Construction of new quantum color codes

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Construction of new quantum color codes

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Abstract

This paper presents a hyperbolic geometry approach to the construction of new quantum color codes. The families of quantum color codes are constructed based on the identification of compact surfaces by hyperbolic tessellations. Codes of these families have 4 and 6 minimum distances and their encoding rate are near to 1. We also provide some quantum color codes with minimum distance of at least eight and a comparison table of quantum codes.

Keywords: quantum color codes; compact surfaces; hyperbolic geometry; tessellations

1. Introduction

Over the past 60 years, digital communication has had a substantial and a growing influence on society. Information theory is the principal foundation of modern digital communication. It provides a theoretical basis for lossless source, lossy source coding and channel coding. Since the inception of information theory with Shannon's ground breaking work [1], there has been a large volume of practical and theoretical research aimed at achieving capacity bounds for various types of information sources and communication channels. After providing error-correcting codes with a transmission rate less than the channel capacity by Shannon, many efforts were made to achieve the optimal codes and known codes such as Hamming codes, Golay codes, Reed-Muller codes, convolutional codes, BCH codes, Reed-Solomon codes, turbo codes, and finally Low Density Parity Check (LDPC) codes were introduced. In addition to extensive research on classical codes, the problem of constructing quantum error-correcting codes (quantum codes, for short) has motivated considerable interest in the literature. This problem was raised within the context

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of quantum computation and quantum information. Since the initial discovery of quantum code by Shor [2], much progress has been made in the development of quantum codes in Engineering [3] and Physics [4]. Over the last two decades, different types of quantum codes have been constructed from classical error-correcting code [5], which can be referred to Kitaev's toric code [6], color codes [7], surface codes [8], toric quantum codes [9] and others [10-13]. Also in the last two years, some binary quantum codes have been constructed in several ways (for instance [14-17]).

The aim of this paper is to provide the families of quantum color codes on compact surfaces with genus greater than or equal to 21. Such quantum codes are associated with identification of these surfaces with hyperbolic tessellations and hyperbolic polygons by pairing the edges. For these quantum codes, the encoding rate is such that $\frac{k}{n}$ tends 1 as n goes to infinity. Moreover, A table of quantum codes which are different parameters in relation to the classes previously presented in [12,13,17], among others is presented.

The structure of this paper is as follows: In Section 2, we focus our attention on preliminaries. Section 3 is related to present families of quantum color codes with distances $d_{min} = 4, 6$ and $d_{min} = 8, 10, 12, 18$. In Section 4, a table of quantum codes comparison is presented. Finally, Section 5 is devoted to conclusion.

2. Preliminaries

In this section, we give some preliminaries such as quantum mechanics and hyperbolic geometry used through the paper. For further information about these concepts, the reader is referred to Refs. [12,13,18,19].

2.1. Review of quantum mechanics on qubits

In the classical physical world, the bit is the fundamental concept of classical computation and information. In quantum computation and quantum information bits are replaced with quantum bits, or qubits for short. Two possible states of a physical system (e.g. charge or no charge) are the states $|0\rangle$ and $|1\rangle$, which correspond to the states 0 and 1 for a classical bit. These two states are known as computational basis states. The difference between bits and qubits is that a qubit can be in a linear combination or superposition of the two states $|0\rangle$ and $|1\rangle$ as:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \qquad (\alpha, \beta \in \mathbb{C},\ |\alpha|^2 + |\beta|^2 = 1). \tag{2.1}$$

In other words, the state of a qubit is a vector in a two-dimensional complex vector space. For example, a qubit can be in the state:

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle, \tag{2.2}$$

So a qubit can exist in a *continuum* of states between $|0\rangle$ and $|1\rangle$.

2.2. Quantum stabilizer codes

We recall quantum stabilizer codes. More detailed information on stabilizer codes may be found in Refs. [20,21]. A quantum codes defined on the complex Hilbert space $\mathcal{H}_2^{\otimes n}$ where $\mathcal{H}_2 = \mathbb{C}^2$ is the two dimensional complex Hilbert space (the unit sphere of $\mathcal{H}_2$ represents the state space of a qubit). A quantum codes, $\mathcal{C}$ with codeword length n, dimension k, and minimum distance d is denoted by $[[n, k, d]]$. Such a code is able to correct all errors up to $\lfloor \frac{d-1}{2} \rfloor$.

A stabilizer code $\mathcal{C}$ is the simultaneous eigenspace, with eigenvalue $+1$, of all the elements of an Abelian subgroup $\mathcal{S}$ of the Pauli group $\mathcal{G}_n$ on n qubits, called stabilizer group. The elements of $\mathcal{S}$ are called stabilizer operators. The Pauli group for one qubit is defined to consists of all Pauli matrices, together with multiplicative factors ± 1, $\pm i$:

$$\mathcal{G}_1 = \{\pm I, \pm iI, \pm X, \pm iX, \pm Y, \pm iY, \pm Z, \pm iZ\} \tag{2.3}$$

where X, Y and Z are the usual Pauli matrices and I is the identity matrix. The set of matrices $\mathcal{G}_1$ forms a group under the operation of matrix multiplication. In general, group $\mathcal{G}_n$ consists of all tensor products of Pauli matrices on n qubits again with multiplicative factors ± 1, $\pm i$.

Thus, the stabilizer code $\mathcal{C}$ is defined as $\mathcal{C} = \{|\psi\rangle : s|\psi\rangle = |\psi\rangle,\ \forall s \in \mathcal{S}\}$, [22]. This code, encodes k logical qubits into n physical qubits. The rate of such code is $\frac{k}{n}$. Since the code space has dimension 2^k so that we can encode k qubits into it. The stabilizer $\mathcal{S}$ has a minimal representation in terms of $n-k$ independent generators $\{g_1, ..., g_{n-k} \mid \forall i \in \{1, ..., n-k\},\ g_i \in \mathcal{S}\}$. The generators are independent in the sense that none of them is a product of any other two (up to a global phase).

Just as in the classical case, there are two bounds which have been established as necessary conditions for quantum codes.

Lemma 2.1.(quantum Hamming bound for binary case [5]) For any pure quantum stabilizer code $[[n, k, d]]_2$, we have the following inequality

$$\sum_{j=0}^{\lfloor \frac{d-1}{2} \rfloor} \binom{n}{j} 3^j 2^k \leq 2^n. \tag{2.4}$$

Lemma 2.2.(quantum Singleton bound [5]) For any quantum stabilizer code $[[n, k, d]]_q$, we have

$$n \geq k + 2d - 2. \tag{2.5}$$

A quantum code that achieves this bound, it is called a quantum maximum-distance-separable (MDS) code. The quantum codes constructed in this paper satisfy the above two bounds.

2.3. CSS codes

The family of CSS (Calderbank-Shor-Steane) codes relies on classical and quantum codes [11,20]. A CSS code of length n is defined by two parity check matrices H_1

and H_2 such that $H_1 H_2^T = 0$. Denote by C_1 the kernel of the matrix H_1 and denote by C_2 the kernel of the matrix H_2. Then the stabilizer code with binary check in the form:

$$A = \left(\begin{array}{c|c} H_1 & 0 \\ 0 & H_2 \end{array} \right), \tag{2.6}$$

is a quantum code with parameters $[[n, k, d_{min}]]$, where

$$k = n - rankH_1 - rankH_2 \tag{2.7}$$

and

$$d_{min} = min\{wt(c) : c \in (C_2 \setminus C_1^{\perp}) \cup (C_1 \setminus C_2^{\perp})\}. \tag{2.8}$$

where $wt(c)$ is the weight of the codeword c.

Notice that

$$C_1^{\perp} = \{y \in \mathbb{Z}_2^n : y.x = 0, \forall x \in C_1\}. \tag{2.9}$$

The ratio $\frac{k}{n}$, called the *rate* of the code, which is presented by R_c. This ratio gives a measure of the cost of error correction. Note that $R_c < 1$. The greater R_c is, the more efficient the code is.

2.4. Surface codes

Let $G = (V, E, F)$ be a tiling whose (V, E) is a graph and F is the set of faces defined by the embedding of (V, E) in a compact surface without overlapping edges. The surface code associated with a tiling $G = (V, E, F)$ is the CSS code defined by two $\mathbb{Z}_2$ matrices H_X and H_Z such that $H_X H_Z^T = 0$, where H_X is the vertex-edge incidence matrix and H_Z is the face-edge incidence matrix. These codes have parameters $[[n, k, d_{min}]]$, where k information qubits are encoded into n physical qubits with minimum distance d_{min}. A quantum code with minimum distance d_{min} can correct all errors up to $\lfloor \frac{d_{min}-1}{2} \rfloor$ qubits.

2.5. Color codes

In this section, we recall of quantum color codes introduced by Bombin and Martin-Delgado in [7,23]. A color code is the CSS code defined by the matrices $H_X = H_Z = H \in \mathcal{M}_{|F|,|V|}(\mathbb{Z}_2)$ such that $H_{i,j} = 1$ if the face f_i includes the vertex v_j. Note that $V = \{v_i\}_{i=1}^{|V|}$ and $F = \{f_i\}_{i=1}^{|F|}$ are the vertices and the faces of the tiling G respectively. Here, assume that the tiling $G = (V, E, F)$ is trivalent, that is every vertex has third degree and the faces of the G can be 3-colored so that two faces that share a common edge do not wear the same color.

In the color codes, unlike the surface codes the qubits are replaced by vertices instead of the edges and the generators of the stabilizers are the face operators. Given a face $f \in F$, the face operator B_f^σ is defined as the tensor product σ_i, $i \in F$ with $\sigma = X, Z$. Equivalently,

$$B_f^\sigma = \bigotimes_{i \in F} \sigma_i \qquad \sigma = X, Z$$

The color code $\mathcal{C}$ consists of the space fixed by the B_f^σ operator is defined by:

$$\mathcal{C} = \{|\psi\rangle : B_f^X|\psi\rangle = |\psi\rangle,\ B_f^Z|\psi\rangle = |\psi\rangle\}$$

The length of the color code associated with G is $n = |V|$, and its dimension is $4 - 2\mathcal{X}$. The $\mathcal{X}$ denotes the Euler characteristic of the surface. When the tiling is orientable, the dimension of the color code is $k = 4g$. The minimum distance of the color code will be the minimum weight of a vector x in $C \setminus C^\perp$. The code Ker$H$ is denoted by C.

In this paper n is the code length, and d_{min} is the minimum distance of the code.

In order to calculate these parameters, we present some basic concepts of hyperbolic geometry. More information on hyperbolic geometry and shrunk lattices may be found in Refs. [7], [12,13,18,19].

Definition 2.1. A hyperbolic polygon P with p edges is a convex closed set consisting of p hyperbolic geodesic segments.

Definition 2.2. A regular tessellation of the hyperbolic plane is a covering of the whole plane by regular polygons, all with the same number of edges, without superposition of such polygons, subject to the constraint that they meet either only at its edges or its vertices.

A regular tessellation is denoted by $\{p, q\}$, where q regular polygons with p edges meet in each vertex. In particular, if $p = q$ the tessellation is said to be self-dual.

Consider a fundamental polygon P' that generates a g-torus. Every possible tiling $\{p, q\}$ of P' satisfies the following equation:

$$\mu(P') = n_f \mu(P), \tag{2.10}$$

In (2.10) $\mu(P')$ denotes the area of P', $\mu(P)$ denotes the area of the polygon with p edges associated with the tiling $\{p, q\}$, and n_f is a positive integer which denotes the number of faces of the tessellation $\{p, q\}$. The n_f is given by, [12],

$$n_f = \frac{4q(g-1)}{pq - 2p - 2q}. \tag{2.11}$$

Here, the length of the code is $n = |V| = n_f \frac{p}{q}$ edges, or qubits.

For a fundamental polygon of $\{4g, 4g\}$, the hyperbolic distance d_h between paired sides is calculated as follows, [19],

$$d_h = 2\text{arccosh}\Big[\frac{\cos(\pi/4\text{g})}{\sin(\pi/4\text{g})}\Big], \tag{2.12}$$

Let $l(p,q)$ denote the hyperbolic length of each edge of a tessellation $\{p,q\}$, thus we have, [19],

$$l(p,q) = \text{arccosh}\Big[\frac{\cos^2(\pi/\text{q}) + \cos(2\pi/\text{p})}{\sin^2(\pi/\text{q})}\Big]. \tag{2.13}$$

Given a regular polygon of $\{p,q\}$, the diameter of its circumscribed circle and an upper bound for an edge of the shrunk lattice are written, respectively as, [13],

$$D(p,q) = 2\text{arccosh}\Big[\frac{\cos(\pi/\text{p})\cos(\pi/\text{q})}{\sin(\pi/\text{p})\sin(\pi/\text{q})}\Big], \tag{2.14}$$

and

$$L(p,q) = l(p,q) + D(p,q). \tag{2.15}$$

Also, a lower bound for the number of the reduced network edges in a non-trivial homology cycle belonging to a shrunk lattice is given as follows, [13],

$$n_e > \frac{d_h}{L(p,q)}, \tag{2.16}$$

Then, the minimum distance of the code is calculated as follows, [13],

$$d_{min} = 2\Big\lceil\frac{d_h}{L(p,q)}\Big\rceil. \tag{2.17}$$

3. Construction of quantum color codes

In this section, new families of quantum color codes from tiling the fundamental polygon P_{4g} by the tessellations $\{\frac{3g+57}{10}, 3\}$ and $\{\frac{g+59}{10}, 3\}$, where $g = 11 + 10m$, $m \geq 3$ are presented.

3.1. Quantum color codes from the tessellation $\{\frac{3g+57}{10}, 3\}$

In this section we present new quantum color codes generated by the approach of Section 2 on compact surfaces of genus $g \geq 41$. For this purpose by considering $q = 3$ and substituting this value in (2.11) for the 41-torus, we have

$$n_f = \frac{480}{p-6};\ p > 6, \tag{3.1}$$

For polygon P_{164} associated with the tessellation $\{18, 3\}$ with the corresponding edge-pairings, using (2.13) and (2.14), we have

$$l(18,3) = \text{arccosh}\Big[\frac{\cos^2(\pi/3) + \cos(2\pi/18)}{\sin^2(\pi/3)}\Big] \approx 1.04,$$

and

$$D(18,3) = 2\text{arccosh}\Big\lceil\frac{\cos(\pi/18)\cos(\pi/3)}{\sin(\pi/18)\sin(\pi/3)}\Big\rceil \approx 3.71,$$

Thus, by (2.15),

$$L(18,3) = l(18,3) + D(18,3) \approx 4.75.$$

Therefore, by using (2.17) the minimum distance of the code is calculated as

$$d_{min} = 2\Big\lceil\frac{d_h}{L(18,3)}\Big\rceil = 2\Big\lceil\frac{9.29}{4.75}\Big\rceil \approx 4.$$

Hence, the quantum color code with parameters $[[240, 164, 4]]$ is constructed.

Now, using a fundamental polygon of $\{4g, 4g\}$ with the corresponding edge-pairings, the quantum color codes will be constructed as follows.

As observed, for the minimum distance $d_{min} = 4$ there are 40 faces of the tessellation. Thus, by using (2.11) with $n_f = 40$ and $q = 3$, we have

$$n_f = \frac{12(g-1)}{p-6}, \tag{3.2}$$

It follows that

$$10p = 3g + 57. \tag{3.3}$$

Since the length of the code is $n = n_f\frac{p}{q}$. Therefore the quantum color codes with parameters $[[4g + 76, 4g, 4]]$, for $g = 11 + 10m,\ m = 3, 5, 7, ..., 2^{30} - 1$ are obtained. For these quantum color codes, the rate is near to 1.

3.2. Quantum color codes from the tessellation $\{\frac{g+59}{10}, 3\}$

Our aim in this section is to provide the quantum color codes based on identification of compact surfaces with hyperbolic tessellations.

In order to construct new quantum color codes, by considering $q = 3$ and substituting this value in (2.11) for the 41-torus, we have

$$n_f = \frac{480}{p-6};\ p > 6, \tag{3.4}$$

For polygon P_{164} associated with the tessellation $\{10, 3\}$ with the corresponding edge-pairings, using (2.13) and (2.14), we have

$$l(10,3) = \text{arccosh}\Big\lceil\frac{\cos^2(\pi/3) + \cos(2\pi/10)}{\sin^2(\pi/3)}\Big\rceil \approx 0.88,$$

and

$$D(10,3) = 2\text{arccosh}\Big\lceil\frac{\cos(\pi/10)\cos(\pi/3)}{\sin(\pi/10)\sin(\pi/3)}\Big\rceil \approx 2.33,$$

Thus, by (2.15),

$$L(10,3) = l(10,3) + D(10,3) \approx 3.21.$$

Therefore, by using (2.17) the minimum distance of the code is calculated as

$$d_{min} = 2\left\lceil \frac{d_h}{L(10,3)} \right\rceil = 2\left\lceil \frac{9.29}{3.21} \right\rceil \approx 6.$$

Hence, the quantum color code with parameters $[[400, 164, 6]]$ is obtained.

Now, using a fundamental polygon of $\{4g, 4g\}$ with the corresponding edge-pairings, the quantum color codes will be constructed as follows.

As mentioned previously, for the minimum distance $d_{min} = 6$ there are 120 faces of the tessellation. Thus, by considering $n_f = 120$ and $q = 3$, and substituting these values in (2.11), we have

$$n_f = \frac{12(g-1)}{p-6}, \tag{3.5}$$

It follows that

$$10p = g + 59. \tag{3.6}$$

Since the length of the code is $n = n_f \frac{p}{q}$. Therefore the quantum color codes with parameters $[[4g+236, 4g, 6]]$, for $g = 11+10m, m = 3, 5, 7, ..., 49$ are obtained. Also, according to the above explanation the quantum color codes from the tessellation $\{\frac{g+59}{10}, 3\}$ and $n_f = 120$ with parameters $[[4g + 236, 4g, 4]]$, for $g = 11 + 10m, m = 51, 53, 55, ..., 1199$ are obtained. For these quantum color codes, the rate is near to 1.

In Table 1 some quantum color codes generated by the above method, on compact surfaces for $g = 21, 41, 121, 1021, 9021$ are provided.

Table 1: Some quantum color codes with $d_{min} = 6, 8, 10, 12, 18$

g	$\{p, q\}$	n_f	$[[n, k, d_{min}]]$
21	$\{12, 3\}$	40	$[[160, 84, 6]]$
21	$\{8, 3\}$	120	$[[320, 84, 8]]$
41	$\{8, 3\}$	240	$[[640, 164, 8]]$
121	$\{8, 3\}$	720	$[[1920, 484, 10]]$
1021	$\{8, 3\}$	6120	$[[16320, 4084, 12]]$
9021	$\{8, 3\}$	54120	$[[144320, 36084, 18]]$

Figure 1, illustrates the polygon P_8 based on the tessellation $\{8, 3\}$ with the corresponding edge-pairings.

For tessellation $\{8, 3\}$ with $g = 2$ using (2.11) we have $n_f = 6$ and $n = 16$. From (2.17), it can be shown that the code distance is $d_{min} = 4$. Hence, we obtain a code with parameters $[[16, 8, 4]]$.

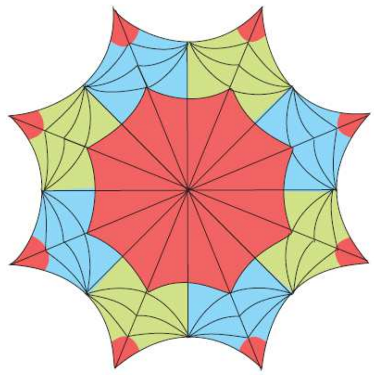

Figure 1: A 8-gon ($g = 2$) surface by the tessellation $\{8, 3\}$ with the corresponding edge-pairings

4. Table of quantum codes comparison

In this section, we present a table of quantum codes which is compared with known classes of quantum codes.

Remark. A list of quantum codes is given in Table 2. The first column shows the value of the length of quantum code. The second column shows the value of k. The third column shows the minimum distance of code. The fourth column shows a list of the quantum codes. All the new quantum color codes is labeled by l in below table. The length of quantum codes having the highest rate $\frac{k}{n}$ is labeled by u.

Table 2: Quantum codes $[[n, k, d_{min}]]$

n	k	d_{min}	$[[n, k, d_{min}]]$
$2r^2$	2	r	$[[2r^2, 2, r]], (r \geq 3)$ [6]
$(\frac{3}{2})r^2$	2	r	$[[(\frac{3}{2})r^2, 2, r]], r = 2m(m \geq 2)$ [11]
r^2	2	r	$[[r^2, 2, r]], r = 2m(m \geq 2)$ [9]
$r^2 + 1$	2	r	$[[r^2 + 1, 2, r]], r = 2m + 1(m \geq 1)$ [8]
$^u 8 + 4s$	$4s$	4	$[[8 + 4s, 4s, 4]], (s \geq 2)$ [13]
$^u 76 + 4s$	$4s$	4	$^l[[76 + 4s, 4s, 4]], (s = 11 + 10m, m = 3, 5, ..., 2^{30} - 1)$
$^u 236 + 4s$	$4s$	6	$^l[[236 + 4s, 4s, 6]], (s = 11 + 10m, m = 3, 5, ..., 49)$
$^u 236 + 4s$	$4s$	4	$^l[[236 + 4s, 4s, 4]], (s = 11 + 10m, m = 51, 53, ..., 1199)$

5. Conclusion

In this paper we have introduced families of quantum color codes on compact surfaces based on identification these surfaces with hyperbolic tessellations. For these quantum color codes the encoding rate is such that $\frac{k}{n} \to 1$ as $n \to \infty$. These quantum color codes are new in the sense that their parameters are not covered by the codes available in the literatures. Future work includes finding quantum color codes with minimum distance greater than or equal to 19.

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