Title

Performance of a single-epoch GNSS integer least-squares ambiguity resolution in the coordinate domain as an ill-posed problem

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Abstract

When a single-epoch observation set is processed, an instantaneous, precise GNSS positioning is an ill-posed problem. To deal with this ill-posed positioning problem, the integer-valued ambiguity parameters can be resolved in the coordinate domain to reduce the dimension of a search space. The ill-posed problem is then smoothed because the unknowns of *n*-dimensional search space are implicitly expressed in the constant three-dimensional space. So, the number of redundant data can now be large enough to perform integer ambiguity resolution. One of the methods which resolve integer ambiguities in the coordinate domain is the Modified Ambiguity Function Approach. This method searches discreetly for optimal integer-valued ambiguity parameters in the constant three-dimensional space using the coordinate function. Due to the ill-conditioning of the single-epoch GNSS mathematical model, the weak approximate GNSS position and its variance-covariance matrix of poor properties still present the problem for well-performing instantaneous integer ambiguity resolution in the coordinate domain. The issue is stabilizing the single-epoch math model by regularization to improve localizing and shaping the search region in the three-dimensional space to contain the actual GNSS position more frequently, representing optimal ambiguities in the sense of integer-least squares principle. The innovation of this contribution is to improve the properties of a single-epoch approximate GNSS position and its variance-covariance matrix in the sense of mean-squared error by regularizing real-valued ambiguity parameters. Thus, the float baseline solution conditioned on regularized float ambiguities is an indirectly regularized approximate GNSS position with the variance-covariance matrix of better precision. A regularized approximate position is promising because its overall accuracy increases thanks to the successful variance reduction. Despite the indirect regularization, an improved initial position is biased, resulting from the biased least-squares estimation. However, the optimal RP values obtained within the quality-based Mean-Squared Error criterion properly govern the contribution of variance reduction in the solution with the regularized bias. In the presence of bias, the regularized single-epoch integer least-squares estimator performs well in the context of correct integer ambiguity resolution in the coordinate domain.

Keywords

Ill-posed problem, single-epoch data set, precise GNSS positioning, ill-conditioned mathematical model, regularization

1 Introduction

Processing of high-quality carrier-phase measurements is essential for precise positioning using satellite technique. In the geodetic community, the double-differenced (DD) observation model is commonly used. Besides three baseline components to estimate, the main problem is the resolution of integer-valued ambiguities. If they are correctly resolved, carrier-phases act as high-precision pseudoranges allowing high-precision GNSS positioning. The problem associated with integer ambiguity resolution (AR) has been under investigation since the early 1980s.

One of the most contributing methods was the ambiguity function (AF). This idea was proposed by Counselman and Gourevitch (Counselman and Gourevitch 1981). It was further developed and applied to different positioning techniques, e.g. kinematic technique (Mader 1992) and attitude determination (Wang et al. 2007). Despite many attempts at AF development, this method is still less efficient than the LAMBDA method (Teunissen 1995). Due to the LAMBDA decorrelation adjustment, LAMBDA is perceived as complete and optimal for fast GNSS positioning. However, the Modified Ambiguity Function Approach (MAFA) presented by (Cellmer et al. 2010) significantly improved the performance of the AF method. In the ambiguity domain, the space dimension for searching the optimal integer ambiguities rise with an increase in satellite number. It is obvious because new ambiguity parameters come to the functional model as unknowns. Here, MAFA makes an advance because the integer AR is performed in the coordinate domain, reducing the computational load. No matter how dimensional the ambiguity space is, integer-valued ambiguities are implicitly expressed in the constant three-dimensional space. In other words, the *n*-dimensional ambiguity space generalizes to a constant three-dimensional space where the coordinate function discreetly resolves integer ambiguities. Their integer nature is preserved. Up-to-date information concerning the MAFA method is presented by Cellmer et al. (2018) and Cellmer et al. (2021). Cellmer et al. (2018) showed a new strategy of the search procedure for an optimal integer solution using the coordinate domain. As a part of the MAFA method, Cellmer et al. (2021) designed an algorithm for determining an optimal step length in the search procedure. However, the weak single-epoch approximate GNSS position and its variance-covariance matrix of poor properties still present the problem for well-performing instantaneous integer AR in the coordinate domain.

MAFA-based instantaneous integer AR is performed by means of the integer-constrained least-squares (ILS) principle in the coordinate domain. In this case, approximate GNSS position and its variance-covariance matrix (VCM) are needed. Since the single-epoch mathematical model is studied, the initial position problem within the least-squares (LS) estimation is ill-posed. The LS estimates of initial position can be meaningless. Thus, an approximate GNSS position and its VCM are of poor properties. The crucial in MAFA is localizing and forming a search region and filling in grid points representing real-valued candidates. They are a'priori positions discreetly expressing some integer-valued ambiguities using the coordinate function. The error ellipsoid of an initial position forms the search region for testing candidates in the sense of ILS principle. The best ILS candidate is the (fixed-)actual GNSS position that discreetly represents the (fixed-)optimal integer ambiguity parameters. A special formulation of a MAFA's algorithm preserves the integer nature of ambiguities in the fixed solution. The search region is centred at the initial position's coordinates. VCM governs the search region's shape. Its size depends on the confidence level assumed in the search procedure. Therefore, the search region is a confidence region in the error ellipsoid of an approximate position. If the search region is constructed based on a weak approximate GNSS position and its VCM of poor properties, an optimal single-epoch integer solution would probably be localized far from the search region centre or outside. The Voronoi cell of a correct solution can non-cover any part of the search region. Therefore, no grid candidates will be inside this cell. So, MAFA-based AR returns an incorrect integer solution in the sense of ILS principle. Note, the term Voronoi cell means here the standardized region containing many real-valued candidates that give the same solution.

Only single-epoch data set processing guarantees an instantaneous integer AR in the coordinate domain. As we know, a problem of single-epoch mathematical modelling is ill-posed. Ill-posedness specifies different classes of problems associated with ill-conditioning, of which linear inverse problem is commonly known (Xu et al. 2006;

Xu 2009). Ill-conditioning results in the single-epoch model instability, usually through a low number of redundant data. Thus, small errors in the poor observation set can disturb the solution accuracy. Besides carrier-phases of well-quality, the code pseudoranges supplement single-epoch data set to guarantee required redundancy. As we know, they are of lower quality. In addition, they are from the same satellites as carrier-phase measurements. Therefore, the geometry construction of positioning is not improved, and the single-epoch math model is still ill-conditioned. A heteroscedastic single-epoch data set does not guarantee that one achieves a sufficient number of redundant data. A reliable solution becomes hard to obtain within LS estimation. Due to the single-epoch model instability, the noise of the LS solution inundates GNSS signals of interest. There is no chance to extract meaningful information from satellite data. In this case, the LS solution is meaningless. The regularization technique can stabilize the ill-conditioned single-epoch math model and improve the accuracy of LS solution in the sense of mean-squared error (MSE) at the cost of regularized bias (Tikhonov 1963). Regularization was successfully applied in different works despite the loss of estimators' unbiasedness (Li et al. 2010; Shen et al. 2012; Teunissen and Khodabandeh 2021; Fischer et al. 2022; Wu and Bian 2022). For instance, Li et al. (2010) proposed AR improvement by baseline parameters regularization or directly ambiguities regularization in fast GNSS positioning. Shen et al. (2012) studied regularized solutions using the bias-corrected approach, leading to increased accuracy. Nevertheless, ionosphere-weighted GNSS parameter estimation is increasingly used to improve the estimators' performance due to atmospheric delays. The general approach concerning regularization of GNSS parameters of interest, especially ionospheric delay, is designed by (Teunissen and Khodabandeh 2021). This approach supports well medium and long GNSS baseline models. In turn, Fischer et al. (2022) designed iterative regularization of LS estimation for single-epoch integer AR. Iterative LS regularization provides a smaller amount of regularized bias while the overall accuracy in the sense of MSE is improved. The impact of regularized bias on the performance of the single-epoch ILS estimator was also analyzed. Teunissen (2001) described the mathematical influence of the general biases on ILS estimation in the ambiguity domain. This work was the ground of the analysis made by Fischer et al. (2022). At the same time, Wu and Bian (2022) contributed the ill-posed integer AR using a promising regularization matrix that reduces to one-parameter regularization thanks to Bayesian statistics. They also analyzed the performance of integer AR because of the biasedness property, however, in the context of lattice theory.

This paper plans to work with the regularization technique to mitigate an ill-conditioning of a single-epoch mathematical model. Based on the shortest observation span, the regularized math model can be sufficiently stable to estimate an improved approximate GNSS position with VCM of good properties at the cost of bias. The innovation of this contribution is to increase the accuracy of float baseline solution in the sense of MSE by regularizing real-valued ambiguity parameters for instantaneous integer AR in the coordinate domain using the MAFA method. The float baseline solution is improved based on the assumption that float ambiguities are regularized. So, the baseline parameters are indirectly regularized because of conditioning on directly regularized ambiguity parameters. For the remainder of this paper, an improved approximate GNSS position is regularized float baseline solution with VCM of better properties. The better properties mean better precision. Despite the bias propagation of regularized float ambiguities on the baseline solution, it can better localize and form the search region in the three-dimensional space. Regularized VCM of better precision can shape error ellipsoid well to contain less number of real-valued MAFA candidates. Therefore, the MAFA method can be more efficient even without applying the decorrelation technique at the stage of ILS estimation. The hypothesis is that calculating an approximate GNSS position within regularized LS estimation increases the probability of the correct integer AR in the coordinate domain when single-epoch data set is processed. The increased probability of the success of instantaneous integer AR is interpreted such that the regularized search region frequently contains the Voronoi cell or its part with the correct integer solution. Note that the current regularization approach can be used for any method resolving integer ambiguities in the coordinate domain. Thus, the regularization of the MAFA method is only an illustration of how the developed approach works. This paper does not intend to analyse any results compared with the LAMBDA solutions. That was a key problem of the earlier work, e.g. Nowel et al. (2018).

2 Mixed integer-real least-squares estimation model

In the concept of GNSS integer ambiguity fixing based on the short baseline model, the starting point is a DD carrier-phase observation equation at a given observational time epoch *t* (Teunissen 1998):

$$\Phi(t) = \frac{1}{\lambda}\rho(t) + a(t) + e, \tag{1}$$

where Φ is DD code-derived carrier-phase measurement. On the right-hand side of the above equation $\frac{1}{\lambda}\rho$ is DD geometric distance between satellite and receiver expressed in cycles, a is the DD integer-valued ambiguity, and e is the carrier-phase measurement random noise. Eq. (1) was simplified by stripping it from its atmospheric and multipath delays. Due to the GNSS short baseline model, we assume appropriate conditions and circumstances to vanish these terms during the differencing process (Teunissen 1998).

To guarantee the required redundancy in the single-epoch model, a code-derived pseudorange measurement intuitively supplements the observation set. The observation equation is as follows:

$$P(t) = \rho(t) + \varepsilon, \tag{2}$$

where P is DD code-derived pseudorange measurement. On the right-hand side of the above equation ε is the pseudorange measurement random noise. After collecting all available carrier-phase observations and code pseudoranges at the given single-epoch, the mixed integer-real model is:

$$\mathbf{y}^{obs} \sim N\left(\mathbf{Aa} + \frac{1}{\lambda}\mathbf{Bb}, \mathbf{Q}_y\right), \mathbf{a} \in \mathbb{Z}^n, \mathbf{b} \in \mathbb{R}^3, \tag{3}$$

where $\mathbf{y}^{obs}_{2nx1}$ is the heteroscedastic vector of ambiguous and non-ambagious observations with the multivariate normal probability distribution as in most GNSS applications. $\mathbf{A}_{2nx(m-3)}$ and $\mathbf{B}_{2nx3}$ are the model matrices containing basis vectors that should be linearly independent. They span the ranges of the subspaces of ambiguity and coordinate. The ambiguity subspace is of *n*-dimensional space $\mathbb{Z}^n$. In turn, the coordinate subspace is of three-dimensional space $\mathbb{R}^3$. $\mathbf{a}_{nx1}$ is the vector of unknown integer ambiguities. $\mathbf{b}_{3x1}$ is the unknown vector of increments to baseline components. $\mathbf{Q}_{\mathbf{y}_{mxm}}$ is the positive-definite VCM representing the a'priori accuracy of DD carrier-phase measurements and DD code pseudoranges. The subscripts mean the dimensions. *n* amounts to one type of DD GNSS observation. In turn, *m* represents the number of parameters. After the linearization of these observation equations, the mixed single-epoch integer-real linear functional model with a stochastic model presents a standard linear ill-posed problem:

$$\mathbf{y} = \mathbf{Aa} + \frac{1}{\lambda}\mathbf{Bb} + \mathbf{e}, \mathbf{Q}_y = \mathrm{diag}\left(\sigma_i^2 \mathbf{U}_i\right) \tag{4}$$

The left-hand side $\mathbf{y}$ is now a vector of differences between observed and computed (based on approximate baseline components) DD observations. In turn, $\mathbf{e}$ is the random error vector of DD carrier-phase measurements and DD code pseudoranges. σ_i^2 are the initial variance factors representing a'priori accuracy of a given type of observation. $\mathbf{U}_i$ are positive (semi-)definite matrices.

In order to solve for (4), the following LS principle with integer ambiguity constraints is given:

$$\left(\breve{\mathbf{a}}, \breve{\mathbf{b}}\right) = \underset{\mathbf{a}\in\mathbb{Z}^n, \mathbf{b}\in\mathbb{R}^3}{\operatorname{argmin}} \left\| \mathbf{y} - \mathbf{A}\mathbf{a} - \frac{1}{\lambda}\mathbf{B}\mathbf{b} \right\|^2_{\mathbf{Q}_\mathbf{y}}, \tag{5}$$

where $\breve{\mathbf{a}} \in \mathbb{Z}^n$ is the ILS estimator of $\mathbf{a}$. In turn, $\breve{\mathbf{b}} \in \mathbb{R}^3$ is the fixed baseline solution conditioned on $\breve{\mathbf{a}}$. *n* is also the number of DD carrier-phase observations amount to the number of integer ambiguities. Since it is essential to consider the relationship between unconstrained LS solution and its integer-constrained counterpart, the objective function of the right-hand side of (5) is orthogonally decomposed into the following sum of squared norms (see Figure 8.1. in Teunissen 1998):

$$\left\| \mathbf{y} - \mathbf{A}\mathbf{a} - \frac{1}{\lambda}\mathbf{B}\mathbf{b} \right\|^2_{\mathbf{Q}_\mathbf{y}} = \left\| \mathbf{y} - \mathbf{A}\hat{\mathbf{a}} - \frac{1}{\lambda}\mathbf{B}\hat{\mathbf{b}} \right\|^2_{\mathbf{Q}_\mathbf{y}} + \left\| \hat{\mathbf{a}} - \mathbf{a} \right\|^2_{\mathbf{Q}_{\hat{\mathbf{a}}}} + \left\| \hat{\mathbf{b}}(\mathbf{a}) - \mathbf{b} \right\|^2_{\mathbf{Q}_{\hat{\mathbf{b}}}} \tag{6}$$

For geometric interpretation of orthogonal decomposition, the reader may refer to Fig. 8.1. in Teunissen (1998). In accordance with Teunissen (2017), the first term on the right-hand side of (6) is constant and the last one is zero when $\mathbf{b} = \hat{\mathbf{b}}(\mathbf{a})$ for any $\mathbf{a}$. Thus, the mixed integer-real constrained LS minimizers of (5) are given:

$$\breve{\mathbf{a}} = \underset{\mathbf{a}\in\mathbb{Z}^n}{\arg\min} \left\| \hat{\mathbf{a}} - \mathbf{a} \right\|_{\mathbf{Q}_{\hat{\mathbf{a}}}} \tag{7}$$

$$\breve{\mathbf{b}} = \hat{\mathbf{b}}(\breve{\mathbf{a}}) = \hat{\mathbf{b}} - \mathbf{Q}_{\hat{\mathbf{b}}\hat{\mathbf{a}}} \mathbf{Q}_{\hat{\mathbf{a}}}^{-1} (\hat{\mathbf{a}} - \breve{\mathbf{a}}) \tag{8}$$

with,

$$\mathbf{Q}_{\hat{\mathbf{b}}(\mathbf{a})} = \mathbf{Q}_{\hat{\mathbf{b}}} - \mathbf{Q}_{\hat{\mathbf{b}}\hat{\mathbf{a}}} \mathbf{Q}_{\hat{\mathbf{a}}}^{-1} \mathbf{Q}_{\hat{\mathbf{a}}\hat{\mathbf{b}}} \tag{9}$$

$\mathbf{Q}_{\hat{\mathbf{a}}}$ is the a'posteriori VCM of float ambiguity solution. $\mathbf{Q}_{\hat{\mathbf{b}}(\mathbf{a})}$ is the a'posteriori VCM of fixed baseline solution conditioned on optimal ILS ambiguities.

3 Modified Ambiguity Function Approach

Carrier-phase measurement noise is about 0.01 cycle (Hofmann-Wellenhof et al. 2008). Thus, the error value of DD carrier-phase measurement should be less than a half cycle. Therefore, the vector of ambiguities at a given single-epoch reads:

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \operatorname{round}\left(\begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_n \end{bmatrix} - \frac{1}{\lambda} \begin{bmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_n \end{bmatrix} \right), \tag{10}$$

where round(.) is a function of rounding to the nearest integer. This equation holds because of the integer nature of ambiguities. Substituting Eq. (10) into (1), the vector-form of error equations based on a single-epoch data set is given as follows:

$$\begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix} = \left(\begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_n \end{bmatrix} - \frac{1}{\lambda} \begin{bmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_n \end{bmatrix} \right) - \mathrm{round}\left(\begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_n \end{bmatrix} - \frac{1}{\lambda} \begin{bmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_n \end{bmatrix} \right) \tag{11}$$

Eq. (11) is linearized using a differentiable function (Cellmer 2012). This function replaces the terms on the right-hand side of the above equation. This process is also presented under (Nowel et al. 2018). The linearized form of (11) can now be formulated in the matrix-vector notation:

$$\mathbf{e} = \boldsymbol{\Delta} - \frac{1}{\lambda}\mathbf{Bb}, \tag{12}$$

where $\boldsymbol{\Delta}$ is the misclosure vector:

$$\boldsymbol{\Delta} = \left(\boldsymbol{\Phi} - \frac{1}{\lambda}\boldsymbol{\rho}\left(\mathbf{b}_0\right) \right) - \mathrm{round}\left(\boldsymbol{\Phi} - \frac{1}{\lambda}\boldsymbol{\rho}\left(\mathbf{b}_0\right) \right) \tag{13}$$

$\mathbf{e} \in \mathbb{R}^n$ is the error vector, $\mathbf{B}_{nx3}$ is the design matrix and $\mathbf{b}_{3x1}$ is the vector of unknown baseline components, i.e. increments to approximate coordinates of GNSS receiver $\mathbf{b}_0$. $\boldsymbol{\Phi}$ is the vector of DD carrier-phase observations. $\boldsymbol{\rho}\left(\mathbf{b}_0\right)$ is the vector of calculated a'priori geometric distances between satellites and receiver based on approximate coordinates $\mathbf{b}_0$. It is clear that (11) does not contain ambiguous parameters. However, (Cellmer et al. 2010) proved that fixed solution considers their integer nature because of expressing integer ambiguities by means of coordinate function. Thus, the criterion of the MAFA method for a fixed baseline solution reads:

$$\breve{\mathbf{b}} = \arg\min_{\mathbf{b}\in\mathbb{R}^3} \left\| \boldsymbol{\Delta} - \frac{1}{\lambda}\mathbf{Bb} \right\|^2_{\mathbf{Q}_{\hat{\mathbf{b}}}} \tag{14}$$

The fixed baseline solution $\breve{\mathbf{b}} \in \mathbb{R}^3$ of $\mathbf{b}$ minimizes the right-hand side criterion of (14). The left-hand side minimizer of (14) can be given:

$$\breve{\mathbf{b}} = \lambda(\mathbf{B}^{\mathrm{T}}\mathbf{Q}_{\hat{\mathbf{b}}}^{-1}\mathbf{B})^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{Q}_{\hat{\mathbf{b}}}^{-1}\boldsymbol{\Delta}, \tag{15}$$

where $\mathbf{Q}_{\hat{\mathbf{b}}}$ is the a'posteriori VCM of float baseline solution which plays the approximate GNSS position role. $\mathbf{Q}_{\hat{\mathbf{b}}}^{-1}$ is the weight matrix. At first glance, (14) seems to be the criterion of unconstrained LS estimation. In fact, Nowel et al. (2018) showed that the criterion of the MAFA method is equivalent to the criterion of constrained LS estimation (5). By orthogonal decomposition, the Fig. 1 links geometrically (14) with (5).

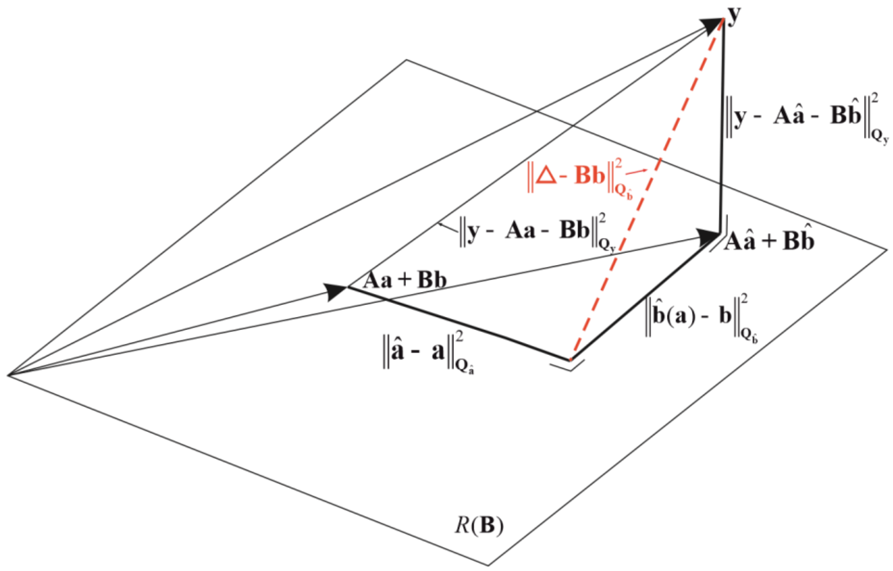

Figure 1 Orthogonal decomposition of the mixed integer-real LS criterion with the principle of the MAFA method

4 Improved approximate GNSS position by regularizing ambiguity parameters

To reveal the value of the criterion of (5), the first term on the right-hand side of (6) must be initially solved. This is the ordinary unconstrained LS problem in GNSS field. The minimizers of an objective function of unconstrained LS criterion together with their a'posteriori VCM are given as follows:

$$\hat{\mathbf{x}}_{\begin{bmatrix}\mathbf{a}\\\mathbf{b}\end{bmatrix}} = \underset{\mathbf{x}_{\begin{bmatrix}\mathbf{a}\\\mathbf{b}\end{bmatrix}}\in\mathbb{R}^m}{\operatorname{argmin}} \left\| \mathbf{y} - \mathbf{N}_{[\mathbf{A}\ \ \mathbf{B}]}\mathbf{x}_{\begin{bmatrix}\mathbf{a}\\\mathbf{b}\end{bmatrix}} \right\|^2_{\mathbf{Q}_y}, \mathbf{Q}_{\hat{\mathbf{x}}_{\begin{bmatrix}\mathbf{a}\\\mathbf{b}\end{bmatrix}}} = \left(\mathbf{N}^{\mathrm{T}}_{[\mathbf{A}\ \ \mathbf{B}]}\mathbf{Q}_y^{-1}\mathbf{N}_{[\mathbf{A}\ \ \mathbf{B}]}\right)^{-1}, \tag{16}$$

where $\hat{\mathbf{x}}_{\begin{bmatrix}\mathbf{a}\\\mathbf{b}\end{bmatrix}}$ is the unconstrained LS estimate of ambiguity and baseline parameters, i.e. $\hat{\mathbf{x}}_{\begin{bmatrix}\mathbf{a}\\\mathbf{b}\end{bmatrix}} = \begin{bmatrix}\hat{\mathbf{a}}\\\hat{\mathbf{b}}\end{bmatrix}$ of $\mathbf{x}_{\begin{bmatrix}\mathbf{a}\\\mathbf{b}\end{bmatrix}} = \begin{bmatrix}\mathbf{a}\\\mathbf{b}\end{bmatrix}$. In the other words, it is the collective vector of LS estimates. $\mathbf{N}_{[\mathbf{A}\ \ \mathbf{B}]} = \begin{bmatrix}\mathbf{A} & \frac{1}{\lambda}\mathbf{B}\end{bmatrix}$ is now the collective model matrix. $\mathbf{Q}_y^{-1}$ is the weight matrix concerning a'priori accuracy of DD carrier-phase measurements and DD code pseudoranges. $\mathbf{Q}_{\hat{\mathbf{x}}_{\begin{bmatrix}\mathbf{a}\\\mathbf{b}\end{bmatrix}}}$ is the a'posteriori VCM of $\hat{\mathbf{x}}_{\begin{bmatrix}\mathbf{a}\\\mathbf{b}\end{bmatrix}}$. Here, ambiguity and baseline parameters are estimated simultaneously. The integer nature of ambiguities is now discarded, so $\mathbf{a}\in\mathbb{R}^n$. The minimizers of unconstrained LS criterion come directly from solving normal equations, and reads:

$$\hat{\mathbf{x}}_{\begin{bmatrix}\mathbf{a}\\\mathbf{b}\end{bmatrix}} = \mathbf{Q}_{\hat{\mathbf{x}}_{\begin{bmatrix}\mathbf{a}\\\mathbf{b}\end{bmatrix}}}\mathbf{N}^{\mathrm{T}}_{[\mathbf{A}\ \ \mathbf{B}]}\mathbf{Q}_y^{-1}\mathbf{y} \tag{17}$$

Since an initial position based on single-epoch data set is weak, regularization can improve its properties. It is important when instantaneous integer AR is discreetly conducted using the coordinate function. To increase the accuracy of float baseline solution, the real-valued ambiguity parameters are enhanced by regularizing. The baseline parameters are then conditioned on regularized float ambiguities. Therefore, the real-valued ambiguities are separately computed by regularized LS estimation. For this purpose, the mixed single-epoch integer-real linear model (4) is multiplied by an orthogonal projector $\mathbf{P}^{\perp}_{\mathbf{B}_{\mathbf{Q}_y^{-1}}} = \mathbf{I}_m - \mathbf{B}(\mathbf{B}^{\mathrm{T}}\mathbf{Q}_y^{-1}\mathbf{B})^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{Q}_y^{-1}$ to eliminate $\mathbf{B}_{\mathbf{Q}_y^{-1}}$:

$$\mathbf{P}_{\mathbf{B}}^{\perp}\mathbf{y} = \mathbf{P}_{\mathbf{B}}^{\perp}\mathbf{A}_{\mathbf{Q}_y^{-1}}\mathbf{a} + \mathbf{P}_{\mathbf{B}}^{\perp}\mathbf{e}, \tag{18}$$

where it is obvious $\mathbf{P}_{\mathbf{B}_{\mathbf{Q}_y^{-1}}}^{\perp}\frac{1}{\lambda}\mathbf{B}_{\mathbf{Q}_y^{-1}}\mathbf{b} = 0$. This matrix orthogonally projects onto a subspace perpendicular to the subspace of the range of $\mathbf{B}_{\mathbf{Q}_y^{-1}}$. Due to the weighted LS problem, only for projection purposes, the ranges of $\mathbf{A}_{\mathbf{Q}_y^{-1}}$ and $\mathbf{B}_{\mathbf{Q}_y^{-1}}$ are expressed together with the weight matrix. This is necessary to show that non-orthonormal base vectors of $\mathbf{Q}_{\mathbf{y}}^{-1}$ visualise the metric in the context of LS. **A** and **B** are used without the weighted subscript in the next parts of paper. In the shorthand notation, (18) has now the following form:

$$\bar{\mathbf{y}} = \bar{\mathbf{A}}_{\mathbf{Q}_y^{-1}}\mathbf{a} + \bar{\mathbf{e}} \tag{19}$$

This is the functional model, projected onto a subspace orthogonal to the subspace of the $\mathbf{B}_{\mathbf{Q}_y^{-1}}$ range. $\bar{\mathbf{y}}$ is the projected vector of differences between observed and computed (based on approximate baseline components) DD observations. $\bar{\mathbf{A}}_{\mathbf{Q}_y^{-1}}$ is the projected model matrix of ambiguities containing basis vectors that should still be linearly independent. Thus, they still span the range of the ambiguity subspace, now orthogonal to a subspace of the $\mathbf{B}_{\mathbf{Q}_y^{-1}}$ range. In turn, $\bar{\mathbf{e}}$ is the projected error vector of DD carrier-phase measurements and DD code pseudoranges. So, the stochastic part is also projected. This can also be defined as Gram-Schmidt orthogonalization. Strang (2004) explains that the first component of the study is assumed to be a reference one. The subspace of the $\mathbf{B}_{\mathbf{Q}_y^{-1}}$ range plays this role. Based on $\mathbf{B}_{\mathbf{Q}_y^{-1}}$ direction, $\mathbf{A}_{\mathbf{Q}_y^{-1}}$ is then orthogonalized.

Since the stochastic part of the model was projected, it is also reasonable to reformulate the VCM representing the a'priori accuracy of heteroscedastic GNSS observations, i.e. ambiguous and non-ambiguous, $\mathbf{Q}_{\mathbf{y}}$. Based on propagation law, the $\bar{\mathbf{Q}}_{\mathbf{y}}$ reads:

$$\bar{\mathbf{Q}}_{\mathbf{y}} = \mathbf{P}_{\mathbf{B}_{\mathbf{Q}_y^{-1}}}^{\perp}\mathbf{Q}_{\mathbf{y}}\mathbf{P}_{\mathbf{B}_{\mathbf{Q}_y^{-1}}}^{\perp\mathrm{T}}, \tag{20}$$

and finally,

$$\bar{\mathbf{Q}}_{\mathbf{y}} = \mathbf{Q}_{\mathbf{y}} - \frac{1}{\lambda^2}\left(\mathbf{B}\mathbf{N}_{\mathbf{b}}^{-1}\mathbf{B}^{\mathrm{T}}\right), \tag{21}$$

where $\mathbf{N}_{\mathbf{b}} = \mathbf{B}^{\mathrm{T}}\mathbf{Q}_{\mathbf{y}}^{-1}\mathbf{B}$. In accordance with Teunissen (2006), the $\mathbf{Q}_{\mathbf{y}}$ must be a symmetric matrix $\left(\mathbf{Q}_{\mathrm{y}}\right)^{\mathrm{T}} = \mathbf{Q}_{\mathrm{y}}$, and a positive-definite matrix, such that $\mathbf{z}^{\mathrm{T}}\mathbf{Q}_{\mathbf{y}}\mathbf{z} > 0$, for all non-zero real column vectors **z**. Therefore, the $\mathbf{Q}_{\mathbf{y}}^{-1}$ inverse exists, and this might be a weight matrix. It keeps when the heteroscedastic data set smooths the single-epoch linear inverse problem up to a sufficient extent. The case does not hold when $\mathbf{Q}_{\mathbf{y}}$ is projected together with its functional model on the range of $\mathbf{B}_{\mathbf{Q}_y^{-1}}$. The projected $\bar{\mathbf{Q}}_{\mathrm{y}}$ is still symmetric. The problem, however, is in the lack of positive definiteness because of some negative and non-negative eigenvalues of $\bar{\mathbf{Q}}_{\mathrm{y}}$. This happens by the projection matrix in (20), which is hardly invertible. The condition $\mathbf{z}^{\mathrm{T}}\bar{\mathbf{Q}}_{\mathbf{y}}\mathbf{z} > 0$ no more holds because some of these eigenvalues occur. Thus, the linear inverse problem of $\bar{\mathbf{Q}}_{\mathrm{y}}$ exists. The Gauss elimination is employed to recover $\bar{\mathbf{Q}}_{\mathrm{y}}$ to be positive-definite, thereby eliminating the inverse problem. Following the Gauss elimination, the row echelon form of $\bar{\mathbf{Q}}_{\mathrm{y}}$ can be achieved. Finding pivot and free columns verifies which ones contribute linear inverse problem. Since negative and non-negative eigenvalues lead matrix's singularity, these eigenvalues can correspond to free columns. They degrade the matrix's positive definiteness. The singular value decomposition (SVD) of $\bar{\mathbf{Q}}_{\mathrm{y}}$ reveals the eigenvalues burdening the invertibility. Discarding these eigenvalues, which correspond to free columns, seems reasonable. Note, this is a single-epoch challenge, so exposing as much meaningful information as possible concerning solutions is hard. Therefore, the generalized inverse should be subsequently admitted. To select the eigenvalues further from zero, the following test is employed:

$$\underset{\lambda\in\mathbb{R}}{\forall}\ \lambda_{1_i} > \kappa \tag{22}$$

The eigenvalues which successfully passed the test are denoted λ_{1_i}. In turn, κ is the small, non-zero, boundary value. The subscript "1" correspond to positive eigenvalues and their eigenvectors. This way, we identify positive eigenvalues and subsequent negative ones are discarded. For the purpose of the generalized inverse, the Moore-Penrose inverse is now employed, and reads:

$$\bar{\mathbf{Q}}_{\mathbf{y}}^{+} = \mathbf{V}_1\boldsymbol{\Sigma}_1^{-1}\mathbf{V}_1^{\mathrm{T}}, \tag{23}$$

where $\mathbf{V}_1$ is a matrix which columns are linearly independent orthonormal vectors. These vectors correspond to positive eigenvalues. The positive eigenvalues are stored on the diagonal of $\boldsymbol{\Sigma}_1$ matrix. In this way, the weight matrix $\bar{\mathbf{Q}}_{\mathbf{y}}^{+}$ is already symmetric and positive-definite, projected inverse of $\mathbf{Q}_{\mathbf{y}}^{-1}$. It is invertible because elimination produces now a full set of pivots. Based on Teunissen (2006), Fig. 2 illustrates the geometric interpretation of the weighted model's projection case.

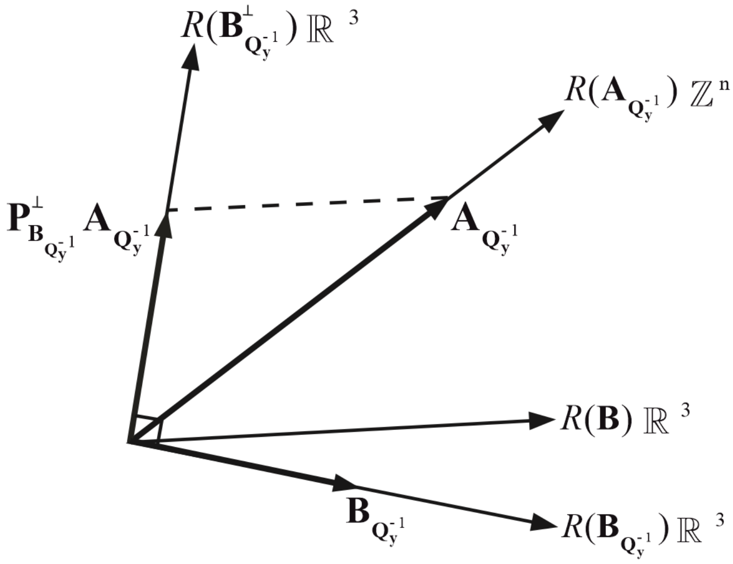

Figure 2 Geometric interpretation of a projection case

Real-valued ambiguity parameters are first estimated based on regularized LS estimation on a subspace orthogonal to the subspace of the **B** range. Discarding the integer nature of ambiguities temporarily in (19), the real-valued ambiguity minimizer of an objective function of unconstrained LS criterion with regularization principle reads:

$$\hat{\mathbf{a}}_{\mathrm{reg}} = \underset{\mathbf{a}\in\mathbb{R}^n}{\operatorname{argmin}}\left\{\left\|\bar{\mathbf{y}} - \bar{\mathbf{A}}\mathbf{a}\right\|^2_{\bar{\mathbf{Q}}_y} + \alpha\left\|\mathbf{a}\right\|^2\right\}, \tag{24}$$

where $\alpha\|\mathbf{a}\|^2$ is the norm of model parameters which offsets the residual norm of observations $\|\bar{\mathbf{y}} - \bar{\mathbf{A}}\mathbf{a}\|^2_{\bar{\mathbf{Q}}_y}$. The optimal balance provides an optimal value of regularization parameter (RP) α. It is essential to satisfy this balance because $\hat{\mathbf{a}}_{\mathrm{reg}}$ must have better properties than its LS counterpart. Taking into account the regularized LS ambiguity estimate $\hat{\mathbf{a}}_{\mathrm{reg}}$ of $\mathbf{a}$, its direct formula is as follows:

$$\hat{\mathbf{a}}_{\mathrm{reg}} = \bar{\mathbf{N}}^{-1}_{a_\alpha}\bar{\mathbf{A}}^{\mathrm{T}}\bar{\mathbf{Q}}^{+}_{\mathbf{y}}\bar{\mathbf{y}}, \tag{25}$$

where $\bar{\mathbf{N}}_{a_\alpha} = \bar{\mathbf{A}}^{\mathrm{T}}\bar{\mathbf{Q}}^{+}_{\mathbf{y}}\bar{\mathbf{A}} + \alpha\mathbf{I}_{\mathrm{m\text{-}3}}$. Based on variance propagation law, the regularized a'posteriori VCM of $\hat{\mathbf{a}}_{\mathrm{reg}}$, which is of better properties, is determined:

$$\mathbf{Q}_{\hat{\mathbf{a}}_{\mathrm{reg}}} = \bar{\mathbf{N}}^{-1}_{a_\alpha}\bar{\mathbf{N}}_{a}\bar{\mathbf{N}}^{-1}_{a_\alpha}, \tag{26}$$

where $\bar{\mathbf{N}}_{a} = \bar{\mathbf{A}}^{\mathrm{T}}\bar{\mathbf{Q}}^{+}_{\mathbf{y}}\bar{\mathbf{A}}$. It is obvious that regularized LS estimation produces biased LS estimate $\hat{\mathbf{a}}_{\mathrm{reg}}$. Using $E\left(\bar{\mathbf{y}}\right) = \bar{\mathbf{A}}\mathbf{a}$, the formula for regularized bias is then derived:

$$\mathbf{d}_{\hat{\mathbf{a}}_{\mathrm{reg}}} = E(\hat{\mathbf{a}}_{\mathrm{reg}}) - \mathbf{a} = -\alpha\bar{\mathbf{N}}^{-1}_{a_\alpha}\mathbf{a} \tag{27}$$

The expectation operator $E(\hat{\mathbf{a}}_{\mathrm{reg}})$ with $\hat{\mathbf{a}}_{\mathrm{reg}}$ is not equal to actual value $\mathbf{a}$. The $\hat{\mathbf{a}}_{\mathrm{reg}}$ is consequently biased. Since $\mathbf{a}$ is never known, the regularized LS ambiguity estimate $\hat{\mathbf{a}}_{\mathrm{reg}}$ can replace $\mathbf{a}$. The formula is now possible to compute. However, bias is not under full control by forced approximation $\hat{\mathbf{a}}_{\mathrm{reg}} \approx \mathbf{a}$:

$$\mathbf{d}_{\hat{\mathbf{a}}_{\mathrm{reg}}} = -\alpha\bar{\mathbf{N}}^{-1}_{a_\alpha}\hat{\mathbf{a}}_{\mathrm{reg}} \tag{28}$$

Since the regularized float ambiguities are contaminated by biasedness, the solution accuracy must be evaluated, including their biased properties. Thus, the MSE matrix (MSEM) is employed to evaluate the accuracy as the sum of VCM and squared matrix of regularized biases:

$$\mathbf{MSE}_{\hat{\mathbf{a}}_{\mathrm{reg}}} = E\left[(\hat{\mathbf{a}}_{\mathrm{reg}} - \mathbf{a})(\hat{\mathbf{a}}_{\mathrm{reg}} - \mathbf{a})^{\mathrm{T}}\right] = \mathbf{Q}_{\hat{\mathbf{a}}_{\mathrm{reg}}} + \mathbf{d}_{\hat{\mathbf{a}}_{\mathrm{reg}}}\mathbf{d}^{\mathrm{T}}_{\hat{\mathbf{a}}_{\mathrm{reg}}} = \bar{\mathbf{N}}^{-1}_{a_\alpha}\left(\bar{\mathbf{N}}_{a} + \alpha^2\mathbf{a}\mathbf{a}^{\mathrm{T}}\right)\bar{\mathbf{N}}^{-1}_{a_\alpha} \tag{29}$$

The MSEM can be interpreted as the matrix which delivers information about improved precision because of the variance reduction in VCM at the expense of regularized bias. In other words, less spread of model parameters around the actual values at the cost of the biased shift of the parameters' geometric centre cloud. This is exactly what accuracy here means. Nevertheless, the problem in (29) lies in an unknown squared matrix of actual values $\mathbf{a}\mathbf{a}^{\mathrm{T}}$. During accuracy analysis, the $\mathbf{a}\mathbf{a}^{\mathrm{T}}$ can be approximated by $\mathbf{Q}_{\hat{\mathbf{a}}_{\mathrm{reg}}}$. However, the quality-based MSEM criterion for estimating an optimal value of α needs computable formula. For this purpose, another approximation of $\mathbf{a}\mathbf{a}^{\mathrm{T}}$ will be proposed.

Conditioning on regularized float ambiguity parameters, the baseline parameters can be now estimated. By separately estimated regularized float ambiguities $\hat{\mathbf{a}}_{\mathrm{reg}}$, the real linear functional model reads:

$$\mathbf{y} = \mathbf{A}\hat{\mathbf{a}}_{\mathrm{reg}} + \frac{1}{\lambda}\mathbf{B}\mathbf{b} + \mathbf{e}, \mathbf{b} \in \mathbb{R}^3 \tag{30}$$

To solve the unknown baseline parameters, but with assumption that they are conditioned on regularized float ambiguities, the following unconstrained LS criterion reads:

$$\hat{\mathbf{b}}_{/\hat{\mathbf{a}}_{\mathrm{reg}}} = \underset{\mathbf{b}\in\mathbb{R}^3}{\operatorname{argmin}} \left\| \left(\mathbf{y} - \mathbf{A}\hat{\mathbf{a}}_{\mathrm{reg}}\right) - \frac{1}{\lambda}\mathbf{B}\mathbf{b} \right\|^2_{\bar{\mathbf{Q}}_{\mathbf{y}}}, \tag{31}$$

where $\hat{\mathbf{b}}_{/\hat{\mathbf{a}}_{\mathrm{reg}}}$ is the float baseline solution, which minimizes the right-hand side LS criterion. The regularized float ambiguities condition float baseline solution, which plays the role of an improved approximate GNSS position. Taking into account the indirectly regularized LS baseline estimate $\hat{\mathbf{b}}_{/\hat{\mathbf{a}}_{\mathrm{reg}}}$ of $\mathbf{b}$, the direct formula reads:

$$\hat{\mathbf{b}}_{/\hat{\mathbf{a}}_{\mathrm{reg}}} = \mathbf{N}_{\mathrm{b}}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{Q}_{\mathbf{y}}^{-1}(\mathbf{y} - \mathbf{A}\hat{\mathbf{a}}_{\mathrm{reg}}) \tag{32}$$

Improvement of baseline parameters can be defined as indirectly conducted by regularizing ambiguity parameters. It is now interesting how more precise the VCM of $\hat{\mathbf{b}}_{/\hat{\mathbf{a}}_{\mathrm{reg}}}$ is, while baseline parameters are not directly regularized. Thus, the impact of variance reduction of ambiguities on baseline parameters precision must be evaluated. It is essential to know that the MAFA method uses information about baseline-ambiguity covariances during the search procedure. Based on variance propagation law, the regularized a'posteriori VCM of $\begin{bmatrix} \hat{\mathbf{a}}_{\mathrm{reg}} \\ \hat{\mathbf{b}}_{/\hat{\mathbf{a}}_{\mathrm{reg}}} \end{bmatrix}$, which is of greater precision, is determined:

$$\begin{bmatrix} \hat{\mathbf{a}}_{\mathrm{reg}} \\ \hat{\mathbf{b}}_{/\hat{\mathbf{a}}_{\mathrm{reg}}} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{N}}_{\mathrm{a}_\alpha}^{-1}\bar{\mathbf{A}}^{\mathrm{T}}\bar{\mathbf{Q}}_{\mathbf{y}}^{+}\mathbf{P}_{\mathbf{B}}^{\perp} \\ \hdashline \mathbf{N}_{\mathrm{b}}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{Q}_{\mathbf{y}}^{-1} - \mathbf{N}_{\mathrm{b}}^{-1}\mathbf{N}_{\mathrm{ba}}\bar{\mathbf{N}}_{\mathrm{a}_\alpha}^{-1}\bar{\mathbf{A}}^{\mathrm{T}}\bar{\mathbf{Q}}_{\mathbf{y}}^{+}\mathbf{P}_{\mathbf{B}}^{\perp} \end{bmatrix}\mathbf{y}, \tag{33}$$

after some mathematical operations, the final formula is derived:

$$\mathbf{Q}_{\begin{bmatrix} \hat{\mathbf{a}}_{\mathrm{reg}} \\ \hat{\mathbf{b}}_{/\hat{\mathbf{a}}_{\mathrm{reg}}} \end{bmatrix}} = \left[\begin{array}{c:c} \mathbf{Q}_{\hat{\mathbf{a}}_{\mathrm{reg}}} & -\mathbf{Q}_{\hat{\mathbf{a}}_{\mathrm{reg}}}\mathbf{N}_{\mathrm{ab}}\mathbf{N}_{\mathrm{b}}^{-1} \\ \hdashline -\mathbf{N}_{\mathrm{b}}^{-1}\mathbf{N}_{\mathrm{ba}}\mathbf{Q}_{\hat{\mathbf{a}}_{\mathrm{reg}}} & \mathbf{N}_{\mathrm{b}}^{-1}\left(\mathbf{N}_{\mathrm{b}} + \mathbf{N}_{\mathrm{ba}}\mathbf{Q}_{\hat{\mathbf{a}}_{\mathrm{reg}}}\mathbf{N}_{\mathrm{ab}}\right)\mathbf{N}_{\mathrm{b}}^{-1} \end{array}\right] \tag{34}$$

Since $\hat{\mathbf{a}}_{\mathrm{reg}}$ is a biased LS estimate of $\mathbf{a}$, the bias propagation on $\hat{\mathbf{b}}_{/\hat{\mathbf{a}}_{\mathrm{reg}}}$ can intuitively be true. Therefore, $\mathbf{d}_{\begin{bmatrix} \hat{\mathbf{a}}_{\mathrm{reg}} \\ \hat{\mathbf{b}}_{/\hat{\mathbf{a}}_{\mathrm{reg}}} \end{bmatrix}}$ is obtained as follows:

$$\mathbf{d}_{\begin{bmatrix} \hat{\mathbf{a}}_{\mathrm{reg}} \\ \hat{\mathbf{b}}_{/\hat{\mathbf{a}}_{\mathrm{reg}}} \end{bmatrix}} = \begin{bmatrix} \bar{\mathbf{N}}_{\mathrm{a}_\alpha}^{-1}\bar{\mathbf{A}}^{\mathrm{T}}\bar{\mathbf{Q}}_{\mathbf{y}}^{+}\mathbf{P}_{\mathbf{B}}^{\perp} \\ \hdashline \mathbf{N}_{\mathrm{b}}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{Q}_{\mathbf{y}}^{-1} - \mathbf{N}_{\mathrm{b}}^{-1}\mathbf{N}_{\mathrm{ba}}\bar{\mathbf{N}}_{\mathrm{a}_\alpha}^{-1}\bar{\mathbf{A}}^{\mathrm{T}}\bar{\mathbf{Q}}_{\mathbf{y}}^{+}\mathbf{P}_{\mathbf{B}}^{\perp} \end{bmatrix} E(\mathbf{y}) - \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}, \tag{35}$$

with $\mathbf{y} = \mathbf{A}\mathbf{a} + \mathbf{B}\mathbf{b}$, the bias formula is derived:

$$\mathbf{d}_{\begin{bmatrix} \hat{\mathbf{a}}_{\mathrm{reg}} \\ \hat{\mathbf{b}}_{/\hat{\mathbf{a}}_{\mathrm{reg}}} \end{bmatrix}} = \begin{bmatrix} -\alpha\bar{\mathbf{N}}_{\mathrm{a}_\alpha}^{-1}\hat{\mathbf{a}}_{\mathrm{reg}} \\ \hdashline \alpha\mathbf{N}_{\mathrm{b}}^{-1}\mathbf{N}_{\mathrm{ba}}\bar{\mathbf{N}}_{\mathrm{a}_\alpha}^{-1}\mathbf{a} \end{bmatrix} \tag{36}$$

However, the problem still lies in the unknown actual values of **a**. For this purpose, the same approximation as in (27) can be used, i.e. $\hat{\mathbf{a}}_{\mathrm{reg}} \approx \mathbf{a}$. Thus, the theoretical formula (36) turns out to be:

$$\mathbf{d}_{\begin{bmatrix}\hat{\mathbf{a}}_{\text{reg}}\\ \hat{\mathbf{b}}_{/\hat{\mathbf{a}}_{\text{reg}}}\end{bmatrix}} = \begin{bmatrix} -\alpha\bar{\mathbf{N}}_{\mathrm{a}_\alpha}^{-1}\hat{\mathbf{a}}_{\text{reg}} \\ \hdashline \alpha\mathbf{N}_{\mathrm{b}}^{-1}\mathbf{N}_{\mathrm{ba}}\bar{\mathbf{N}}_{\mathrm{a}_\alpha}^{-1}\hat{\mathbf{a}}_{\text{reg}} \end{bmatrix} \tag{37}$$

Due to the $E\left(\hat{\mathbf{b}}_{/\hat{\mathbf{a}}_{\text{reg}}}\right) \neq \mathbf{b}$, the float baseline solution is biased. Conditioning on regularized float ambiguities contributes to biasedness property. However, the actual values of **b** does not exist in (37). This can be the confirmation that baseline parameters are not directly regularized but improved on regularized float ambiguities. In other words, $\mathbf{a}$ stands in the bias formula of $\hat{\mathbf{b}}_{/\hat{\mathbf{a}}_{\text{reg}}}$ wherein **b** does not. To evaluate the solution accuracy, the MSEM must be formulated. According to the MSEM property, which consists of VCM and squared matrix of regularized biases, the MSEM formula reads:

$$\mathbf{MSE}_{\begin{bmatrix}\hat{\mathbf{a}}_{\text{reg}}\\ \hat{\mathbf{b}}_{/\hat{\mathbf{a}}_{\text{reg}}}\end{bmatrix}} = E\left[\left(\begin{bmatrix}\hat{\mathbf{a}}_{\text{reg}}\\ \hat{\mathbf{b}}_{/\hat{\mathbf{a}}_{\text{reg}}}\end{bmatrix} - \begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix}\right)\left(\begin{bmatrix}\hat{\mathbf{a}}_{\text{reg}}\\ \hat{\mathbf{b}}_{/\hat{\mathbf{a}}_{\text{reg}}}\end{bmatrix} - \begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix}\right)^{\mathrm{T}}\right] = \left[\begin{array}{c:c} \mathbf{MSE}_{\hat{\mathbf{a}}_{\text{reg}}} & -\mathbf{MSE}_{\hat{\mathbf{a}}_{\text{reg}}}\mathbf{N}_{\mathrm{ab}}\mathbf{N}_{\mathrm{b}}^{-1} \\ \hdashline -\mathbf{N}_{\mathrm{b}}^{-1}\mathbf{N}_{\mathrm{ba}}\mathbf{MSE}_{\hat{\mathbf{a}}_{\text{reg}}} & \mathbf{N}_{\mathrm{b}}^{-1}\left(\mathbf{N}_{\mathrm{b}} + \mathbf{N}_{\mathrm{ba}}\mathbf{MSE}_{\hat{\mathbf{a}}_{\text{reg}}}\mathbf{N}_{\mathrm{ab}}\right)\mathbf{N}_{\mathrm{b}}^{-1} \end{array}\right], \tag{38}$$

where $\mathbf{MSE}_{\hat{\mathbf{a}}_{\text{reg}}} = \bar{\mathbf{N}}_{\mathrm{a}_\alpha}^{-1}\left(\bar{\mathbf{N}}_{\mathrm{a}} + \alpha^2\mathbf{a}\mathbf{a}^{\mathrm{T}}\right)\bar{\mathbf{N}}_{\mathrm{a}_\alpha}^{-1}$. The problem of unknown squared matrix of ambiguity actual values $\mathbf{a}\mathbf{a}^{\mathrm{T}}$ can be approximated exactly by $\mathbf{Q}_{\hat{\mathbf{a}}_{\text{reg}}}$. Due to no need for information for actual values of baseline components in the bias formula, its unknown squared matrix $\mathbf{b}\mathbf{b}^{\mathrm{T}}$ does not exist in the MSEM.

5 Quality-based MSEM criterion for one-parameter ambiguity regularization

Since many good methods are designed for estimating the optimal value of RP, the quality-based MSEM criterion seems most advantageous. This advantage includes the relation between the point, representing the minimum of MSE function with the regularized estimator's statistical quality. However, other approaches are also used, e.g. L-curve (Lawson and Hanson 1995) or generalized cross-validation (Golub et al. 1979). For instance, the L-curve method is attractive because it well balances norms of observation residuals and model parameters. It is good to know that there is a lack of relation between the minimizing point and the optimality of the estimator. Therefore, the MSEM method can be called optimal because it is a quality-based method. This method is technically based on minimizing the trace of MSEM. Thus, the regularized estimator's accuracy is taken into account. However, accuracy properties cannot be fully included because non-diagonal elements are still omitted. Now, we are in position to formulate the criterion of MSEM trace minimization to determine an optimal value of RP for ambiguity parameters:

$$\mathrm{tr}\left(\mathbf{MSE}_{\hat{\mathbf{a}}_{\text{reg}}}(\alpha)\right) = \min, \tag{39}$$

where $\alpha > 0$ because second-order derivative holds $\dfrac{\partial^2 \mathrm{tr}\left(\mathbf{MSE}_{\hat{\mathbf{a}}_{\text{reg}}}(\alpha)\right)}{\partial\alpha^2} > 0$. It is nothing else a computing problem. The optimal value of RP can now be introduced as the real-valued minimizer of the criterion (39):

$$\hat{\alpha} = \arg\min_{\alpha\in\mathbb{R}}\left\{\mathrm{tr}\left(\bar{\mathbf{N}}_{\mathrm{a}_\alpha}^{-1}\left(\bar{\mathbf{N}}_{\mathrm{a}} + \alpha^2\mathbf{a}\mathbf{a}^{\mathrm{T}}\right)\bar{\mathbf{N}}_{\mathrm{a}_\alpha}^{-1}\right)\right\} \tag{40}$$

The biggest problem lies in the squared matrix of actual values of ambiguities $\mathbf{aa}^{\mathrm{T}}$. It is clear that actual values of ambiguity model parameters are unknown. There are many ideas for reformulating this problem, such as proposed by Li et al. (2010) or replacing $\mathbf{aa}^{\mathrm{T}}$ with the classic LS VCM (Fischer et al. 2022). Here, we suggest to use the ambiguity cofactor matrix presented by Cellmer et al. (2022):

$$\mathbf{Q}_{\mathbf{a}_0} = \mathbf{Q}_{\mathbf{y}} + \tfrac{1}{\lambda^2}\mathbf{A}\mathbf{Q}_{\hat{\mathbf{b}}}\mathbf{A}^{\mathrm{T}} \tag{41}$$

$\mathbf{Q}_{\hat{\mathbf{b}}}$ is the VCM of baseline solution obtained within the single point positioning (SPP) technique, for instance, using an algorithm designed by Fischer et al. (2021). We introduce the whole process for clarity, which results in formula (41). The formula (41) is designed based on relation between ambiguity vector and both observation and position vectors:

$$\mathbf{a}(\boldsymbol{\Phi},\mathbf{b}) = \boldsymbol{\Phi} - \frac{1}{\lambda}\boldsymbol{\rho}(\mathbf{b}) \tag{42}$$

By linearizing the right-hand side, the formula (42) turns out to be:

$$\mathbf{a}(\boldsymbol{\Phi},\mathbf{b}) = \boldsymbol{\Phi} - \frac{1}{\lambda}\mathbf{Adx} - \boldsymbol{\rho}(\mathbf{b}^0) \tag{43}$$

Before employing the variance propagation law to (43), its stochastic part is defined. Hence, the following initial cofactor matrix of input data which are of random properties is presented:

$$\mathbf{Q}_{\begin{bmatrix}\boldsymbol{\Phi}\\ \mathbf{b}\end{bmatrix}} = \begin{bmatrix}\mathbf{Q}_{\mathbf{y}} & \\ & \mathbf{Q}_{\hat{\mathbf{b}}}\end{bmatrix} \tag{44}$$

Next, the formula (41) can be derived. Successfully, $\mathbf{aa}^{\mathrm{T}}$ can be approximated by $\mathbf{Q}_{\mathbf{a}_0}$, i.e. $\mathbf{Q}_{\mathbf{a}_0} \approx \mathbf{aa}^{\mathrm{T}}$, for estimating an optimal value of RP. This way, the real-valued minimizer of (39) solves the following criterion by minimizing the MSEM trace:

$$\hat{\alpha} = \arg\min_{\alpha\in\mathbb{R}}\left\{\mathrm{tr}\left(\bar{\mathbf{N}}_{\mathrm{a}_\alpha}^{-1}\left(\bar{\mathbf{N}}_{\mathrm{a}} + \alpha^2\mathbf{Q}_{\mathbf{a}_0}\right)\bar{\mathbf{N}}_{\mathrm{a}_\alpha}^{-1}\right)\right\} \tag{45}$$

$\hat{\alpha}$ is an optimal value of RP that best minimizes the trace of MSEM. Within this paper, (45) is solved using the highest gradient slope technique (HGS). Figure 3 illustrates the search for an optimal value of RP based on the HGS technique.

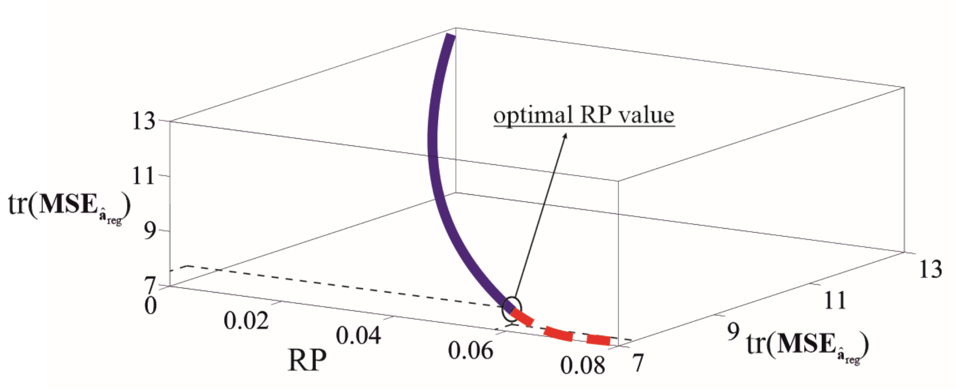

Figure 3 Determining an optimal RP value based on HGS technique

6 Regularized search region in the coordinate space

The parameters of the error ellipsoid form the search region for an optimal integer solution using the coordinate function in the constant three-dimensional space. These parameters are the ellipsoid centre in the coordinates of an approximate GNSS position, length of its axes and their orientation. Right now, we need an initial position coordinates $\mathbf{b}_0$ and its VCM $\mathbf{Q}_{\mathbf{b}_0}$. Instead $\mathbf{b}_0$, vector of improved approximate GNSS position conditioned on regularized float ambiguities $\hat{\mathbf{b}}_{/\hat{\mathbf{a}}_{\mathrm{reg}}}$ is an initial position. The regularized float baseline solution coordinates determine the centre of a regularized search region in the constant three-dimensional space. In turn, regularized VCM of better properties $\mathbf{Q}_{\hat{\mathbf{b}}/\hat{\mathbf{a}}_{\mathrm{reg}}}$ replaces $\mathbf{Q}_{\mathbf{b}_0}$. This is a VCM of regularized float baseline solution. Before setting the length of error ellipsoid axes, the eigenvalue decomposition of $\mathbf{Q}^{-1}_{\hat{\mathbf{b}}/\hat{\mathbf{a}}_{\mathrm{reg}}}$ is carried out:

$$\mathbf{Q}^{-1}_{\hat{\mathbf{b}}/\hat{\mathbf{a}}_{\mathrm{reg}}} = \mathbf{S}^{\mathrm{T}} \begin{bmatrix} \lambda_x & & \\ & \lambda_y & \\ & & \lambda_z \end{bmatrix} \mathbf{S} \tag{46}$$

Since $\mathbf{Q}^{-1}_{\hat{\mathbf{b}}/\hat{\mathbf{a}}_{\mathrm{reg}}}$ is symmetric and positive-definite, so this is a diagonalizable matrix where we have the full set of non-repeated eigenvalues and their corresponding linearly independent eigenvectors. $\mathbf{S}$ is the orthonormal matrix whose columns are independent eigenvectors. In the middle, the diagonal matrix of eigenvalues λ_i is. Thanks to such diagonalization of $\mathbf{Q}^{-1}_{\hat{\mathbf{b}}/\hat{\mathbf{a}}_{\mathrm{reg}}}$, the axes length of the ellipsoid error can be determined as follows (see left part of Fig. 4 for geometric interpretation) :

$$r_x = \sqrt{\frac{\chi_c^2}{\lambda_x}}, r_y = \sqrt{\frac{\chi_c^2}{\lambda_y}}, r_z = \sqrt{\frac{\chi_c^2}{\lambda_z}}, \tag{47}$$

, where r_x, r_y, r_z are the axis lengths of error ellipsoid, and χ_c^2 is the critical value of the random variable of noncentral Chi-square distribution $\chi^2\left(f, \left\|\mathbf{d}_{\hat{\mathbf{b}}/\hat{\mathbf{a}}_{\mathrm{reg}}}\right\|^2_{\mathbf{Q}_{\hat{\mathbf{b}}/\hat{\mathbf{a}}_{\mathrm{reg}}}}\right)$. Forming the regularized search region in the coordinate space, thereby using the noncentral Chi-square distribution, is an innovation in the search procedure of the MAFA method. The realization of the random variable is a vector of baseline components. χ_c^2 is a positive-constant scalar evaluated at the corresponding probability represented by the confidence level. The critical value is calculated as the inverse of the non-central Chi-square cumulative distribution function (CDF) with *f* degrees of freedom and a non-centrality parameter obtained based on squared norm: $\left\|\mathbf{d}_{\hat{\mathbf{b}}/\hat{\mathbf{a}}_{\mathrm{reg}}}\right\|^2_{\mathbf{Q}_{\hat{\mathbf{b}}/\hat{\mathbf{a}}_{\mathrm{reg}}}} = \mathbf{d}^{\mathrm{T}}_{\hat{\mathbf{b}}/\hat{\mathbf{a}}_{\mathrm{reg}}} \mathbf{Q}^{-1}_{\hat{\mathbf{b}}/\hat{\mathbf{a}}_{\mathrm{reg}}} \mathbf{d}_{\hat{\mathbf{b}}/\hat{\mathbf{a}}_{\mathrm{reg}}}$. In other words, the non-centrality parameter is approximated by regularized bias. The Chi-square distribution must be considered non-central because LS baseline estimates lose their unbiasedness due to conditioning on (biased-)regularized LS ambiguities. We are now in a position to formulate the base of formula (47):

$$P\left(\chi_f^2 \le \chi_c^2\right) = p_c, \tag{48}$$

where $\chi_f^2 = \left(\mathbf{b} - \hat{\mathbf{b}}_{/\hat{\mathbf{a}}_{\mathrm{reg}}}\right)^{\mathrm{T}} \mathbf{Q}^{-1}_{\hat{\mathbf{b}}/\hat{\mathbf{a}}_{\mathrm{reg}}} \left(\mathbf{b} - \hat{\mathbf{b}}_{/\hat{\mathbf{a}}_{\mathrm{reg}}}\right)$ can be rewritten as follows:

$$\left(\mathbf{b}-\hat{\mathbf{b}}_{/\hat{\mathbf{a}}_{\mathrm{reg}}}\right)^{\mathrm{T}}\mathbf{Q}_{\hat{\mathbf{b}}/\hat{\mathbf{a}}_{\mathrm{reg}}}^{-1}\left(\mathbf{b}-\hat{\mathbf{b}}_{/\hat{\mathbf{a}}_{\mathrm{reg}}}\right)\leq\chi_{\mathrm{c}}^{2} \tag{49}$$

We obtained the theoretical formulation for the search region size using the confidence region with an assumed confidence level $\mathrm{p_c}$. Factorizing $\mathbf{Q}_{\hat{\mathbf{b}}/\hat{\mathbf{a}}_{\mathrm{reg}}}^{-1}$ in the sense of eigenvalue decomposition, the left-hand side term of (49) turns out to be:

$$\left(\mathbf{b}-\hat{\mathbf{b}}_{/\hat{\mathbf{a}}_{\mathrm{reg}}}\right)^{\mathrm{T}}\mathbf{S}^{\mathrm{T}}\mathbf{S}\mathbf{Q}_{\hat{\mathbf{b}}/\hat{\mathbf{a}}_{\mathrm{reg}}}^{-1}\mathbf{S}^{\mathrm{T}}\mathbf{S}\left(\mathbf{b}-\hat{\mathbf{b}}_{/\hat{\mathbf{a}}_{\mathrm{reg}}}\right)=\mathbf{b}_{\mathrm{s}}^{\mathrm{T}}\boldsymbol{\Lambda}\mathbf{b}_{\mathrm{s}} \tag{50}$$

where $\boldsymbol{\Lambda}=\mathbf{S}\mathbf{Q}_{\hat{\mathbf{b}}/\hat{\mathbf{a}}_{\mathrm{reg}}}^{-1}\mathbf{S}^{\mathrm{T}}=\begin{bmatrix}\lambda_x & & \\ & \lambda_y & \\ & & \lambda_z\end{bmatrix}$ and $\mathbf{b}_{\mathrm{s}}=\mathbf{S}\left(\mathbf{b}-\hat{\mathbf{b}}_{/\hat{\mathbf{a}}_{\mathrm{reg}}}\right)$. $\mathbf{b}_{\mathrm{s}}$ is a vector of point coordinates inside the error ellipsoid. Bear in mind that these coordinates are expressed in the local coordinate system of an error ellipsoid (LCSEE) (see Cellmer et al. 2021). The origin of this system, and the orientation of the main axes, are identified with the origin and axes orientation of the error ellipsoid. The back transformation can be made by multiplying both sides of $\mathbf{b}_{\mathrm{s}}$ equation by $\mathbf{S}^{\mathrm{T}}$. Now, we are in a position to express the size of the regularized search region along every axes using the canonical form:

$$\frac{x_s^2}{r_x^2}+\frac{y_s^2}{r_y^2}+\frac{z_s^2}{r_z^2}<1 \tag{51}$$

For more information the reader may refer to Cellmer et al. (2018). This contribution will also study the impact of improved precision VCM, i.e. $\mathbf{Q}_{\hat{\mathbf{b}}/\hat{\mathbf{a}}_{\mathrm{reg}}}$, on forming up the search region at the cost of regularized bias. The essential is to check how regularized approximate GNSS position and its VCM of superior properties influence the search region biased shift due to regularized bias. Regularized search region can be defined as reduced compared to the original one because of less spread of baseline model parameters around the actual values at the expense of bias. Figure 4 presents an example of an original search region and regularized one.

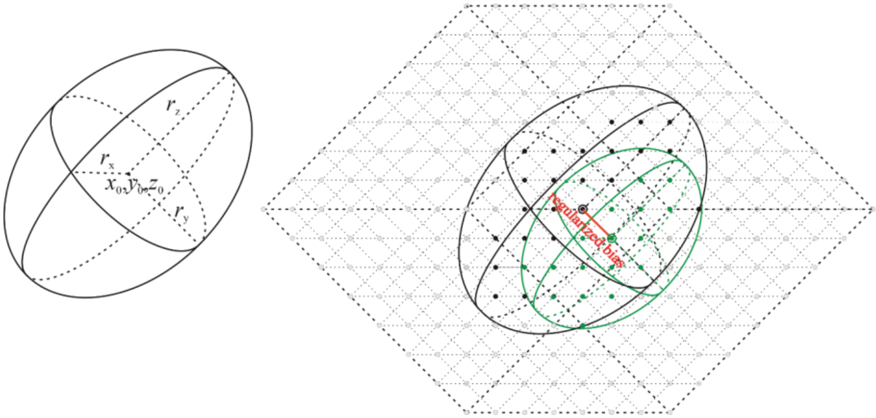

Figure 4 Search region in the shape of an error ellipsoid of an approximate GNSS position (left part). An example of an original (black) and regularized (green) search region with the presence of bias (right part)

7 Designing the numerical experiment

The simulation experiment provides many independent repetitions of single-epoch data samples, which can hardly be achieved using real GNSS data (Wieser 2004). The GNSS simulation study is flexible and guarantees different geometric constructions of positioning in each single-epoch, including highly favourable and unfavourable variants. The biggest advantage is full control of the computation environment and allowance in omitting the factors that are not the study's subject. If the simulation is well prepared, there is no way to choose the computational window in which our solutions work good. Experiments based on real GNSS data make it possible to choose the specified computational window, favouring testing ideas. In this way, the solutions could not be verified. The thing is excluded in the simulation.

To guarantee the high level of single-epoch GNSS data variability, the following way of GNSS data simulation is presented:

1. Satellite positions for simulating carrier-phase and code pseudorange in a single-epoch are not necessary. Only azimuths and elevation angles are required for forming the model matrix. In contrast, the carrier-phase and code random errors are necessary for creating disclosure vector in the linearized system of observation equations. This can be easily explained based on the undifferenced observation equations for a given single-epoch at time *t*:

$$\begin{gathered} \Phi_i^j(t) = \frac{1}{\lambda}\rho_i^j(t) + a_i^j(t) + \varepsilon_i^j(t) + \text{other terms}(t) \\ P_i^j(t) = \rho_i^j(t) + e_i^j(t) + \text{other terms}(t) \end{gathered} \tag{52}$$

The *other terms* contain terms such as atmospheric corrections, clock errors, etc., that are vanished during differentiation. It is possible because of simulation experiment. They are non-essential for the experiment viewpoint purposes. The subscript *rov* means rover's station. In turn, *ref* means reference station.

2. Let's imagine that GNSS observation can be identified with the actual distance between *ith* station and *jth* satellite:

$$\begin{gathered} \Phi_i^j = \frac{1}{\lambda}\rho_{i\,(\text{actual})}^j \\ P_i^j = \rho_{i\,(\text{actual})}^j \end{gathered} \tag{53}$$

Then, it is assumed that undifferenced observations of carrier-phase and code pseudorange consist of the following components:

$$\begin{gathered} \frac{1}{\lambda}\rho_{i\,(\text{actual})}^j(t) = \frac{1}{\lambda}\rho_i^j(t) + a_i^j(t) + \varepsilon_i^j(t) + \text{other terms}(t), \\ \rho_{i\,(\text{actual})}^j(t) = \rho_i^j(t) + e_i^j(t) + \text{other terms}(t), \end{gathered} \tag{54}$$

where $\rho_{i(\text{actual})}^j$ is the actual distance between *ith* station and *jth* satellite. The carrier-phase observation error $e_{\Phi i}^j$ is generated as a variable of the normal distribution N[0; 0.01 cycle], N[0; 0.02 cycle], and N[0; 0.03 cycle]. Based on the accuracy of carrier-phase measurement, which is assumed to be one hundred times higher than that of code pseudorange, the code pseudorange observation error $e_{\Phi i}^j$ is simulated. The ambiguities are randomly generated as integer-valued after double-differencing.

3. Each of the geometric distance on the right-hand-side of equation (54) can be expressed as a sum of two components:

$$\rho_i^j = \rho_{i\,(\text{actual})}^j + \Delta\rho_i^j, \tag{55}$$

where $\Delta\rho_i^j$ is a part of a computed geometric distance, resulting from a shift of approximated GNSS position concerning the actual one. The term $\rho_{i\,(\text{actual})}^j$ appears on both sides of equations (54). Therefore, it can be removed from the equations. Then, the following formulation of residuals can express the undifferenced observations at the given observational time epoch *t*:

$$
\begin{aligned}
-\varepsilon_i^j(t) &= \frac{1}{\lambda}\Delta\rho_i^j(t) + a_i^j(t) + \text{other terms}(t) \\
-e_i^j(t) &= \Delta\rho_i^j(t) + \text{other terms}(t)
\end{aligned}
\tag{56}
$$

4. The DD linearized system of the observation equations of carrier phase and code pseudorange read:

$$
\begin{aligned}
-\mathbf{e}_\Phi &= \frac{1}{\lambda}\mathbf{Adx} + \mathbf{a}, \\
-\mathbf{e}_P &= \mathbf{Adx},
\end{aligned}
\tag{57}
$$

where $\mathbf{e}_\Phi$ and $\mathbf{e}_P$ are theoretical DD error vectors of the carrier phase measurements and code pseudoranges, respectively. $\mathbf{a}$ is the vector of DD integer ambiguities. The term $\mathbf{Adx}$ is a DD linearized version of the terms $\Delta\rho_i^j$ of (56). The model matrix **A** is multiplied by the shift of approximate GNSS position vector **dx** to be estimated. The DD errors and ambiguities in formula (57) are formed by differencing respective vectors with generated entries. Thus, the last problem in data simulation, besides generating errors and ambiguities, is preparing the appropriate model matrix **A**. As was mentioned before, the satellite positions for doing it are not necessary. The *jth* row of the undifferenced model matrix $\mathbf{A_u}$ related to the block of baseline components consist of the ratios:

$$
\mathbf{a}_{ui} = \left[-\frac{\Delta x_{rov}^j}{\rho_{rov}^j}, -\frac{\Delta y_{rov}^j}{\rho_{rov}^j}, -\frac{\Delta z_{rov}^j}{\rho_{rov}^j} \right]
\tag{58}
$$

The $\Delta x_{rov}^j, \Delta y_{rov}^j, \Delta z_{rov}^j$ are satellite-rover vector components: $\rho_{rov}^j = \sqrt{\left(\Delta x_{rov}^j\right)^2, \left(\Delta y_{rov}^j\right)^2, \left(\Delta z_{rov}^j\right)^2}$. We can compute a_{uj} entries based on the *jth* satellite's azimuth and elevation angle in a local, horizontal coordinate system:

$$
\mathbf{a}_{uj} = \left[-\cos\left(El_j\right)\cos\left(Az_j\right), -\cos\left(El_j\right)\sin\left(Az_j\right), -\sin\left(El_j\right) \right]
\tag{59}
$$

Thus, it is enough to generate satellites' azimuths and elevation angles to form a model matrix for undifferenced observations. The model matrix for DD observations is created by subtracting the row related to the reference satellite from the other rows of the $\mathbf{A_u}$ matrix:

$$
\mathbf{a}_j = \mathbf{a}_{uj} - \mathbf{a}_{uref}\text{, for all } j \neq \text{ref.}
\tag{60}
$$

The values of azimuths and elevation angles have been generated as random numbers from ranges: (0°;180°) and (10°;90°), respectively. The simulation has been repeated 10000 times for data generated by different measurement noises. The complex algorithm concerning the improvement of an approximate GNSS position by regularizing real-valued ambiguity parameters has been summarized in Tab. 1.

Table 1 Research scheme

Solution	Formula
1) Estimation of RP	$\hat{\alpha} = \arg\min_{\alpha\in\mathbb{R}} \left\{ \mathrm{tr}\left(\bar{\mathbf{N}}_{a_\alpha}^{-1} \left(\bar{\mathbf{N}}_a + \alpha^2 \mathbf{Q}_{\mathbf{a_0}} \right) \bar{\mathbf{N}}_{a_\alpha}^{-1} \right) \right\}$ (45); $\mathbf{Q}_{\mathbf{a_0}} = \mathbf{Q_y} + \frac{1}{\lambda^2}\mathbf{A}\mathbf{Q}_{\hat{\mathbf{b}}}\mathbf{A}^T$ (41)
2) Regularized float ambiguity solution	$\hat{\mathbf{a}}_{reg} = \bar{\mathbf{N}}_{a_\alpha}^{-1}\bar{\mathbf{A}}^T\bar{\mathbf{Q}}_{\mathbf{y}}^{+}\bar{\mathbf{y}}$ (25)
3) Improved float baseline solution by regularizing ambiguity parameters	$\hat{\mathbf{b}}_{/\hat{\mathbf{a}}_{reg}} = \mathbf{N}_b^{-1}\mathbf{B}^T\mathbf{Q}_{\mathbf{y}}^{-1}(\mathbf{y} - \mathbf{A}\hat{\mathbf{a}}_{reg})$ (32)

4) Regularized VCM	$$\mathbf{Q}_{\begin{bmatrix}\hat{\mathbf{a}}_{\mathrm{reg}}\\ \hat{\mathbf{b}}_{/\hat{\mathbf{a}}_{\mathrm{reg}}}\end{bmatrix}}=\left[\begin{array}{c:c}\mathbf{Q}_{\hat{\mathbf{a}}_{\mathrm{reg}}} & -\mathbf{Q}_{\hat{\mathbf{a}}_{\mathrm{reg}}}\mathbf{N}_{\mathrm{ab}}\mathbf{N}_{\mathrm{b}}^{-1}\\ \hdashline -\mathbf{N}_{\mathrm{b}}^{-1}\mathbf{N}_{\mathrm{ba}}\mathbf{Q}_{\hat{\mathbf{a}}_{\mathrm{reg}}} & \mathbf{N}_{\mathrm{b}}^{-1}\left(\mathbf{N}_{\mathrm{b}}+\mathbf{N}_{\mathrm{ba}}\mathbf{Q}_{\hat{\mathbf{a}}_{\mathrm{reg}}}\mathbf{N}_{\mathrm{ab}}\right)\mathbf{N}_{\mathrm{b}}^{-1}\end{array}\right] \quad (34)$$
5) Regularized bias	$$\mathbf{d}_{\begin{bmatrix}\hat{\mathbf{a}}_{\mathrm{reg}}\\ \hat{\mathbf{b}}_{/\hat{\mathbf{a}}_{\mathrm{reg}}}\end{bmatrix}}=\left[\begin{array}{c}-\alpha\bar{\mathbf{N}}_{\mathrm{a}_\alpha}^{-1}\hat{\mathbf{a}}_{\mathrm{reg}}\\ \hdashline \alpha\mathbf{N}_{\mathrm{b}}^{-1}\mathbf{N}_{\mathrm{ba}}\bar{\mathbf{N}}_{\mathrm{a}_\alpha}^{-1}\hat{\mathbf{a}}_{\mathrm{reg}}\end{array}\right] \quad (37)$$
6)MSEM	$$\mathbf{MSE}_{\begin{bmatrix}\hat{\mathbf{a}}_{\mathrm{reg}}\\ \hat{\mathbf{b}}_{/\hat{\mathbf{a}}_{\mathrm{reg}}}\end{bmatrix}}=\left[\begin{array}{c:c}\mathbf{MSE}_{\hat{\mathbf{a}}_{\mathrm{reg}}} & -\mathbf{MSE}_{\hat{\mathbf{a}}_{\mathrm{reg}}}\mathbf{N}_{\mathrm{ab}}\mathbf{N}_{\mathrm{b}}^{-1}\\ \hdashline -\mathbf{N}_{\mathrm{b}}^{-1}\mathbf{N}_{\mathrm{ba}}\mathbf{MSE}_{\hat{\mathbf{a}}_{\mathrm{reg}}} & \mathbf{N}_{\mathrm{b}}^{-1}\left(\mathbf{N}_{\mathrm{b}}+\mathbf{N}_{\mathrm{ba}}\mathbf{MSE}_{\hat{\mathbf{a}}_{\mathrm{reg}}}\mathbf{N}_{\mathrm{ab}}\right)\mathbf{N}_{\mathrm{b}}^{-1}\end{array}\right] \quad (38)$$

7.1 Mitigating the single-epoch GNSS mathematical model ill-conditioning

The shortest observation span leads to GNSS mathematical model instability. Thus, it is hard to obtain reliable solutions with high accuracy level when there is not enough good-quality redundant data. To smooth this ill-posed problem, the meaningful information from single-epoch data sets must be exploited. However, they are sometimes more contaminated by observation noises and do not deliver enough good information. To not abandon this problem and, as a consequence, deal with it, the regularized LS estimation is designed and developed. The improvement by regularizing can make the math model stable as a result of eliminating its ill-conditioning. Then, the accuracy of an approximate GNSS position is increased in the sense of MSE. The regularized VCM of improved float baseline solution is of superior properties, which provide a better metric for performing single-epoch ILS ambiguity estimation in the coordinate domain even when bias is large. Figure 5 presents the improved condition number of the regularized single-epoch math model against the original one depending on PDOP growth.

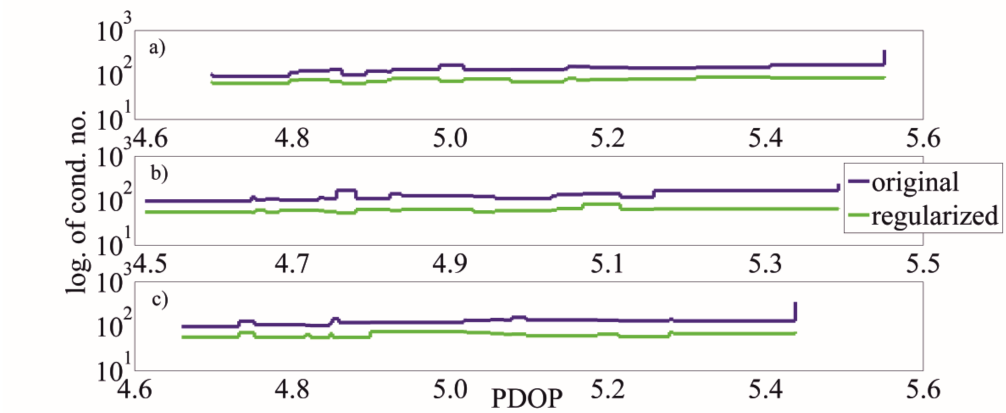

Figure 5 Condition number of the original and regularized single-epoch model: a) $\sigma = 0.01$cycle , b) $\sigma = 0.02$cycle , c) $\sigma = 0.03$cycle

The condition number was computed as the highest-to-lowest singular values ratio using the SVD of the normal equations matrix. The singular values were obtained as the squared root of the eigenvalues of the normal matrix because of the non-squared model matrix. Here, the condition numbers are the mean values from the slot of 500 single-epoch data samples. In the same way, the values of PDOP were averaged. Thanks to that, the visual presentation is more readable. Based on the obtained results, the condition numbers of the single-epoch GNSS mathematical model are improved when regularization is used. Thus, the model ill-conditioning is mitigated. So, the instability is reduced to some extent. Due to the extreme ill-conditioning of a single-epoch model, the instability cannot be fully eliminated. Rather than using the original model, the regularized single-epoch model provides

GNSS positioning as a well-posed problem at the cost of regularized bias. The regularized model is better conditioned. This well-posedness holds when we compare it to the original single-epoch math model.

7.2 Accuracy of improved single-epoch approximate GNSS position at the cost of regularized bias

Table 2 Accuracy of an approximate GNSS position

Carrier-phase nominal accuracy/Solution of an approximate GNSS position	Variance [m^2]			**Overall regularized bias** [m^2]	Mean-squared error [m^2]		
	$\overline{V}_{\hat{x}}$	$\overline{V}_{\hat{y}}$	$\overline{V}_{\hat{z}}$		$m\overline{s}e_{\hat{x}}$	$m\overline{s}e_{\hat{y}}$	$m\overline{s}e_{\hat{z}}$
		Overall precision [m^2]				**Overall accuracy** [m^2]	
σ = 0.01cycle							
original float baseline solution (OFBS)	0.508	0.605	2.435	-	0.508	0.605	2.435
		3.548		-		**3.548**	
regularized float baseline solution (RFBS)	0.397	0.421	1.672	-	0.406	0.432	1.725
		2.490		**0.073**		**2.563**	
σ = 0.02cycle							
OFBS	1.859	1.790	7.375	-	1.859	1.790	7.375
		11.024		-		**11.024**	
RFBS	0.509	0.545	1.911	-	0.601	0.640	2.286
		2.965		**0.562**		**3.527**	
σ = 0.03cycle							
OFBS	4.599	5.262	20.819	-	4.599	5.262	20.819
		30.680		-		**30.680**	
RFBS	0.713	0.730	2.733	-	0.916	0.942	3.565
		4.176		**1.247**		**5.423**	

Table 2 presents the specific and general accuracy of an approximate GNSS position for precise positioning using the MAFA method. Here, the original float baseline solution (OFBS) and regularized float baseline solution (RFBS) are an original and regularized approximated positions. Precision, regularized bias and final accuracy are studied for different variants of carrier-phase noise. Here, the overall precision is calculated as the mean trace of the single traces from the VCMs of float baseline solutions. The overall regularized bias is determined as the mean trace of the single traces from the squared matrix of regularized baseline biases. The overall accuracy is the mean trace of the single traces from the MSEMs of float baseline solutions. Despite the OFBS being of unbiasedness property, the precision of RFBS is superior at the cost of regularized bias. The accuracy of RFBS increases in the sense of MSE. This conclusion holds in each variant concerning nominal accuracy of carrier-phase measurement. Although the RFBS's accuracy improves, the estimator's bias is large. It grows if measurement noise also rises, what's is obvious. The bias formula of the regularized float baseline solution did not contain the unknown baseline components. The RP of ambiguity parameters and approximated by regularized solution actual ambiguities propagate regularized bias to baseline solution to a large extent. However, the question is whether discarding the estimator's unbiased nature to such an extent is worth the better accuracy. The profits of the better MSEM properties are evident. An optimal RP value balances the contributions of variance reduction in the sense of better precision with a regularized bias to the regularized solution. Although the improved float baseline solution is conditioned on regularized real-valued ambiguities, Fig. 6 shows that this balance is well-governed and well-controlled by the optimal RP values of ambiguity parameters. Therefore, it seems trustworthy to abandon the unbiasedness because in estimating an optimal RP based on quality-based MSEM criterion, the further accuracy of a float solution is considered.

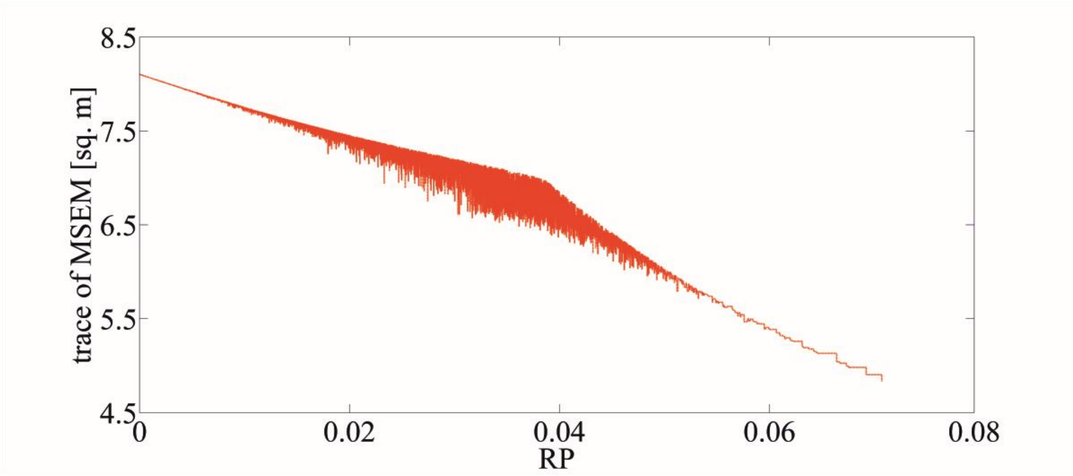

Figure 6 Minimized MSEM traces by the optimal RP values

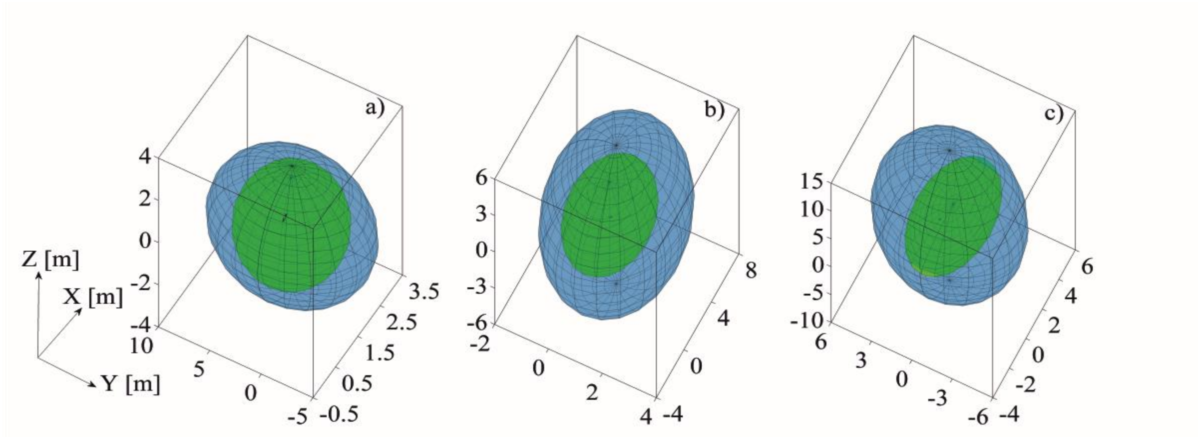

Figure 7 Search regions: a) $\sigma = 0.01$cycle , b) $\sigma = 0.02$cycle , c) $\sigma = 0.03$cycle

Considering Fig. 7 and solutions accuracy (Tab. 2), the search region at each variant are of smaller volume when regularization is used. The blue ones represent the original error ellipsoids. In turn, the greens are regularized. The error ellipsoids created by the original VCM include the error ellipsoids of regularized VCM. So, the variance reduction in the sense of better precision is beneficial, providing the better-sized search regions at the cost of regularized bias. Despite the relatively large bias, there is no significant biased shift concerning the regularized search regions' centers. Considering the regularized MAFA candidates in the coordinate domain, their amounts are supposed to be smaller. This will be an advantage in the context of optimization of the search procedure. In addition, rather than using the parameterization technique at the ILS stage of the MAFA method, such as decorrelation, the regularized approximate GNSS position with VCM of better properties can reduce candidates' numbers immediately when determining the initial position. Note that the single-epoch GNSS math model is studied, and a reduction in candidates' numbers cannot be significant expected.

7.3 Performance of regularized single-epoch ILS estimation in the coordinate domain

This test was intended to compare the simulated SR of correct instantaneous integer AR in the coordinate domain of the MAFA method's classic and regularized approach. The simulation method is used within this experiment to evaluate the performance of regularized ILS estimator in the coordinate function with the presence of regularized bias. Therefore, the simulated SR of correct integer AR is then approximated by:

$$P(\hat{\mathbf{a}} = \mathbf{a}) \approx \frac{\mathrm{N_s}}{\mathrm{N}}, \tag{61}$$

where $P(\hat{\mathbf{a}} = \mathbf{a})$ is the probability operator which indicates that real-valued ambiguities will deliver actual-integer ones. $\mathrm{N_s}$ is the number of correct ILS ambiguities. N represents the number of all single-epoch data samples. The ILS vector is correct if it coincidence actual ambiguities. The essential is to analyze the SR of correct ILS estimation concerning the changes in geometry construction of the positioning. This is crucial because single-epoch GNSS positioning is conducted based on the shortest observation span. Note that any single-epoch as an independent data sample has another geometry construction of the positioning. So, lack of stability.

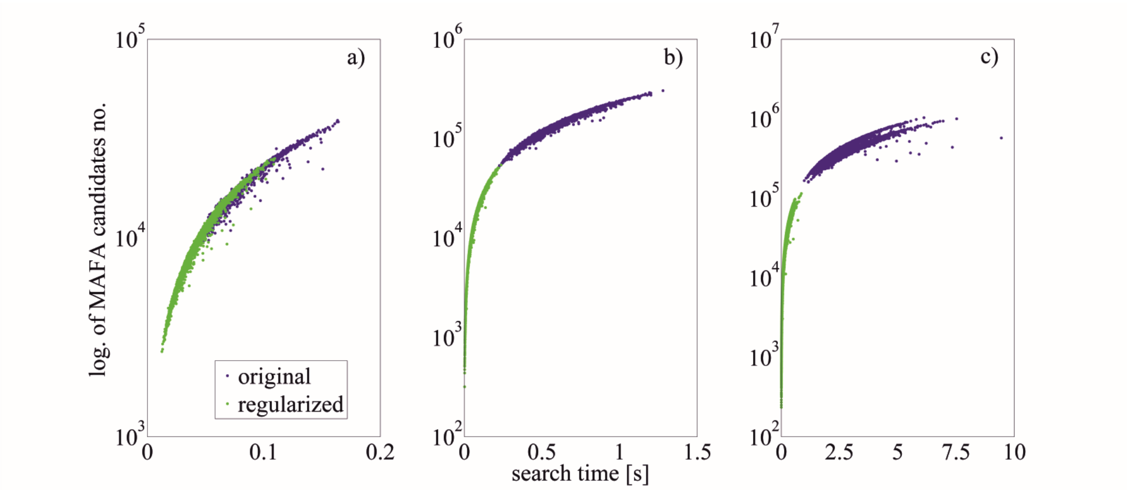

Figure 8 MAFA candidates and search times: a) σ = 0.01cycle , b) σ = 0.02cycle , c) σ = 0.03cycle

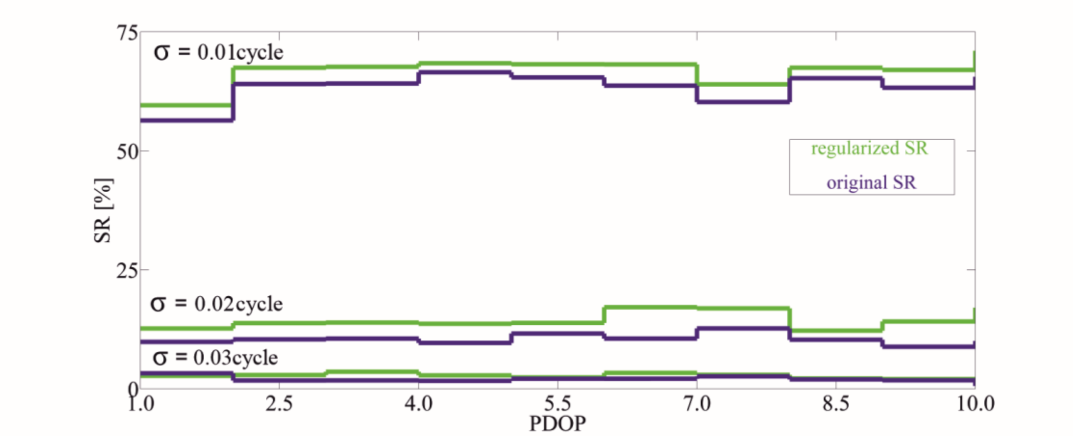

Figure 9 SR dependency on PDOP changes

Figure 8 presents the number of real-valued candidates in the sense of MAFA method in relation to the search time for an optimal ILS solution. It is obvious that number of MAFA's candidates decrease at each experiment variant when regularization is used at the cost of regularized bias. Therefore, the search time is also successfully decreased. The number of candidates and the search time are the best reduced if regularized search region is created based on variant (c). Instead of using the decorrelation technique in the MAFA method, regularization can be a promising alternative immediately when determining an approximate GNSS position. Fig. 9 shows the SR of the single-epoch ILS estimator in the coordinate function of the MAFA method. At each experiment variant the performance of regularized ILS estimator is improved at the cost of regularized bias compared to original one. For preparing Fig. 9, the PDOP was averaged from the slot of 500 single-epochs. In this computational window, the simulated SR was determined. Table 2 describes the mean SRs and search times at each variant. Mean values of the absolute differences between fixed baseline components and the actual ones in the NEU (North East Up) system complement this analysis.

Table 3 Performance of a single-epoch precise positioning in the coordinate domain

Carrier-phase nominal accuracy/Solution of an approximate GNSS position	**Mean simulated SR [%]**	**Mean search time [s]**	**Mean difference in the fixed coordinates [m]**		
			ΔN	ΔE	ΔU
σ = 0.01cycle					
original ILS estimator (OILSE)	63.88	0.068	0.261	0.168	0.493
regularized ILS estimator (RILSE)	67.16	0.042	0.224	0.149	0.408
σ = 0.02cycle					
OILSE	10.62	0.501	1.062	0.556	1.695
RILSE	14.56	0.062	0.791	0.477	1.269
σ = 0.03cycle					
OILSE	1.96	2.600	1.787	0.705	2.688
RILSE	3.00	0.137	1.279	0.621	1.938

Despite conducting the search procedure with the presence of regularized bias, the SR grows if regularized ILS estimator in the coordinate function is employed at each research variant. Simultaneously, the search time for the optimal integer solution decreases because of the lowering number of MAFA candidates, thanks to the variance reduction at the cost of bias. Since a single-epoch data set is challenged, the results of correct integer AR and the fixed position's accuracy may not satisfy. However, regularization can be a promising tool for optimizing ILS estimation in the coordinate domain of the MAFA method. When the search procedure process is in the constant three-dimensional space, there is no matter how many new ambiguities burden the functional model because integer ambiguities are resolved discreetly using the coordinate function. So, precise GNSS positioning in the coordinate domain performs well without loss of generality.

8 Summary and conclusions

This paper analyzed instantaneous integer AR in the coordinate domain as an ill-posed problem of a single-epoch precise GNSS positioning. To illustrate the performance of a single-epoch ILS estimator in the coordinate function, the MAFA method was employed. In this method, the solution of precise GNSS position expresses optimal integer-valued ambiguities implicitly. However, an approximate GNSS position and its VCM are needed to form the search region for integer AR based on ILS principle. Since the single-epoch data set was challenged, the initial position problem within the least-squares (LS) estimation was ill-posed. Therefore, the original LS solution with its VCM was poor for forming the search region to frequently contain the Voronoi cell of an actual GNSS position expressing optimal ILS solution in the coordinate function. It came from the ill-posed problem of a single-epoch challenge associated with the ill-conditioned mathematical model. To reduce the single-epoch model's ill-conditioning, regularization technique was used. By regularizing real-valued ambiguity parameters, an approximate GNSS position and its VCM were improved in the sense of MSE. Thus, the float baseline solution conditioned on regularized float ambiguities was an indirectly regularized approximate GNSS position with VCM of better precision. Despite the indirect regularization, the properties of initial position were improved at the expense of regularized bias, resulting from the biased least-squares estimation. Based on a numerical experiment, the empirical profits exceed the costs, which are summarized as follows:

(i) Single-epoch GNSS mathematical model is stabilized by eliminating its ill-conditioning at the cost of regularized bias.

(ii) With the improved precision, the overall accuracy of a single-epoch approximate position increases in the sense of MSE at the expense of bias. Thereby, the regularized VCM is of superior precision. The quality-based MSEM criterion provides optimal RP values for ambiguity parameters that guarantee the proper balance between the contribution in variance reduction and bias in the regularized solution.

(iii) The regularized search region is better localized and shaped in the constant three-dimensional space to contain more frequently an actual integer solution expressed in the coordinate function. The number of MAFA candidates for searching optimal ILS ambiguities of the actual GNSS position simultaneously reduces. Thus, the search time diametrically decreases. The biased shift of a regularized search region centre resulting from regularized bias is minimal. The regularization technique can optimize the search procedure in the coordinate domain of the MAFA method rather than the decorrelation technique at the ILS stage. It only holds when we consider the single-epoch case of MAFA-based integer AR.

(iv) Based on the SR of correct instantaneous integer AR in the coordinate domain, the regularized single-epoch ILS estimator in the coordinate function performs well with the presence of bias. The optimal RP values govern the regularized bias well with improved ambiguity precision.

Data availability

All data generated during this study are included in this published article, section 7. In justified cases, the single-epoch datasets generated during the current study are available from the corresponding author only on reasonable request.

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References

Cellmer S, Wielgosz P, Rzepecka Z (2010) Modified ambiguity function approach for GPS carrier phase positioning. J. Geodesy 84:264-275. https://doi.org/10.1007/s00190-009-0364-8

Cellmer S (2012) A graphic representation of the necessary condition for the MAFA method. IEEE Trans Geosci Remote 50(2):482–488.

Cellmer S, Nowel K, Kwaśniak D (2018) The New Search Method in Precise GNSS Positioning. IEEE Trans. on Aerospace and Elec. Systems 54:404-415. 10.1109/TAES.2017.2760578

Cellmer S, Nowel K, Fischer A (2021) A search step optimization in an ambiguity function-based GNSS precise positioning. Survey Review 54:117-124. https://doi.org/10.1080/00396265.2021.1885947

Cellmer S, Nowel K, Fischer A (2022) Reduction as an improvement of a precise satellite positioning based on an ambiguity function. J. Applied Geodesy, under review since 30th January, 2022. ID: JAG.2022.0005

Counselman C, Gourevitch S (1981) Miniature interferometer terminals for earth surveying: Ambiguity and multipath with the global positioning system. IEEE Trans. Geosci. Remote Sens. GE-19:244-252.

Fischer A, Cellmer S, Nowel K (2021) Single point positioning with vertical total electron content estimation based on single-epoch data. Geosci. Instrum. Method. Data Syst. 10:1–12. https://doi.org/10.5194/gi-10-1-2021

Fischer A, Cellmer S, Nowel K (2022) Regularizing ill-posed problem of single-epoch precise GNSS positioning using an iterative procedure. Published online: 28 March, 2022. https://doi.org/10.1515/jag-2021-0031

Golub G, Heath M, Wahba G (1979) Generalized cross-validation as a method for choosing a good ridge parameter. Technometrics 21:215–223.

Hofmann-Wellenhof B, Lichtenegger H, Wasle E (2008) GNSS Global Navigation Satellite Systems. Springer-Verlag, Wien.

Lawson C, Hanson R (1995) Solving Least Squares Problems. SIAM. 10.1137/1.9781611971217

Li B, Shen Y, Feng Y (2010) Fast GNSS ambiguity resolution as an ill-posed problem. J. Geodesy 84:683–698. 10.1007/s00190-010-0403-5

Mader G (1992) Rapid static and kinematic global positioning system solutions using the ambiguity function technique. J. Geophys. Res. 97:3271-3283.

Nowel K, Cellmer S, Kwaśniak D (2018) Mixed integer–real least squares estimation for precise GNSS positioning using a modified ambiguity function approach. GPS Solut. 22:31. 10.1007/s10291-017-0694-6

Shen Y, Xu P, Li B (2012) Bias-corrected regularized solution to inverse ill-posed models. J. Geodesy 86:597–608. 10.1007/s00190-012-0542-y

Strang G (2016) Introduction to Linear Algebra. Wellesley - Cambridge Press, Wellesley.

Teunissen P (1995) The least-squares ambiguity decorrelation adjustment: A method for fast GPS integer ambiguity estimation. J. Geodesy 70:65-82. https://doi.org/10.1007/BF00863419

Teunissen P, Kleusberg A (1998) GPS for Geodesy. Springer, Berlin.

Teunissen P (2001) Integer estimation in the presence of biases. J. Geodesy 75:399–407.

Teunissen P (2006) Adjustment theory. VSSD, Delft

Teunissen P, Montenbruck O (2017) Global Navigation Satellite Systems. Springer Handbook.

Teunissen P, Khodabandeh A (2021) A mean-squared-error condition for weighting ionospheric delays in GNSS baselines. J. Geodesy 95:118. 10.1007/s00190-021-01569-7

Tikhonov A (1963) Regularization of ill-posed problems. Dokl. Akad. Nauk SSSR 151:49–52.

Wang Y, Zhan X, Zhang Y (2007) Improved ambiguity function method based on analytical resolution for GPS attitude determination. Meas. Sci. Technol. 18:2985-2990.

Wieser A (2004) Reliability checking for GNSS baseline and network processing. GPS Solutions 8:55-66. 10.1007/s10291-004-0091-9

Wu Z, Bian S (2022) Regularized integer least-squares estimation: Tikhonov's regularization in a weak GNSS model. J Geod 96:22. https://doi.org/10.1007/s00190-021-01585-7

Xu P, Shen Y, Fukuda Y, Liu Y (2006) Variance components estimation in linear inverse ill-posed models. J. Geodesy 80:69–81. 10.1007/s00190-006-0032-1

Xu P (2009) Iterative generalized cross-validation for fusing heteroscedastic data of inverse ill-posed problems. Geophys. Journal Inter.179:182–200. 10.1111/j.1365-246X.2009.04280.x