-1}. In addition, it is assumed that the domain of the operator L_θ is not a compactum at Y. The dependence of the operator L_θ on its parameter θ indicates that the sought quantity θ is indirectly mapped to the known element f. It is assumed that a finite representation of the operator Z of the corresponding direct problem, $Z\theta = y$, exists but it is unknown. Therefore, the direct problem, $LZ\theta = f$, is not formulated. Because of this, we cannot construct a direct representation of the mapping from the sought quantity space to the observation space and examine the properties of the corresponding operators [3, 14, 27]. The absence of the above-mentioned operators affects the design method and imposes critical conditions on the regularization strategy.

Let a finite sample $y^{(\delta)} = \left\{y_{i,j}^{(\delta)}\right\}_{i=\overline{1,m}}^{j=\overline{1,n}}$, $p \leq m < \infty$, $1 \leq n < \infty$ be given that is burdened by a perturbation function ϵ(including random noise),

$$y_{i,j}^{(\delta)} = y\left(x_i, t_j\right)\Big|_{\bar{\theta}} + \epsilon\left(x_i, t_j\right), i = \overline{1, m}, j = \overline{1, n},$$

2.2

$$\max_{1\le j\le n}\left|\epsilon(x_i, t_j)\right| \le \delta_i,\ i=1,\bar{m}$$

2.3

and that deviates from its prototype $\bar{y}$ by a known amount,

$$\max_{1\le j\le n}\left|y_{i,j}^{(\delta)} - \bar{y}(x_i, t_j)\Big|_{\bar{\theta}}\right| \le \delta_i,\ i=1,\bar{m}$$

2.4

on each statistically independent *m* group of observations, where each group has *n* options of measurements for a design $\Xi = \left\{x_i, t_j\right\}_{i=1,m}^{j=1,\bar{n}}$. The observations are obtained in a region $\mathrm{D}(x,t) \subseteq Q(x) \times (0,T)$ with a finite number of measurements *m* and *n*. Each *i*-th sensor has a known interval of noise concentration $\left\{\pm\delta_i\right\}_{i=1,\bar{m}}$. *The condition of a zero mean value* of the observation errors is not assumed. The prototype $\bar{y}$ is the solution to Eq. (2.1) for the actual ('true') values $\bar{\theta} = \left\{\bar{\theta}_k\right\}_{k=1,\bar{p}}$ of the desired parameters $\bar{y} = L_{\bar{\theta}}^{-1}f$.

Among all design problems, we consider a signal observational design. Namely, the design $\Xi^{(opt)} = \left\{x_i^{(opt)}, t_j^{(opt)}\right\}_{i=1,m}^{j=1,\bar{n}} \in \mathrm{D}$ should be determined such that the error $\nu = \|\bar{\theta}\Big|_{\Xi} - \theta^{(\nu)}\Big|_{\Xi}\|$ of the estimated quantity $\theta^{(\nu)}\Big|_{\Xi}$ is minimized, and the effect of the perturbation function ε is reduced by finding the best observation areas in \mathfrak{D}.

The introduction of *the design error* notion instead of *the estimation error* emphasizes the distinction between design and estimation problems in connection with the type of sample {y}^{\left(\delta \right)} used. The sample should be simulated for the design problem, while the estimation problem processes the experimentally observed data.

To ultimately formulate the design problem, it is necessary to specify how the sample (2.2) with properties (2.3) and (2.4) should be simulated. Equations (2.2) – (2.4) require that the proximity between the reconstructed signal {\left.y\right|}_{{\theta }^{\left(\nu \right)}} and the sample {y}^{\left(\delta \right)} to be within an interval of noise concentration. The upper bound

\delta =\underset{1\le i\le m}{\text{m}\text{a}\text{x}}\left|{\delta }_{i}\right|
2.5

determines this interval. Based on this estimator, in [37] it was proposed that the sample ${y}^{\left(\delta \right)}$ required for the design to be represented by the model with *the worst observation errors*. Such a model assumes that every observation contains one of the two worst-case measurement errors, $\pm \delta$, and all combinations of errors are taken into account regarding the estimated signal ${\left.y\right|}_{\theta }$ (Fig. 1). Accordingly, condition (2.4) is transformed into *the consistency equations with observations*:

$$\underset{1\le j\le n}{\text{m}\text{a}\text{x}}\left|{\left.\stackrel{-}{y}({x}_{i},{t}_{j})\right|}_{\stackrel{-}{\theta }}-{\left.y({x}_{i},{t}_{j})\right|}_{\theta }\pm \delta \right|\le \delta ,i=\overline{1,m} \quad (2.6)$$

Their solvability requires the consistency of the number of matching equations m and the option n with the number of unknowns. Generally, it is necessary to require $m \geq p$ and $n \geq 1$. The solutions ${\left\{{\theta }_{k}\right\}}_{k=\overline{1,p}}$ express the approximation of the prototype ${\left.\stackrel{-}{y}\right|}_{\stackrel{-}{\theta }}$ by the model-calculated signal ${\left.y\right|}_{\theta }$, whose deviation from the sample ${y}^{\left(\delta \right)}$ should belong to the scatter band (2.5). Introducing the value δ into the design formulation is not a burdensome requirement since the absolute norm (2.5) is the standard estimator for each observation.

By introducing the variables ${\nu }_{k}={\stackrel{-}{\theta }}_{k}-{\theta }_{k},k=\stackrel{-}{1,p}$, Eq. (2.6) are reduced to determine the design errors for a given ${\Xi }$ and δ. The reduction is helpful because the transition to the variable ν explicitly recognizes in the observation space the points with $\nu \to \infty$. This property of the consistency equations with observations permits examining the one-to-one correspondence between the desired quantities and a given sample and determining the conditions of the signal observability.

The combination number of the pair $\pm \delta$ in Eq. (2.6) is 2^m (Fig. 1). Therefore, there exist 2^m elements $\tilde{\theta }$ that satisfy Eq. (2.6) and differ from the prototype $\stackrel{-}{\theta }$ by no more than the value ν. In accordance with this, a principle of unique element determination among the 2^m solutions is required. Such a choice expresses the best solution in the frame of all possible measurement error variants. For this purpose, let us introduce a measure $\Omega[\theta]$ of the sought quantity complexity and require its minimization. Accordingly, the solution to the variational problem

$$\widehat{\theta }=Arg\underset{\theta \in {A}^{\left(\nu \right)}}{\text{i}\text{n}\text{f}}{\Omega }\left[\theta \right],{A}^{\left(\nu \right)}=\{\theta :\|\stackrel{-}{\theta }-\tilde{\theta }\|\le \nu \}\subset A, \quad (2.7)$$

determines the value $\widehat{\theta }$ that should satisfy the consistency equations with observations (2.6) for the current design ${\Xi }$ and a given δ. The reduction in Eq. (2.6) to the variable ν by introducing the parameter ${\theta }^{\left(\nu \right)}=\stackrel{-}{\theta }-\nu$ simplifies the solution to the problem (2.7).

It is proposed to define the estimate $\widehat{\theta }$ as the solution with *a guaranteed design error* to emphasize the constrained solution under the maximum permissible observation errors in the consistency corridor $\pm 2\delta$ that is generated by the scatter band $\pm \delta$.

The measure $\Omega[\theta]$ is a continuous and nonnegative functional in which the domain involves the sought quantities θ. Its choice depends on the functional properties of the parameter space A and is one of the standard norms for a Banach space. Formally, the minimum value of $\Omega[\widehat{\theta }]$ is less than the measure $\Omega[\stackrel{-}{\theta }]$ of the actual parameters $\stackrel{-}{\theta }$. Hence, a regularized design will define optimal observations that give better estimation accuracy than the FIM and statistical designs due to direct accounting for design errors.

To complete the solution construction, the sought optimum design should minimize a specific criterion. Since the proposed design is based on the formulation of the consistency equations with observations where the actual $\stackrel{-}{\theta }$ and reconstructed ${\theta }^{\left(\nu \right)}$ quantities are compared, the sought optimal design ${{\Xi }}^{\left(opt\right)}$ can be defined as the direct minimization of the residual with the actual quantity $\stackrel{-}{\theta }$:

$${{\Xi }}^{\left(opt\right)}=Arg\underset{x,t\in \mathfrak{D}}{ \text{i}\text{n}\text{f}}\|\stackrel{-}{\theta }-\widehat{\theta }\| \tag{2.8}$$

Minimizing design errors should be performed for every ${\Xi }\in\mathfrak{D}$ concerning the element $\widehat{\theta }$ with the minimum measure Ω. By this means, the desired optimal solution is constructed as *the regularized minimum value of the residual with the prototype*. This indicates that design is reduced to the bilevel problem minimizing both the sought quantity complexity and the norm of the design errors.

Generally, the optimal design by the proposed regularization (2.6) – (2.8) includes four steps.

I. Introducing two quantities – the prototype $\stackrel{-}{\theta }$ and the error burdened ${\theta }^{\left(\nu \right)}$ with the deviation ν from $\stackrel{-}{\theta }$ – and determining, analytically or numerically, the function y that satisfies Eq. (2.1) for a given θ.
II. Obtaining the 2^m solutions to the consistency equations with observations on a given ${\Xi }$ relative to the design error ν to determine parameters $\tilde{\theta }$ that differ from the actual quantities $\stackrel{-}{\theta }$ by no more than the value v among all options of the worst approximation of the prototype $\overline{y}$ in the consistency corridor $\pm 2\delta$.
III. Minimizing the complexity measure ${\Omega }\left[\tilde{\theta }\right]$ on a matching subset to obtain the reconstruction $\widehat{\theta }$ with minimum numerical variations.
IV. Constructing the optimal design ${{\Xi }}^{\left(opt\right)}$ with a minimal measure of the design errors v in a given design region $\mathfrak{D}$.

The details of the algorithmic implementations are described below, examining classical evolutionary signals.

Ultimately, *the model of the worst observation errors* proposes considering the design problem as *the solution to the consistency equations*. From the viewpoint of ill-posed problems, their solutions require stabilization to account for model discretization and noise effects. For this purpose, the determination of *the guaranteed design error*, rather than examining arbitrary estimations, suggests inspecting only those that are bounded due to the minimization of the complexity measure $\Omega[\theta]$. *Combining the solution to the consistency equations with the guaranteed design error minimization* ensures the solution adjustment to the noise level and constrains a feasible solution range. This stabilization follows *the Tikhonov regularization paradigm* [44]. However, its implementation described above differs from *the Tikhonov algorithm*, which requires the determination of *a regularization parameter* (see [30] for the design problem). The absence of this parameter in Step III offers advantages in noisy data processing [35]. In contrast to the FIM and statistical optimality criteria, *the proposed approach minimizes the design error directly*.

Thus, the proposed *regularized-minimal-residual design* is the best reconstruction accuracy with the minimum complexity of the sought quantities among all possible variants of the worst observation errors simulated at every measurement.

4 Regularized Design Features

Note the essential features of the optimal design obtained under the formulation (2.6)– (2.8).

First, from Eq. (2.6), it follows that the model-calculated signal ${\left.y\right|}_{\theta }$ should be matched with the prototype ${\left.\stackrel{-}{y}\right|}_{\stackrel{-}{\theta }}$ in *the consistency corridor* $\pm 2\delta$ (Fig. 1). The doubling of the value δ indicates that the absence of *a priori* information on the nature of sampling bias requires accounting for the additional scatter band $\pm \delta$ at every observation. The model of the worst observation errors does not exclude some of the observations that may be the actual signal regarding which the scatter band $\pm \delta$ should be considered (Fig. 1).

Second, Eq. (2.6) requires considering *the 2^m options of the worst observation fitting*. Each option should be accounted for to cover all possible cases of the actual signal approximation by the model-calculated function (Fig. 1). The heterogeneity of the consistency equations with observations is significant. Accounting for the interval of the noise concentration is crucial to the problem formulation, which should reflect the existence of the deviation between a finite sample and its prototype.

In particular, note that the option ${\left.y\right|}_{\theta }=\overline{y}$ is *a trivial solution to* Eq. (2.6). This case corresponds to the interpolation of the prototype $\overline{y}$ and does not express the worst-case approximation. Because of this, the possible samples without the observation errors are excluded from the design construction. Optimal designs for the latter case are obtained by the approximation in the consistency corridor $\pm 2\delta$ for which $\delta \to 0$ holds.

In practice, the perturbation function $\epsilon (x,t)$ is usually postulated with *a zero mean value.* This requires appropriate samples for which $\underset{n\to \infty }{\text{lim}}{\left\{{y}_{i,j}^{\left(\delta \right)}\right\}}_{i=\stackrel{-}{1,m}}^{j=\stackrel{-}{1,n}}={\left\{{\overline{y}}_{i}\right\}}_{i=\stackrel{-}{1,m}}$. For such a formulation, the design problem express the interpolation case and does not facilitate the prototype approximation in a nonzero corridor. After this, the generalization of the results obtained is usually proposed to the arbitrary noise distribution. By this means, the solution to the approximation problem is changed to the interpolation problem. The transition from the signal approximation to its interpolation is a substantial limitation in the design formulation. In the framework of the developed approach, the optimal design is determined under a small noisy sample and the 2^m variants of a prototype approximation into the consistency corridor $\pm 2\delta$.

Third, the model of the worst observations (2.5) and (2.6) facilitates studying a noise effect *without sample numerical modeling* by the Monte Carlo method [5]. This allows for an increase in the degree of inference generality, extends the resulting design to a more general case, and identifies the specific features of the model rather than obtaining partial solutions.

Fourth, *the requirement to minimize the stabilizer* is crucial in the framework of nonunique roots of the consistency equations. By this means, the filtration of roots and numerical stabilization are ensured. In the following, we demonstrate how this requirement improves the solution precision even for constant model parameters for which a domain of feasible solutions is not extended.

The observation models (2.2) – (2.4) convey a broad class of perturbation sources and facilitate the study of many practically essential cases of observations. One of the important cases is *a small sample*, and another is *a nonzero mean of the noise distribution*. The models (2.2) – (2.4) express direct measurements of a state function. If necessary, any other representations can be contemplated, for example, when the observables are the functional of the prototype $\overline{y}$. The observation model does not require any hypothesis regarding the noise distribution. The expected value and covariance matrices are not specified. The formulation can easily be changed from the general noise bound estimate δ to the bounds of each observation groups ${\{\delta }_{i}{\}}_{i=\stackrel{-}{1,m}}$. This is helpful for the simulation where the observations have small and large magnitudes.

In addition to estimator (2.5), other norms of the observation space can be considered, for example, the *rms*-estimator $\delta_{rms} = \left(\sum_{i=1}^{m} \delta_i^2/m\right)^{1/2}$. For the latter, the consistency equations

$$\left\{\sum_{j=1}^{n}[\bar{y}(x_i,t_j)\big|_{\bar{\theta}} - y(x_i,t_j)\big|_{\theta} \pm \delta_{rms}]^2/n\right\}^{1/2} \le \delta_{rms}, i=\overline{1,m} \qquad (3.1)$$

corresponds to it. Similarly, *other consistency equations with observations* can be introduced to account for the known properties of the perturbation function $\epsilon (x,t)$.

The design under schemes (2.6) – (2.8) and (2.7) – (3.1) can be supplemented with *any qualitative or quantitative information* about observations and unknowns. A similar possibility provides a mechanism for formally incorporating prior information into the design process when the improvement of the solution accuracy is studied.

Essentially, regularization (2.6) – (2.8) or (2.7) – (3.1) is provided *without a regularization parameter* [44]. It is necessary to distinguish between *the Tikhonov regularization paradigm* and its numerical implementation with *the regularization parameter* as the basic idea from its one possible realization. Different implementations can often be proposed within the same idea that are suitable for the problem in question. Regarding the Tikhonov regularization paradigm, the algorithm without the regularization parameter was proposed in [20, 35, 45]. This algorithm is applied here to exploit its advantages [35], which ensures better solution properties when a few unknowns are reconstructed simultaneously. It is theoretically essential to understand how the solution to the design problem behaves when the number of the sought quantities $p > 1$ and $\delta \to 0$. The question of solution behavior is significant because the set of unknowns should correspond to a system of equations, not a single equation. The well-known complexity of the regularization parameter determination is excluded for the proposed regularization (2.6) – (2.8). The nonconvexity of the nonlinear programming problem (2.6) – (2.8) can be solved by moving to the scheme (2.7) – (3.1) and choosing convex stabilizers [35, 36].

The key feature of the proposed design is the investigation of the properties of the consistency equations with observations. By this means, the existence and uniqueness of the design can be studied. There may be initial data for which the solvability of the simultaneous equations is absent. For a given $\stackrel{-}{\theta}$ and ${\Xi}$, the solvability of the consistency equations is determined by the value $\delta \neq 0$. Here, the existence of such ${{\Xi}}^{*}$ is not excluded, for which *the solution to the consistency equations is absent*. This indicates that for some $\delta \neq 0$ there exists such a design ${{\Xi}}^{*}$ that will not allow for the reconstruction. This case is related to the inability to approximate the prototype in the consistency corridor $\pm 2\delta$ under a certain placement of the sensor. The solution absence and the multimodal character of the design criterion are not the aggravating circumstances of the proposed design regularization. The investigation of similar cases facilitates revealing the singularities of observational designs.

5 A Regularized Solution To A Design Problem

According to the regularized design (Section 2), the best observations are sought as a solution to a nonlinear programming problem. From the view point of abstract theory, substantiating the existence and uniqueness has known difficulties. However, it is possible to obtain solvability conditions of nonlinear formulations for the specific processes studied and determine local and global extrema of the design criterion.

From this viewpoint, consider the classical evolutionary models with constant parameters. Their investigations began in [42] and continued in a series of studies [7, 9, 12, 13, 16, 22, 33, 34] and many others. If the sought quantities $\theta = {\left\{{\theta }_{k}\right\}}_{k=\overline{1,p}}$ belong to the Euclidean space, and *the signal in question depends on one variable*, $y = y(x)$, $x \in Q$, then Steps I – IV described in Section 2 are executed as follows.

Step I is a solution to a direct problem:

$$y={L}_{\theta }^{-1}f$$
4.1

In general, its obtaining can include a wide range of simulation/solution errors that increase the impact of the perturbation $\epsilon (x,t)$. In the present investigation, such errors are excluded because of research with the precisely known prototype $\overline{y}$. This is important in theoretical terms, as maximum attention will be paid to the peculiarities of the design, which does not depend on a solution method of an original direct problem.

For the constant parameters, ${\theta }_{k}=const$, the design ${\Xi }=\{{x}_{i}\}_{i=\overline{1,m}}\in \mathfrak{D}\subseteq Q$, $n = 1$, is sufficient. The influence of the specifying of the design region $\mathfrak{D}$ on the optimal solution is discussed below for the concrete model L. The design error ν is introduced by the replacement ${{\theta }_{k}^{\left(\nu \right)}=\stackrel{-}{\theta }}_{k}-{\nu }_{k},k=\stackrel{-}{1,p}$.

Step II is the determination of the design error ν from the consistency equations for the current design. If $m = p$, then the consistency equations are reduced to the form:

$${\left.\stackrel{-}{y}\left({x}_{i}\right)\right|}_{\stackrel{-}{\theta }}-{\left.y\left({x}_{i}\right)\right|}_{{\theta }^{\left(\nu \right)}}={{\Delta }}_{l},i=\overline{1,m},l=\overline{\text{1,2}},$$
4.2

where ${\Delta }=(+2\delta ,-2\delta {})^{\text{T}}$. For definiteness, we refer to ${\Delta }$ as *the consistency corridor matrix*.

The solutions to Eqs. (4.2) should be found for the 2^m variants of the worst prototype fits (Fig. 1). Denote these solutions as $\tilde{\nu }={\left\{{\tilde{\nu }}_{k,l}\right\}}_{k=\stackrel{-}{1,p}}^{l=\stackrel{-}{1,{2}^{m}}}$ to reflect selecting from the total set of estimates θ a subset $\tilde{\theta }$ whose elements are guaranteed *to be coherent with the noise level* in the context of the consistency corridor matrix ${\Delta }$. As a result, we obtain the set ${\tilde{\theta }=\{\stackrel{-}{\theta }}_{k}-{\tilde{\nu }}_{k,l}{\}}_{k=\stackrel{-}{1,p}}^{l=\stackrel{-}{1,{2}^{m}}}$ whose elements are assigned a measure ${\Omega }$. For the overdetermined samples, $m > p$, the condition (2.6) or (3.1) should be used.

Step III is the minimization of the measure ${\Omega }\left[\tilde{\theta }\right]$ *among all matching options* $l=\stackrel{-}{1,{2}^{m}}$. The result is the best reconstruction $\widehat{\theta }$ with the minimum estimation complexity measure Ω (in other words, the stabilizer). For the sought quantities belonging to the Euclidean space, the stabilizer is chosen as the spherical norm of the residuals from the actual quantities:

$$\widehat{\nu }\left(x\right)=Arg\underset{1\le l\le {2}^{m}}{\text{m}\text{i}\text{n}}{\sum }_{k=1}^{p}{\left[{\stackrel{-}{\theta }}_{k}-{\tilde{\nu }}_{k,l}\left(x\right)\right]}^{2} , x\in \mathfrak{D}. \quad (4.3)$$

The value $\widehat{\nu }$ of the matching subset $\tilde{\nu }$ is the element with the smallest measure Ω. This selection of all possible fits over the worst-case approximation into the given noise interval of concentration can limit numerical variations in the parameters sought. From this point of view, the value $\widehat{\nu }$ can be defined as *the guaranteed design error* on the set of the worst approximation cases. Other forms of the measure $\Omega[\theta]$, such as the absolute norm, can also be considered to study various mathematical aspects of the approximation.

Step IV is the minimization of the function $\widehat{\nu }\left(x\right)$ under the *R*-optimal criterion:

$$\Xi^{(R)} = Arg \min_{x\in\mathfrak{D}}\left\{ \sum_{k=1}^{p} [\hat{\nu}_k(x)]^2/p \right\}^{1/2} \qquad (4.4)$$

or the *M*-optimal criterion:

$${\Xi }^{\left(M\right)}=Arg\underset{x\mathfrak{D}}{\text{m}\text{i}\text{n}}\underset{1\le k\le p}{\text{m}\text{a}\text{x}}\left|{\widehat{\nu }}_{k}\left(x\right)\right|$$
to determine the best observational points following a different design error measure.

Further details of the proposed regularization are described in the following. Note that more general functional dependencies of the sought parameters, including space-, time-, and state-dependent object properties, can also be considered. The regularized formulation remains the same, and the design errors should satisfy the consistency equations and minimize the stabilizer. For such cases, each of the m samples requires a nonunique number of measurements, $n > 1$.

6 Optimal Designs Of Exponential Regressions

Determine the regularized design for the commonly known class of exponential regressions. This class of model functions is widely used in practice, for example, in the life sciences. It describes typical growth dynamics, such as exponential, *S*-shaped, spiking and other object behavior. Classical model functions are chosen because their features are well studied. Therefore, the effectiveness of the proposed design approach is maximally expressed, and the mathematical features of the resulting solutions are revealed to define further research directions.

Theoretically important, the model functions under study will express explicit solutions to Eq. (2.1). This eliminates in design construction both the affect of the direct problem error and the possible effects of the solution discretization.

This Section aims to reveal *what occurs in the observational design, accounting for the effect of noise.* From a theoretical viewpoint, the following essential issues are investigated: (i) the character of the noise influence on the design error distribution, (ii) factors that can reduce the noise effect, and (iii) peculiarities of the optimal design that are generated by the noise. From a practical viewpoint, the focus is on how the 2^m options of the observation fitting (Fig. 1) are applied in the optimal design. Everywhere below the sample with the minimum size ($m = p$) is considered.

6.1 One-parameter model

Consider the case of a nonlinear model function that facilitates analytically determining the optimal observation and its design error. Such a study represents each step of the regularized solution to the design problem and reveals its main features in detail.

Let the following exponential function describes the received signal:

$y(x)|_\theta = Y_0 \exp(\theta x)$, $-{\infty }<x<+{\infty }$ (5.1)

with a known value ${\left.y\right|}_{x=0}={Y}_{0}\ne 0$ and an unknown parameter $\theta \ne 0$ that is to be estimated. The optimal one-point design ${{\Xi }}^{\left(M\right)}= \left\{{x}^{\left(opt\right)}\right\}$ is sought in design space R^1 using the observations that satisfy (2.2) – (2.4). The desired solution is constructed in four steps (Section 4).

Step I is simple since the model function (5.1) is a solution to the well-known differential equation. Accordingly, the signals ${\left.\stackrel{-}{y}\right|}_{\stackrel{-}{\theta }}$ and ${\left.y\right|}_{{\theta }^{\left(\nu \right)}}$ with the parameters $\stackrel{-}{\theta }$ and ${\theta }^{\left(\nu \right)}=\stackrel{-}{\theta }-\nu$, which are needed to obtain the consistency equations, are expressed by the function (5.1) directly. Their explicit types eliminate the influence of the prototype simulation errors on the design.

Step II, using the functions ${\left.\stackrel{-}{y}\right|}_{\stackrel{-}{\theta }}$ and ${\left.y\right|}_{{\theta }^{\left(\nu \right)}}$, gives the following consistency equations (4.2) relative to the design error $\nu =\stackrel{-}{\theta }-{\theta }^{\left(\nu \right)}$:

$${Y}_{0}\text{exp}(\stackrel{-}{\theta }x)\left[1-\text{exp}(-\nu x)\right]=\pm 2\delta .$$
5.2

Eq. (5.2) has two nonzero solutions. Regarding the relative error $\mu =\nu /\stackrel{-}{\theta }$ *the elements from the matching subset* are as the following roots:

$${\tilde{\mu }}_{1}=1-\frac{1}{\stackrel{-}{\theta }x}\text{l}\text{n}\left[\text{exp}\left(\stackrel{-}{\theta }x\right)+2\delta /{Y}_{0}\right],-{\infty }<x<+{\infty },$$
5.3

$${\tilde{\mu }}_{2}=1-\frac{1}{\stackrel{-}{\theta }x}\text{l}\text{n}\left[\text{exp}\left(\stackrel{-}{\theta }x\right)-2\delta /{Y}_{0}\right],-{\infty }<x<+{\infty }$$
5.4

Step III requires choosing *the guaranteed design error* $\widehat{\mu }$ in design space R^1 between two solutions, $\tilde{\mu }={\tilde{\mu }}_{1}\left(x\right)$ or $\tilde{\mu }={\tilde{\mu }}_{2}\left(x\right)$, that minimize the stabilizer ${\Omega }\left[\tilde{\theta }\right]= {\stackrel{-}{\theta }}^{2}[1-\tilde{\mu }{\left(x\right)]}^{2}$. Accordingly, if $\stackrel{\sim}{\theta }<0$, then for $x < 0$, the stabilizer Ω has the minimum on the branch $\widehat{\mu }={\tilde{\mu }}_{2}\left(x\right)$, and for $x > 0$, the stabilizer Ω has the minimum value on the branch $\widehat{\mu }={\tilde{\mu }}_{1}\left(x\right)$. If $\stackrel{\sim}{\theta }>0$, then $\widehat{\mu }= {\tilde{\mu }}_{1}\left(x\right)$ for $x < 0$, and $\widehat{\mu }={\tilde{\mu }}_{2}\left(x\right)$ for $x > 0$.

Step IV requires determining the global minimum of the relative design error, ${\mu }^{\left(M\right)}=\underset{x}{\text{m}\text{i}\text{n}}\left|\widehat{\mu }\left(x\right)\right|$. In design space R^1, the global minimum is determined by the asymptotic decay of functions (5.3) and (5.4). From this, it follows that for $\stackrel{-}{\theta }<0$ the global minimum is ${\left.{\mu }^{\left(M\right)}\right|}_{{x}^{\left(opt\right)}\to -\infty }=0$, and for $\stackrel{-}{\theta }>0$ is ${\left.{\mu }^{\left(M\right)}\right|}_{{x}^{\left(opt\right)}\to +\infty }=0$. The sought optimal design is

$${\left.{\Xi }^{\left(M\right)}\right|}_{x\in {\text{R}}^{1}}=\left\{\begin{array}{c}{x}^{\left(opt\right)}\to -{\infty },\stackrel{-}{\theta }<0\\ {x}^{\left(opt\right)}\to +{\infty },\stackrel{-}{\theta }>0\end{array}\right\}$$

In addition to the global minimum, it is necessary to consider the existence of a local minimum. The function ${\mu }_{1}\left(x\right)$ has the local minimum $\widehat{\mu }$, whose value is determined below. The position ${x}^{*}=0$ is unidentifiable due to the violation of the one-to-one correspondence.

From (5.3), the desired position ${x}^{\left(opt\right)}$ from R^1 is defined for every $\overline{\theta }\ne 0$ as the expression

$${x}^{\left(opt\right)}=\frac{1}{\stackrel{-}{\theta }\widehat{\mu }}\text{l}\text{n}\left[1-\widehat{\mu }\right] \quad (5.5)$$

where the value $\widehat{\mu }$ satisfies the equation

$$\widehat{\mu }{\left[1-\widehat{\mu }\right]}^{\frac{1-\widehat{\mu }}{\widehat{\mu }}}=2\delta /\left|{Y}_{0}\right| \quad (5.6)$$

If $\stackrel{-}{\theta }<0$, then the local minimum becomes global, and ${\mu }^{\left(M\right)}= \widehat{\mu }$. If $\stackrel{\sim}{\theta }>0$, then the global minimum in the design region $\mathfrak{D} = \{0 < x \le \mathcal{l}<\mathcal{\infty }\}$ will be located at the end of the observation interval, ${x}^{\left(opt\right)}=\mathcal{l}$. Thus, *the guaranteed design error* is given by

$${\mu }^{\left(M\right)}=\left\{\begin{array}{cc}\widehat{\mu },& \stackrel{-}{\theta }<0,\mathcal{l}\ge {x}^{\left(opt\right)};\\ 1-\frac{1}{\stackrel{-}{\theta }\mathcal{l}}\text{l}\text{n}\left[\text{exp}\left(\stackrel{-}{\theta }\mathcal{l}\right)-2\delta /{Y}_{0}\right],& \stackrel{-}{\theta }<0,\mathcal{l}<{x}^{\left(opt\right)},\stackrel{-}{\theta }>0,\mathcal{l}>0,\\ & \end{array}\right.$$

which corresponds to the optimal design

$${\left.{{\Xi }}^{\left(M\right)}\right|}_{x\in [0,\mathcal{l}]}=\left\{\begin{array}{c}{x}^{\left(opt\right)},\\ \\ \mathcal{l},\\ \end{array}\begin{array}{c}\stackrel{-}{\theta }<0,\mathcal{l}\ge {x}^{\left(opt\right)}\\ \\ \stackrel{-}{\theta }<0,\mathcal{l}<{x}^{\left(opt\right)},\stackrel{-}{\theta }>0,\mathcal{l}>0\\ \end{array}\right\}$$

Let us describe the features of the obtained solution.

Expressions (5.5) and (5.6) demonstrate that *the optimal design depends not only on the sought parameter* $\stackrel{-}{\theta }$ *but also on the noise scatter band δ*. Notably, the optimal solutions (5.5) and (5.6) for a given δ are constructed under the bilevel scheme $\delta \Rightarrow \widehat{\mu }\Rightarrow {x}^{\left(opt\right)}$. Additionally, Eqs. (5.5) and (5.6) show the existence of the factor determining the behavior of the design accuracy and solution existence. The value $\mathcal{R}=2\delta /\left|{Y}_{0}\right|$ expresses that Eq. (5.6) has a solution only for $\mathcal{R}<1$. Accordingly, the value ${\delta }^{*}= \left|{Y}_{0}\right|/2$ determines the upper bound of the noise, and when $\delta \ge {\delta }^{*}$, the desired parameter θ cannot be reconstructed. Since such experimental conditions exist, it is suggested to introduce a factor that expresses conditions under which there are no estimates and indicates model characteristics to reduce noise effects. The above-determined factor $\mathcal{R}$ describes the model properties that guarantee the solution existence for a given δ and therefore, it can be called *a reduction factor of noise effects.*

Furthermore, from (5.5) and (5.6), it follows that the values x*=0 and ${x}^{*}=1/\left|\stackrel{-}{\theta }\right|$ determine the segment $[0, 1/\left|\overline{\theta }\right|]$ for $\overline{\theta }<0$ and $[-1/\overline{\theta }, 0]$ for $\overline{\theta }>0$, where it is not recommended to observe the signal $y(x)$ because the samples will give rise to inadmissible design errors, $\mu \gg 1$. For any $\delta \neq 0$, the optimal observation is displaced at the position $\left|{x}^{\left(opt\right)}\right|>1/\left|\overline{\theta }\right|$. If $\delta \to 0$, then $\left|{x}^{\left(opt\right)}\right|\to 1/\left|\overline{\theta }\right|+0$. The offset is determined by the value δ as well as by choice of the design space. The obtained result demonstrates that identifying *the best and worst regions* in the observation space is an essential feature of an experimental design.

Compare the results obtained with the investigations of the input-output model (5.1) that have been conducted by many authors using the FIM paradigm.

The commonly known design ${{\Xi }}_{1}=\{1/\stackrel{-}{\theta }\}$ was determined in [7]. As shown above, a similar design is valid in the case of a sufficiently small noise level. From (5.5), it is easy to define the corresponding relative design error ${\mu }^{\left(m\right)}$. Then from (5.6), it follows that the upper boundary of the noise should be $\delta < 0.02\,Y_0$. The latter expresses the condition that describes the boundary of the design ${{\Xi }}_{1}$ correct application. As seen, the interval is very narrow. Outside this interval, the estimation will give significant errors.

Another well-known design ${{\Xi }}_{2}=\{1.6/\stackrel{-}{\theta }\}$ was obtained in [9, 13, 16, 33]. This design is more precise in terms of noise impact than the previous Ξ_1. The design ${{\Xi }}_{2}$ expresses the approximately optimal position for $\delta < 0.18 Y_0$. This condition determines the boundary of the design ${{\Xi }}_{2}$ correct application. For such a case, the best relative design error is ${\mu }^{\left(m\right)} < 0.639$. As seen, the design is the approximated solution to the optimal design problem for small noise dispersion.

As demonstrated above, it is essential that for $\delta \neq 0$, the position ${x}^{*}= 1/\left|\stackrel{-}{\theta }\right|$ defines not an optimal sensor placement but the boundary of the unsatisfactory estimation zone. In this regard, the optimal observation for the case $\delta \neq 0$ is correctly positioned with a positive offset from the point ${x}^{*}=1/\left|\stackrel{-}{\theta }\right|$.

It is very often claimed that for the signal (5.1), there always exists a locally optimal one-point design. In particular, the designs ${{\Xi }}_{\text{1,2}}$ do not express the constraints on the noise level and they are formally valid for every δ. The existence of the value δ* demonstrates that this statement is not correct.

Thus, the solutions obtained under the regularized paradigm describe all the factors that determine the value and behavior of the design error. *The noise significantly affects the design*, and *the reduction factor* reflect the condition for reducing the noise impact. Substantial distortions in the optimal solution can occur if the deviation between the actual signal and its finite sample is not considered. The solution to the optimal design problem does not exist for all noisy data. *A threshold of noise* exists that causes the solution to the design problem to be absent at every point of the observation space. The new designs obtained comprise the previously known optimal solutions as individual cases. The boundaries of the applications of the known optimal designs are determined.

6.2 Two-parameter model

An essential part of the optimization of signal observations is to determine *the design error map* concerning the sensor's positions. For this purpose, let us show how the regularized solution (4.1) – (4.4) can express *the best and worst observation regions*. This option is not available with other known design approaches.

Consider the following exponential function:

$$y(x{\left.)\right|}_{{\theta }_{\text{1,2}}}={\theta }_{1}\text{exp}({\theta }_{2}x) , -{\infty }<x<+{\infty }, \quad (5.7)$$

described an observed signal. Here, the unknown parameters ${\theta }_{\text{1,2}}\ne 0\in R^2$ are to be estimated. The optimal design $\Xi^{(opt)} = \{{x}_{k}^{\left(opt\right)}\}_{k=\overline{\text{1,2}}}$ is sought in the design space R^2, using the sample (2.2) – (2.4).

Step I consists in using the explicitly known prototype (5.7) to define the functions ${\left.\stackrel{-}{y}\right|}_{{\stackrel{-}{\theta }}_{\text{1,2}}}$ and ${\left.y\right|}_{{\theta }_{\text{1,2}}^{\left(\nu \right)}}$ with the parameters ${\stackrel{-}{\theta }}_{\text{1,2}}$ and ${\theta }_{\text{1,2}}^{\left(\nu \right)}= {\stackrel{-}{\theta }}_{\text{1,2}}-{\nu }_{\text{1,2}}$. Based on the functions {\left.\stackrel{-}

{y}\right|}_{{\stackrel{-}{\theta }}_{\text{1,2}}} and {\left.y\right|}_{{\theta }_{\text{1,2}}^{\left(\nu \right)}}, the consistency equations are constructed relative to the design errors {\nu }_{\text{1,2}}={\stackrel{-}{\theta }}_{\text{1,2}}-{\theta }_{\text{1,2}}^{\left(\nu \right)} for the two-point design as the system of equations:

{\stackrel{-}{\theta }}_{1}\text{exp}({\stackrel{-}{\theta }}_{2}{x}_{i})-[{\stackrel{-}{\theta }}_{1}-{\nu }_{1,l}^{}\left]\text{exp}[\right({\stackrel{-}{\theta }}_{2}-{\nu }_{2,l}^{}\left){x}_{i}\right]={{\Delta }}_{i,l},i=\text{1,2},l=\overline{\text{1,4}}.
5.8
The following consistency corridor matrix describes the worst observations:

{\Delta }=\left(\begin{array}{cc}+2\delta ,& +2\delta ,\\ +2\delta ,& -2\delta ,\end{array}\begin{array}{cc} -2\delta ,& -2\delta \\ +2\delta ,& -2\delta \end{array}\right).
Step II gives the solutions to the Eqs. (5.8) regarding the relative errors {\mu }_{k,l}={\nu }_{k,l}^{}/{\stackrel{-}{\theta }}_{k},k=\text{1,2},l=\stackrel{-}{\text{1,4}} as follows

{\tilde{\mu }}_{1,l}=1-\left[1-\text{exp}(-{\stackrel{-}{\theta }}_{2}{x}_{1})\right]{\left[\frac{1-{{\Delta }}_{1,l}/{\stackrel{-}{\theta }}_{1}\text{ exp}(-{\stackrel{-}{\theta }}_{2}{x}_{1})}{1-{{\Delta }}_{2,l}/{\stackrel{-}{\theta }}_{1}\text{ exp}(-{\stackrel{-}{\theta }}_{2}{x}_{2})}\right]}^{\frac{{x}_{1}}{{x}_{2}-{x}_{1}}},l=\overline{\text{1,4}},
5.9
{\tilde{\mu }}_{2,l}=\frac{1}{{\stackrel{-}{\theta }}_{2}({x}_{2}-{x}_{1})}\text{ln}\left[\frac{1-{{\Delta }}_{1,l}/{\stackrel{-}{\theta }}_{1 }\text{ exp}(-{\stackrel{-}{\theta }}_{2}{x}_{1})}{1-{{\Delta }}_{2,l}/{\stackrel{-}{\theta }}_{1} \text{exp}(-{\stackrel{-}{\theta }}_{2}{x}_{2})}\right],l=\overline{\text{1,4}}
5.10
Step III requires determining among four options {\left\{{\tilde{\mu }}_{k,l}^{}\right\}}_{k=\text{1,2}}^{l=\stackrel{-}{\text{1,4}}} the best one, which is named as *the guaranteed design error* \widehat{\mu }={\left\{{\widehat{\mu }}_{k}^{}\right\}}_{k=\text{1,2}}^{}. The latter is defined for each {\Xi }=\left\{{x}_{\text{1,2}}\right\} by minimizing the stabilizer (4.3)

\widehat{\mu }\left({x}_{\text{1,2}}\right)=Arg\underset{1\le l\le 4}{\text{min}}{\sum }_{k=\text{1,2}}{\left\{{\stackrel{-}{\theta }}_{k}[1-{\tilde{\mu }}_{k,l}\left({x}_{\text{1,2}}\right)]\right\}}^{2}
The optimal design {{\Xi }}^{\left(R\right)} will be determined based on the minimization of the root-mean-square norm of the guaranteed design error

{{\Xi }}^{\left(R\right)}=Arg \underset{{x}_{\text{1,2}}}{\text{m}\text{i}\text{n}}{ \mu }_{rms}\left({x}_{\text{1,2}}\right),
where

$$\mu_{rms}(x_{1,2}) = \left\{ [\hat{\mu}_1^2(x_{1,2}) + \hat{\mu}_2^2(x_{1,2})]/2 \right\}^{1/2}.$$

Step IV realization depends on the different cases of the parameters ${\overline{\theta }}_{\text{1,2}}$. Having omitted the cumbersome but straightforward calculations, let us describe the main functional features of the design errors (5.9) and (5.10).

5.2.1 The case ${\overline{\theta }}_{\text{1,2}}> 0$

If the design region has not are the constraints, then the global minimum is defined by the asymptotic properties of the function ${\widehat{\mu }}_{\text{1,2}}\left({x}_{\text{1,2}}\right)$.

For the positive value ${\overline{\theta }}_{2}> 0$, the asymptotic behavior is expressed as follows. If ${x}_{\text{1,2}}\to -\infty$, then ${\widehat{\mu }}_{1}=1\pm 2\delta /{\stackrel{̄}{\theta }}_{1}$, ${\widehat{\mu }}_{2}=1$. If ${x}_{\text{1,2}}\to +\infty$, then ${\widehat{\mu }}_{\text{1,2}}=0$. For the case ${x}_{1}=0$ and ${x}_{2}\to +\infty$, the design errors have the asymptotic values

$${\left.{\widehat{\mu }}_{1}\right|}_{{x}_{1}=0,{x}_{2}\to {\infty }}=2\delta /{\stackrel{̄}{\theta }}_{1},{\left.{\widehat{\mu }}_{2}\right|}_{{x}_{1}=0,{x}_{2}\to {\infty }}=0 \tag{5.11}$$

As a result, in the absence of any constraints on design space R^2 the global minimum of the function ${\mu }_{rms}\left({x}_{\text{1,2}}\right)$ belongs to the first quadrant (${x}_{\text{1,2}}>0$) and ${\left.{\mu }^{\left(R\right)}\right|}_{{x}_{\text{1,2}}\to +\infty }=0$. The third quadrant (${x}_{\text{1,2}}<0$) is the worst area from the viewpoint of asymptotic behaviour. Into the second (${x}_{1}<0,{x}_{2}>0$) and fourth (${x}_{1}>0,{x}_{2}<0$) quadrants the best measurements exist only for the limited interval of ${x}_{2}$. From (5.9), (5.10), it follows that if ${x}_{2}>0$ and ${x}_{1}\to -\infty$ or ${x}_{2}<0$ and ${x}_{1}\to +\infty$, then ${\widehat{\mu }}_{1}\to - \infty$.

In restricting the further study to only the first quadrant, the determination of the local minima of the functions (5.9), (5.10) will be made for the case ${x}_{2}>{x}_{1}$. The option ${x}_{1}>{x}_{2}$ is symmetric. It follows from the condition ${\mu }_{rms}\left({x}_{\text{1,2}}\right)= {\mu }_{rms}\left({x}_{\text{2,1}}\right)$.

For ${x}_{\text{1,2}}\ge 0$ the largest values of the function ${\mu }_{rms}\left({x}_{\text{1,2}}\right)$ are allocated on the curve

$$\chi \left({x}_{1}\right)=Arg\underset{0<{x}_{2}<{\infty }}{\text{m}\text{a}\text{x}}{\mu }_{rms}\left({x}_{\text{1,2}}\right) \tag{5.12}$$

We omit a cumbersome description of the general form $\chi \left({x}_{1}\right)$ for the arbitrary ${x}_{1}\ge 0$, however, indicate that this curve passes through the point $(0,{\chi }_{0})$. The value ${\chi }_{0}$ is the solution to the equation

$$\left[\text{exp}({\stackrel{-}{\theta }}_{2}{\chi }_{0})+2\delta /{\stackrel{-}{\theta }}_{1}\right]\left[\text{exp}({\stackrel{-}{\theta }}_{2}{\chi }_{0})-2\delta /{\stackrel{-}{\theta }}_{1}\right]={\left(1+2\delta /{\stackrel{-}{\theta }}_{1}\right)}^{2}$$

5.13

for $\delta < {\overline{\theta }}_{1}/2$ and ${\overline{\theta }}_{1}<{\overline{\theta }}_{2}$, or for $\delta > {\overline{\theta }}_{1}/2$ and any ${\overline{\theta }}_{\text{1,2}}$. For the case $\delta < {\overline{\theta }}_{1}/2$ and ${\overline{\theta }}_{1}\ge {\overline{\theta }}_{2}$ the value ${\chi }_{0}$ is determined by

$$\text{exp}(-4\delta {\chi }_{0}){\left[\text{exp}({\stackrel{-}{\theta }}_{2}{\chi }_{0})-2\delta /{\stackrel{-}{\theta }}_{1}\right]}^{2}=\left(1+2\delta /{\stackrel{-}{\theta }}_{1}\right)\left(1-2\delta /{\stackrel{-}{\theta }}_{1}\right)$$

5.14

Due to the existence of the curve $\chi \left({x}_{1}\right)$, the observation area is divided into two zones, $0\le {x}_{2}< \chi \left({x}_{1}\right)$ and ${x}_{2}\ge \chi \left({x}_{1}\right)$. In each zone, $\mu_{1,2}\to 0$ with unlimited growth of both ${x}_{1}$ and ${x}_{2}$. In this case, the partition line $\chi \left({x}_{1}\right)$ tends to the line ${x}_{2} = {x}_{1}$, if ${x}_{1}\to \infty$. The condition means that the size of the first zone is reduced, and the best observations in this zone should be very close to each other. Expressions (5.10) and (5.11) facilitate determining the maximum width of the first zone.

The position of the global minimum in these areas is determined by the constraints that are imposed on the design region. If the constraints are given as the condition $0\le {x}_{\text{1,2}}\le \mathcal{l}$, then, due to the asymptotic properties of the function ${\mu }_{rms}\left({x}_{\text{1,2}}\right)$ described earlier, the neighbourhood of the point ${x}_{\text{1,2}}=\mathcal{l}$ is optimal for any $\mathcal{l}$:

$${\left.{{\Xi }}_{3}^{\left(R\right)}\right|}_{{\stackrel{-}{\theta }}_{\text{1,2}}>0, {x}_{\text{1,2}}\in \left[0,\mathcal{l}\right]}=\{\mathcal{l}-0,\mathcal{l}\}$$

5.15

The latter expression demonstrates that the first optimal measurement should be performed from the left side of the border neighborhood, and the second one is the border itself.

If the value $\mathcal{l}$ is increased, then the *R*-optimal design error is decreased, and the optimal design in ${\text{R}}^{2}$ is expressed as

$${\left.{{\Xi }}_{4}^{\left(R\right)}\right|}_{{\stackrel{-}{\theta }}_{\text{1,2}}>0,{x}_{\text{1,2}}\in {\text{R}}^{2}}=\{{x}_{\text{1,2}}^{\left(opt\right)}\to +{\infty }\}$$

5.16

In connection with this optimal solution, it is well known that the value $\mathcal{l}$ should be as maximal as possible.

If the designs (5.15) and (5.16) are not admissible from a practical viewpoint, then the case of the local minima at the cross-section ${x}_{2}=\mathcal{l}$ can be considered. Here the principal significance is provided by the topology of the surface ${\mu }_{rms}\left({x}_{\text{1,2}}\right)$. For the arbitrary ${\stackrel{\sim}{\theta }}_{\text{1,2}}$ and δ the cross-sections for the function ${\mu }_{rms}\left({x}_{\text{1,2}}\right)$ at ${x}_{2}=\mathcal{l}$ do not have anyone specific functional character, and there are regions of monotonicity, unimodality, and multimodality.

Compare the obtained results with the known solutions. The design

$$
{{\Xi }}_{5}=\{\mathcal{l}-1/{\stackrel{-}{\theta }}_{2},\mathcal{l}\} \tag{5.17}
$$

has been found in [7] and also by many other authors. It holds for $\mathcal{l}\gg \text{max}(1/{\stackrel{-}{\theta }}_{2},1)$ and correlates with the design (5.15) for the large $\mathcal{l}$. However, for the case $\mathcal{l}\le 1/{\stackrel{-}{\theta }}_{2}$, the design (5.17), even for its refinement into the form $\Xi_6 = \max(0, \mathcal{l}-1/{\stackrel{-}{\theta }}_{2})$, turns out to be the rough approximation of the position of the minimal design error due to the existence of the above-described variants of the optimal solution multimodality. The result means that if the value $\mathcal{l}$ is not enormous, it is always possible to determine the optimal solution better than the design (5.17).

For example, for the signal (5.7) with ${\overline{\theta }}_{1} = 1.15$, ${\overline{\theta }}_{2} = 1.28$, $\mathcal{l} = 1$, $\delta = 0.5$, the design $\Xi_7 = \{0.054,1\}$ has been determined in [22] by the over-approximation of the set of guaranteed parameter estimates. For this design, it is easy to obtain from (5.9), (5.10) that ${\left.{\mu }_{rms}\right|}_{{{\Xi }}_{7}}=0.8168$. The design (5.17) yields ${\left.{\mu }_{rms}\right|}_{{{\Xi }}_{5}=\left\{\text{0.219,1}\right\}}= 0.4983$. For the design (5.15), its error is ${\left.{\mu }_{rms}\right|}_{{{\Xi }}_{3}=\left\{\text{0.9986,1}\right\}} =0.4170$. The latter is the best solution among others mentioned above. The regularized designing under the scheme (4.1) – (4.4) additionally determines the local minimum for $\Xi_8 = \{0.419662, 0.419664\}$, for which ${\left.{\mu }_{rms}\right|}_{{{\Xi }}_{8}}=0.6186$ and for $\Xi_9 = \{0, 1\}$, for which ${\left.{\mu }_{rms}\right|}_{{{\Xi }}_{9}}= 0.7916$. As can be seen, the regularized paradigm depicts the functional properties of the design more widely.

5.2.2 The case ${\overline{\theta }}_{1} > 0$, ${\overline{\theta }}_{2} < 0$

For the negative parameter ${\overline{\theta }}_{2} < 0$, the asymptotic behavior is asymmetric to the case ${\overline{\theta }}_{2} > 0$. Therefore, the global minimum of the optimality criterion belongs in the third quadrant (${\overline{\theta }}_{2}> 0$) and the first quadrant (${\overline{\theta }}_{2} < 0$). The second and fourth quadrants are the worst observation areas.

Asymptotic properties, as in the previous case, play a decisive role in the existence of both the global and the local minima. If $x_{1,2}\to -\infty$, then ${\widehat{\mu }}_{\text{1,2}}=0$ and if $x_{1,2}\to+\infty$, then

$$
{\widehat{\mu }}_{1}=1\pm 2\delta /{\stackrel{-}{\theta }}_{1},{\widehat{\mu }}_{2}=1 \tag{5.18}
$$

For ${x}_{1} = 0$ and ${x}_{2} \to +\infty$, the design errors have asymptotic values (5.11). Therefore, if the constraints on the design space are absent, $\mathfrak{D} \subseteq R^2$, then the global minimum of the design error is maintained for the third quadrant (${x}_{\text{1,2}}\le 0$)

$$
{\left.{\mu }^{\left(R\right)}\right|}_{{x}_{\text{1,2}}\to -{\infty }}=0
$$

Therefore, the optimal design has the following form ${\left.{{\Xi }}^{\left(R\right)}\right|}_{{\bar{\theta }}_{1}>0,{\bar{\theta }}_{2}<0,{x}_{\text{1,2}}\in {R}^{2}}=\{{x}_{\text{1,2}}^{\left(opt\right)}\to -{\infty }\}$. The reduction factor of the model (5.7) is $\mathcal{R}=2\delta /{\bar{\theta }}_{1}$. Consequently, if $\mathcal{R}\ge 1$, then ${\widehat{\mu }}_{\text{1,2}} > 1$.

As in the previous case ${\overline{\theta }}_{\text{1,2}} > 0$, there is a partition curve $\chi \left({x}_{1}\right)$ expressing the maximal design error of (5.12) at a given point ${x}_{1}$ among all ${x}_{2}$. It passes through the point $(0, {\chi }_{0})$, where the value ${\chi }_{0}$ satisfies the equation

$$\text{exp}(-4\delta {\chi }_{0}){\left[\text{exp}({\bar{\theta }}_{2}{\chi }_{0})+\frac{2\delta }{{\bar{\theta }}_{1}}\right]}^{2}=\left(1+\frac{2\delta }{{\bar{\theta }}_{1}}\right)\left(1-\frac{2\delta }{{\bar{\theta }}_{1}}\right)$$

for $\delta <{\overline{\theta }}_{1}/2$, ${\overline{\theta }}_{1}\ge {\overline{\theta }}_{2}$ and the equation

$$\left[\text{exp}({\bar{\theta }}_{2}{\chi }_{0})+\frac{2\delta }{{\bar{\theta }}_{1}}\right]\left[\text{exp}({\bar{\theta }}_{2}{\chi }_{0})-\frac{2\delta }{{\bar{\theta }}_{1}}\right]={\left(1-\frac{2\delta }{{\bar{\theta }}_{1}}\right)}^{2}$$

for $\delta <{\overline{\theta }}_{1}/2$, ${\overline{\theta }}_{1}<{\overline{\theta }}_{2}$.

Taking into account the existence of the partition curve χ and using the asymptotic values (5.18), the global minimum of the function ${\mu }_{rms}$ is determined into the interval $0<{x}_{2}<{\left.\chi \right|}_{{x}_{1}=0}$. For this zone, the *R*-optimal design is $\Xi^{(R)} = \{0,{x}_{2}^{\left(opt\right)}\}$, where ${x}_{2}^{\left(opt\right)}$ is defined by the equation

$$\text{ln}\frac{\text{exp}[{\bar{\theta }}_{2}{x}_{2}^{\left(opt\right)}]-\frac{2\delta }{{\bar{\theta }}_{1}}}{1-\frac{2\delta }{{\bar{\theta }}_{1}}}=\frac{{\bar{\theta }}_{2}{x}_{2}^{\left(opt\right)}}{1-\frac{2\delta }{{\bar{\theta }}_{1}}\text{exp}[-{\bar{\theta }}_{2}{x}_{2}^{\left(opt\right)}]}$$

for ${\overline{\theta }}_{1}>\left|{\overline{\theta }}_{2}\right|$, $\delta <{\overline{\theta }}_{1}/2$, and the equation

$$\text{ln}\frac{\text{exp}[{\bar{\theta }}_{2}{x}_{2}^{\left(opt\right)}]+\frac{2\delta }{{\bar{\theta }}_{1}}}{1+\frac{2\delta }{{\bar{\theta }}_{1}}}=\frac{{\bar{\theta }}_{2}{x}_{2}^{\left(opt\right)}}{1+\frac{2\delta }{{\bar{\theta }}_{1}}\text{exp}[-{\bar{\theta }}_{2}{x}_{2}^{\left(opt\right)}]}$$

for ${\overline{\theta }}_{1}\le \left|{\overline{\theta }}_{2}\right|$, $\delta <{\overline{\theta }}_{1}/2$.

The feature of the first zone is its strong flatness. Here, the variations of the optimal measurement into the interval $0<{x}_{2}<\chi \left({x}_{1}\right)$ lead to slight changes ${\mu }_{rms}$. In this connection, the value of the global minimum can be majorized from the values ${\mu }_{\text{1,2}}$ at the point ${x}_{1}=0$, ${x}_{2}\to 0$. As a result, if ${\overline{\theta }}_{1}>\left|{\overline{\theta }}_{2}\right|$ on the condition $\delta <{\overline{\theta }}_{1}/2$, then ${\widehat{\mu }}_{1}=-2\delta /{\bar{\theta }}_{1}$, {\widehat{\mu

${\stackrel{-}{\theta }}_{2}$ $=2\delta /(2\delta +{\stackrel{-}{\theta }}_{1})$, and if ${\overline{\theta }}_{1}\le \left|{\overline{\theta }}_{2}\right|$ on the condition $\delta <{\overline{\theta }}_{1}/2$, then ${\widehat{\mu }}_{1}=2\delta /{\stackrel{-}{\theta }}_{1}$, ${\widehat{\mu }}_{2}=2\delta /(2\delta -{\stackrel{-}{\theta }}_{1})$.

For the case ${x}_{2}>\chi \left({x}_{1}\right)$, the local minimum exists at the position $\{0, {x}_{2}^{\left(opt\right)}\}$, where ${x}_{2}^{\left(opt\right)}$ satisfies the equation

$$\text{ln}\frac{\text{exp}[{\stackrel{-}{\theta }}_{2}{x}_{2}^{\left(opt\right)}]+\frac{2\delta }{{\stackrel{-}{\theta }}_{1}}}{1-\frac{2\delta }{{\stackrel{-}{\theta }}_{1}}}=\frac{{\stackrel{-}{\theta }}_{2}{x}_{2}^{\left(opt\right)}}{1+\frac{2\delta }{{\stackrel{-}{\theta }}_{1}}\text{exp}[-{\stackrel{-}{\theta }}_{2}{x}_{2}^{\left(opt\right)}]}$$

for any ${\overline{\theta }}_{\text{1,2}}\ne 0$. From this equation, it follows that ${x}_{2}^{\left(opt\right)}\to \infty$ for the small δ and the large ${\overline{\theta }}_{1}$. Because of this, the function ${\mu }_{rms}$ asymptotically decreases. The relationship between the values $\mathcal{l}$ and ${x}_{2}^{\left(opt\right)}$ establishes the position of the optimal observation in this region.

As in the previous case ${\overline{\theta }}_{\text{1,2}} > 0$, the noise is of crucial importance, whose scatter band 2δ determines the position of the optimal observation. There is a limit value ${\delta }^{*}={\stackrel{-}{\theta }}_{1}/2$, after which ${\left.{\mu }^{\left(R\right)}\right|}_{\delta \ge {\delta }^{*}}>1$ for any ${x}_{\text{1,2}}>0$ and ${\overline{\theta }}_{\text{1,2}}\ne 0$. This condition means that the signal (5.7) is not identifiable beginning from a specific threshold value ${\delta }^{*}$. It is also noteworthy that for any $\delta \ne 0$, the optimal design depends on both ${\overline{\theta }}_{1}$ and ${\overline{\theta }}_{2}$. Only for $\delta =0$, the optimal solution does not depend on the value ${\overline{\theta }}_{1}$.

5.2.3 The case ${\overline{\theta }}_{\text{1,2}}<0$

To complete the signal (5.7) study, we briefly describe the case of the negative ${\overline{\theta }}_{\text{1,2}}$.

If ${\overline{\theta }}_{\text{1,2}}<0$, then the position of the global minimum $\mu^{(R)}$ is not limited to a neighborhood along with the point $(0,{x}_{2}\to +0)$. Besides, the global minimum can significantly shift from this point along the line ${x}_{2} = {x}_{1}$. This shift occurs for fixed ${\overline{\theta }}_{2}$ and δ, where the value ${\overline{\theta }}_{1}$ is increasing. The local minimum is on the line ${x}_{1}=0$.

In summary, the investigation of the functional properties of the signal in question yields *the map of the best and worst estimation zones for various δ and* ${\overline{\theta }}_{\text{1,2}}$. A multimodal character is typical for the optimal design. The determination of the alternative observational areas extends the regions of the best measurements and seems to be an essential part of the design. Accordingly, the design problem can be formulated not as seeking strictly defined sensor positions at the observation region but as areas with the best measurements covering the design criterion's global and local minima. The transition to the regularized paradigm clarifies the scope of the commonly known FIM designs for the signal (5.7) and brings the new features of the optimal solution to light. As it turns out, the well-known FIM inferences are mathematically valid only for the exactly known data.

6.3 Three-parameter model

Previously, the commonly known dependence of the optimum design on the desired parameters for a nonlinear signal – a solution to the direct problem (2.1) – was confirmed. Mathematically, this dependency indicates that the design problem is solved locally, making it impossible to extend the resulting solution ${\Xi }^{\left(opt\right)}$ in a global sense and generalize it to a broad class of desired quantities. Accordingly, the theoretically important question is, *what are the best design solutions in a global sense for nonlinear cases*?

As demonstrated above, this complex situation is manageable if we can determine *the optimal design structure* for various sought quantities and construct the correspondence 'set of quantities – best observation areas'. As a result, design optimization can be considered globally [39] to determine a Pareto efficient solution [11].

Let us deal separately with the design dependence on the model parameters and demonstrate how *the investigation of the consistency equations* facilitates revealing the structure of the optimal solution. It is demonstrated that in a nonlinear case, the regularized paradigm can define the optimal solution structure independently of the model parameters and bounded perturbations. In addition, as in the above-studied cases, the change in the design paradigm will reveal new features of the classical model.

Consider the Verhulst equation

$$\frac{dy}{dx}={\theta }_{1}y-{\theta }_{2}{y}^{2},x>0,$$
5.19
$${\left.y\right|}_{x=0}={\theta }_{0}$$
5.20
that is well known in mathematical biology [32], and y expresses the typical dynamics of the population spread whose parameters ${\theta }_{\text{0,1},2}$ are the unknown quantities to be estimated. The following four steps determine the optimal design.

Step I. The solution to Eqs. (5.19), (5.20) is commonly known as an *S*-shaped curve. Its original form for the case ${\theta }_{\text{1,2}}=const$ is expressed as an exponential regression

$$y=\frac{{\theta }_{0}{\theta }_{1}\text{e}\text{x}\text{p}\left({\theta }_{1}x\right)}{{\theta }_{1}+{\theta }_{0}{\theta }_{2}\left[\text{exp}\left({\theta }_{1}x\right)-1\right]}.$$
Regarding the number of the sought parameters of the latter function, note that the way in which the optimal design can express the sensitivity of the observed signal concerning the initial state ${\theta }_{0}$ is theoretically important. It is known that the state function of lumped and distributed systems is often locally insensitive to variations in the initial conditions [43]. Therefore, the determination of the best observations with the minimum volume $m = 3$ is considered to reconstruct all the parameters ${\theta }_{\text{0,1},2}$ of the model (5.19), (5.20).

Using the analytical solution to Eqs. (5.19) and (5.20) for two sets of the model parameters ${\stackrel{-}{\theta }}_{\text{0,1},2}$ and ${\theta }_{\text{0,1},2}^{\left(\nu \right)}={\stackrel{-}{\theta }}_{\text{0,1},2}-{\nu }_{\text{0,1},2}$, the consistency equations (4.2) are obtained as the expression

$$\frac{{\stackrel{-}{\theta }}_{0}{\stackrel{-}{\theta }}_{1}\text{e}\text{x}\text{p}\left({\stackrel{-}{\theta }}_{1}{x}_{i}\right)}{{\stackrel{-}{\theta }}_{1}+{\stackrel{-}{\theta }}_{0}{\stackrel{-}{\theta }}_{2}\left[\text{exp}\left({\stackrel{-}{\theta }}_{1}{x}_{i}\right)-1\right]}-\frac{\left({\stackrel{-}{\theta }}_{0}-{\nu }_{0,l}^{}\right)\left({\stackrel{-}{\theta }}_{1}-{\nu }_{1,l}^{}\right)\text{exp}\left[\left({\stackrel{-}{\theta }}_{1}-{\nu }_{1,l}^{}\right){x}_{i}\right]}{\left({\stackrel{-}{\theta }}_{1}-{\nu }_{1,l}^{}\right)+\left({\stackrel{-}{\theta }}_{0}-{\nu }_{0,\text{l}}^{}\right)\left({\stackrel{-}{\theta }}_{2}-{\nu }_{2,l}^{}\right)\left\{\text{exp}\left[\left({\stackrel{-}{\theta }}_{1}-{\nu }_{1,l}^{}\right){x}_{i}\right]-1\right\}}={{\Delta }}_{i,l}, i=\stackrel{-}{\text{1,3}},l=\stackrel{-}{\text{1,8}},} (5.21)$$

where ${\nu }_{k,l}^{}$ denotes the absolute design error of the *k*-th parameter for the *l*-th matching option, which is described by the consistency corridor matrix

$${\Delta }=\left(\begin{array}{c}+2\delta ,\\ +2\delta ,\\ +2\delta ,\end{array}\begin{array}{c}-2\delta ,\\ +2\delta ,\\ +2\delta ,\end{array}\begin{array}{c}+2\delta ,\\ -2\delta ,\\ +2\delta ,\end{array}\begin{array}{c}+2\delta ,\\ +2\delta ,\\ -2\delta ,\end{array}\begin{array}{c}+2\delta ,\\ -2\delta ,\\ -2\delta ,\end{array}\begin{array}{c}-2\delta ,\\ +2\delta ,\\ -2\delta ,\end{array}\begin{array}{c}-2\delta ,\\ -2\delta ,\\ +2\delta ,\end{array}\begin{array}{c}-2\delta \\ -2\delta \\ -2\delta \end{array}\right)$$

5.22

Step II. Taking into account the separability of the matching equation terms, the design error *v* in Eqs. (5.21) decreases when the expressions with exponential terms tend to zero or infinity. This indicates that at the optimum observation positions, the conditions $\text{exp}\left[\left({\stackrel{-}{\theta }}_{1}-{\nu }_{1,l}^{}\right){x}_{i}\right]\to 1 \text{a}\text{n}\text{d} \text{exp}\left[\left({\stackrel{-}{\theta }}_{1}-{\nu }_{1,l}^{}\right){x}_{i}\right]\to \infty$ occur for any *θ*. From this, it follows that ${x}_{1}^{\left(opt\right)}=0$ and ${x}_{3}^{\left(opt\right)}\to \infty$. As a result, we have

$${\tilde{\nu }}_{0,l}^{}={{\Delta }}_{1,l},l=\stackrel{-}{\text{1,8}},}$$

5.23

$${\tilde{\nu }}_{1.l}^{}={\stackrel{-}{\theta }}_{1}-\left(\frac{{\stackrel{-}{\theta }}_{1}}{{\stackrel{-}{\theta }}_{2}}-{\varDelta }_{3,l}\right)\left({\stackrel{-}{\theta }}_{2}-{\tilde{\nu }}_{2,l}^{}\right),l=\stackrel{-}{\text{1,8}}$$

5.24

By substituting expressions (5.23) and (5.24) in (5.21), the following design error is finally determined

$${\tilde{\nu }}_{2,l}^{}={\stackrel{-}{\theta }}_{2}+\frac{1}{\left({\stackrel{-}{\theta }}_{1}/{\stackrel{-}{\theta }}_{2}-{\varDelta }_{3,l}\right){x}_{2}}\text{l}\text{n}\left\{\frac{({\stackrel{-}{\theta }}_{0}-{\varDelta }_{1,l}))}{\left({\stackrel{-}{\theta }}_{0}-{\varDelta }_{1,l}\right)-\left({\stackrel{-}{\theta }}_{1}/{\stackrel{-}{\theta }}_{2}-{\varDelta }_{3,l}\right)}\left[1-\frac{{\stackrel{-}{\theta }}_{1}/{\stackrel{-}{\theta }}_{2}-{\varDelta$$

}_{3,l}}{ \frac{{\stackrel{-}{\theta }}_{0}{\stackrel{-}{\theta }}_{1}\text{e}\text{x}\text{p}\left({\stackrel{-}{\theta }}_{1}{x}_{2}\right)}{{\stackrel{-}{\theta }}_{1}+{\stackrel{-}{\theta }}_{0}{\stackrel{-}{\theta }}_{1}\left[\text{exp}\left({\stackrel{-}{\theta }}_{1}{x}_{2}\right)-1\right]}-{\varDelta }_{2,l }}\right]\right\},l=\stackrel{-}{\text{1,8}}
5.25
Step III. The guaranteed design errors are defined by minimizing the stabilizer (4.3),

\widehat{\nu }\left({x}_{2}\right)=Arg\underset{1\le l\le 8}{\text{m}\text{i}\text{n}}\sum _{k=0}^{2}[{\stackrel{-}{\theta }}_{k}-{\tilde{\nu }}_{k,l}^{}({x}_{2}){]}^{2}
among all eight cases (5.23) – (5.25) for every l=\stackrel{-}{\text{1,8}} that express the worst approximation.

Step IV. The optimal position {x}_{2}^{\left(opt\right)} is defined by minimizing criterion (4.4). This gives the following structure of the optimal solution. Three positions of the measurements determine the minimum volume of the sample. Two positions among the three optimal sensor placements do not depend on the sought model parameters for any {\stackrel{-}{\theta }}_{\text{0,1},2} and \delta <{\delta }^{*}. The initial and as long as possible positions are the best sensor placements, {x}_{1}^{\left(opt\right)}=0, {x}_{3}^{\left(opt\right)}\to \infty . The sensitivity of the signal *y* to the initial condition {\theta }_{0} does not bring the design error outside the interval \pm 2\delta.

The position of the second optimum observation {x}_{2}^{\left(opt\right)} is always higher than the position {x}_{infl}at which the growth function *y* has an inflection point, {d}^{2}y/{\left.d{x}^{2}\right|}_{x={x}_{infl}}=0. This point determines the basic properties of the *S*-shaped curve [32]. For the case study (Table 1), the best second observational region is determined by the interval {1.5{x}_{infl}<x}_{2}^{\left(opt\right)}<2{x}_{infl}.

The proposed regularized-minimal-residual design for the *S*-shaped curve facilitates determining the features, which express the global character of the solution obtained.

First, the obtained optimal solution demonstrates the property $\nu_{1,2,3}\to 0$ for $\delta\to 0$. From the existence of the design error ν_2 it follows that the threshold level of the noise has the value $\delta^{*}=\bar{\theta}_0/2$. If $\bar{\theta}_0 \geq {}^{\bar{\theta}_1}/_{2\bar{\theta}_2}$, then the solution to the design problem does not exist. Table 1 expresses the effect of these conditions. The detected design properties demonstrate how the initial condition impacts the estimation of the growth process parameters.

The values of the design errors {\nu }_{\text{0,1},2}^{} can be defined explicitly. If \delta >{\stackrel{-}{\theta }}_{1}/{2/\stackrel{-}{\theta }}_{2}, then {\nu }_{1}>{\stackrel{-}{\theta }}_{1}. The value of the absolute design error of the parameter {\stackrel{-}{\theta }}_{0}does not depend on its value and is determined only by the noise.

For small ${\stackrel{-}{\theta }}_{\text{1,2}}$ the main design error appears in reconstructing the initial condition ${\stackrel{-}{\theta }}_{0}$. Taking into account the existence of the threshold value, ${\delta }^{*}={\stackrel{-}{\theta }}_{0}/2$, it is better to define the initial condition by direct measurements. In practice, the following inference has significance: *a shift in the observations from the starting point of the growth process leads to an increase in the estimation error*.

During the monotonic change in the variable *x*, the stabilizer's minimum value can be achieved at different variants of the matching conditions. This leads to a jump in the error ${\nu }_{2}$ and consequently to the appearance of the local minima of the total design error. In general, the design is not unimodal.

The numerical analysis of the estimation behavior at the observation positions that deviated from the optimal positions shows a significant decrease in the obtained design errors. These errors can exceed the minimum design errors by order of magnitude.

Compare the obtained results with the inferences of the previously conducted studies of the model (5.19) and (5.20).

Bayesian optimal designs for the logistic model with two unknown parameters were found in [8]. It has been shown that the optimum positions should be close to the boundaries of the observation interval. In [24], it was demonstrated that minimax *D*-optimal designs could be quite efficient under a Bayesian setup.

In [11], the design was regularized by Tikhonov's algorithm with the regularization parameter. The optimal solution was referred to as the Pareto efficient. However, the proposed formulation does not restrict the solution domain and does not take into account the existence of the noise scatter band because the noise is considered to have a zero mean value (see Section 2).

The previously determined optimal position ${x}_{2}^{\left(opt\right)}$ was specified near the inflection point ${x}_{infl}$. The design (4.1) – (4.4) demonstrates that such an estimation can only be considered as a lower bound of the optimal position. In [5], the best observations were obtained by the Monte Carlo method. Formulation (4.1) – (4.4) does not require numerical modelling of the sample. Because of this, the more general specifications of the problem formulation were studied. The sufficiency of the three measurement positions is proven for the more general conditions of the problem in question.

The above-described conditions for the optimal design existence are determined for the first time. These conditions express the boundaries of the admissible signal observations.

6.4 Numerical analysis of an observational design structure

Consider a case with no analytical solutions to the consistency equations. For such formulations, it is necessary to construct a correspondence between the desired quantities and observation areas by introducing some reference parameters. In what follows, the idea of determining the best and worst observation areas is examined in detail numerically. It is demonstrated that the global and local minima can extend the design solution by the notion of *the best observation areas*.

The following exponential regression is given:

y(x{\left.)\right|}_{{\theta }_{\text{1,2},3}}=\frac{{\theta }_{1}}{{\theta }_{1}+{\theta }_{2}-{\theta }_{3}}\left\{\text{exp}\left(-{\theta }_{3}x\right)-\text{e}\text{x}\text{p}[-({\theta }_{1}+{\theta }_{2}\left)x\right]\right\} , 0\le x<+{\infty }. (5.26)

This function is considered in life sciences [2, 23] to describe the spiking growth dynamics.

For the directly defined prototype (5.26), it is desired to estimate {\theta }_{\text{1,2},3}=const\ne 0. The optimal design $\Xi^{(R)}$ = \{{x}_{i}^{\left(opt\right)}{\}}_{i=\overline{\text{1,3}}}is sought in the region \mathfrak{D}\subseteqR^3[0,∞) using the observations (2.2) – (2.4). The design steps are executed as follows.

Step I. The required signal representations {\left.\stackrel{-}{y}\right|}_{{\stackrel{-}{\theta }}_{\text{1,2},3}} and {\left.y\right|}_{{\theta }_{\text{1,2},3}^{\left(\nu \right)}} with the parameters {\stackrel{-}{\theta }}_{\text{1,2},3} and {\theta }_{\text{1,2},3}^{\left(\nu \right)}={\stackrel{-}{\theta }}_{\text{1,2},3}-{\nu }_{\text{1,2},3} is performed analytically by the function (5.26). The consistency equations (4.2) relative to the design errors {\nu }_{\text{1,2},3} for the three-point design ($m = p = 3$) are of the form

\frac{{\stackrel{-}{\theta }}_{1}}{{\stackrel{-}{\theta }}_{1}+{\stackrel{-}{\theta }}_{2}-{\stackrel{-}{\theta }}_{3}}\{\text{exp}(-{\stackrel{-}{\theta }}_{3}{x}_{i})-\text{exp}[-({\stackrel{-}{\theta }}_{1}+{\stackrel{-}{\theta }}_{2}){x}_{i}\left]\right\}-
\frac{{\stackrel{-}{\theta }}_{1}-{\nu }_{1,l}^{}}{{\stackrel{-}{\theta }}_{1}+{\stackrel{-}{\theta }}_{2}-{\stackrel{-}{\theta }}_{3}-{\nu }_{1,l}^{}-{\nu }_{2,l}^{}+{\nu }_{3,l}^{}}\{\text{exp}[-({\stackrel{-}{\theta }}_{3}-{\nu }_{3,l}^{}\left){x}_{i}\right]-
\text{exp}[-({\stackrel{-}{\theta }}_{1}+{\stackrel{-}{\theta }}_{2}-{\nu }_{1,l}^{}-{\nu }_{2,l}^{}){x}_{i}\left]\right\}={{\Delta }}_{i,l},i=\overline{\text{1,3}},l=\overline{\text{1,8},}
5.27
where the worst observations are described by the consistency corridor matrix (5.22).

Step II. The simultaneous equations (5.27) cannot be solved analytically, so its solution for the given \stackrel{~}{\theta }, δ and \{{x}_{k}{\}}_{k=\overline{\text{1,3}}} is sought numerically. For such a case, it is necessary to introduce the reference parameters \stackrel{~}{\theta } against which the optimal solution is sought. Similarly to [23], the first option is selected as {\stackrel{-}{\theta }}^{\left(1\right)} = (6.85, 1.70, 0.55)^T. The second option {\stackrel{-}{\theta }}^{\left(2\right)}= (92.41, – 88.115, 0.059)^T is introduced as in [2].

The selected reference parameters {\stackrel{-}{\theta }}^{\left(1\right)} and {\stackrel{-}{\theta }}^{\left(2\right)} determine the different behaviours of the signal (5.26). Due to the limited scope of the publication, we will not present the results of the sensitivity analysis of the signal (5.26) relative to its parameters. It should only be noted that the variations of the parameters from the option ${\stackrel{-}{\theta }}^{\left(2\right)}$ are sharply limited due to the absence of the solution $\left\{{\nu }_{\text{1,2},3}\right\}$ to Eqs. (5.27). The solvability violation of the consistency equations will undoubtedly affect the design behavior. The option ${\stackrel{-}{\theta }}^{\left(1\right)}$ has no limitation in a similar sense. The design dependence on the parameters ${\stackrel{-}{\theta }}_{\text{1,2},3}$ will be considered, whereby the variations of the parameters should cover a wide range of the functional properties of the signal (5.26).

Step III. Among eight solutions to Eqs. (5.27) $\tilde{\nu }$ the one is determined that minimizes the stabilizer (4.3)

$$\widehat{\nu }\left({x}_{\text{1,2},3}\right)=Arg\underset{1\le l\le 8}{\text{m}\text{i}\text{n}}\sum _{k=1}^{3}[{\stackrel{-}{\theta }}_{k}-{\tilde{\nu }}_{k,l}^{}({x}_{\text{1,2},3}){]}^{2}$$

The latter expresses *the guaranteed design error* at each point of the design region.

Step IV. The optimal design $\Xi^{(R)}$ is determined as the minimum of the criterion

$$\mu_{rms}(x_{1,2,3}) = \left[\sum_{k=1}^{3} \hat{\mu}_k^2(x_{1,2,3})/3\right]^{1/2},\ 0 < x_{1,2,3} < +\infty,$$

where ${\widehat{\mu }}_{k}={\stackrel{}{\nu }}_{k}/{\stackrel{-}{\theta }}_{k},k=\overline{\text{1,3}}$ denotes the relative design errors.

The obtained results reveal the following properties of the optimal design:

- Tables 2 and 3 express the noise level effect on the optimal design,
- Table 4 shows the difference between the global and local minima, and
- Table 5 depicts the dependence of the optimal solution on the sought-for quantities.

Then the features of the optimal observational design are as follows.

First, the noise effect on the optimal design can be broken down into four grades (Tables 2 and 3). The asymptotic of the obtained solutions demonstrate the tendency ${\nu }_{\text{1,2},3}\to 0$for $\delta \to 0$.

Second, the solutions express the existence of *the strictly defined structure of the optimal design*. A similar structure reflects *the best observation areas*. For signal (5.26), the structure is expressed in three areas. They are (i) the ascending branch, (ii) the region of the specific point of the state function, and (iii) the descending branch (Tables 2 and 3). The sought parameters' variations (Tables 4 and 5) express the structure boundaries and the conditions under which the structure is changed. The deviations of the optimal solutions within the structure are not significant. The comparison of the optimal solutions for the cases ${\stackrel{-}{\theta }}^{\left(1\right)}$ and ${\stackrel{-}{\theta }}^{\left(2\right)}$ shows the dependence of the structure character on the functional properties of the model in question. What matters here is determined by the differences noted above between options ${\stackrel{-}{\theta }}^{\left(1\right)}$ and ${\stackrel{-}{\theta }}^{\left(2\right)}$.

Third, the characteristic feature of the optimal observation is *its multimodal nature*. For the signal (5.26), the differences between the global and local minima of the design errors are not substantial and are often small (Table4). The design structure changes from one optimal area to another as the noise scatter band increases in size.

The existence of the optimal solution structure indicates that it is possible to specify the strongly determined areas of the best observations. Summarizing the results in Tables 2–5, it is found that for the signal (5.26) with a range of parameters such as ${\stackrel{-}{\theta }}^{\left(1\right)}\pm 30\%$ and for any noise with $0<\delta <{\delta }^{*}$, the best observations belong to the intervals $0.07 < x < 0.19$, $0.48 < x < 0.73$, $1.4 < x < 1.74$ and $2.8 < x < 3.5$.

The last recommendation demonstrates how the dependence of the design on the initial data ${\stackrel{-}{\theta },\delta}$ can be reduced. Recommendations of this kind are exempted from the need to specify the initial guesses of the unknowns accurately. It is theoretically important that the developed approach can define the correspondence between a set of desired quantities and a specific set of the best observation areas. Therefore, the search for the optimal solution at certain points in the observation space can be replaced by determining the best region.

Additionally, this outcome indicates a direction of further investigation of the signal (5.26). It is the study of the design formulation with overdetermined measurements. Examining the consistency equations ensures the determination of the regions of optimal multi-point measurements.

Fourth, the existence of the significant errors in the optimal solution, ${\mu }^{\left(R\right)}> 0.1$, is associated, first of all, with the poor scalability of the sought quantities ${\stackrel{-}{\theta }}_{\text{1,2},3}$. Their values differ from each other by more than an order of magnitude (compare the results in Tables 4 and 5). The relationship between the value δ and the maximum value of the observed signal, ${y}_{max}=\underset{x\ge 0}{\text{m}\text{a}\text{x}} y\left(x\right)$, is also relevant (Tables 4 and 5). The sensitivity of the signal $y(x)$ doubtless influences the design error ${\mu }^{\left(R\right)}$. The case of ${\stackrel{-}{\theta }}_{1}=7$, ${\stackrel{-}{\theta }}_{2}=5$ demonstrates that the reconstruction accuracy improves with increasing ${\stackrel{-}{\theta }}_{3}$, although the value ${y}_{max}$ is decreased (Table 5). Here, the rate of the signal change becomes the main factor. Notably, there exists a case of the sought parameters in which their optimal design errors are lower than the noise level, ${\mu }^{\left(R\right)}<\delta$ (Table 3).

Based on the solvability of the design problem, the following property of the model function (5.26) should be highlighted: for a given $\stackrel{-}{\theta }$, the solvability is determined by the observation positions ${\{{x}_{k}\}}_{k=\overline{\text{1,3}}}\in \mathfrak{D}$ and the value $\delta > 0$. There may exist ${\left\{{x}_{k}^{*}\right\}}_{k=\overline{\text{1,3}}}$ and δ^* for which a solution to the matching equations is absent. Because of this, there are points ${\left\{{x}_{k}^{*}\right\}}_{k=\overline{\text{1,3}}}$ that do not allow signal reconstruction, even for a small δ. From there, it follows that only minor variations in the sought parameters are acceptable. Because of this, the design behavior will not change significantly when the noise level varies. This property is wholly manifested in the case of ${\stackrel{-}{\theta }}^{\left(2\right)}$ where the solvability violation of the consistency equations occurs in a wide range of the desired parameters. As a result, the structure of the optimal solution for option ${\stackrel{-}{\theta }}^{\left(2\right)}$ is less varied than the structure of the optimal solution for the reference parameter ${\stackrel{-}{\theta }}^{\left(1\right)}$.

Let us compare the obtained results with the known solutions under the FIM paradigm. For ${\stackrel{-}{\theta }}^{\left(1\right)}$ the design Ξ_{10} = {0.1, 0.5, 2.0} was determined in [23] and for the case of ${\stackrel{-}{\theta }}^{\left(2\right)}$ the design Ξ_{11} = {0.2288, 1.3886, 18.417} was found in [2].

The error of the design Ξ_{10} significantly exceeds the design error of the regularized design (see Table 2). The FIM results in an optimal design that is close to the regularized design only for $\delta \to 0$. Even for a small δ, the reconstruction accuracy of the FIM optimum design is worse than that of the regularized design. The difference increases with the growth of δ. The optimal design essentially depends on the noise.

The design Ξ_{11} does not provide a solution to Eqs. (5.27) for all eight options (5.22). For this reason, the results of the estimation with design Ξ_{11} are not included in Table 3.

For the high-noise scatter band, the FIM designs Ξ_{10} and Ξ_{11} manifest unacceptable design errors, ${\mu }_{rms}>1$. At the same time, the regularization ensures the solvability of the design problem even for large δ and, in fact, accomplishes the required minimization of the noise influence.

7 Discussion

Let us look at the results in terms of the principal features of ill-posed problems, where there are not enough observations *to ensure a zero mean, and this cannot be unimproved* (in other words, the mean does not tend to zero).

As shown above, the noise affects both the estimation accuracy and the design existence, and it primarily defines the nature and behavior of the optimal solution. Investigating the solvability of the consistency equations demonstrates the existence of vast areas of nonidentifiability connected with the existence of the scatter band of observations. These cases are not due to the invariant properties of the mathematical models. They arise because of *the inability to approximate a sample in the framework of the scatter band* under given observation conditions. The design problem offers no solution starting with a specific noise threshold level. By this means, the proposed regularized-minimal-residual design facilitates investigating the existence and uniqueness of solutions to inverse problems. Such a possibility expands the study of many essential theoretical questions in practice.

Because of this, the recommendation to eliminate the effect of errors ‘by adopting high resolution computational methods and alleviate difficulties that may arise from using computational representation’ [4] is valid if there are no differences between a sample and a prototype. In other words, when the noise is zero-filled.

The design solution should be regularized to take advantage of noisy data and take into account the low sensitivity of the signal to the desired quantities. Regularization can be achieved in many ways. The proposed regularization facilitates reconstructing a set of model parameters simultaneously when only the general functional properties of the signal model are known. All the optimal solutions obtained above demonstrate that the proposed method does not have the disadvantage inherent in using the regularization parameter [44].

The proposed regularized-minimal-residual design facilitates introducing *a wide range of optimal criteria* to limit the total or individual variations of the desired parameters. All of these variations directly reflect the estimation errors. Their minimization has a clear advantage over minimizing any norm of the FIM because the latter expresses the design errors implicitly. As a result, *a comprehensive analytical and numerical examination of the design error behavior* can be executed. Additionally, the dependence of the optimal design on the conditions of the experiment is defined. The best and worst observation areas are established, and the functional features of the problem in question are revealed. The obtained results demonstrate that *the minimum of the implicit criteria does not refer to the minimum of the design errors*. For this reason, such designs can always be improved.

The unique solution of the design problem can be implemented with *a minimally sized sample*. If the noise scatter band does not change, and the signal model is adequate, then refining the observation grids with small steps to carry out numerous measurements and performing many experiments to estimate all model parameters is not necessary. Narrowing the interval of noise concentration is the main reason to use any additional observations.

Globally and locally optimal observation regions express a design structure. The dependence of the optimal measurements on the sought parameters occurs within the framework of such a structure. This permits groups of optimal sensor placements and sampling times to be sought as a feature of the model in question. *The optimal solution structure and its determination seem* to be an essential property of design.

There exist nonunique areas of the estimation error minimum. In many cases, the local minimum values differ slightly from the global minimum. The determination of all the optimal areas doubtless has practical significance. The multimodality of the optimal solution allows for considering alternative observation schemes. Additionally, the multimodal character of the design demonstrates that *the best observations can be defined as the multipoint measurements in the regions of the global and local minima*. The transformation of the local minimum to the global minimum occurs due to an increase in the noise scatter band. It is possible to rank the noise variation by relating it to groups of optimal sensor placements.

An essential part of the design is to define the conditions that will compensate for the noise effect. *Optimal sensor placements* reflect the first kind of compensation: choosing the best measurement schemes. In addition, there are the *best experimental conditions* under which the noise effect can be reflected by the reduction factor of the model in question. The third influential factor of noise compensation is *solution regularization*. The proposed regularized schemes ensure the stable and optimal solution even at significant noise where the others cannot be applied.

The FIM and statistical paradigms yield biased optimal designs compared to those that account for the noise scatter band. The regularized design demonstrates that the reconstruction accuracy is also improved when the noise is near zero.

The FIM and statistical paradigms do not determine all the best areas of the observations. Nevertheless, finding these areas is an essential design feature because it facilitates applying the optimal solutions without having specified initial guesses regarding the sought parameters. The regions can be indicated quite definitely for a wide range of unknowns and noise scatter bands. The change from seeking the optimal observational positions to determining the best areas in the observation space reduces the influence of initial data uncertainties.

Although the case of the exponential regression under study is specific, the investigations demonstrate the main design features: the optimal solution structure, the factors in the noise effect reduction, the behavioral features of the optimal estimation, and the existence of a noise threshold. These features are difficult to identify for the abstract operator model (2.1). However, they are analytically or numerically revealed when a specific process is considered. The study of the parabolic models [37–40] also demonstrates the same features of the optimal regularized design. The identified features of the particular signals determine directions for further research on abstract models.

Thus, incorporating noise effects in the design problem solution provides a more complex picture of the optimal observations and a more comprehensive investigation of the experiment’s features.

8 Conclusions

The best sensor statements are conducted under *the regularization paradigm with the model of the worst observation errors*. The specification of the upper boundary of the noise is sufficient to find the optimal sensor placements. The assumptions regarding noise properties turn out to be the most common conditions. The noise distribution, error variance, zero mean, and correlation are not required. However, the absence of knowledge regarding the noise, including its scatter band, leads to design ambiguity. The proposed observation model shows that postulating the noise action as white leads to biased estimates, and the offset quickly grows with the interval of noise concentration.

The regularized-minimal-residual design gives a wide possibility to study and determine optimal experimental conditions. The best observational schemes and their estimation errors, factors of noise effect reduction, design peculiarities, and singular experiment’s conditions can be revealed.

It was proven that the existence of the noise scatter band generates *a case of nonidentifiable observations*. The design problem offers no solution for any desired parameters, starting with a specific threshold value. The proposed regularization of the design problem ensures a stable reconstruction until the threshold value of noise is reached.

The existence of the structure of the optimal solutions to the design problem was verified. The design can have several structures, which can be divided into substructures. Changes in the sought quantities and noise within a structure or substructure do not lead to significant variations in the optimal design. Functional properties of the model in question and the deviation between a sample and its prototype state function determine the design structure.

It was shown how the determination of global and local minima can change the character of the optimal design solution. *The determination of the best observation areas*, rather than the specific points of the optimal observations, is more independent of the initial guesses regarding the sought quantities.

The attainment of the novel features of the known design problem solutions indicates the effectiveness of the regularization paradigm.

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Nomenclature

Nomenclature

f	driving force
m	number of samples
n	sample volume
p	number of parameters
t	time
$t^{(opt)}$	optimal time of observations
x	sensor location (measurement point)
$x^{(opt)}$	optimal sensor statement
x^*	exceptional point
y	signal (state function)
$\bar{y}$	actual signal
$y\|_\theta$	model-calculated signal with current values of model parameters
$y^{(\delta)}$	sample
ℓ	length of the measurement interval
$\mathcal{A}, \mathcal{F}, \mathcal{Y}$	functional spaces
$\mathbf{R}^{1,2,3}$	Euclidean spaces
L_θ	mathematical operator depended on parameters θ
Y_0	initial state
δ	upper bound of noise
δ_{rms}	*rms* estimator of noise
δ^*	threshold level of noise
ε	noise
θ	sought parameters
$\bar{\theta}$	actual parameters
$\theta^{(\nu)}$	parameters with design errors ν
$\tilde{\theta}$	parameters, satisfied by consistency equations with observations
$\hat{\theta}$	regularized parameters
μ	relative design error
$\tilde{\mu}$	relative design error satisfied by consistency equations for given design point
$\hat{\mu}$	best relative design error among all worst fitting options (relative guaranteed design error)
$\mu^{(M)}$	minimum relative design error according to *min-max* criterion
$\mu^{(R)}$	minimum relative design error according to *rms* criterion
ν	absolute design error
$\tilde{\nu}$	solutions to consistency equations with observations
$\hat{\nu}$	best absolute design error among all worst fitting options (absolute guaranteed design error)
χ	partition line
Δ	consistency corridor matrix
Ξ	design of observations
$\Xi^{(M)}$	optimal design according to *min-max* criterion
$\Xi^{(R)}$	optimal design according to *rms* criterion
Ω	stabilizer
$\mathcal{R}$	reduction factor

Declarations

Competing interests:

The author declares no competing interests.

Tables

Tables 1-5 are available in the Supplementary Files section

Figures

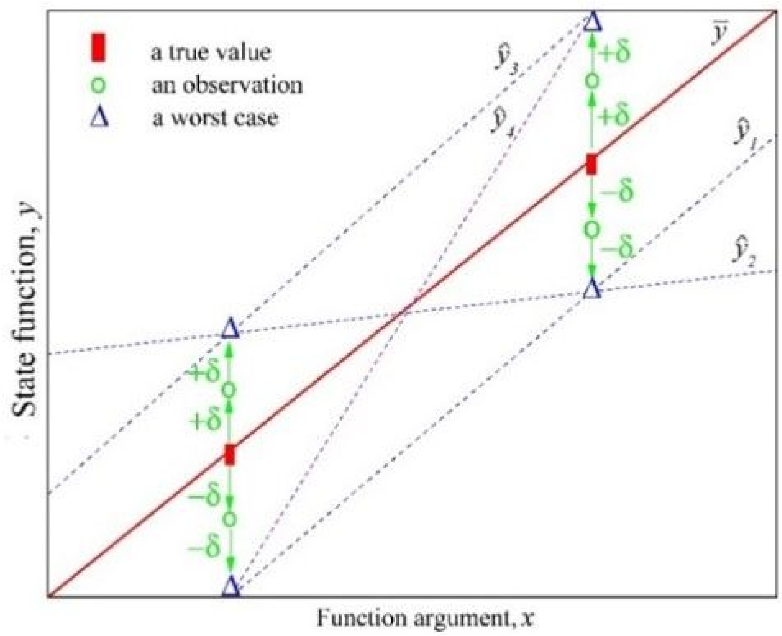

Figure 1. A model of the worst observation errors: a two-point prototype $\overline{y}$ and four possible options $\hat{y}_{1-4}$ of worst approximations into the consistency corridor $\pm 2\delta$.

Figure 1

See image above for figure legend.

Supplementary Files

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