

The Nonlinear Analysis in p -Vector Spaces by Applying Best Approximation Method

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Research Article

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Nonlinear Analysis by Applying Best Approximation Method in p -Vector Spaces

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Abstract

It is known that the class of p -vector spaces ($0 < p \leq 1$) is an important generalization of usual norm spaces with rich topological and geometrical structure, but the most tools and general principles with nature in nonlinearity have not been developed yet, the goal of this paper is to develop some useful tools in nonlinear analysis by applying the best approximation approach for the classes of semiclosed 1-set contractive set-valued mappings in p -vector spaces. In particular, we first develop the general fixed point theorems of condensing mappings which provide answer to Schauder conjecture in 1930's in the affirmative way under the setting of p -vector spaces by taking $p = 1$ for a p -vector space being a topological vector space. Then one best approximation result for upper semi-continuous and 1-set contractive set-valued is established, which is used as a useful tool to establish fixed points of non-self set-valued mappings with either inward or outward set conditions. Finally, we develop fixed points and related principle of nonlinear alternative for the classes of semiclosed set-valued mappings including nonexpansive set-valued mappings as special cases under uniformly convex Banach spaces, or locally convex topological spaces with Opial condition. The results in this paper under the category of nonlinear analysis not only include the corresponding results in the existing literature as special cases, but also expected to be useful tools for the study of nonlinear problems arising from theory to practice under the general framework of p -vector spaces for $0 < p \leq 1$.

Keywords: Nonlinear analysis, Best approximation, Fixed points, Measure of noncompactness, 1-set contractive mapping, Semiclosed 1-set contractive, condensing mapping, nonlinear alternative, Leray-Schauder alternative, Nonexpansive mapping, p -Inward and p -Outward set, p -vector space, Uniform convex space, Opial condition, Demiclosedness principle

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1. Introduction

It is known that the class of p -semi-norm spaces ($0 < p \leq 1$) is an important generalization of usual normed spaces with rich topological and geometrical structures, and related study has received a lot of attention (e.g., see Alghamdi et al.[4], Balachandran [6], Bayoumi [7], Bayoumi et al.[8], Bernuées and Pena [10], Ding [29], Ennassik and Taoudi [31], Ennassik et al.[32], Gal and Goldstein [38], Gholizadeh et al.[39], Jarchow [51], Kalton [53]-[54], Kalton et al.[55], Machrafi and Oubbi [71], Park [88], Qiu and Rolewicz [97], Rolewicz [101], Silva et al.[111], Simons [108], Tabor et al.[113], Tan [114], Wang [117], Xiao and Lu [120], Xiao and Zhu [121]-[122], Yuan [131], and many others). However, to the best of our knowledge, the corresponding basic tools and associated results in the category of nonlinear functional analysis have not been well developed, thus the goal of this paper is to develop some important tools for nonlinear analysis for 1-set contractive mappings under the framework of p -(semi)norm spaces.

In particular, we first develop the general fixed point theorems for 1-set contractive mappings which provide answer to Schauder conjecture since 1930's in the affirmative way under the general framework of p -vector spaces (when $p = 1$ reduced to a setting of topological vector spaces); then the one best approximation result for upper semi-continuous and 1-set contractive mappings is given with the boundary condition, which is used as a tool to establish fixed points for non-self set-valued mappings with either inward or outward set conditions; and finally we give existence results for solutions of Birkhoff-Kellogg problems, general principle of nonlinear alternative by including Leray-Schauder alternative and related results as special classes. The results given in this paper do not only include the corresponding results in the existing literature as special cases, but also expected to be useful tools for the study of nonlinear problems arising from social science, engineering, applied mathematics and related topics and areas.

Before discussing the study of the best approximation and related nonlinear analysis tools under the framework of p -vector spaces, we like first to share with readers that that though most of results in nonlinear analysis are normally highly associated with the convexity hypotheses under the local convex topological vector spaces (of course, including normed spaces, and Banach spaces, nice metric spaces), it seems surprise that p -vector spaces which in general do not have the local convex structure comparing with locally convex spaces, but they are provide some nice properties in the nature way with some kinds of nice approximation and better (i.e., the bigger) structures for the so-called the convexities of p -convex subset play very important roles for us to describe Birkhoff and Kellogg problems, and related nonlinear problems (such as fixed point problem and so on) in topological vector spaces (TVS) based on p -vector spaces's behaviors for p in $(0, 1)$ (the p -vector space E reduces to TVS when $p = 1$), and also see the corresponding results and properties as pointed by the Remark 2.1 (1), Lemma 2.1(ii) and Lemma 2.3 below.

Here, we also like to recall that since the first Birkhoff-Kellogg problem was introduced and associated theorem was proved by Birkhoff and Kellogg [11] in 1922 in discussing the existence of solutions for the equation $x = \lambda F(x)$, where λ is a real parameter, and F is a general nonlinear non-self mapping defined on an open convex subset U

of a topological vector space E , now the general form of the Birkhoff-Kellogg problem is to find the so-called an invariant direction for the nonlinear set-valued mappings F , i.e., to find $x_0 \in \bar{U}$ and $\lambda > 0$ such that $\lambda x_0 \in F(x_0)$ (may also considering if x_0 is from the boundary \bar{U}).

On the other hand, after Birkhoff and Kellogg theorem given by Birkhoff and Kellogg in 1920's, the study on Birkhoff-Kellogg problem has been received a lot of attention by scholars since then. For example, in 1934, one of the fundamental results in nonlinear functional analysis, famously called the Leray-Schauder alternative by Leray and Schauder [64] was established via topological degree. Thereafter, certain other types of Leray-Schauder alternatives were proved using different techniques other than topological degree, see works given by Granas and Dugundji [46], Furi and Pera [37] in the Banach space setting and applications to the boundary value problems for ordinary differential equations in noncompact problems, a general class of mappings for nonlinear alternative of Leray-Schauder type in normal topological spaces, and some Birkhoff-Kellogg type theorems for general class mappings in topological vector spaces by Agarwal et al.[1], Agarwal and O'Regan [2]-[3], Park [86], O'Regan [80] by using the Leray-Schauder type coincidence theory applying to establish Birkhoff-Kellogg problem, Furi-Pera type result for a general class of set-valued mappings. In this paper, based on the application of our best approximation
50 as a tool for general 1-set contractive set-valued mappings, we will develop the general principle for existence of solutions for Birkhoff-Kellogg problems and related nonlinear alternatives will be established, which then also allows us to give general existence of Leray-Schauder type and related fixed point theorems for non-self mappings in general p -vector spaces for $(0 < p \leq 1)$. These new results given in this paper not only include the corresponding results in the existing literature as special cases, but also expected to be useful tools for the study of nonlinear problems arising from theory to practice.

Now we give a brief introduction on the best approximation theorem related to the development of the nonlinear analysis as a powerful tool with some background.

We all know that the best approximation in nature related to fixed points for non-self mappings, which tightly link with the classical Leray-Schauder alternative based on the Leray-Schauder continuation theorem by Leray and Schauder [64], which is a remarkable result in nonlinear analysis; and in addition, there exist several continuation theorems, which have many applications to the study of nonlinear functional equations (see O'Regan and Precup [81]). Historically, it seems that the continuation theorem is based on the idea of obtaining a solution of a given equation, starting from one of the solutions of a simpler equation. The essential part of this theorem is the "Leray-Schauder boundary condition". It seems the "continuation method" was initiated by Poincare [95], Bernstein [9]. Certainly, Leray and Schauder [64] in 1934 gave the first abstract formulation of "continuation principle" using the topological degree (see also Granas and Dugundji [46], Isac [52], Rothe [102]-[103], Zeidler [133]). But in this paper, we will see how the best approximation method could be used for the study of fixed point theorems in p -vector space for $0 < p \leq 1$ which as a basic tool, will help us to develop the principle of nonlinear alternative, Leray-Schauder alternative, fixed point theorems of Rothe, Petryshyn, Atzman type for non-self mappings, and related fixed point, nonlinear alternative with different boundary conditions. Moreover, the new results given in this paper are highly expected to become useful tools for the study on optimization, nonlinear programming, variational inequality,

complementarity, game theory, mathematical economics, and related other social science area.

It is well-known that the best approximation is one of very important aspects for the study of nonlinear problems related to the problems on their solvability for partial differential equations, dynamic systems, optimization, mathematical program, operation research; and in particularly, the one approach well accepted for study of nonlinear problems in optimization, complementarity problems and of variational inequalities problems and so on, strongly based on today called Fan's best approximation theorem given by Fan [35] in 1969 which acts as a very powerful tool in nonlinear analysis, and see the book of Singh et al.[109] for the related discussion and study on the fixed point theory and best approximation with the KKM-map principle), among them, the related tools are Rothe type and principle of Leray-Schauder alternative in topological vector spaces (TVS), and local topological vector spaces (LCS) which are comprehensively studied by Chang et al.[22], Chang et al.[23]-[25], Carbone and Conti [18], Ennassik and Taoudi [31], Ennassik et al.[32], Isac [52], Granas and Dugundji [46], Kirk and Shahzad [58], Liu [70], Park [89], Rothe [102]-[103], Shahzad [105]-[107], Xu [123], Yuan [129]-[131], Zeidler [133], and references therein.

On the other hand, since the celebrated so-called KKM principle established in 1929 in [59], was based on the celebrated Sperner combinatorial lemma and first applied to a simple proof of the Brouwer fixed point theorem. Later it became clear that these three theorems are mutually equivalent and they were regarded as a sort of mathematical trinity (Park [89]). Since Fan extended the classical KKM theorem to infinite-dimensional spaces in 1961 by Fan [34]-[36], there have been a number of generalizations and applications in numerous areas of nonlinear analysis, and fixed points in TVS and LCS as developed by Browder [12]-[17] and related references therein. Among them, Schauder's fixed point theorem [110] in normed spaces is one of the powerful tools in dealing with nonlinear problems in analysis. Most notably, it has played a major role in the development of fixed point theory and related nonlinear analysis and mathematical theory of partial and differential equations and others. A generalization of Schauder's theorem from normed space to general topological vector spaces is an old conjecture in fixed point theory which is explained by the Problem 54 of the book "The Scottish Book" by Mauldin [73] as stated as Schauder's conjecture: "Every nonempty compact convex set in a topological vector space has the fixed point property, or in its analytic statement, does a continuous function defined on a compact convex subset of a topological vector space to itself have a fixed point?" Recently, this question has been recently answered by the work of Górniewicz [44], L. Górniewicz et al. [45], Ennassik and Taoudi [31], Ennassik et al.[32] by using the p -seminorm methods under p -vector spaces; and also significant contribution by Cauty [20], plus corresponding contributions by Askoura and
100 Godet-Thobie [5], Cauty [19], Chang [21], Chang et al.[22], Chen [26], Dobrowolski [30], Gholizadeh et al.[39], Isac [52], Li [68], Li et al. [67], Liu [70], Nhu [75], Okon [77], Park [88]-[90], Reich [98], Smart [112], Weber [118]-[119], Xiao and Lu [120], Xiao and Zhu [121]-[122], Xu [126], Xu et al.[127], Yuan [129]-[132] and related references therein under the general framework of p -vector spaces for even non-self set-valued mappings ($0 < p \leq 1$).

The goal of this paper is to establish the general new tools of nonlinear analysis under the framework of general p -(semi)norm spaces for 1-set contractive mappings (here $0 < p \leq 1$), but we do wish these new results such as best approximation, theorems of Birkhoff-Kellogg type, nonlinear alternative, fixed point theorems for non-self set-valued with boundary conditions, Rothe, Petryshyn type, Altman type, Leray-Schedule types, related others

nonlinear problems would play important roles for the nonlinear analysis of p -seminorm spaces for $0 < p \leq 1$. In addition, our results on the fixed point theorem for compact upper semi-continuous mappings also provide solutions for Schauder's conjecture since 1930's in the affirmative way under the general setting of p -vector spaces (which may not locally convex, see Kalton [53]-[54], Kalton et al.[55], Jarchow [51], Roloewicz [101], Fan [33]-[36], Singh et al.[109], and related references).

The paper has seven sections. Section 1 is the introduction. Section 2 describes general concepts for the p -convex subsets of topological vector spaces ($0 < p \leq 1$). In Section 3, then some basic results of KKM principle related to abstract convex spaces are given. In Section 4, as the application of the KKM principle in abstract convex spaces which including p -convex vector spaces as a special class ($0 < p \leq 1$) by combining the embedding lemma for compact p -convex subsets from topological vector spaces into locally p -convex spaces, we provide general fixed point theorems for upper semi-continuous self-mappings defined on p -convex compact, and 1-set contractive upper semi-continuous set-valued mappings defined on non-compact p -convex subsets in locally p -convex vector spaces. In section 5, the general best approximation result for 1-set contractive upper semi-continuous mappings is first given under the framework of p -vector spaces, which is used as a tool to establish the general existence theorems for solutions of Birkhoff-Kellogg (problem) alternative, general principle of nonlinear alternative, and including Leray-Schauder alternative, Rothe type, Altman type associated with different boundary conditions. In section 6, we give a number of new results based on general principles of Birkhoff-Kellogg theorems, Leray-Schauder alternative established in Section 5 for 1-set contractive mappings with different boundary conditions. In Section 7, by considering p -(semi)norm spaces for $p \in (0, 1]$, as application of best approximation, we unify and improve the corresponding results in the existing literature under the general framework of p -(semi)norm spaces. In section 7, we focus on the study of fixed points theorems for classes of 1-Set contractive set-valued mappings under various boundary conditions. Finally, from section 8 to section 10, we develop fixed points and related principle of nonlinear alternative for the classes of semiclosed set-valued mappings including nonexpansive set-valued mappings under uniformly convex Banach spaces or locally convex topological spaces with Opial condition.

For the convenience of our discussion, throughout this paper, all p -convex topological vector spaces and the compact p -convex sets are always assumed to be Hausdorff, and p satisfying the the condition for $0 < p \leq 1$ unless specified, and also we denote by \mathbb{N} the set of all positive integers, i.e., $\mathbb{N} := \{1, 2, \dots, \}$.

2. The Basic Results of p -Vector Spaces

We now recall some notion and definitions for p -convex topological vector spaces which will be used below (see Balachandran [6], Bayoumi [7], Jarchow [51], Kalton [53], Rolewicz [101], Gholizadeh et al.[39], Ennassik et al.[32], Ennassik and Taoudi [31], Xiao and Lu [120], Xiao and Zhu [121] and references therein).

Definition 2.1. A set A in a vector space X is said to be p -convex if for any $x, y \in A$, $s, t \geq 0$, we have $(1-t)^{1/p}x + t^{1/p}y \in A$, whenever $0 \leq t \leq 1$. If A is 1-convex, it is simply called convex (for $p = 1$) in general vector spaces.

Definition 2.2. If A is a subset of a topological vector space X , the closure of A is denoted by \bar{A} , then the p -convex

hull of A and its closed p -convex hull denoted by $C_p(A)$, and $\overline{C}_p(A)$, respectively, which is the smallest p -convex set containing A , and the smallest closed p -convex set containing A , respectively.

Definition 2.3. Let A be p -convex and $x_1, \dots, x_n \in A$, and $t_i \geq 0$, $\sum_1^n t_i^p = 1$. Then $\sum_1^n t_i x_i$ is called a p -convex combination of $\{x_i\}$ for $i = 1, 2, \dots, n$. If $\sum_1^n |t_i|^p \leq 1$, then $\sum_1^n t_i x_i$ is called an absolutely p -convex combination.

It is easy to see that $\sum_1^n t_i x_i \in A$ for a p -convex set A .

Definition 2.4. A subset A of a vector space X is called circled (or balanced) if $\lambda A \subset A$ holds for all scalars λ satisfying $|\lambda| \leq 1$. We say that A is absorbing if for each $x \in X$, there is a real number $\rho_x > 0$ such that $\lambda x \in A$ for all $\lambda > 0$ with $|\lambda| \leq \rho_x$.

By the definition 2.4, it is easy to see that the system of all circled subsets of X is easily seen to be closed under the formation of linear combinations, arbitrary unions, and arbitrary intersections. In particular, every set $A \subset X$ determines a smallest circled subset \hat{A} of X in which it is contained: \hat{A} is called the circled hull of A . It is clear that $\hat{A} = \cup_{|\lambda| \leq 1} \lambda A$ holds, so that A is circled if and only if (in short, iff) $\hat{A} = A$. We use $\overline{\hat{A}}$ to denote for the closed circled hull of $A \subset X$.

In addition, if X is a topological vector space, we use the $\text{int}(A)$ to denote the interior of set $A \subset X$ and if $0 \in \text{int}(A)$, then $\text{int}(A)$ is also circled, and using ∂A to denote the boundary of A in X unless specified.

Definition 2.5. A topological vector space is said to be locally p -convex if the origin has a fundamental set of absolutely p -convex 0-neighborhoods. This topology can be determined by p -seminorms which are defined in the obvious way (see P.52 of Bayoumi [7], Jarchow [51] or Rolewicz [101]).

Definition 2.6. Let X is a vector space and \mathbb{R}^+ is a non-negative part of a real line \mathbb{R} . Then a mapping $P : X \rightarrow \mathbb{R}^+$ is said to be a p -seminorm if it satisfies the requirements for $(0 < p \leq 1)$

- (i) $P(x) \geq 0$ for all $x \in X$;
- (ii) $P(\lambda x) = |\lambda|^p P(x)$ for all $x \in X$ and $\lambda \in \mathbb{R}$;
- (iii) $P(x + y) \leq P(x) + P(y)$ for all $x, y \in X$.

An p -seminorm P is called a p -norm if $x = 0$ whenever $P(x) = 0$, so a vector space with a specific p -norm is called an p -normed space, and of course if $p = 1$, X is a normed space as discussed before (e.g., see Jarchow [51]).

By Lemma 3.2.5 of Balachandran [6], the following proposition gives a necessary and sufficient condition for an p -seminorm to be continuous.

Proposition 2.1. Let X be a topological vector space, P is a p -seminorm on X and $V := \{x \in X : P(x) < 1\}$. Then P is continuous if and only if $0 \in \text{int}(V)$, where $\text{int}(V)$ is the interior of V .

Now given an p -seminorm P , the p -seminorm topology determined by P (in short, the p -topology) is the class of unions of open balls $B(x, \epsilon) := \{y \in X : P(y - x) < \epsilon\}$ for $x \in X$ and $\epsilon > 0$.

Definition 2.7. A topological vector space X is said to be locally p -convex if it has a 0-basis consisting of p -convex neighborhoods for $(0 < p \leq 1)$. If $p = 1$, X a usual locally convex space.

We also need the following notion for the so-called p -gauge (see Balachandran [6]).

Definition 2.8. Let A be an absorbing subset of a vector space X . For $x \in X$ and $0 < p \leq 1$, set $P_A = \inf\{\alpha > 0 : x \in \alpha^{\frac{1}{p}}A\}$, then the non-negative real-valued function P_A is called the p -gauge (gauge if $p = 1$). The p -gauge of A is also known as the Minkowski p -functional.

By Proposition 4.1.10 of Balachandran [6], we have the following proposition.

Proposition 2.2. Let A be an absorbing subset of X . Then p -gauge P_A has the following properties:

- (i) $P_A(0) = 0$;
- (ii) $P_A(\lambda x) = |\lambda|^p P_A(x)$ if $\lambda \geq 0$;
- (iii) $P_A(\lambda x) = |\lambda|^p P_A(x)$ for all $\lambda \in R$ provided A is circled;
- (iv) $P_A(x + y) \leq P_A(x) + P_A(y)$ for all $x, y \in A$ provided A is p -convex.

In particular, P_A is a p -seminorm if A is absolutely p -convex (and also absorbing).

As mentioned above, a given p -seminorm is said to be an p -norm if $x = 0$ whenever $P(x) = 0$. A vector space with a specific p -norm is called a p -normed space. The p -norm of an element $x \in E$ will usually be denoted by $\|x\|_p$. If $p = 1$, X is a usual normed space. If X is an p -normed space, then (X, d_p) is a metric linear space with a translation invariant metric d_p such that $d_p = d_p(x, y) = \|x - y\|_p$ for $x, y \in X$. We point out that p -normed spaces are very important in the theory of topological vector spaces. Specifically, a Hausdorff topological vector space is locally bounded if and only if it is a p -normed space for some p -norm $\|\cdot\|_p$, where $0 < p \leq 1$ (see p.114 of Jarchow [51]). We also note that examples of p -normed spaces include such as $L^p(\mu)$ - spaces and Hardy spaces H_p , $0 < p < 1$, endowed with their usual p -norms.

Remark 2.1. We like to make the following important two points as follows:

(1) First, by the fact that (e.g., see Kalton et al.[55], or Ding [29]), there is no open convex non-void subset in $L^p[0, 1]$ (for $0 < p < 1$) except $L^p[0, 1]$ itself, this means that p -normed paces with $0 < p < 1$ are not necessarily locally convex. Moreover, we know that every p -normed space is locally p -convex; and incorporating Lemma 2.3 below, it seems that p -vector spaces (for $0 < p \leq 1$) is a nicer space as we can use p -vector space to approximate (Hausdorff) topological vector spaces (TVS) in terms of Lemma 2.1 (ii) below for the convex subsets in TVS by using a bigger p -convex subsets in p -vector spaces for $p \in (0, 1)$ by also considering Lemma 2.3 below, in this way, it seems P -vector spaces seems having better properties in terms of p -convexity than the usually (1-) convex subsets used in TVS with $p = 1$.

(2) Second, it is worthwhile noting that a 0-neighborhood in a topological vector space is always absorbing by Lemma 2.1.16 of Balachandran [6], or Proposition 2.2.3 of Jarchow [51]).

Now by Proposition 4.1.12 of Balachandran [6], we also have the following Proposition 2.3 and Remark 2.2 (which is the Remark 2.3 of Ennassik and Taoudi [31]).

Proposition 2.3. Let A be a subset of a vector space X , which is absolutely p -convex ($0 < p \leq 1$) and absorbing. Then, we have that

- (i) The p -gauge P_A is a p -seminorm such that if $B_1 := \{x \in X : P_A(x) < 1\}$, and $\overline{B_1} = \{x \in X : P_A(x) \leq 1\}$. then $B_1 \subset A \subset \overline{B_1}$; in particular, $\ker P_A \subset A$, where $\ker P_A := \{x \in X : P_A(x) = 0\}$.

(ii) $A = B_1$ or $\overline{B_1}$ according as A is open or closed in the P_A -topology.

Remark 2.2. Let X be a topological vector space and let U be an open absolutely p -convex neighborhood of the origin, and let ϵ be given. If $y \in \epsilon^{\frac{1}{p}}U$, then $y = \epsilon^{\frac{1}{p}}u$ for some $u \in U$ and $P_U(y) = P_U(\epsilon^{\frac{1}{p}}u) = \epsilon P_U(u) \leq \epsilon$ (as $u \in U$ implies that $P_U(u) \leq 1$). Thus, P_U is continuous at zero, and therefore, P_U is continuous everywhere. Moreover, we have $U = \{x \in X : P_U(x) < 1\}$.

Indeed, since U is open and the scalar multiplication is continuous, we have that for any $x \in U$, there exists $0 < t < 1$ such that $x \in t^{\frac{1}{p}}U$ and so $P_U(x) \leq t < 1$. This shows that $U \subset \{x \in X : P_U(x) < 1\}$. The conclusion follows by Proposition 2.3 above.

The following result is a very important and useful result which allows use to make the approximation for convex subsets in topological vector spaces by p -convex subsets in p -convex vector spaces. For the reader's self-contained in reading, we provide a sketch of proof below (see also Lemma 2.1 of Ennassik and Taoudi [32], Remark 2.1 of Qiu and Rolewicz [97]).

Lemma 2.1. Let A be a subset of a vector space X , then we have

- (i) If A is p -convex, with $0 < p < 1$, then $\alpha x \in A$ for any $x \in A$ and any $0 < \alpha \leq 1$.
- (ii) If A is convex and $0 \in A$, then A is p -convex for any $p \in (0, 1]$.
- (iii) If A is p -convex for some $p \in (0, 1)$, then A is s -convex for any $s \in (0, p]$.

Proof. (i) As $r \leq 1$, by the fact that “for all $x \in A$ and all $\alpha \in [2^{(n+1)(1-\frac{1}{p})}, 2^{n(1-\frac{1}{p})}]$, we have $\alpha x \in A$ ” is true for all integer $n \geq 0$. Taking into account the fact that $(0, 1] = \cup_{n \geq 0} [2^{(n+1)(1-\frac{1}{p})}, 2^{n(1-\frac{1}{p})}]$, thus the result is obtained.

(ii) Assume that A is a convex subset of X with $0 \in A$ and take a real number $s \in (0, 1]$. we show that A is s -convex. Indeed, let $x, y \in A$ and $\alpha, \beta > 0$ with $\alpha^p + \beta^p = 1$. Since A is convex, then $\frac{\alpha}{\alpha+\beta}x + \frac{\beta}{\alpha+\beta}y \in A$. Keeping in mind that $0 < \alpha + \beta < \alpha^p + \beta^p = 1$, it follows that $\alpha x + \beta y = (\alpha + \beta)(\frac{\alpha}{\alpha+\beta}x + \frac{\beta}{\alpha+\beta}y) + (1 - \alpha - \beta)0 \in A$.

(iii) Now, assume that A is r -convex for some $p \in (0, 1)$ and pick up any real $s \in (0, p]$. We show that A is s -convex. To see this, let $x, y \in A$ and $\alpha, \beta > 0$ such that $\alpha^s + \beta^s = 1$. First notice that $0 < \alpha^{\frac{p-s}{p}} \leq 1$ and $0 < \beta^{\frac{p-s}{p}} \leq 1$, which imply that $\alpha^{\frac{p-s}{p}}x \in A$ and $\beta^{\frac{p-s}{p}}y \in A$. By the p -convexity of A and the equality $(\alpha^{\frac{s}{p}})^p + (\beta^{\frac{s}{p}})^p = 1$, it follows that $\alpha x + \beta y = \alpha^{\frac{s}{p}}(\alpha^{\frac{p-s}{p}}x) + \beta^{\frac{s}{p}}(\beta^{\frac{p-s}{p}}y) \in A$. This completes the sketch of the proof. \square

Remark 2.3. We like to point out that the results (i) and (iii) of Lemma 2.1 do not hold for $p = 1$. Indeed, any singleton $\{x\} \subset X$ is convex in topological vector spaces; but if $x \neq 0$, then it is not p -convex for any $p \in (0, 1)$.

We also need the following Proposition which is proposition 6.7.2 of Jarchow [51].

Proposition 2.4. Let K be compact in a topological vector X and ($1 < p \leq 1$). Then the closure $\overline{C}_p(K)$ of the p -convex hull, and the closure $\overline{AC}_p(K)$ of absolutely p -convex hull of K are compact if and only if $\overline{C}_p(K)$ and $\overline{AC}_p(K)$ are complete, respectively.

We also need following fact, which is a special case of Lemma 2.4 of Xiao and Zhu [121].

Lemma 2.2. Let C be a bounded closed p -convex subset of p -seminorm X with $0 \in \text{int}C$, where ($0 < p \leq 1$). For every $x \in X$ define an operator by $r(x) := \frac{x}{\max\{1, (P_C(x))^{\frac{1}{p}}\}}$, where P_C is the Minkowski p -functional of C . Then C

is a retract of X and $r : X \rightarrow C$ is a continuous such that

- (1) if $x \in C$, then $r(x) = x$;
- (2) if $x \notin C$, then $r(x) \in \partial C$;
- (3) if $x \notin C$, then the Minkowski p -functional $P_C(x) > 1$.

Proof. Taking $s = p$ in Lemma 2.4 of Xiao and Zhu [121], Proposition 2.3 and Remark 2.2, thus the proof is complete. \square

Remark 2.4. As discussed by Remark 2.2, Lemma 2.2 still holds if “the bounded closed p -convex subset C of the p -normed space $(X, \|\cdot\|_p)$ ” is replaced by “ X is a p -seminorm vector space and C is a bounded closed absorbing p -convex subset with $0 \in \text{int}C$ of X ”.

Before we close this section, we like to point out that the structure of p -convexity when $p \in (0, 1)$ is really different from what we normally have for the concept of “convexity” used in topological vector spaces (TVS), in particular, maybe the following fact is one of reasons for us to use better (p -convex) structures in p -vector spaces to approximate the corresponding structure of the convexity used in TVS (i.e., the p -vector space when $p = 1$). Based on the discussion in P.1740 of Xiao and Zhu [121](see also Bernués and Pena [10] and Sezer et al.[134]), we have the following fact which indicates that each p -convex subset is “bigger” than the convex subset in topological vector spaces.

Lemma 2.3. Let x be a point of p -vector space E , where assume $0 < p < 1$, then the p -convex hull and the closure of $\{x\}$ is given by

$$C_p(\{x\}) = \begin{cases} \{tx : t \in (0, 1]\}, & \text{if } x \neq 0, \\ \{0\}, & \text{if } x = 0; \end{cases} \quad (1)$$

and

$$\overline{C_p(\{x\})} = \begin{cases} \{tx : t \in [0, 1]\}, & \text{if } x \neq 0, \\ \{0\}, & \text{if } x = 0. \end{cases} \quad (2)$$

But note that if x is a given one point in p -vector space E , when $p = 1$, we have that $\overline{C_1(\{x\})} = C_1(\{x\}) = \{x\}$, This shows significantly different for the structure of p -convexity between $p = 1$ and $p \neq 1$!

As an application of Lemma 2.3, we have the following fact for (set-valued) mappings with non-empty closed p -convex values in p -vector spaces for $p \in (0, 1)$, which are truly different from any (set-valued) mappings defined in topological vector spaces (i.e., for a p -vector space with $p = 1$).

Lemma 2.4. Let U be a non-empty subset of a p -vector space E (where $0 < p < 1$), with zero $0 \in U$, and assume a (set-valued) mapping $T : U \rightarrow 2^E$ is with non-empty closed p -convex values. Then T has at least one fixed point in U , which is the element zero, i.e., $0 \in \bigcap_{x \in U} T(x) \neq \emptyset$.

Proof. For each $x \in U$, as $T(x)$ is non-empty closed p -convex, by Lemma 2.3, we have at least $0 \in T(x)$. It implies that $0 \in \bigcap_{x \in U} T(x)$ and thus zero of E is a fixed point of T . This completes the proof. \square

Remark 2.5. By following Definitions 2.5 and 2.6, the discussion given by Proposition 2.3 and remarks thereafter,

each given (open) p -convex subset U in a p -vector space E with the zero $0 \in \text{int}(U)$ always corresponds to a p -seminorm P_U , which is indeed the Minkowski p -functional of U in E , and P_U is continuous in E . In particular, a topological vector space is said to be locally p -convex if the origin 0 of E has a fundamental set (denoted by) \mathfrak{U} , which is a family of absolutely p -convex 0 -neighborhoods (each denoted by U). This topology can be determined by p -seminorm P_U , which are indeed the family $\{P_U\}_{U \in \mathfrak{U}}$, where P_U is just the Minkowski p -functional for each $U \in \mathfrak{U}$ in E (see also P.52 of Bayoumi [7], Jarchow [50] or Rolewicz [99]).

Throughout this paper, by following Remark 2.5, without loss of generality unless specified, for a given p -vector space E , where $p \in (0, 1]$, we always denote by \mathfrak{U} the base of the p -vector space E 's topology structure, which is the family of its 0 -neighborhoods. For each $U \in \mathfrak{U}$, its corresponding P -seminorm P_U is the Minkowski p -functional of U in E . For a given point $x \in E$ and a subset $C \subset E$, we denote by $d_{P_U}(x, C) := \inf\{P_U(x - y) : y \in C\}$ for the distance of x and C by the seminorm P_U , where P_U is the Minkowski p -functional for each $U \in \mathfrak{U}$ in E .

3. The KKM Principle in Abstract Convex Spaces

As mentioned in the introduction, since Knaster, Kuratowski and Mazurkiewicz (in short, KKM)[59] in 1929 obtained the so-called KKM principle (theorem) to give a new proof for the Brouwer fixed point theorem in finite dimensional spaces; and later in 1961, Fan [34] (see also Fan [36]) extended the KKM principle (theorem) to any topological vector spaces and applied it to various results including the Schauder fixed point theorem. Since then there have appeared a large number of works devoting applications of the KKM principle (theorem). In 1992, such research field was called the KKM theory first time by Park [83], then the KKM theory has been extended to general abstract convex spaces by Park [87](see also Park [88] and [89]) which actually include locally p -convex spaces ($0 < p \leq 1$) as a special class.

Here we first give some notion and a brief introduction on the abstract convex spaces which play important role for the development of KKM principle and related applications. Once again, the corresponding comprehensive discussion on KKM theory and its various applications to nonlinear analysis and related topics, we refer to Mauldin [73], Granas and Dugundji [46], Park [89] and [90], Yuan [130]-[131] and related comprehensive reference there.

Let $\langle D \rangle$ denote the set of all nonempty finite subsets of a given non-empty set D , and 2^D denotes the family of all subsets of D . We have the following definition for abstract convex spaces essentially by Park [87].

Definition 3.1. An abstract convex space $(E, D; \Gamma)$ consists of a topological space E , a nonempty set D , and a set-valued mapping $\Gamma : \langle D \rangle \rightarrow 2^E$ with nonempty values $\Gamma_A := \Gamma(A)$ for each $A \in \langle D \rangle$, such that the Γ -convex hull of any $D' \subset D$ is denoted and defined by $\text{co}_\Gamma D' := \cup\{\Gamma_A | A \in \langle D' \rangle\} \subset E$.

A subset X of E is said to be a Γ -convex subset of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D' \rangle$, we have $\Gamma_N \subseteq X$, that is, $\text{co}_\Gamma D' \subset X$. For the convenience of our discussion, in the case $E = D$, the space $(E, E; \Gamma)$ is simply denoted by $(E; \Gamma)$ unless specified.

Definition 3.2. Let $(E, D; \Gamma)$ be an abstract convex space and Z a topological space. For a set-valued mapping (or say, multimap) $F : E \rightarrow 2^Z$ with nonempty values, if a set-value mapping $G : D \rightarrow 2^Z$ satisfies $F(\Gamma_A) \subset G(A) :=$

$\bigcup_{y \in A} G(y)$ for all $A \in \langle D \rangle$, then G is called a KKM mapping with respect to F . A KKM mapping $G : D \rightarrow 2^E$ is a KKM mapping with respect to the identity map 1_E .

Definition 3.3. The partial KKM principle for an abstract convex space $(E, D; \Gamma)$ is that, for any closed-valued KKM mapping $G : D \rightarrow 2^E$, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. The KKM principle is that, the same property also holds for any open-valued KKM mapping.

An abstract convex space is called a (partial) KKM space if it satisfies the (partial) KKM principle (resp.). We now gave some known examples of (partial) KKM spaces (see Park [87], and also [88]) as follows.

Definition 3.4. A ϕ_A -space $(X, D; \{\phi_A\}_{A \in \langle D \rangle})$ consists of a topological space X , a nonempty set D , and a family of continuous functions $\phi_A : \Delta_n \rightarrow 2^X$ (that is, singular n -simplices) for $A \in \{D\}$ with $|A| = n + 1$. By putting $\Gamma_A := \phi_A(\Delta_n)$ for each $A \in \langle D \rangle$, the triple $(X, D; \Gamma)$ becomes an abstract convex space.

Definition 3.5. For a ϕ_A -space $(X, D; \{\phi_A\})$, we recall that any set-valued mapping $G : D \rightarrow 2^X$ satisfying $\phi_A(\Delta_J) \subset G(J)$ for each $A \in \langle D \rangle$ and $J \in \langle A \rangle$ is called a KKM mapping.

By the definition, it is clear that every ϕ_A -space is a KKM space, thus we have the following fact (see Lemma 1 of Park [88]).

Lemma 3.1. Let $(X, D; \Gamma)$ be a ϕ_A -space and $G : D \rightarrow 2^X$ a set-valued (multimap) with nonempty closed [resp. open] values. Suppose that G is a KKM mapping, then $\{G(a)\}_{a \in D}$ has the finite intersection property.

By following Definition 2.6, we recall that a topological vector space is said to be locally p -convex if the origin has a fundamental set of absolutely p -convex 0-neighborhoods. This topology can be determined by p -seminorms which are defined in the obvious way (see Jarchow [51], or P.52 of Bayoumi [7]).

Now we have a new KKM space as follows inducted by the concept of p -convexity (see Park [88]).

Lemma 3.2. Suppose that X is a subset of topological vector space E and $p \in (0, 1]$, and D is a nonempty subset of X such that $C_p(D) \subset X$. Let $\Gamma_N := C_p(N)$ for each $N \in \langle D \rangle$. Then $(X, D; \Gamma)$ is a ϕ_A -space.

Proof. Since $C_p(D) \subset X$, Γ_N is well-defined. For each $N = \{x_0, x_1, \dots, x_n\} \subset D$, we define $\phi_N : \Delta_n \rightarrow \Gamma_N$ by $\sum_{i=0}^n t_i e_i \mapsto \sum_{i=0}^n (t_i)^{\frac{1}{p}} x_i$. Then clearly $(X, D; \Gamma)$ is a ϕ_A -space. This completes the proof. \square

4. Fixed Point Theorems for Condensing Set-Valued Mappings in p -Vector Spaces

In this section, we will establish fixed point theorems for upper semi-continuous and condensing mappings for p -convex subsets under the general framework of p -vector spaces, which will be a tool used in section 5 and section 6 to establish the best approximation, fixed points, the principle of nonlinear alternative, Birkhoff-Kellogg problems, Leray-Schauder alternative which would be useful tools in nonlinear analysis for the study of nonlinear problems arising from theory to the practice. Here, we first gather together necessary definitions, notations, and known facts needed in this section.

Definition 4.1. Let X and Y be two topological spaces. A set-valued mapping (also saying, multifunction) $T : X \rightarrow 2^Y$ is a point to set function such that for each $x \in X$, $T(x)$ is a subset of Y . The mapping T is said to be upper semi-continuous (USC) if the subset $T^{-1}(B) := \{x \in X : T(x) \cap B \neq \emptyset\}$ (resp., the set $\{x \in X : T(x) \subset B\}$)

is closed (resp., open) for any closed (resp., open) subset B in Y . The function $T : X \rightarrow 2^Y$ is said to be lower semi-continuous (LSC) if the set $T^{-1}(A)$ is open for any open subset A in Y .

As an application of KKM principle for general abstract convex spaces with the help of embedding lemma for Hausdorff compact p -convex subsets from topological vector spaces (TVS) into locally p -convex vector spaces, we have the following general existence result for the ‘‘approximation’’ of fixed points for upper and lower semi-continuous set-valued mappings in p -convex vector spaces for $0 < p \leq 1$ (see the corresponding related results given by Theorem 2.7 of Gholizadeh et al. [39], Theorem 5 of Park [88] and related discussion therein).

350 The following result is originally given by given by Yuan [131], here we provide the sketch of its proof for the purpose of reading’s self-containing.

Theorem 4.1. Let A be a p -convex compact subset of a locally p -convex vector space X , where $0 < p \leq 1$. Suppose that $T : A \rightarrow 2^A$ is lower (resp. upper) semi-continuous with non-empty p -convex values. Then for any given U which is a p -convex neighborhood of zero in X , there exists $x_U \in A$ such that $T(x_U) \cap (x_U + U) \neq \emptyset$.

Proof. Suppose U is any given element of \mathfrak{U} , there is a symmetric open neighborhood V of zero for which $\overline{V} + \overline{V} \subset U$ in locally p -convex neighborhood of zero, we prove the results by two cases for T is lower semicontinuous (LSC) and upper semicontinuous (USC).

Case 1, by assuming T is lower semi-continuous: As X is locally p -convex vector space, suppose that \mathfrak{U} is the family of neighborhoods of 0 in X . For any element U of \mathfrak{U} , there is a symmetric open neighborhood V of zero for which $\overline{V} + \overline{V} \subset U$. Since A is compact, so there exist x_0, x_1, \dots, x_n in A such that $A \subset \cup_{i=0}^n (x_i + V)$. By using the fact that A is p -convex, we find $D := \{b_0, b_2, \dots, b_n\} \subset A$ for which $b_i - x_i \in V$ for all $i \in \{0, 1, \dots, n\}$ and we define C by $C := C_p(D) \subset A$. By the fact that T is LSC, it follows that the subset $F(b_i) := \{c \in C : T(c) \cap (x_i + V) = \emptyset\}$ is open in C (as the set $x_i + V$ is open) for each $i \in \{0, 1, \dots, n\}$. For any $c \in C$, we have $\emptyset \neq T(c) \cap A \subset T(c) \cap \cup_{i=0}^n (x_i + V)$, it follows that $\cap_{i=0}^n F(b_i) = \emptyset$. Now applying Lemma 3.1 and Lemma 3.2, which implies that that there is $N := \{b_{i_0}, b_{i_1}, \dots, b_{i_k}\} \in \langle D \rangle$ and $x_U \in C_p(N) \subset A$ for which $x_U \notin F(N)$, and so $T(x_U) \cap (x_{i_j} + V) \neq \emptyset$ for all $j \in \{0, 1, \dots, k\}$. As $b_i - x_i \in V$ and $\overline{V} + \overline{V} \subset U$, which imply that $x_{i_j} + \overline{V} \subset b_{i_j} + U$, which means that $T(x_U) \cap ((b_{i_j} + U) \neq \emptyset$, it follows that $N \subset \{c \in C : T(x_U) \cap (c + U) \neq \emptyset\}$. By the fact that the subsets $C, T(x_U)$ and U are p -convex, we have that $x_U \in \{c \in C : T(x_U) \cap (c + U) \neq \emptyset\}$, which means that $T(x_U) \cap (x_U + U) \neq \emptyset$.

Case 2, by assuming T is upper semi-continuous: We define $F(b_i) := \{c \in C : T(c) \cap (x_i + \overline{V}) = \emptyset\}$, which is then closed in C (as the subset $x_i + \overline{V}$ is closed) for each $i = 0, 1, \dots, n$. Then the argument is similar to the proof for the case T is USC, and by applying Lemma 3.1 and Lemma 3.2 again, it follows that there exists $x_U \in A$ such that $T(x_U) \cap (x_U + U) \neq \emptyset$. This completes the proof. \square

By Theorem 4.1, we have the following Fan-Glicksberg fixed point theorems (Fan [33]) in locally p -convex vector spaces for ($0 < p \leq 1$), which also improve or generalize the corresponding results given by Yuan [130], Xiao and Lu [120], Xiao and Zhu [121]-[122] into locally p -convex vector spaces.

Theorem 4.2. Let A be a p -convex compact subset of a locally p -convex vector space X , where $0 < p \leq 1$. Suppose

that $T : A \rightarrow 2^A$ is upper semi-continuous with non-empty p -convex closed values. Then T has at least one fixed point.

Proof. Assume \mathfrak{U} is the family of neighborhoods of 0 in X , and $U \in \mathfrak{U}$, by Theorem 4.1, there exists $x_U \in A$ such that $T(x_U) \cap (x_U + U) \neq \emptyset$. Then there exists $a_U, b_U \in A$ for which $b_U \in T(a_U)$ and $b_U \in a_U + U$. Now, two nets $\{a_U\}$ and $\{b_U\}$ in $\text{Graph}(T)$, which is a compact graph of mapping T as A is compact and T is semi-continuous, we may assume that a_U has a subnet converging to a , and $\{b_U\}$ has a subnet converging to b . As \mathfrak{U} is the family of neighborhoods for 0, we should have $a = b$ (e.g., by the Hausdorff separation property), and $a = b \in T(b)$ due to the fact that $\text{Graph}(T)$ is close (e.g., see Lemma 1.1 of Yuan [129]), thus the proof is complete. \square

For a given set A in vector space X , we denote by “ $\text{lin}(A)$ ” the “linear hull” of A in X .

Definition 4.2. Let A be a subset of a topological vector space X and let Y be another topological vector space. We shall say that A can be linearly embedded in Y if there is a linear map $L : \text{lin}(A) \rightarrow Y$ (not necessarily continuous) whose restriction to A is a homeomorphism.

The following embedded Lemma 4.1 is a significant result due to Theorem 1 of Kalton [53], which says though not every compact convex set can be linearly imbedded in a locally convex space (e.g., see Roberts [100] and Kalton et al.[55]), but for p -convex sets when $0 < p < 1$, every compact p -convex set in topological vector spaces can be considered as a subset of a locally p -convex vector space, hence every such set has sufficiently many p -extreme points.

Secondly, by the property (ii) of Lemma 2.1 above, each convex subset of a topological vector space containing zero is always p -convex for $0 < p \leq 1$, thus it is possible for us to transfer the problem involved p -convex subsets from topological vector spaces into the locally p -convex vector spaces, which indeed allows us to establish the existence of fixed points for upper semi-continuous set-valued mappings for compact p -convex subsets in topological vector spaces ($0 < p \leq 1$) to cover case when the underlying is just a topological vector space, which provides the answer for Schauder’s conjecture in the affirmative.

Lemma 4.1. Let K be a compact p -convex subset ($0 < p < 1$) of a topological vector space X . Then, K can be linearly embedded in a locally p -convex topological vector space.

Proof. It is Theorem 1 of Kalton [53], which completes the proof. \square

Remark 4.1. At this point, it is important to note that Lemma 4.1 does not hold for $p = 1$. By Theorem 9.6 of Kalton et al.[55], it was shown that the spaces $L_p = L_p(0, 1)$, where $0 < p < 1$, contain compact convex sets with no extreme points, which thus cannot be linearly embedded in a locally convex space, see also Roberts [100].

Now we have the following fixed point theorems of upper semi-continuous set-valued mappings for compact p -convex subsets in topological vector spaces (which may not be locally convex). The single-valued version was first given by Theorem 3.3 of Ennassik and Taoudi [31] applying the p -seminorm argument in p -vector spaces for $0 < p \leq 1$. Here we apply KKM theory to establish the set-valued versions for upper semi-continuous mappings, and the technique skill used here called “functional method”.

Theorem 4.3. If K is a nonempty compact p -convex subset of a topological vector space X , with $(0 < p \leq 1)$, then any upper semi-continuous set-valued mappings $T : K \rightarrow 2^K$ with non-empty p -convex closed value, has at least a fixed point.

Proof. We complete the argument by following two steps.

First step, by assuming that K is p -convex, with $0 < p < 1$. By Lemma 4.1, it follows that K can be linearly embedded in a locally p -convex space E , which means that there exists a linear map $L : \text{lin}(K) \rightarrow E$ whose restriction to K is a homeomorphism. Define the mapping $S : L(K) \rightarrow L(K)$ by $(Lx) := L(Tx)$ for $x \in X$. This mapping is easily checked to be well defined. It is also continuous since L is a homeomorphism and T is continuous on K . Furthermore, the set $L(K)$ is compact, being the image of a compact set under a continuous mapping. It is also p -convex since it is the image of an p -convex set under a linear mapping. Then, by Theorem 4.2, there exists $x \in K$ such that $S(Lx) = Lx$. As $L(Tx) = Lx$, which implies (since L is a homeomorphism) that $Tx = x$.

Second step, consider $p = 1$, by the fact K is convex. Choose an arbitrary $x_0 \in K$ and put $K_0 := \{x - x_0 : x \in K\}$. Then we have that, K_0 is a compact convex subset of X which contains the zero element. By Lemma 2.1 (ii), we also conclude that K_0 is p -convex for any $p \in (0, 1)$. Now, define the mapping $R : K_0 \rightarrow K_0$ by $R(x - x_0) := Tx - x_0$ for each $x \in K$. Clearly, R is continuous. Now applying the result from the first step to R , we conclude that there exists $x \in K$ such that $R(x - x_0) = x - x_0$, which implies that $Tx = x$. Thus the proof is complete. \square

Remark 4.2. Theorem 4.3 says that each compact p -convex subsets $(0 < p \leq 1)$ in topological vector spaces, has the fixed point property, which does not only include or improve most available results for fixed point theorems in the existing literature as special cases. In particular, the single-valued version of Theorem 4.3 was first given by Ennassik and Taoudi (Theorem 3.3 of [31]) by using the p -seminorm method, here in this paper it was given by combing both KKM theory and p -convex structure. Here we also mention a number of related works and authors, see Mauldin [73], Granas and Dugundji [46], Ennassik and Taoudi [32], Park [89]-[90] and references therein). Indeed, Theorem 4.3 provides an answer to Schauder conjecture for set-valued version under the general vector spaces which include p -vector spaces as a special of classes.

We recall that for two given topological spaces X and Y , and a set-valued mapping $T : X \rightarrow 2^Y$ is said to be compact if there is compact subset set C in Y such that $F(X) (= \{y \in F(X), x \in X\})$ is contained in C , i.e., $F(X) \subset C$. Now we have the following non-compact version of fixed point theorems for compact set-valued mappings defined on a general p -convex subset in p -vector spaces for $0 < p \leq 1$.

Theorem 4.4 (Schauder Fixed Point Theorem for compact mappings). If C is a nonempty closed p -convex subset of a topological vector space E with $(0 < p \leq 1)$ and $T : C \rightarrow 2^C$ being an upper semi-continuous set-valued mapping with non-empty closed p -convex values, if $F(C)$ is contained in a compact subset of C , then T has at least one fixed point.

Proof. As T is compact, there exists a compact subset A in C such that $T(C) \subset A$. Let $K := \overline{C}_p(A)$, the closure of the p -convex hull of set A in C . Then K is compact p -convex by Proposition 2.4, and the mapping $T : K \rightarrow 2^K$ is upper semi-continuous with non-empty closed p -convex values. Now by Theorem 4.3, it follows that T has a fixed

point $x \in K \subset C$ such that $x \in T(x)$. This completes the proof. \square

As an immediate consequence of Theorem 4.3, we have following result which gives an affirmative answer to Schauder's conjecture for upper semi-continuous version in topological vector spaces (TVS).

Corollary 4.1. If K is a nonempty compact convex subset of a topological vector space X , then any upper semi-continuous set-valued mappings $T : K \rightarrow 2^K$ with non-empty closed convex values has at least a fixed point.

Proof. Apply Theorem 4.3 with $p = 1$, this completes the proof. \square

Corollary 4.1 also improves or unifies corresponding results given by Askoura and Godet-Thobie [5], Cauty [19], Cauty [20], Chen [26], Isac [52], Li [68], Nhu [75], Okon [77], Park [90], Reich [98], Smart [112], Yuan [130], Theorem 3.3 of Ennassik and Taoudi [32], Theorem 3.14 of Gholizadeh et al.[39], Xiao and Lu [120], Xiao and Zhu [121]-[122] under the framework of topological vector spaces for set-valued mappings instead of single-valued functions.

In order to establish fixed point theorems for the classes of 1-set contractive and condensing mappings in p -vector spaces by using the concept of the measure of noncompactness (or saying, the noncompactness measures) which were introduced and widely accepted in mathematical community by Kuratowski [63], Darbo [28] and related references therein, we first need to have a brief introduction for the concept of non-compactness measures for the so-called Kuratowski or Hausdorff measures of noncompactness in normed spaces (see Alghamdi et al.[4], Machrafi and Oubbi [71], Nussbaum [76], Sadovskii [104], Silva et al.[111], Xiao and Lu [120] for the general concepts under the framework of p -seminorm or, just for locally convex p -convex settings for $0 < p \leq 1$ for which will be discussed below, too).

For a given metric space (X, d) (or a p -normed space $(X, \|c \cdot\|_p)$), we recall notions of completeness, boundedness, relatively compactness and compactness as follows. Let (X, d) and (Y, d) be two metric spaces and $T : X \rightarrow Y$ is a mapping (or saying, operator). Then: 1) T is said to be bounded if for each bounded set $A \subset X$, $T(A)$ is bounded set of Y ; 2) T is said to be continuous if for every $x \in X$, the $\lim_{n \rightarrow \infty} x_n = x$ implies that $\lim_{n \rightarrow \infty} T(x_n) = T(x)$; and 3) T is said to be completely continuous if T is continuous and $T(A)$ is relatively compact for each bounded subset A of X .

Let $A_1, A_2 \subset X$ be bounded of a metric space (X, d) , we also recall that the Hausdorff metric $d_H(A_1, A_2)$ between A_1 and A_2 is defined by

$$d_H(A_1, A_2) := \max\left\{ \sup_{x \in A_1} \inf_{y \in A_2} d(x, y), \sup_{y \in A_2} \inf_{x \in A_1} d(x, y) \right\}$$

The Hausdorff and Kuratowski measures of noncompactness (denoted by β_H and β_K , respectively) for nonempty bounded subset D in X are the nonnegative real numbers $\beta_H(D)$ and $\beta_K(D)$ defined by

$$\beta_H(D) := \inf\{\epsilon > 0 : D \text{ has a finite } \epsilon\text{-net}\},$$

and

$$\beta_K(D) := \inf\{\epsilon > 0 : D \subset \cup_{i=1}^n D_i, \text{ where } D_i \text{ is bounded and } \text{diam} D_i \leq \epsilon, n \text{ is an integer}\},$$

here $\text{diam} D_i$ means the diameter of the set D_i , and it is well known that $\beta_H \leq \beta_K \leq 2\beta_H$. We also point out that the notions above can be well defined under the framework of p -seminorm spaces $(E, \|\cdot\|_p)_{p \in \mathfrak{R}}$ by following the

similar idea and method used by Chen and Singh [27], Ko and Tasi [61], Kozlov et al.[62] and references therein for more in details.

Let T is a mapping from $D \subset X$ to X . Then we have that: 1) T is said to be a k -set contraction with respect to β_K (or β_H) if there is a number $k \in [0, 1)$ such that $\beta_K(T(A)) \leq k\beta_K(A)$ (or $\beta_H(T(A)) \leq k\beta_H(A)$) for all bounded sets A in D ; and 2) T is said to be β_K -condensing (or β_H -condensing) if $(\beta_K(T(A)) < \beta_K(A))$ (or $\beta_H(T(A)) < \beta_H(A)$) for all bounded sets A in D with $\beta_K(A) > 0$ (or $\beta_H(A) > 0$).

For the convenience of our discussion, throughout the rest part of this paper, if a mapping “is β_K -condensing (or β_H -condensing)”, we simply say it is “a condensing mapping” unless specified.

Moreover, it is easy to see that: 1) if T is a compact operator, then T is a k -set contraction; and 2) if T is a k -set contraction for $k \in (0, 1)$, then T is condensing.

In order to establish the fixed points of set-valued condensing mappings in p -vector spaces for $0 < p \leq 1$, we need to recall some notions introduced by Machrafi and Oubbi [71] for the measure of noncompactness in locally p -convex vector spaces, which also satisfies some necessary (common) properties of the classical measures of noncompactness such as β_K and β_H mentioned above introduced by Kuratowski [63], Sadovskii [104](see, also related discussion by Alghamdi et al.[4], Nussbaum [76], Silva et al.[111], Xiao and Lu [120] and references therein). In particular, the measures of noncompactness in locally p -vector spaces (for $0 < p \leq 1$) should have the stable property which means the measure of noncompactness A is the same by transition to the (closure) for the p -convex hull of subset A .

For the convenience of discussion, we follow up to use α and β to denote the Kuratowski and the Hausdorff measures of noncompactness in topological vector spaces, respectively (see the same way used by Machrafi and Oubbi [71]), unless otherwise stated. The E is used to denote a Hausdorff topological vector space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{Q}\}$, here \mathbb{R} denotes for all real numbers, and \mathbb{Q} for all complex numbers, and $p \in (0, 1]$. Here, the base set of family of all balanced zero neighborhoods in E is denoted by \mathfrak{B}_0 .

We recall that $U \in \mathfrak{B}_0$ is said to be shrinkable, if it is absorbing, balanced, and $rU \subset U$ for all $r \in (0, 1)$, and we know that any topological vector space admits a local base at zero consisting of shrinkable sets (see Klee [60], or Jarchow [51] for details).

Recalling that a topological vector space E is said to be a locally p -convex space, if E has a local base at zero consisting of p -convex sets. The topology of a locally p -convex space is always given by an upward directed family P of p -semi-norms, where a p -semi-norm on E is any non negative real-valued and subadditive functional $\|\cdot\|_p$ on E such that $\|\lambda x\|_p = |\lambda|^p \|x\|_p$ for each $x \in E$ and $\lambda \in \mathbb{R}$ (i.e., the real number line). When E is Hausdorff, then for every $x \neq 0$, there is some $p \in P$ such that $P(x) \neq 0$. Whenever the family P is reduced to a singleton, one says that $(E, \|\cdot\|)$ is a p -semi-normed space. A p -normed space is a Hausdorff p -seminormed space, and when $p = 1$ which is the usual locally convex case. Furthermore, a p -normed space is a metric vector space with the translation invariant metric $d_p(x, y) := \|x - y\|_p$ for all $x, y \in E$, which is the same notation used above.

By Remark 2.5, if P is a continuous p -seminorm on E , then the ball $B_p(0, s) := \{x \in E : P(x) < s\}$ is shrinkable for each $s > 0$. Indeed, if $r \in (0, 1)$ and $x \in \overline{rB_p(0, s)}$, then there exists a net $(x_i)_{i \in I} \subset B_p(0, s)$ such that rx_i converges to x . By continuity of P , we get $P(x) \leq r^p s < s$, which means that $\overline{rB_p(0, s)} \subset B_p(0, s)$. In generally, it

can be shown that every p -convex $U \in \mathfrak{V}_0$ is shrinkable.

We recall that a given such neighborhood U , a subset $A \subset E$ is said to be U -small if $A - A \subset U$ (or saying, small of order U by Robertson [99]). Now by following the idea of Kaniok [56] in the setting of a topological vector space E to use zero neighborhoods in E instead of semi-norms to define the measure of noncompactness in (local convex) p -vector spaces ($0 < p \leq 1$) as follows: For each $A \subset E$, the U -measures of noncompactness $\alpha_U(A)$ and $\beta_U(A)$ for A are defined by:

$$\alpha_U(A) := \inf\{r > 0 : A \text{ is covered by a finite number of } rU\text{-small sets } A_i \text{ for } i = 1, 2, \dots, n\},$$

and

$$\beta_U(A) := \inf\{r > 0 : \text{there exists } x_1, \dots, x_n \in E \text{ such that } A \subset \cup_{i=1}^n (x_i + rU)\},$$

here we set $\inf \emptyset := \infty$.

By the definition above, it is clear that when E is a normed space and U is the closed unit ball of E , α_U and β_U are nothing else but the Kuratowski measure β_K and Hausdorff measure β_H of noncompactness, respectively. Thus, if \mathfrak{U} denotes a fundamental system of balanced and closed zero neighborhoods in E and $\mathfrak{F}_{\mathfrak{U}}$ is the space of all functions $\phi : \mathfrak{U} \rightarrow \mathbb{R}$, endowed with the pointwise ordering, then, the α_U (resp., β_U) measures for noncompactness introduced by Kaniok [56] can be expressed by the Kuratowski (resp., the Hausdorff) measure of noncompact $\alpha(A)$ (resp., $\beta(A)$) for a subset A of E as the function defined from \mathfrak{U} into $[0, \infty)$ by

$$\alpha(A)(U) := \alpha_U(A) \text{ (resp., } \beta(A)(U) := \beta_U(A)).$$

By following Machrafi and Oubbi [71], in order to define the measure of noncompactness in (locally convex) p -vector space E , we need the following notions of basic and sufficient collections for zero neighborhoods in a topological vector space. To do this, let us introduce an equivalence relation on \mathfrak{V}_0 by saying that U is related to V , written $U \mathfrak{R} V$, if and only if there exist $r, s > 0$ such that $rU \subset V \subset sU$. We now have the following definition.

Definition 4.2 (BCZN). We say that $\mathfrak{B} \subset \mathfrak{V}_0$ is a basic collection of zero neighborhoods (in short, BCZN) if it contains at most one representative member from each equivalence class with respect to \mathfrak{R} . It will be said to be sufficient (in short, SCZN) if it is basic and, for every $V \in \mathfrak{V}_0$, there exists some $U \in \mathfrak{B}$ and some $r > 0$ such that $rU \subset V$.

Remark 4.3. By Remark 2.5, it follows that for a locally p -convex space E , its base set \mathfrak{U} , the family of all open p -convex subsets for 0 is BCZB. We also note that: 1) In the case if E is a normed space, if f is a continuous functional on E , $U := \{x \in E : |f(x)| < 1\}$, and V is the open unit ball of E , then $\{U\}$ is basic but not sufficient, but $\{V\}$ is sufficient; 2) Secondly, if (E, τ) is a locally convex space, whose topology is given by an upward directed family P of seminorms, so that no two of them are equivalent, the collection $(B_p)_{p \in P}$ is a SCZN, where B_p is the open unit ball of p . Further, if \mathfrak{W} is a fundamental system of zero neighborhoods in a topological vector space E , then there exists an SCZN consisting of \mathfrak{W} members; and 3) By following Oubbi [82], we recall that a subset A of E is called uniformly bounded with respect to a sufficient collection \mathfrak{B} of zero neighborhoods, if there exists

$r > 0$ such that $A \subset rV$ for all $V \in \mathfrak{B}$. Note that in the locally convex space $C_c(X) := C_c(X, \mathbb{K})$, the set $B_\infty := \{f \in C(X) : \|f\|_\infty \leq 1\}$ is uniformly bounded with respect to the SCZN $\{B_k, k \in \mathbb{K}\}$, where B_k is the (closed or) open unit ball of the seminorm P_k , where $k \in \mathbb{K}$.

Now we are ready to give the definition for the measure of non-compactness in (locally p -convex) topological vector space E as follows.

Definition 4.3. Let \mathfrak{B} be a SCZN in E . For each $A \subset E$, we define the measure of noncompactness of A with respect to \mathfrak{B} by $\alpha_{\mathfrak{B}}(A) := \sup_{U \in \mathfrak{B}} \alpha_U(A)$.

By the definition above, it is clear that: 1) The measure of noncompactness α_B holding the semi-additivity, i.e., $\alpha_B(A \cup B) = \max\{\alpha_B(A), \alpha_B(B)\}$; and 2) $\alpha_B(A) = 0$ if and only if A is a precompact subset of E (for more properties in details, see Proposition 1 and related discussion by Machraf and Oubbi [82]).

As we know, under the normed spaces (and even semi-normed spaces), Kuratowski [63], Darbo [28] and Sadovskii [104] introduced the notions of k -set-contractions for $k \in (0, 1)$, and the condensing mappings to establish fixed point theorems in the setting of Banach spaces, normed or semi-norm spaces. By following the same idea, if E is a Hausdorff locally p -convex space, we have the following definition for general (nonlinear) mappings.

Definition 4.4. A mapping $T : C \rightarrow 2^C$ is said to be a k -set contraction (resp., condensing), if there is some SCZN \mathfrak{B} in E consisting of p -convex sets, such that (resp., condensing) for any $U \in \mathfrak{B}$, there exists $k \in (0, 1)$ (resp., condensing) such that $\alpha_U(T(A)) \leq k\alpha_U(A)$ for $A \subset C$ (resp., $\alpha_U(T(A)) < \alpha_U(A)$ for each $A \subset C$ with $\alpha_U(A) > 0$).

It is clear that a contraction mapping on C is a k -set contraction mapping (where we always mean $k \in (0, 1)$), and a k -set contraction mapping on C is condensing; and they all reduce to the usually cases by the definitions for the β_K and β_H which are the Kuratowski measures and Hausdorff measure of noncompactness, respectively in normed spaces (see Kuratowski [63]).

From now on, we denote by \mathfrak{V}_0 the set of all shrinkable zero neighborhoods in E , we have the following result which is Theorem 1 of Machraf and Oubbi [71], saying that in the general setting of locally p -convex spaces, the measure of noncompactness α for U given by Definition 4.3 above is stable from U to its p -convex hull $C_p(A)$ of the subset A in E , which is key for us to establish the fixed points for condensing mappings in locally p -convex spaces for $0 < p \leq 1$. This also means that the key property for the measures due to the Kuratowski and Hausdorff measures of noncompactness in normed (or p -semi-norm) spaces, which also holds for the measure of noncompactness by Definition 4.3 in the setting of locally p -convex spaces with $(0 < p \leq 1)$ (see more similar and related discussion in details by Alghamdi et al.[4] and Silva et al.[111]).

Lemma 4.2. If $U \in \mathfrak{V}_0$ is p -convex for some $0 < p \leq 1$, then $\alpha(C_p(A)) = \alpha(A)$ for every $A \subset E$.

Proof. It is Theorem 1 of Machraf and Oubbi [71]. The proof is complete. \square

Now based on the definition for the measure of noncompactness given by Definition 4.3 (originally from Machraf and Oubbi [71]), we have the following general extension version of Schauder, Darbo and Sadovskii type fixed point

theorems in the context of locally p -convex vector spaces for condensing mappings.

Theorem 4.5 (Schauder Fixed Point Theorem for condensing mappings). Let $C \subset E$ be a complete p -convex subset of a Hausdorff locally p -convex space E , with $0 < p \leq 1$. If $T : C \rightarrow 2^C$ is an upper semi-continuous and (α) condensing set-valued mappings with non-empty p -convex closed values, then T has a fixed point in C and the set of fixed points of T is compact.

Proof. Let \mathfrak{B} be a sufficient collection of p -convex zero neighborhoods in E with respect to which T is condensing and for any given $U \in \mathfrak{B}$. We choose some $x_0 \in C$ and let \mathfrak{F} be the family of all closed p -convex subsets A of C with $x_0 \in A$ and $T(A) \subset A$. Note that \mathfrak{F} is not empty since $C \in \mathfrak{F}$. Let $A_0 = \bigcap_{A \in \mathfrak{F}} A$. Then A_0 is a non empty closed p -convex subset of C , such that $T(A_0) \subset A_0$. We shall show that A_0 is compact. Let $A_1 = \overline{C_p(T(A_0) \cup \{x_0\})}$. Since $T(A_0) \subset A_0$ and A_0 is closed and p -convex, $A_1 \subset A_0$. Hence, $T(A_1) \subset T(A_0) \subset A_1$. It follows that $A_1 \in \mathfrak{F}$ and therefore $A_1 = A_0$. Now by Proposition 1 of Machrafi and Oubbi [71] and Lemma 4.2 above (i.e., Theorem 1 and Theorem 2 in [71]), we get $\alpha_U(T(A_0)) = \alpha_U(A)$. Our assumption on T shows that $\alpha_U(A_0) = 0$ since T is condensing. As U is arbitrary from the family \mathfrak{B} , thus A_0 is p -convex and compact (see Proposition 4 in [71]). Now, the conclusion follows by Theorem 4.3 above. Secondly, let C_0 be the set of fixed points of T in C . Then it follows that $C_0 \subset T(C_0)$ and the upper semi-continuity of T implies that its graph is closed, so is the set C_0 . As T is condensing, we have $\alpha_U(T(C_0)) \leq \alpha_U(C_0)$, which implies that $\alpha_U(C_0) = 0$. As U is arbitrary from the family \mathfrak{B} , which implies that C_0 is compact (by the Proposition 4 in [71] again). The proof is complete. \square

As applications of Theorem 4.5, we have the following a few of fixed points for condensing mappings in locally p -convex spaces.

Corollary 4.2 (Darbo type fixed point theorem). Let C be a complete p -convex subset of a Hausdorff locally p -convex space E with $0 < p \leq 1$. If $T : C \rightarrow 2^C$ is a (k) -set-contraction (where $k \in (0, 1)$) with closed and p -convex values, then T has a fixed point.

Corollary 4.3 (Sadovskii type fixed point theorem). Let $(E, \|\cdot\|)$ be a complete p -normed space and C be a bounded, closed and p -convex subset of E , where $0 < p \leq 1$. Then, every continuous and condensing mapping $T : C \rightarrow 2^C$ with closed and p -convex values has a fixed point.

Proof. In Theorem 4.5, let $\mathfrak{B} := \{B_p(0, 1)\}$, where $B_p(0, 1)$ stands for the closed unit ball of E , and by the fact that it is clear that $\alpha(A) = (\alpha_{\mathfrak{B}}(A))^p$ for each $A \subset E$. Then that T satisfies all conditions of Theorem 4.5. This completes the proof. \square

Corollary 4.4 (Darbo type). Let $(E, \|\cdot\|)$ be a complete p -normed space and C be a bounded, closed and p -convex subset of E , where $0 < p \leq 1$. Then each mapping $T : C \rightarrow C$ which is continuous and a set-contraction has a fixed point.

Theorem 4.5 improves Theorem 5 of Machrafi and Oubbi [71] for general condensing mappings which are general upper semi-continuous mappings with closed p -convex values, and also unify corresponding the results in the existing literature, e.g., see Alghamdi et al.[4], Górniewicz [44], Górniewicz et al.[45], Nussbaum [76], Silva et al.[111], Xiao

and Lu [120], Xiao and Zhu [121]-[122] and references therein.

Secondly, as an application of KKM principle for abstract convex spaces with Kalton's remarkable embedded lemma [53] for compact p -convex sets in topological vector spaces, we also establish the general fixed point theorems for upper semi-continuous set-valued mappings, which also allow us to answer Schauder's conjecture in the affirmative way under the framework of p -vector spaces for $0 < p \leq 1$.

Before the ending of this section, we also like to remark that by comparing with topological method or related arguments used by Askoura et al.[5], Cauty [19]-[20], Nhu [75], Reich[98], the fixed points given in this section improve or unify the corresponding ones given by Alghamdi et al.[4], Darbo [28], Liu[70], Machrafi and Oubbi [71], Sadovskii [104], Silva et al.[111], Xiao and Lu [120] and those from references therein.

5. The Best Approximation for the classes of 1-Set Contractive Mappings and related Applications

The goal of this section is first to establish one general best approximation results for 1-set upper semi-continuous and hemicompact (see its definition below) non-self set-valued mappings, which in turn is used as a tool to derive the general principle for the existence of solutions for Birkhoff-Kellogg Problems (see Birkhoff and Kellogg [11]) and fixed points for non-self 1-set contractive set-valued mappings.

Here, we recall that since the Birkhoff-Kellogg theorem was first introduced and proved by Birkhoff and Kellogg [11] in 1922 in discussing the existence of solutions for the equation $x = \lambda F(x)$, where λ is a real parameter, and F is a general nonlinear non-self mapping defined on an open convex subset U of a topological vector space E , now the general form of the Birkhoff-Kellogg problem is to find the so-called an invariant direction for the nonlinear set-valued mappings F , i.e., to find $x_0 \in \bar{U}$ (or $x_0 \in \partial\bar{U}$) and $\lambda > 0$ such that $\lambda x_0 \in F(x_0)$.

Since Birkhoff and Kellogg theorem given by Birkhoff and Kellogg in 1920's, the study on Birkhoff-Kellogg problem has been received a lot of attention by scholars since then, for example, one of the fundamental results in nonlinear functional analysis, called the Leray-Schauder alternative by Leray and Schauder [64] in 1934, was established via topological degree. Thereafter, certain other types of Leray-Schauder alternatives were proved using different techniques other than topological degree, see work given by Granas and Dugundji [46], Furi and Pera [37] in the Banach space setting and applications to the boundary value problems for ordinary differential equations, and a general class of mappings for nonlinear alternative of Leray-Schauder type in normal topological spaces, and also Birkhoff-Kellogg type theorems for general class mappings in TVS by Agarwal et al.[1], Agarwal and O'Regan [2]-[3], Park [86]; in particular, recently O'Regan [80] using the Leray-Schauder type coincidence theory to establish some Birkhoff-Kellogg problem, Furi-Pera type results for a general class of set-valued mappings, too.

In this section, one best approximation result for 1-set contractive mappings in p -seminorm spaces is first established, which is then used to the general principle for solutions of Birkhoff-Kellogg problems and related nonlinear alternatives, then they allows us to give general existence results for the Leray-Schaduer type and related fixed point theorems of non-self mappings in p -seminorm spaces for $(0 < p \leq 1)$. The new results given in this part not only include the corresponding results in the existing literature as special cases, but also expected to be useful tools for the study of nonlinear problems arising from theory to practice for 1-set contractive mappings.

We also note that the general nonlinear alternative related to Leray-Schauder alternative under the framework of p -seminorm spaces for $(0 < p \leq 1)$ given in this section would be useful tools for the study of nonlinear problems. In addition, we also note that corresponding results in the existing literature for Birkhoff-Kellogg problems and the Leray-Schauder alternative have been studied comprehensively by Granas and Dugundji [46], Isac [52], Park [87]-[89], Carbone and Conti [18], Chang and Yen [23], Chang et al.[24]-[25], Kim et al.[57], Shahzad [107]-[105], Singh [109]; and in particular, many general forms recently obtained by O'Regan [79], and references therein.

In order to study the existence of fixed points for non-self mappings in p -vector spaces, we need the following definitions.

Definition 5.1 (Inward and Outward Sets in p -vector spaces). Let C be a subset of a p -vector space E and $x \in E$ for $0 < p \leq 1$. Then the p -Inward set $I_C^p(x)$ and p -Outward set $O_C^p(x)$ are defined by

$I_C^p(x) := \{x + r(y - x) : y \in C, \text{ for any } r \geq 0 \text{ (1) if } 0 \leq r \leq 1, \text{ with } (1 - r)^p + r^p = 1; \text{ or (2) if } r \geq 1, \text{ with } (\frac{1}{r})^p + (1 - \frac{1}{r})^p = 1\}$; and

$O_C^p(x) := \{x + r(y - x) : y \in C, \text{ for any } r \leq 0 \text{ (1) if } 0 \leq |r| \leq 1, \text{ with } (1 - |r|)^p + |r|^p = 1; \text{ or (2) if } |r| \geq 1, \text{ with } (\frac{1}{|r|})^p + (1 - \frac{1}{|r|})^p = 1\}$.

From the definition, it is obviously that when $p = 1$, the both Inward and Outward sets $I_C^p(x)$, $O_C^p(x)$ are reduced to the definition for the Inward set $I_C(x)$ and the Outward set $O_C(x)$, respectively in topological vector spaces introduced by Halpern and Bergman [48] and used for the study of non-self mappings related to nonlinear functional analysis in the literature. In this paper, we will mainly focus on the study of the p -Inward set $I_C^p(x)$ for the best approximation and related to the boundary condition for the existence of the fixed points in p -vector spaces. By the special property of p -convex concept when $p \in (0, 1)$ and $p = 1$, we have the following fact.

Lemma 5.1. Let C be a subset of a p -vector space E and $x \in E$, where for $0 < p \leq 1$. Then for both p -Inward and Outward sets $I_C^p(x)$ and $O_C^p(x)$ defined above, we have

(I) when $p \in (0, 1)$, $I_C^p(x) = [\{x\} \cup C]$, and $O_C^p(x) = [\{x\} \cup \{2x\} \cup -C]$,

(II) when $p = 1$, in general $[\{x\} \cup C] \subset I_C^p(x)$, and $[\{x\} \cup \{2x\} \cup -C] \subset O_C^p(x)$.

Proof. First, when $p \in (0, 1)$, by the definitions of $I_C^p(x)$, the only real number $r \geq 0$ satisfying the equation $(1 - r)^p + r^p = 1$ for $r \in [0, 1]$ is $r = 0$ or $r = 1$, and when $r \geq 1$, the equation $(\frac{1}{r})^p + (1 - \frac{1}{r})^p = 1$ implies that $r = 1$. The same reason for $O_C^p(x)$, it follows that $r = 0$ and $r = -1$.

Secondly when $p = 1$, all $r \geq 0$, and all $r \leq 0$ satisfy the requirement of definition for $I_C^p(x)$ and $O_C^p(x)$, respectively, thus the proof is complete. \square

By following the original idea by Tan and Yuan [115] for hemicompact mappings in metric spaces, we introduce the following definition for a mapping being hemicompact in p -seminorm spaces for $p \in (0, 1]$, which is indeed the “**(H) condition**” below used in Theorem 5.1 to prove the existence of best approximation results for 1-set contractive set-valued mappings in p -seminorm vector spaces for $p \in (0, 1]$.

Definition 5.2 (Hemicompact mapping). Let E be a p -vector space with p -seminorm for $1 < p \leq 1$. For a given bonded (closed) subset D in E , a mapping $F : D \rightarrow 2^E$ is said to be hemicompact if each sequence $\{x_n\}_{n \in \mathbb{N}}$

in D has a convergent subsequence with limit x_0 such that $x_0 \in F(x_0)$, whenever $\lim_{n \rightarrow \infty} d_{P_U} P(x_n, F(x_n)) = 0$ for each $U \in \mathfrak{U}$, where $d_{P_U} P(x, C) := \inf\{P_U(x - y) : y \in C\}$ is the distance of a single point x with the subset C in E based on P_U , P_U is the Minkowski p -functional in E for $U \in \mathfrak{U}$, which is the base of the family consisted by all subsets of 0-neighborhoods in E .

Remark 5.1. We like to point that the Definition 5.2 is indeed an extension for a “hemicompact mapping” defined from a metric space to a p -vector space with the p -seminorm, where $p \in (0, 1]$ (see Tan and Yuan [115]). By the monotonicity of Minkowski p -functionals, i.e., the bigger 0-neighborhoods, the smaller Minkowski p -functionals’ values (see also p.178 of Balachandran [6]), the Definition 5.2 describes the converge for the distance between x_n and $F(x_n)$ by using the language of seminorms in terms of Minkowski p -functionals for each 0-neighborhood in \mathfrak{U} (the base), which is the family consisted of its 0-neighborhoods in p -vector space E .

Now we have the following Schauder fixed point theorem for 1-set contractive mappings in locally p -convex spaces for $p \in (0, 1]$.

Theorem 5.1 (Schauder Fixed Point Theorem for 1-set contractive mappings). Let U be a non-empty bounded open subset of a (Hausdorff) locally p -convex space E and its zero $0 \in U$, and $C \subset E$ be a closed p -convex subset of E such that $0 \in C$, with $0 < p \leq 1$. If $F : C \cap \bar{U} \rightarrow 2^{C \cap \bar{U}}$ is an upper semi-continuous and 1-set contractive set-valued mappings with non-empty p -convex closed values and satisfying the following (H) or (H1) condition:

(H) Condition: The sequence $\{x_n\}_{n \in \mathbb{N}}$ in \bar{U} has a convergent subsequence with limit $x_0 \in \bar{U}$ such that $x_0 \in F(x_0)$, whenever $\lim_{n \rightarrow \infty} d_{P_U}(x_n, F(x_n)) = 0$, where, $d_{P_U}(x_n, F(x_n)) := \inf\{P_U(x_n - z) : z \in F(x_n)\}$, where P_U is the Minkowski p -functional for any $U \in \mathfrak{U}$, which is the family of all non-empty open p -convex subset containing the zero in E .

(H1) Condition: There exists x_0 in \bar{U} with $x_0 \in F(x_0)$ if there exists $\{x_n\}_{n \in \mathbb{N}}$ in \bar{U} such that $\lim_{n \rightarrow \infty} d_{P_U}(x_n, F(x_n)) = 0$, where, P_U is the Minkowski p -functional for any $U \in \mathfrak{U}$, which is the family of all non-empty open p -convex subset containing the zero in E .

Then F has at least one fixed point in $C \cap \bar{U}$.

Proof. Let \mathfrak{U} be the family of all non-empty open p -convex subset containing the zero in E , and U be any element in \mathfrak{U} . As the mapping T is 1-set contractive, taking an increasing sequence $\{\lambda_n\}$ such that $0 < \lambda_n < 1$ and $\lim_{n \rightarrow \infty} \lambda_n = 1$, where $n \in \mathbb{N}$. Now we define a mapping $F_n : C \rightarrow 2^C$ by $F_n(x) := \lambda_n F(x)$ for each $x \in C$ and $n \in \mathbb{N}$. Then it follows that F_n is a λ_n -set-contractive mapping with $0 < \lambda_n < 1$. By Theorem 4.5 on the condensing mapping F_n in p -vector space with p -seminorm P_U for each $n \in \mathbb{N}$, there exists $x_n \in C$ such that $x_n \in F_n(x_n) = \lambda_n F(x_n)$. Thus there exists $y_n \in F(x_n)$ such that $x_n = \lambda_n y_n$. As P_U is the Minkowski p -functional of U in E , it follows that P_U is continuous as $0 \in \text{int}(U) = U$. Note that for each $n \in \mathbb{N}$, $\lambda_n x_n \in \bar{U} \cap C$, which imply that $x_n = r(\lambda_n y_n) = \lambda_n y_n$, thus $P_U(\lambda_n y_n) \leq 1$ by Lemma 2.2. Note that

$$P_U(y_n - x_n) = P_U(y_n - \lambda_n y_n) = P_U(y_n - \lambda_n y_n) = P_U\left(\frac{(1 - \lambda_n)\lambda_n y_n}{\lambda_n}\right) \leq \left(\frac{1 - \lambda_n}{\lambda_n}\right)^p P_U(\lambda_n y_n) \leq \left(\frac{1 - \lambda_n}{\lambda_n}\right)^p,$$

which implies that $\lim_{n \rightarrow \infty} P_U(y_n - x_n) = 0$ for all $U \in \mathfrak{U}$.

Now (1) if F satisfies the (H) condition, it implies that the consequence $\{x_n\}_{n \in \mathbb{N}}$ has a convergent subsequence which converges to x_0 such that $x_0 \in F(x_0)$. Without loss of the generality, we assume that $\lim_{n \rightarrow \infty} x_n = x_0$, here $y_n \in F(x_n)$ is with $x_n = \lambda_n y_n$, and $\lim_{n \rightarrow \infty} \lambda_n = 1$, it implies that $x_0 = \lim_{n \rightarrow \infty} (\lambda_n y_n)$, which means $y_0 := \lim_{n \rightarrow \infty} y_n = x_0$. There exists $y_0 (= x_0) \in F(x_0)$.

(ii) if F satisfies the (H1) condition, then by the (H1) condition, it follows that there exists x_0 in \bar{U} such that $x_0 \in F(x_0)$, which is a fixed point of F . We complete the proof. \square

Theorem 5.2 (Best approximation for 1-set-contractive mappings). Let U be a bounded open p -convex subset of a locally p -convex space E ($0 \leq p \leq 1$) the zero $0 \in U$, and C a (bounded) closed p -convex subset of E with also zero $0 \in C$. Assume $F : \bar{U} \cap C \rightarrow 2^C$ is a 1-set contractive and upper semi-continuous mapping with non-empty closed p -convex values, and for each $x \in \partial_C U$ with $y \in F(x) \cap (C \setminus \bar{U})$, $(P_U^{\frac{1}{p}}(y) - 1)^p \leq P_U(y - x)$ for $0 < p \leq 1$ (this is trivial when $p = 1$). In addition, if F satisfies the following (H) or (H1) condition:

(H) Condition: The sequence $\{x_n\}_{n \in \mathbb{N}}$ in \bar{U} has a convergent subsequence with limit $x_0 \in \bar{U}$ such that $x_0 \in F(x_0)$, whenever $\lim_{n \rightarrow \infty} d_{P_U}(x_n, F(x_n)) = 0$, where, $d_{P_U}(x_n, F(x_n)) := \inf\{P_U(x_n - z) : z \in F(x_n)\}$, where P_U is the Minkowski p -functional for any $U \in \mathfrak{U}$, which is the family of all non-empty open p -convex subset containing the zero in E .

(H1) Condition: There exists x_0 in \bar{U} with $x_0 \in F(x_0)$ if there exists $\{x_n\}_{n \in \mathbb{N}}$ in \bar{U} such that $\lim_{n \rightarrow \infty} d_{P_U}(x_n, F(x_n)) = 0$, where, P_U is the Minkowski p -functional for any $U \in \mathfrak{U}$, which is the family of all non-empty open p -convex subset containing the zero in E .

Then we have that there exist $x_0 \in C \cap \bar{U}$ and $y_0 \in F(x_0)$ such that

$$P_U(y_0 - x_0) = d_P(y_0, \bar{U} \cap C) = d_p(y_0, \overline{I_U^p(x_0)} \cap C),$$

where P_U is the Minkowski p -functional of U . More precisely, we have the following either (I) or (II) holding:

(I) F has a fixed point $x_0 \in \bar{U} \cap C$, i.e., $0 = P_U(y_0 - x_0) = d_P(y_0, \bar{U} \cap C) = d_p(y_0, \overline{I_U^p(x_0)} \cap C)$,

(II) there exists $x_0 \in \partial_C(U)$ and $y_0 \in F(x_0) \setminus \bar{U}$ with

$$P_U(y_0 - x_0) = d_P(y_0, \bar{U} \cap C) = d_p(y_0, \overline{I_U^p(x_0)} \cap C) = (P_U^{\frac{1}{p}}(y_0) - 1)^p > 0.$$

Proof. As E is p -convex space and U is a bounded open p -convex subset of E , it suffices to prove that there exists a sequence $(x_n)_{n \in \mathbb{N}}$ in \bar{U} and $y_n \in F(x_n)$ such that $\lim_{n \rightarrow \infty} P_U(y_n - x_n) = 0$, and the conclusion follows by applying the (H) condition.

Let $r : E \rightarrow U$ be a retraction mapping defined by $r(x) := \frac{x}{\max\{1, (P_U(x))^{\frac{1}{p}}\}}$ for each $x \in E$, where P_U is the Minkowski p -functional of U . Since the space E 's zero $0 \in U (= \text{int}U$ as U is open), it follows that r is continuous by Lemma 2.2. As the mapping F is 1-set contractive, taking an increasing sequence $\{\lambda_n\}$ such that $0 < \lambda_n < 1$ and $\lim_{n \rightarrow \infty} \lambda_n = 1$, where $n \in \mathbb{N}$. Now for each $n \in \mathbb{N}$, we define a mapping $F_n : C \cap \bar{U} \rightarrow 2^C$ by $F_n(x) := \lambda_n F \circ r(x)$ for each $x \in C \cap \bar{U}$. By the fact that C and \bar{U} are p -convex, it follows that $r(C) \subset C$ and $r(\bar{U}) \subset \bar{U}$, thus $r(C \cap \bar{U}) \subset C \cap \bar{U}$. Therefore F_n is a mapping from $\bar{U} \cap C$ to itself. Nor each $n \in \mathbb{N}$, by the fact that F_n is a λ_n -set-contractive mapping with $0 < \lambda_n < 1$ it follows by Theorem 4.5 for the condensing mapping that there exists

$z_n \in C \cap \bar{U}$ such that $z_n \in F_n(z_n) = \lambda_n F \circ r(z_n)$. As $r(C \cap \bar{U}) \subset C \cap \bar{U}$, let $x_n = r(z_n)$. Then we have that $x_n \in C \cap \bar{U}$. and there exists $y_n \in F(x_n)$ with $x_n = r(\lambda_n y_n)$ such that the following (1) or (2) holding for each $n \in \mathbb{N}$: (1) $\lambda_n y_n \in C \cap \bar{U}$; or (2) $\lambda_n y_n \in C \setminus \bar{U}$.

Now we prove the conclusion by considering the following two cases under the (H) condition and (H1) condition.

Case (I) For each $n \in \mathbb{N}$, $\lambda_n y_n \in C \cap \bar{U}$; or

Case (II) There there exists a positive integer n such that $\lambda_n y_n \in C \setminus \bar{U}$.

First, by the case (I), for each $n \in \mathbb{N}$, $\lambda_n y_n \in \bar{U} \cap C$, which imply that $x_n = r(\lambda_n y_n) = \lambda_n y_n$, thus $P_U(\lambda_n y_n) \leq 1$ by Lemma 2.2. Note that

$$P_U(y_n - x_n) = P_U(y_n - x_n) = P_U(y_n - \lambda_n y_n) = P_U\left(\frac{(1 - \lambda_n)\lambda_n y_n}{\lambda_n}\right) \leq \left(\frac{1 - \lambda_n}{\lambda_n}\right)^p P_U(\lambda_n y_n) \leq \left(\frac{1 - \lambda_n}{\lambda_n}\right)^p,$$

which implies that $\lim_{n \rightarrow \infty} P_U(y_n - x_n) = 0$. Now for any $V \in \mathcal{U}$, without loss of generality, let $U_0 = V \cap U$. Then we have the following conclusion:

$$P_{U_0}(y_n - x_n) = P_{U_0}(y_n - x_n) = P_{U_0}(y_n - \lambda_n y_n) = P_{U_0}\left(\frac{(1 - \lambda_n)\lambda_n y_n}{\lambda_n}\right) \leq \left(\frac{1 - \lambda_n}{\lambda_n}\right)^p P_{U_0}(\lambda_n y_n) \leq \left(\frac{1 - \lambda_n}{\lambda_n}\right)^p,$$

which implies that $\lim_{n \rightarrow \infty} P_{U_0}(y_n - x_n) = 0$, where P_{U_0} is the Minkowski p -functional of U_0 in E .

Now if F satisfies the (H) condition, it follows that the consequence $\{x_n\}_{n \in \mathbb{N}}$ has a convergent subsequence which converges to x_0 such that $x_0 \in F(x_0)$. Without loss of the generality, we assume that $\lim_{n \rightarrow \infty} x_n = x_0$, where $y_n \in F(x_n)$ is with $x_n = \lambda_n y_n$, and $\lim_{n \rightarrow \infty} \lambda_n = 1$, and as $x_0 = \lim_{n \rightarrow \infty} (\lambda_n y_n)$, which implies that $y_0 = \lim_{n \rightarrow \infty} y_n = x_0$. Thus there exists $y_0 (= x_0) \in F(x_0)$, we have $0 = d_p(x_0, F(x_0)) = d(y_0, \bar{U} \cap C) = d_p(y_0, \overline{I_{\bar{U}}^p(x_0)} \cap C)$ as indeed $x_0 = y_0 \in F(x_0) \in \bar{U} \cap C \subset \overline{I_{\bar{U}}^p(x_0)} \cap C$.

If F satisfies the (H1) condition, it follows that there exists $x_0 \in \bar{U} \cap C$ with $x_0 \in F(x_0)$. Then we have $0 = P_U(y_0 - x_0) = d_P(y_0, \bar{U} \cap C) = d_p(y_0, \overline{I_{\bar{U}}^p(x_0)} \cap C)$.

Second, by the case (II) there exists a positive integer n such that $\lambda_n y_n \in C \setminus \bar{U}$. Then we have that $P_U(\lambda_n y_n) > 1$, and also $P_U(y_n) > 1$ as $\lambda_n < 1$. As $x_n = r(\lambda_n y_n) = \frac{\lambda_n y_n}{(P_U(\lambda_n y_n))^{\frac{1}{p}}}$, which implies that $P_U(x_n) = 1$, thus $x_n \in \partial_C(U)$. Note that

$$P_U(y_n - x_n) = P_U\left(\frac{(P_U(y_n))^{\frac{1}{p}} - 1}{P_U(y_n)^{\frac{1}{p}}}\right) = (P_U^{\frac{1}{p}}(y_n) - 1)^p.$$

By the assumption, we have $(P_U^{\frac{1}{p}}(y_n) - 1)^p \leq P_U(y_n - x)$ for $x \in C \cap \partial \bar{U}$, it follows that

$$P_U(y_n) - 1 \leq P_U(y_n) - \sup\{P_U(z) : z \in C \cap \bar{U}\} \leq \inf\{P_U(y_n - z) : z \in C \cap \bar{U}\} = d_p(y_n, C \cap \bar{U}).$$

Thus we have the best approximation: $P_U(y_n - x_n) = d_P(y_n, \bar{U} \cap C) = (P_U^{\frac{1}{p}}(y_n) - 1)^p > 0$.

Now we want to show that $P_U(y_n - x_n) = d_P(y_n, \bar{U} \cap C) = d_p(y_n, \overline{I_{\bar{U}}^p(x_0)} \cap C) > 0$.

By the fact that $(\bar{U} \cap C) \subset I_{\bar{U}}^p(x_n) \cap C$, let $z \in I_{\bar{U}}^p(x_n) \cap C \setminus (\bar{U} \cap C)$, we first claim that $P_U(y_n - x_n) \leq P_U(y_n - z)$. If not, we have $P_U(y_n - x_n) > P_U(y_n - z)$. As $z \in I_{\bar{U}}^p(x_n) \cap C \setminus (\bar{U} \cap C)$, there exists $y \in \bar{U}$ and a non-negative number c (actually $c \geq 1$ as shown soon below) with $z = x_n + c(y - x_n)$. Since $z \in C$, but $z \notin \bar{U} \cap C$, it implies that $z \notin \bar{U}$. By the fact that $x_n \in \bar{U}$ and $y \in \bar{U}$, we must have the constant $c \geq 1$; otherwise, it implies that $z (= (1 - c)x_n + cy) \in \bar{U}$, this is impossible by our assumption, i.e., $z \notin \bar{U}$. Thus we have that $c \geq 1$, which implies

that $y = \frac{1}{c}z + (1 - \frac{1}{c})x_n \in C$ (as both $x_n \in C$ and $z \in C$). On the other hand, as $z \in I_{\overline{U}}^p(x_n) \cap C \setminus (\overline{U} \cap C)$, and $c \geq 1$ with $(\frac{1}{c})^p + (1 - \frac{1}{c})^p = 1$, combing with our assumption that for each $x \in \partial_C \overline{U}$ and $y \in F(x_n) \setminus \overline{U}$, $P_U^{\frac{1}{p}}(y) - 1 \leq P_U^{\frac{1}{p}}(y - x)$ for $0 < p \leq 1$, it then follows that

$$P_U(y_n - y) = P_U[\frac{1}{c}(y_n - z) + (1 - \frac{1}{c})(y_n - x_n)] \leq [(\frac{1}{c})^p P_U(y_n - z) + (1 - \frac{1}{c})^p P_U(y_n - x_n)] < P_U(y_n - x_n),$$

which contradicts that $P_U(y_n - x_n) = d_P(y_n, \overline{U} \cap C)$ as shown above we know that $y \in \overline{U} \cap C$, we should have $P_U(y_n - x_n) \leq P_U(y_n - y)$! This helps us to complete the claim: $P_U(y_n - x_n) \leq P_U(y_n - z)$ for any $z \in I_{\overline{U}}^p(x_n) \cap C \setminus (\overline{U} \cap C)$, which means that the following best approximation of Fan's type (see [35]-[36]) holding:

$$0 < d_P(y_n, \overline{U} \cap C) = P_U(y_n - x_n) = d_p(y_n, I_{\overline{U}}^p(x_n) \cap C).$$

Now by the continuity of P_U , it follows that the following best approximation of Fan type is also true:

$$0 < P_U(y_n - x_n) = d_P(y_n, \overline{U} \cap C) = d_p(y_n, I_{\overline{U}}^p(x_n) \cap C) = d_p(y_n, \overline{I_{\overline{U}}^p(x_n)} \cap C).$$

The proof is complete. \square

Remark 5.2. Based on the Proof of Theorem 5.2, we have that 1): For the condition “ $x \in \partial_C U$ with $y \in F(x)$, $P_U^{\frac{1}{p}}(y) - 1 \leq P_U^{\frac{1}{p}}(y - x)$ for $0 < p \leq 1$ ”, indeed we only need that for “ $x \in \partial_C U$ with $y \in F(x) \cap (C \setminus \overline{U})$ ”, $P_U^{\frac{1}{p}}(y) - 1 \leq P_U^{\frac{1}{p}}(y - x)$ for $0 < p \leq 1$ ”; 2): Theorem 5.2 also improve the corresponding best approximation for 1-set contractive mappings given by Li et al.[67], Liu [70], Xu [126], Xu et al.[127], and results from the references therein; and 3): When $p=1$, we have the similar best approximation result for the mapping F in the locally convex spaces with outward set boundary condition below (see Theorem 3 of Park [85] and related discussion by references therein).

For the p -vector space with $p = 1$ being a topological vector space E , we have the following best approximation for the outward set $\overline{O_{\overline{U}}(x_0)}$ based on the point $\{x_0\}$ respect the convex subset U in E .

Theorem 5.3 (Best approximation for outward sets). Let U be a bounded open convex subset of a locally convex space E (i.e., $p = 1$) with zero $0 \in \text{int}U = U$ (the interior $\text{int}U = U$ as U is open), and C a closed p -convex subset of E with also zero $0 \in C$. Assume that $F : \overline{U} \cap C \rightarrow C$ is a 1-set-contractive continuous mapping, and satisfying the condition (H) or (H1) above. Then there exist $x_0 \in \overline{U} \cap X$ and $y_0 \in F(x_0)$ such that $P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{O_{\overline{U}}(x_0)} \cap C)$, where P_U is the Minkowski p -functional of U . More precisely, we have the following either (I) or (II) holding:

- (I) F has a fixed point $x_0 \in U \cap C$, i.e., $P_U(y_0 - x_0) = P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{O_{\overline{U}}(x_0)} \cap C) = 0$,
- (II) there exists $x_0 \in \partial_C(U)$ and $y_0 \in F(x_0) \setminus \overline{U}$ with

$$P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, O_{\overline{U}}(x_0) \cap C) = d_p(y_0, \overline{O_{\overline{U}}(x_0)} \cap C) > 0.$$

Proof. We define a new mapping $F_1 : \overline{U} \cap C \rightarrow 2^C$ by $F_1(x) := \{2x\} - F(x)$ for each $x \in \overline{U} \cap C$, then F_1 is also compact and upper semi-continuous mapping with non-empty closed convex values, and F_1 satisfies all hypothesis

of Theorem 5.2 with $p = 1$. It follows by Theorem 5.2 that there exists $x_0 \in \bar{U} \cap X$ and $y_1 \in F_1(x_0)$ such that $P_U(y_1 - x_0) = d_P(y_1, \bar{U} \cap C) = d_p(y_1, \overline{I_{\bar{U}}(x_0)} \cap C)$. More precisely, we have the following either (I) or (II) holding:

- (I) F_1 has a fixed point $x_0 \in U \cap C$ (so $0 = P_U(y_1 - x_0) = P_U(y_1 - x_0) = d_P(y_1, \bar{U} \cap C) = d_p(y_1, \overline{I_{\bar{U}}(x_0)} \cap C)$);
- (II) there exists $x_0 \in \partial_C(U)$ and $y_1 \in F_1(x_0) \setminus \bar{U}$ with

$$P_U(y_1 - x_0) = d_P(y_1, \bar{U} \cap C) = d_p(y_1, \overline{O_{\bar{U}}(x_0)} \cap C) > 0.$$

Now for any $x \in O_{\bar{U}}(x_0)$, there exist $r < 0, u \in \bar{U}$ such that $x = x_0 + r(u - x_0)$. Let $x_1 = 2x_0 - x$, then $x_1 = 2x_0 - x_0 - r(u - x_0) = x_0 + (-r)(u - x_0) \in I_{\bar{U}}(x_0)$. Let $y_1 = 2x_0 - y_0$, for some $y_0 \in F(x_0)$. As we have $P_U(y_1 - x_0) = d_P(y_1, \bar{U} \cap C) = d_p(y_1, \overline{I_{\bar{U}}(x_0)} \cap C)$, it follows that $P_U(y_1 - x_0) \leq P_U(y_1 - x_1)$, which implies that

$$P_U(x_0 - y_0) = P_U(y_1 - x_0) \leq P_U(y_1 - x_1) = P_U(2x_0 - y_0 - (2x_0 - x)) = P_U(y_0 - x)$$

for all $x \in O_{\bar{U}}(x_0)$. Thus we have $P_U(y_0 - x_0) = d_P(y_0, \bar{U} \cap C) = d_p(y_0, O_{\bar{U}}(x_0) \cap C)$ and by the continuity of P_U , it follows that

$$P_U(y_0 - x_0) = d_P(y_0, \bar{U} \cap C) = d_p(y_0, \overline{O_{\bar{U}}(x_0)} \cap C)(P_U^{\frac{1}{p}}(y_0) - 1)^p > 0.$$

This completes the proof. \square

Now by the application of Theorem 5.2 and Theorem 5.3, the Remark 5.2 and the argument used in Theorem 5.2, we have the following general principle for the existence of solutions for Birkhoff-Kellogg Problems in p -seminorm spaces, where $(0 < p \leq 1)$.

Theorem 5.4 (Principle of Birkhoff-Kellogg alternative). Let U be a bounded open p -convex subset of a locally p -convex space E ($0 \leq p \leq 1$) with zero $0 \in \text{int}U = (U)$ (the interior $\text{int}U$ as U is open), and C a closed p -convex subset of E with also zero $0 \in C$. Assume that $F : \bar{U} \cap C \rightarrow C$ is a 1-set-contractive continuous mapping, and satisfying the (H) or (H1) condition above. Then F has at least one of the following two properties:

- (I) F has a fixed point $x_0 \in U \cap C$ such that $x_0 \in F(x_0)$,

- (II) there exist $x_0 \in \partial_C(U)$, $y_0 \in F(x_0) \setminus \bar{U}$, and $\lambda = \frac{1}{(P_U(y_0))^{\frac{1}{p}}} \in (0, 1)$ such that $x_0 = \lambda y_0 \in \lambda F(x_0)$; In addition

if for each $x \in \partial_C U$, $P_U^{\frac{1}{p}}(y) - 1 \leq P_U^{\frac{1}{p}}(y - x)$ for $0 < p \leq 1$ (this is trivial when $p = 1$), then the best approximation between x_0 and y_0 given by

$$P_U(y_0 - x_0) = d_P(y_0, \bar{U} \cap C) = d_p(y_0, \overline{I_{\bar{U}}^p(x_0)} \cap C) = (P_U^{\frac{1}{p}}(y_0) - 1)^p > 0.$$

Proof. If (I) is not the case, then (II) is proved by the Remark 5.2 and by following the proof in Theorem 5.2 for the case (ii): $y_0 \in C \setminus \bar{U}$, with $y_0 = f(x_0) \in f(x_0)$. Indeed, as $y_0 \notin \bar{U}$, it follows that $P_U(y_0) > 1$, and $x_0 = f(y_0) = y_0 \frac{1}{(P_U(y_0))^{\frac{1}{p}}}$. Now let $\lambda = \frac{1}{(P_U(y_0))^{\frac{1}{p}}}$, we have $\lambda < 1$ and $x_0 = \lambda y_0$ with $y_0 \in F(x_0)$. Finally, the additionally assumption in (II) allows us to have the best approximation between x_0 and y_0 obtained by following the proof of Theorem 5.2 as $P_U(y_0 - x_0) = d_P(y_0, \bar{U} \cap C) = d_p(y_0, \overline{I_{\bar{U}}^p(x_0)} \cap C) > 0$. This completes the proof. \square

As an application of Theorem 5.2 for the non-self set-valued mappings discussed in Theorem 5.3 with outward set condition, we have the following general principle of Birkhoff-Kellogg alternative in topological vector spaces.

Theorem 5.5 (Principle of Birkhoff-Kellogg alternative in TVS). Let U be a bounded open p -convex subset of the topological vector space E with the zero $0 \in U$, and C a closed convex subset of E with also zero $0 \in C$. Assume the $F : \bar{U} \cap C \rightarrow 2^C$ is a 1-set contractive and upper semi-continuous mapping with non-empty closed convex values, and satisfying the (H) or (H1) condition (H) above. Then it has at least one of the following two properties:

(I) F has a fixed point $x_0 \in U \cap C$ such that $x_0 \in F(x_0)$,

(II) there exists $x_0 \in \partial_C(U)$ and $y_0 \in F(x_0) \setminus \bar{U}$ and $\lambda \in (0, 1)$ such that $x_0 = \lambda y_0$, and the best approximation between x_0 and y_0 is given by $P_U(y_0 - x_0) = d_P(y_0, \bar{U} \cap C) = d_p(y_0, \overline{I_U^p(x_0)} \cap C) > 0$.

On the other hand, by the Proof of Theorems 5.2, we note that for case (II) of Theorem 5.2, the assumption “each $x \in \partial_C U$ with $y \in F(x)$, $P_U^{\frac{1}{p}}(y) - 1 \leq P_U^{\frac{1}{p}}(y - x)$ ” is only used to guarantee the best approximation “ $P_U(y_0 - x_0) = d_P(y_0, \bar{U} \cap C) = d_p(y_0, \overline{I_U^p(x_0)} \cap C) > 0$ ”, thus we have the following Leray-Schauder alternative in p -vector spaces, which, of course, includes the corresponding results in locally convex spaces as special cases.

Theorem 5.6 (The Leray-Schauder Nonlinear Alternative). Let C a closed p -convex subset of p -seminorm space E with $0 \leq p \leq 1$ and the zero $0 \in C$. Assume the $F : C \rightarrow 2^C$ is a 1-set contractive and upper semi-continuous mapping with non-empty closed p -convex values, and satisfying the (H) or (H1) condition above. Let $\varepsilon(F) := \{x \in C : x \in \lambda F(x), \text{ for some } 0 < \lambda < 1\}$. Then either F has a fixed point in C or the set $\varepsilon(F)$ is unbounded.

Proof. We prove the conclusion by assuming that F has no fixed point, then we claim that the set $\varepsilon(F)$ is unbounded. Otherwise, assume the set $\varepsilon(F)$ is bounded. and assume P is the continuous p -seminorm for E , then there exists $r > 0$ such that the set $B(0, r) := \{x \in E : P(x) < r\}$, which contains the set $\varepsilon(F)$, i.e., $\varepsilon(F) \subset B(0, r)$, which means for any $x \in \varepsilon(F)$, $P(x) < r$. Then $B(0, r)$ is an open p -convex subset of E and the zero $0 \in B(0, r)$ by Lemma 2.2 and Remark 2.4. Now let $U := B(0, r)$ in Theorem 5.4, it follows that for the mapping $F : B(0, r) \cap C \rightarrow 2^C$ satisfies all general conditions of Theorem 5.4, and we have that any $x_0 \in \partial_C B(0, r)$, no any $\lambda \in (0, 1)$ such that $x_0 = \lambda y_0$, where $y_0 \in F(x_0)$. Indeed, for any $x \in \varepsilon(F)$, it follows that $P(x) < r$ as $\varepsilon(F) \subset B(0, r)$, but for any $x_0 \in \partial_C B(0, r)$, we have $P(x_0) = r$, thus the conclusion (II) of Theorem 5.4 does not have hold. By Theorem 5.4 again, F must have a fixed point, but this contradicts with our assumption that F is fixed point free. This completes the proof. \square

800 Now assume a given p -vector space E equipped with the P -seminorm (by assuming it is continuous at zero) for $0 < p \leq 1$, then we know that $P : E \rightarrow \mathbb{R}^+$, $P^{-1}(0) = 0$, $P(\lambda x) = |\lambda|^p P(x)$ for any $x \in E$ and $\lambda \in \mathbb{R}$. Then we have the following useful result for fixed points due to Rothe and Altman types in p -vector spaces, which plays important roles for optimization problem, variational inequality, complementarity problems (see isac [52], or Yuan [130] and references therein for related study in details).

Corollary 5.1. Let U be a bounded open p -convex subset of a locally p -convex space E and zero $0 \in U$, plus C is a closed p -convex subset of E with $U \subset C$, where $0 < p \leq 1$. Assume that $F : \bar{U} \rightarrow 2^C$ is a 1-set contractive upper semicontinuous mapping with non-empty closed p -convex values, and satisfying the (H) or (H1) condition above.

If one of the following is satisfied,

- (1) (Rothe type condition): $P_U(y) \leq P_U(x)$ for $y \in F(x)$, where $x \in \partial U$,
- (2) (Petryshyn type condition): $P_U(y) \leq P_U(y - x)$ for $y \in F(x)$, where $x \in \partial U$,
- (3) (Altman type condition): $|P_U(y)|^{\frac{2}{p}} \leq [P_U(y - x)]^{\frac{2}{p}} + [P_U(x)]^{\frac{2}{p}}$ for $y \in F(x)$, where $x \in \partial U$,

then F has at least one fixed point.

Proof. By the conditions (1), (2) and (3), it follows that the conclusion of (II) in Theorem 5.4 “there exist $x_0 \in \partial_C(U)$ and $\lambda \in (0, 1)$ such that $x_0 \notin \lambda F(x_0)$ ” does not hold, thus by the alternative of Theorem 5.4, F has a fixed point. This completes the proof. \square .

By the fact that when $p = 1$, each p -vector space is a topological vector space, and thus we have the following classical Fan’s best approximation (see [35]) as a powerful tool for the study in the optimization, mathematical programming, games theory, and mathematical economics, and others related topics in applied mathematics.

Corollary 5.2 (Fan’s best approximation). Let U be a bounded open convex subset of a topological vector space E with the zero $0 \in U$, and C a closed convex subset of E with also zero $0 \in C$, and assume $F : \bar{U} \cap C \rightarrow 2^C$ is a 1-set contractive and upper semi-continuous mapping with non-empty closed convex values, and satisfying the (H) or (H1) condition above. Assume P_U being the Minkowski p -functional of U in E . Then there exist $x_0 \in \bar{U} \cap X$ and $y_0 \in T(x_0)$ such that $P_U(y_0 - x_0) = d_P(y_0, \bar{U} \cap C) = d_p(y_0, \overline{I_{\bar{U}}(x_0)} \cap C)$. More precisely, we have the following either (I) or (II) holding, where $W_{\bar{U}}(x_0)$ is either inward set $I_{\bar{U}}(x_0)$, or the outward set $O_{\bar{U}}(x_0)$:

- (I) F has a fixed point $x_0 \in U \cap C$, $0 = P_U(y_0 - x_0) = P_U(y_0 - x_0) = d_P(y_0, \bar{U} \cap C) = d_p(y_0, \overline{W_{\bar{U}}(x_0)} \cap C)$,
- (II) there exists $x_0 \in \partial_C(U)$ and $y_0 \in F(x_0) \setminus \bar{U}$ with

$$P_U(y_0 - x_0) = d_P(y_0, \bar{U} \cap C) = d_p(y_0, \overline{W_{\bar{U}}(x_0)} \cap C) = P_U(y_0) - 1 > 0.$$

Proof. When $p = 1$, then it automatically satisfies that the inequality: $P_U^{\frac{1}{p}}(y) - 1 \leq P_U^{\frac{1}{p}}(y - x)$, and indeed we have that for $x_0 \in \partial_C(U)$, with $y_0 \in F(x_0)$, we have $P_U(y_0 - x_0) = d_P(y_0, \bar{U} \cap C) = d_p(y_0, \overline{W_{\bar{U}}(x_0)} \cap C) = P_U(y_0) - 1$. The conclusions are given by Theorem 5.2 (or Theorem 5.3). The proof is complete. \square

We like to point out the similar results on Rothe and Leray-Schauder alternative have been developed by Isac [52], Park [84], Potter [96], Shahzad [105]-[107], Xiao and Zhu [121], and related references therein as tools of nonlinear analysis in topological vector spaces. As mentioned above, when $p = 1$ and take F as a continuous mapping, then we obtain the version of Leray-Schauder in TVS and this we omit it statement in details.

6. Principle of Nonlinear Alternatives for classes of 1-Set Contractive Mappings

As applications of results in Section 5 above, we new establish general results for the existence of solutions for Birkhoff-Kellogg problem, and the principle of Leray-Schauder alternatives in p -vector spaces for $0 < p \leq 1$.

Theorem 6.1 (Birkhoff-Kellogg alternative in p -vector spaces). Let U be a bounded open p -convex subset of a locally p -convex space E (where, $0 \leq p \leq 1$) with the zero $0 \in U$, and C a closed p -convex subset of E with

also zero $0 \in C$, and assume $F : \bar{U} \cap C \rightarrow 2^C$ is a 1-set contractive and upper semi-continuous mapping with non-empty closed p -convex values, and satisfying the condition (H) or (H1) above. In addition, for each $x \in \partial_C(U)$ with $y \in F(x)$, $P_U^{\frac{1}{p}}(y) - 1 \leq P_U^{\frac{1}{p}}(y - x)$ for $0 < p \leq 1$ (this is trivial when $p = 1$), where P_U is the Minkowski p -functional of U . Then we have that either (I) or (II) holding below:

(I) there exists $x_0 \in \bar{U} \cap C$ such that $x_0 \in F(x_0)$,

(II) there exists $x_0 \in \partial_C(U)$ with $y_0 \in F(x_0) \setminus \bar{U}$ and $\lambda > 1$ such that $\lambda x_0 = y_0 \in F(x_0)$, i.e., $F(x_0) \cap \{\lambda x_0 : \lambda > 1\} \neq \emptyset$.

Proof. By following the argument and symbols used in the proof of Theorem 5.2, we have that either

(1) F has a fixed point $x_0 \in U \cap C$; or

(2) there exists $x_0 \in \partial_C(U)$ and $y_0 \in F(x_0)$ with $x_0 = f(y_0)$ such that

$$P_U(y_0 - x_0) = d_P(y_0, \bar{U} \cap C) = d_p(y_0, \overline{I_{\bar{U}}(x_0)} \cap C) = P_U(y_0) - 1 > 0,$$

where $\partial_C(U)$ denotes the boundary of U relative to C in E , and f is the restriction of the continuous retraction r respect to the set U in E .

If F has no fixed point, then above (2) holds and $x_0 \notin F(x_0)$. As given by the proof of Theorem 5.2, we have that $y_0 \in F(x_0)$ and $y_0 \notin \bar{U}$, thus $P_U(y_0) > 1$ and $x_0 = f(y_0) = \frac{y_0}{(P_U(y_0))^{\frac{1}{p}}}$, which means $y_0 = (P_U(y_0))^{\frac{1}{p}} x_0$. Let $\lambda = (P_U(y_0))^{\frac{1}{p}}$, then $\lambda > 1$ and we have $\lambda x_0 = y_0 \in F(x_0)$. This completes the proof. \square

Theorem 6.2 (Birkhoff-Kellogg alternative in TVS). Let U be a bounded open convex subset of a topological space E with the zero $0 \in U$, and C a closed convex subset of E with also zero $0 \in C$, and assume $F : \bar{U} \cap C \rightarrow 2^C$ is a 1-set contractive and upper semi-continuous mapping with non-empty closed convex values, and satisfying the condition (H) or (H1) above. Then we have the following either (I) or (II) holding, where $W_{\bar{U}}(x_0)$ is either inward set $I_{\bar{U}}(x_0)$, or the outward set $O_{\bar{U}}(x_0)$:

(I) there exists $x_0 \in \bar{U} \cap C$ such that $x_0 \in F(x_0)$,

(II) there exists $x_0 \in \partial_C(U)$ with $y_0 \in F(x_0) \setminus \bar{U}$ and $\lambda > 1$ such that $\lambda x_0 = y_0 \in F(x_0)$, i.e., $F(x_0) \cap \{\lambda x_0 : \lambda > 1\} \neq \emptyset$.

Proof. When $p = 1$, then it automatically satisfies that the inequality: $P_U^{\frac{1}{p}}(y) - 1 \leq P_U^{\frac{1}{p}}(y - x)$, and indeed we have that for $x_0 \in \partial_C(U)$, with $y_0 \in F(x_0)$, we have $P_U(y_0 - x_0) = d_P(y_0, \bar{U} \cap C) = d_p(y_0, \overline{W_{\bar{U}}(x_0)} \cap C) = P_U(y_0) - 1$. The conclusions are given by Theorems 5.3 and 5.4. The proof is complete. \square

Indeed, we have the following fixed points for non-self mappings in p -vector spaces for $0 < p \leq 1$ under different boundary conditions.

Theorem 6.3 (Fixed Points of non-self mappings). Let U be a bounded open p -convex subset of a locally p -convex space E (where, $0 \leq p \leq 1$) with the zero $0 \in U$, and C a closed p -convex subset of E with also zero $0 \in C$, and assume $F : \bar{U} \cap C \rightarrow 2^C$ is a 1-set contractive and upper semi-continuous mapping with non-empty closed p -convex values, and satisfying the condition (H) or (H1) above. In addition, for each $x \in \partial_C(U)$ with $y \in F(x)$, $P_U^{\frac{1}{p}}(y) - 1 \leq P_U^{\frac{1}{p}}(y - x)$ for $0 < p \leq 1$ (this is trivial when $p = 1$), where P_U is the Minkowski p -functional of U . If

F satisfies any one of the following conditions for any $x \in \partial_C(U) \setminus F(x)$

- (i) For each $y \in F(x)$, $P_U(y - z) < P_U(y - x)$ for some $z \in \overline{I_{\overline{U}}(x)} \cap C$;
- (ii) For each $y \in F(x)$, there exists λ with $|\lambda| < 1$ such that $\lambda x + (1 - \lambda)y \in \overline{I_{\overline{U}}(x)} \cap C$;
- (iii) $F(x) \subset \overline{I_{\overline{U}}(x)} \cap C$;
- (iv) $F(x) \cap \{\lambda x : \lambda > 1\} = \emptyset$;
- (v) $F(\partial U) \subset \overline{U} \cap C$;
- (vi) For each $y \in F(x)$, $P_U(y - x) \neq ((P_U(y))^{\frac{1}{p}} - 1)^p$;

then F must has a fixed point.

Proof. By following the argument and symbols used in the proof of Theorem 5.2 (see also Theorem 5.4), we have that either

- (1) F has a fixed point $x_0 \in U \cap C$; or
- (2) there exists $x_0 \in \partial_C(U)$ and $y_0 \in F(x_0)$ with $x_0 = f(y_0)$ such that

$$P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_P(y_0, \overline{I_{\overline{U}}(x_0)} \cap C) = P_U(y_0) - 1 > 0,$$

where $\partial_C(U)$ denotes the boundary of U relative to C in E , and f is the restriction of the continuous retraction r respect to the set U in E .

First, suppose that F satisfies the condition (i), if F has no fixed point, then above (2) holds and $x_0 \notin F(x_0)$. Then by the condition (i), it follows that $P_U(y_0 - z) < P_U(y_0 - x_0)$ for some $z \in \overline{I_{\overline{U}}(x_0)} \cap C$, this contradicts with the best approximation equations given by (2) above, thus F must have a fixed point.

Second, suppose that F satisfies the condition (ii), if F has no fixed point, then above (2) holds and $x_0 \notin F(x_0)$. Then by the condition (ii), there exists $\lambda > 1$ such that $\lambda x_0 + (1 - \lambda)y_0 \in \overline{I_{\overline{U}}(x_0)} \cap C$. It follows that

$$P_U(y_0 - x_0) \leq P_U(y_0 - (\lambda x_0 + (1 - \lambda)y_0)) = P_U(\lambda(y_0 - x_0)) = |\lambda|^p P_U(y_0 - x_0) < P_U(y_0 - x_0)$$

this is impossible and thus F must have a fixed point in $\overline{U} \cap C$.

Third, suppose that F satisfies the condition (iii), i.e., $F(x) \subset \overline{I_{\overline{U}}(x)} \cap C$, then the (2), we have that $P_U(y_0 - x_0)$ and thus $x_0 = y_0 \in F(x_0)$, which means F has a fixed point.

Forth, suppose that F satisfies the condition (iv), if F has no fixed point, then above (2) holds and $x_0 \notin F(x_0)$. As given by the proof of Theorem 5.2, we have that $y_0 \notin \overline{U}$, thus $P_U(y_0) > 1$ and $x_0 = f(y_0) = \frac{y_0}{(P_U(y_0))^{\frac{1}{p}}}$, which means $y_0 = (P_U(y_0))^{\frac{1}{p}} x_0$, where $(P_U(y_0))^{\frac{1}{p}} > 1$, this contradicts with the assumption (iv), thus F must have a fixed point in $\overline{U} \cap C$.

Fifth, suppose that F satisfies the condition (v), then $x_0 \notin F(x_0)$. As $x_0 \in \partial_C U$, now by the condition (v), we have that $F(\partial U) \subset \overline{U} \cap C$, it follows that for any $y_0 \in F(x_0)$, we have $y_0 \in \overline{U} \cap C$, thus $y_0 \notin \overline{U} \setminus \cap C$, which implies that $0 < P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = 0$, this is impossible, thus F must have a fixed point. Here, like pointed out by Remark 5.2, we know that based on the condition (v), the mapping F has a fixed point by applying $F(\partial U) \subset \overline{U} \cap C$ is enough, not needing the general hypothesis: “for each $x \in \partial_C(U)$ with $y \in F(x)$, $P_U^{\frac{1}{p}}(y) - 1 \leq P_U^{\frac{1}{p}}(y - x)$ for $0 < p \leq 1$ ”.

Finally, suppose that F satisfies the condition (vi), if F has no fixed point, then above (2) holds and $x_0 \notin F(x_0)$.
 900 Then the condition (v) implies that $P_U(y_0 - x_0) \neq ((P_U(y))^\frac{1}{p} - 1)^p$, but the our proof in Theorem shows that $P_U(y_0 - x_0) = ((P_U(y))^\frac{1}{p} - 1)^p$, this is impossible, thus F must have a fixed point. Then the proof is complete. \square

Now by taking the set C in Theorem 6.1 as the whole p -vector space E itself, we have the following general results for non-self upper semi-continuous set-valued mappings which include results of Rothe, Petryshyn, Altman and Leray-Schauder types' fixed points as special cases.

Taking $p = 1$ and $C = E$ in Theorem 6.3, we have the following fixed points for non-self upper semi-continuous set-valued mappings associated with inward or outward sets in topological vector spaces (TVS) as follows.

Theorem 6.4 (Fixed Points of non-self mappings with boundary conditions). Let U be a bounded open convex subset of the topological vector spaces E with the zero $0 \in U$, and assume $F : \bar{U} \rightarrow 2^E$ is a 1-set contractive and upper semi-continuous mapping with non-empty closed convex values, and satisfying the condition (H) or (H1) above. If F satisfies any one of the following conditions for any $x \in \partial(U) \setminus F(x)$

- (i) For each $y \in F(x)$, $P_U(y - z) < P_U(y - x)$ for some $z \in \overline{I_{\bar{U}}(x)}$ (or $z \in \overline{O_{\bar{U}}(x)}$)
 - (ii) For each $y \in F(x)$, there exists λ with $|\lambda| < 1$ such that $\lambda x + (1 - \lambda)y \in \overline{I_{\bar{U}}(x)}$ (or, $\overline{O_{\bar{U}}(x)}$).
 - (iii) $F(x) \subset \overline{I_{\bar{U}}(x)}$ (or $\overline{O_{\bar{U}}(x)}$)
 - (iv) $F(x) \cap \{\lambda x : \lambda > 1\} = \emptyset$;
 - (v) $F(\partial(U)) \subset \bar{U}$;
 - (vi) For each $y \in F(x)$, $P_U(y - x) \neq P_U(y) - 1$;
- then F must has a fixed point.

In what follow, based on the best approximation theorem in p -seminorm space, we will also give some fixed point theorems for non-self set-valued mappings with various boundary conditions which are related to the study for the existence of solutions for PDE and differential equations with boundary problems (see, Browder [15], Petryshyn [91]-[92], Reich [98]), which would play roles in nonlinear analysis for p -seminorm space as shown below.

First, as discussed by Remark 5.2, the proof of Theorem 5.2, with the strongly boundary condition " $F(\partial(U)) \subset \bar{U} \cap C$ " only, we can prove that F has a fixed point, thus we have the following fixed point theorem of Rothe type in p -vector spaces.

Theorem 6.5 (Rothe Type). Let U be a bounded open p -convex subset of a locally p -convex space E (where, $0 \leq p \leq 1$) with the zero $0 \in U$. Assume $F : \bar{U} \rightarrow 2^E$ is a 1-set contractive and upper semi-continuous mapping with non-empty closed p -convex values, satisfying the condition (H) or (H1) above, and such that $F(\partial(U)) \subset \bar{U}$, then F must has a fixed point.

Now as applications of Theorem 6.5, we give the following Leray-Schauder Alternative in p -vector spaces for non-self set-valued mappings associated with the boundary condition which often appear in the applications (see Isac [52] and references therein for the study of complementary problems and related topics in optimization).

Theorem 6.6 (Leray-Schauder Alternative in p -Vector Spaces). Let E be a locally p -convex space E , where $0 < p \leq 1$, $B \subset E$ a bounded closed p -convex such that $0 \in \text{int}B$. Let $F : [0, 1] \times B \rightarrow E$ be 1-set contractive and upper semi-continuous set-valued with non-empty closed p -convex values, satisfying the condition (H) or (H1) above, and such that the set $F([0, 1] \times B)$ be relatively compact in E . If the following assumptions are satisfied:

- (1) $x \notin F(t, x)$ for all $x \notin \partial B$ and $t \in [0, 1]$,
- (2) $F(\{0\} \times \partial B) \subset B$,

then there is an element $x^* \in B$ such that $x^* \in F(1, x^*)$.

Proof. For any $n \in N$, we consider the mapping

$$F_n(x) = \begin{cases} F\left(\frac{1 - P_B(x)}{\epsilon_n}, \frac{x}{P_B(x)}\right), & \text{if } 1 - \epsilon_n \leq P_B(x) \leq 1, \\ F\left(1, \frac{x}{1 - \epsilon_n}\right), & \text{if } P_B(x) < 1 - \epsilon_n, \end{cases} \quad (3)$$

where P_B is the Minkowski p -functional of B and $\{\epsilon_n\}_{n \in N}$ is a sequence of real numbers such that $\lim_{n \rightarrow \infty} \epsilon_n = 0$ and $0 < \epsilon_n < \frac{1}{2}$ for any $n \in N$. We observe that for each $n \in N$, the mapping F_n is 1-set contractive upper semi-continuous with non-empty closed p -convex values on B . From assumption (2), we have that $F_n(\partial B) \subset B$, and the assumptions of Theorem 6.5 are satisfied, then for each $n \in N$, there exists an element $u_n \in B$ such that $u_n \in F_n(u_n)$.

We first prove the following statement: “It is impossible to have an infinite number of the elements u_n satisfy the following inequality: $1 - \epsilon_n \leq P_B(u_n) \leq 1$. ”

If not, we assume to have an infinite number of the elements u_n satisfy the following inequality:

$$1 - \epsilon_n \leq P_B(u_n) \leq 1.$$

As $F_n(B)$ is relatively compact and by the definition of mappings F_n , we have that $\{u_n\}_{n \in N}$ is contained in a compact set in E . Without loss of the generality (indeed, each compact set is also countably compact), we define the sequence $\{t_n\}_{n \in N}$ by $t_n := \frac{1 - P_B(u_n)}{\epsilon_n}$ for each $n \in N$. Then we have that $\{t_n\}_{n \in N} \subset [0, 1]$ and we may assume that $\lim_{n \rightarrow \infty} t_n = t \in [0, 1]$. The corresponding subsequence of $\{u_n\}_{n \in N}$ is denoted again by $\{u_n\}_{n \in N}$ and it also satisfies the inequality: $1 - \epsilon_n \leq P_B(u_n) \leq 1$, which implies that $\lim_{n \rightarrow \infty} P_B(u_n) = 1$.

Now let u^* be an accumulation point of $\{u_n\}_{n \in N}$, thus have $\lim_{n \rightarrow \infty} (t_n, \frac{u_n}{P_B(u_n)}, u_n) = (t, u^*, u^*)$. By the fact that F is compact, we have assume that $u_n \in F(t_n, \frac{u_n}{P_B(u_n)})$ for each $n \in N$, it follows that $u^* \in F(t, u^*)$, this contradicts with the assumption (1) as we have $\lim_{n \rightarrow \infty} P_B(u_n) = 1$ (which means that $u^* \in \partial B$, this is impossible).

Thus it is impossible to have that “to have an infinite number of elements u_n satisfy the inequality: $1 - \epsilon_n \leq P_B(u_n) \leq 1$ ”, which means that there is only a finite number of elements of sequence $\{u_n\}_{n \in N}$ satisfying the inequality: $1 - \epsilon_n \leq P_B(u_n) \leq 1$. Now, without loss of the generality, for $n \in N$, we have the following inequality:

$$P_B(u_n) < 1 - \epsilon_n.$$

By the fact that $\lim_{n \rightarrow \infty} (1 - \epsilon_n) = 1$, $u_n \in F(1, \frac{u_n}{1 - \epsilon_n})$ for all $n \in N$ and assume that $\lim_{n \rightarrow \infty} u_n = u^*$, then the upper semi-continuity of F with non-empty closed values implies that the graph of F is closed, and by the fact $u_n \in F(1, \frac{u_n}{1 - \epsilon_n})$, it implies that $u^* \in F(1, u^*)$. This completes the proof. \square

As a special case of Theorem 6.6, we have the following principle for the implicit form of Leray-Schauder type alternative for set-valued mappings in p -vector spaces for $0 < p \leq 1$.

Corollary 6.1 (The Implicit Leray-Schauder Alternative). Let E be a locally p -convex space E , where $0 < p \leq 1$, $B \subset E$ a bounded closed p -convex such that $0 \in \text{int}B$. Let $F : [0, 1] \times B \rightarrow E$ be 1-set contractive and continuous with non-empty closed p -convex values, satisfying the condition (H) or (H1) above, and the set $F([0, 1] \times B)$ be relatively compact in E . If the following assumptions are satisfied:

- (1) $F(\{0\} \times \partial B) \subset B$,
- (2) $x \notin F(0, x)$ for all $x \in \partial B$,

then at least one of the following properties is satisfied:

- (i) there exists $x^* \in B$ such that $x^* \in F(1, x^*)$; or
- (ii) there exists $(\lambda^*, x^*) \in (0, 1) \times \partial B$ such that $x^* \in F(\lambda^*, x^*)$.

Proof. The result is an immediate consequence of Theorem 6.6, this completes the proof. \square

We like to point out the similar results on Rothe and Leray-Schauder alternative have been developed by Furi and Pera [37], Granas and Dugundji [46], Górniewicz [44], Górniewicz et al.[45], Isac [52], Li et al.[67], Liu [70], Park [84], Potter [96], Shahzad [105]-[107], Xu [126], Xu et al.[127], and related references therein as tools of nonlinear analysis in the Banach space setting and applications to the boundary value problems for ordinary differential equations in noncompact problems, a general class of mappings for nonlinear alternative of Leray-Schauder type in normal topological spaces, and some Birkhoff-Kellogg type theorems for general class mappings in topological vector spaces are also established by Agarwal et al.[1], Agarwal and O'Regan [2]-[3], Park [86], and references therein for more in detail; and in particular, recently O'Regan [80] uses the Leray-Schauder type coincidence theory to establish some Birkhoff-Kellogg problem, Furi-Pera type results for a general class of mappings.

Before closing this section, we like to share that as the application of the best approximation result for 1-set contractive mappings, we can establish the fixed point theorems and general principle of Leray-Schauder alternative for non-self mappings, which seem would play important roles for the nonlinear analysis under the framework of p -seminorm spaces, as the achievement of nonlinear analysis for the underling being locally topological vector spaces, normed spaces, or in Banach spaces.

7. Fixed Points for classes of 1-Set Contractive Mappings

In this section, based on the best approximation Theorem 5.2 for classes of 1-set contractive mappings developed in section 5, we will show how it can be used as a useful tool to establish fixed point theorems for non-self upper semi-continuous mappings in p -seminorm spaces (for $p \in (0, 1]$, and including norm spaces, uniformly convex Banach spaces as special classes).

By following Browder [15], Li [66], Goebel and Kirk [41], Petryshyn [91]-[92], Tan and Yuan [115], Xu [126] and references therein, we recall some definitions as follows for p -seminorm spaces, where $p \in (0, 1]$.

Definition 7.1. Let D be a non-empty (bounded) closed subset of p -vector spaces $(E, \|\cdot\|_p)$ with p -seminorm,

where $p \in (0, 1]$. Suppose $f : D \rightarrow X$ is a (single-valued) mapping, then: (1) f is said to be nonexpansive if for each $x, y \in D$, we have $\|f(x) - f(y)\|_p \leq \|x - y\|_p$; (2) f (actually, $(I - f)$) is said to be demiclosed (see Browder [15]) at $y \in X$ if for any sequence $\{x_n\}_{n \in \mathbb{N}}$ in D , the conditions $x_n \rightarrow x_0 \in D$ weakly, and $(I - f)(x_n) \rightarrow y_0$ strongly imply that $(I - f)(x_0) = y_0$, where I is the identity mapping; (3) f is said to be hemicompact (see p.379 of Tan and Yuan [115]) if each sequence $\{x_n\}_{n \in \mathbb{N}}$ in D has a convergent subsequence with the limit x_0 such that $x_0 = f(x_0)$, whenever $\lim_{n \rightarrow \infty} d_p(x_n, f(x_n)) = 0$, here $d_p(x_n, f(x_n)) := \inf\{P_U(x_n - z) : z \in f(x_n)\}$, and P_U is the Minkowski p -functional for any $U \in \mathfrak{U}$, which is the family of all non-empty open p -convex subset containing the zero in E ; (4) f is said to be demicompact (by Petryshyn [91]) if each sequence $\{x_n\}_{n \in \mathbb{N}}$ in D has a convergent subsequence whenever $\{x_n - f(x_n)\}_{n \in \mathbb{N}}$ is a convergent sequence in X ; (5) f is said to be a semi-closed 1-set contractive mapping if f is 1-set contractive mapping, and $(I - f)$ is closed, where I is identity mapping (by Li [66]); and (6) f is said to be semicontractive (see Petryshyn [92] and Browder [15]) if there exists a mapping $V : D \times D \rightarrow 2^X$ such that $f(x) = V(x, x)$ for each $x \in D$, with (a) for each fixed $x \in D$, $V(\cdot, x)$ is nonexpansive from D to X ; and (b) for each fixed $x \in D$, $V(x, \cdot)$ is completely continuous from D to X , uniformly for u in a bounded subset of D (which means if v_j converges weakly to v in D and u_j is a bounded sequence in D , then $V(u_j, v_j) - V(u_j, v) \rightarrow 0$, strongly in D).

From the definition above, we first observe that the definitions (1) to (6) for set-valued mappings can be given by the similar way with the Hausdorff metric H (we omit their definitions here in details by saving spaces); Secondly, if f is a continuous demicompact mapping, then $(I - f)$ is closed, where I is the identity mapping on X . It is also clear from definitions that every demicompact map is hemicompact in seminorm spaces, but the converse is not true by the example in p.380 by Tan and Yuan [115]. It is evident that if f is demicompact, then $I - f$ is demiclosed. It is known that for each condensing mapping f , when D or $f(D)$ is bounded, then f is hemicompact; and also f is demicompact in metric spaces by Lemma 2.1 and Lemma 2.2 of Tan and Yuan [115], respectively. In addition, it is known that every nonexpansive map is a 1-set-contractive mapping; and also if f is a hemicompact 1-set-contractive mapping, then f is a 1-set-contractive mapping satisfying the following **(H1) condition** (which is the same as the “condition (H1)” in Section 5, but slightly different from the condition (H) used there in the Section 5):

(H1) condition: Let D be a nonempty bounded subset of a space E and assume $F : \overline{D} \rightarrow 2^E$ a set-valued mapping. If $\{x_n\}_{n \in \mathbb{N}}$ is any sequence in D such that for each x_n , there exists $y_n \in F(x_n)$ with $\lim_{n \rightarrow \infty} (x_n - y_n) = 0$, then there exists a point $x \in \overline{D}$ such that $x \in F(x)$.

We first note that the “(H1) Condition” above is actually the same one as the “Condition (C)” used by Theorem 1 of Petryshyn [92]. Secondly, it was shown by Browder [15] that indeed the nonexpansive mapping in a uniformly convex Banach X enjoys the condition (H1) as shown below.

Lemma 7.1. Let D be a nonempty bounded convex subset of a uniformly convex Banach space E . Assume $F : \overline{D} \rightarrow E$ is a nonexpansive (single-valued) mapping, then the mapping $P := I - F$ defined by $P(x) := (x - F(x))$ for each $x \in \overline{D}$ is demiclosed, and in particular, the “(H1) Condition” holds.

Proof. By following the argument given in p.329 (see the proof of Theorem 2.2 and Corollary 2.1) by Petryshyn [92], the mapping F is demiclosed (which actually is called Browder's demiclosedness principle), which says that by the assumption of (H1) condition, If $\{x_n\}_{n \in \mathbb{N}}$ is any sequence in D such that for each x_n , there exists $y_n \in F(x_n)$ with $\lim_{n \rightarrow \infty} (x_n - y_n) = 0$, then we have $0 \in (I - F)(\overline{D})$, which means that there exists $x_0 \in \overline{D}$ with $0 \in (I - F)(x_0)$, this implies that $x_0 \in F(x_0)$. The proof is complete. \square .

Remark 7.1. When a p -vector space E is with a p -norm, then “(H) condition” satisfies the “(H1) condition”. The (H1) condition mainly supported by the so-called demiclosedness principle after the work by Browder [15].

By applying Theorem 5.2, we have the following result for non-self mappings in p -seminorm spaces for $p \in (0, 1]$.

Theorem 7.1. Let U be a bounded open p -convex subset of a p -(semi)norm space E ($0 < p \leq 1$) the zero $0 \in U$. Assume $F : \overline{U} \rightarrow 2^E$ is a 1-set contractive and upper semi-continuous mapping with non-empty closed p -convex values, satisfying the condition (H) or (H1) above. In addition, for any $x \in \partial \overline{U}$ and $y \in F(x)$, we have $\lambda x \neq y$ for any $\lambda > 1$ (i.e., the “Leray-Schauder boundary condition”). Then F has at least one fixed point.

Proof. By Theorem 5.2 with $C = E$, it follows that we have the following either (I) or (II) holding:

(I) F has a fixed point $x_0 \in U$, i.e., $P_U(y_0 - x_0) = 0$,

(II) there exists $x_0 \in \partial(U)$ and $y_0 \in F(x_0)$ with $P_U(y_0 - x_0) = (P_U^{\frac{1}{p}}(y_0) - 1)^p > 0$.

If F has no fixed point, then above (II) holds and $x_0 \notin F(x_0)$. By the proof of Theorem 5.2, we have that $x_0 = f(y_0)$ and $y_0 \notin \overline{U}$. Thus $P_U(y_0) > 1$ and $x_0 = f(y_0) = \frac{y_0}{(P_U(y_0))^{\frac{1}{p}}}$, which means $y_0 = (P_U(y_0))^{\frac{1}{p}} x_0$, where $(P_U(y_0))^{\frac{1}{p}} > 1$, this contradicts with the assumption. Thus F must have a fixed point. The proof is complete. \square

By following the idea used and developed by Browder [15], Li [66], Li et al.[67], Goebel and Kirk [41], Petryshyn [91]-[92], Tan and Yuan [115], Xu [126], Xu et al.[127] and references therein, we have the following a number of existence theorems for the principle of Leray-Schauder type alternatives in p -seminorm spaces $(E, \|\cdot\|_p)$ for $p \in (0, 1]$.

Theorem 7.2. Let U be a bounded open p -convex subset of a p -(semi)norm space $(E, \|\cdot\|_p)$ ($0 < p \leq 1$) the zero $0 \in U$. Assume $F : \overline{U} \rightarrow 2^E$ is a 1-set contractive and upper semi-continuous mapping with non-empty closed p -convex values, satisfying the condition (H) or (H1) above. In addition, there exist $\alpha > 1$, $\beta \geq 0$, such that for each $x \in \partial \overline{U}$, we have that for any $y \in F(x)$, $\|y - x\|_p^{\alpha/p} \geq \|y\|_p^{(\alpha+\beta)/p} \|x\|_p^{-\beta/p} - \|x\|_p^{\alpha/p}$. Then F has at least one fixed point.

Proof. We prove the conclusion by showing the Leray-Schauder boundary condition in Theorem 7.1 does not hold. If we assume F has no fixed point, by the boundary condition of Theorem 7.1, there exist $x_0 \in \partial \overline{U}$, $y_0 \in F(x_0)$ and $\lambda_0 > 1$ such that $y_0 = \lambda_0 x_0$.

Now, consider the function f defined by $f(t) := (t - 1)^\alpha - t^{\alpha+\beta} + 1$ for $t \geq 1$. We observe that f is a strictly decreasing function for $t \in [1, \infty)$ as the derivative of $f'(t) = \alpha(t - 1)^{\alpha-1} - (\alpha + \beta)t^{\alpha+\beta-1} < 0$ by the differentiation, thus we have $t^{\alpha+\beta} - 1 > (t - 1)^\alpha$ for $t \in (1, \infty)$. By combining the boundary condition, we have that $\|y_0 - x_0\|_p^{\alpha/p} = \|\lambda_0 x_0 - x_0\|_p^{\alpha/p} = (\lambda_0 - 1)^\alpha \|x_0\|_p^{\alpha/p} < (\lambda_0^{\alpha+\beta} - 1) \|x_0\|_p^{(\alpha+\beta)/p} \|x_0\|_p^{-\beta/p} = \|y_0\|_p^{(\alpha+\beta)/p} \|x_0\|_p^{-\beta/p} - \|x_0\|_p^{\alpha/p}$, which

contradicts the boundary condition given by Theorem 7.2. Thus, the conclusion follows and the proof is complete.

□

Theorem 7.3. Let U be a bounded open p -convex subset of a p -(semi)norm space $(E, \|\cdot\|_p)$ ($0 < p \leq 1$) the zero $0 \in U$. Assume $F : \bar{U} \rightarrow 2^E$ is a 1-set contractive and upper semi-continuous mapping with non-empty closed p -convex values, satisfying the condition (H) or (H1) above. In addition, there exist $\alpha > 1$, $\beta \geq 0$, such that for each $x \in \partial\bar{U}$, we have that for any $y \in F(x)$, $\|y + x\|_p^{(\alpha+\beta)/p} \leq \|y\|_p^{\alpha/p} \|x\|_p^{\beta/p} + \|x\|_p^{(\alpha+\beta)/p}$. Then F has at least one fixed point.

Proof. We prove the conclusion by showing the Leray-Schauder boundary condition in Theorem 7.1 does not hold. If we assume F has no fixed point, by the boundary condition of Theorem 7.1, there exist $x_0 \in \partial\bar{U}$, $y_0 \in F(x_0)$ and $\lambda_0 > 1$ such that $y_0 = \lambda_0 x_0$.

Now, consider the function f defined by $f(t) := (t+1)^{\alpha+\beta} - t^\alpha - 1$ for $t \geq 1$. We then can show that f is a strictly increasing function for $t \in [1, \infty)$, thus we have $t^\alpha + 1 < (t+1)^{\alpha+\beta}$ for $t \in (1, \infty)$. By the boundary condition given in Theorem 7.3, we have that

$$\|y_0 + x_0\|_p^{(\alpha+\beta)/p} = (\lambda_0 + 1)^{\alpha+\beta} \|x_0\|_p^{(\alpha+\beta)/p} > (\lambda_0^\alpha + 1) \|x_0\|_p^{(\alpha+\beta)/p} = \|y_0\|_p^{\alpha/p} \|x_0\|_p^{\beta/p} + \|x_0\|_p^{(\alpha+\beta)/p},$$

which contradicts the boundary condition given by Theorem 7.3. Thus, the conclusion follows and the proof is complete. □

Theorem 7.4. Let U be a bounded open p -convex subset of a p -(semi)norm space $(E, \|\cdot\|_p)$ ($0 < p \leq 1$) the zero $0 \in U$. Assume $F : \bar{U} \rightarrow 2^E$ is a 1-set contractive and upper semi-continuous mapping with non-empty closed p -convex values, satisfying the condition (H) or (H1) above. In addition, there exist $\alpha > 1$, $\beta \geq 0$ (or alternatively, $\alpha > 1$, $\beta \geq 0$) such that for each $x \in \partial\bar{U}$, we have that for any $y \in F(x)$, $\|y - x\|_p^{\alpha/p} \|x\|_p^{\beta/p} \geq \|y\|_p^{\alpha/p} \|y + x\|_p^{\beta/p} - \|x\|_p^{(\alpha+\beta)/p}$. Then F has at least one fixed point.

Proof. The same as above, we prove the conclusion by showing the Leray-Schauder boundary condition in Theorem 7.1 does not hold. If we assume F has no fixed point, by the boundary condition of Theorem 7.1, there exist $x_0 \in \partial\bar{U}$, $y_0 \in F(x_0)$ and $\lambda_0 > 1$ such that $y_0 = \lambda_0 x_0$.

Now, consider the function f defined by $f(t) := (t-1)^\alpha - t^\alpha(t-1)^\beta + 1$ for $t \geq 1$. We then can show that f is a strictly decreasing function for $t \in [1, \infty)$, thus we have $(t-1)^\alpha < t^\alpha(t+1)^\beta - 1$ for $t \in (1, \infty)$.

By the boundary condition given in Theorem 7.4, we have that

$$\|y_0 - x_0\|_p^{\alpha/p} \|x_0\|_p^{\beta/p} = (\lambda_0 - 1)^\alpha \|x_0\|_p^{(\alpha+\beta)/p} < (\lambda_0^\alpha (\lambda_0 + 1)^\beta - 1) \|x_0\|_p^{(\alpha+\beta)/p} = \|y_0\|_p^{\alpha/p} \|y_0 + x_0\|_p^{\beta/p} - \|x_0\|_p^{(\alpha+\beta)/p},$$

which contradicts the boundary condition given by Theorem 7.4. Thus, the conclusion follows and the proof is complete. □

Theorem 7.5. Let U be a bounded open p -convex subset of a p -(semi)norm space $(E, \|\cdot\|_p)$ ($0 < p \leq 1$) the zero $0 \in U$. Assume $F : \bar{U} \rightarrow 2^E$ is a 1-set contractive and upper semi-continuous mapping with non-empty closed p -convex values, satisfying the condition (H) or (H1) above. In addition, there exist $\alpha > 1$, $\beta \geq 0$, we have that for any $y \in F(x)$, $\|y + x\|_p^{(\alpha+\beta)/p} \leq \|y - x\|_p^{\alpha/p} \|x\|_p^{\beta/p} + \|y\|_p^{\beta/p} \|x\|_p^{\alpha/p}$. Then F has at least one fixed point.

Proof. The same as above, we prove the conclusion by showing the Leray-Schauder boundary condition in Theorem 7.1 does not hold. If we assume F has no fixed point, by the boundary condition of Theorem 7.1, there exist $x_0 \in \overline{\partial U}$, $y_0 \in F(x_0)$ and $\lambda_0 > 1$ such that $y_0 = \lambda_0 x_0$.

Now, consider the function f defined by $f(t) := (t+1)^{\alpha+\beta} - (t-1)^\alpha - t^\beta$ for $t \geq 1$. We then can show that f is a strictly increasing function for $t \in [1, \infty)$, thus we have $(t+1)^{\alpha+\beta} > (t-1)^\alpha + t^\beta$ for $t \in (1, \infty)$.

By the boundary condition given in Theorem 7.5, we have that $\|y_0 + x_0\|_p^{(\alpha+\beta)/p} = (\lambda_0 + 1)^{\alpha+\beta} \|x_0\|_p^{(\alpha+\beta)/p} > ((\lambda_0 - 1)^\alpha + \lambda_0^\beta) \|x_0\|_p^{(\alpha+\beta)/p} = \|\lambda_0 x_0 - x_0\|_p^{\alpha/p} \|x_0\|_p^{\beta/p} + \|\lambda_0 x_0\|_p^{\beta/p} \|x_0\|_p^{\alpha/p} = \|y_0 - x_0\|_p^{\beta/p} \|x_0\|_p^{\alpha/p} + \|y_0\|_p^{\beta/p} \|x_0\|_p^{\alpha/p}$, which implies that

$$\|y_0 + x_0\|_p^{(\alpha+\beta)/p} > \|y_0 - x_0\|_p^{\beta/p} \|x_0\|_p^{\alpha/p} + \|y_0\|_p^{\beta/p} \|x_0\|_p^{\alpha/p},$$

this contradicts the boundary condition given by Theorem 7.5. Thus, the conclusion follows and the proof is complete. \square

As an application of Theorems 7.1 by testing the Leray-Schauder boundary condition, we have the following conclusion for each special case, and thus we omit their proofs in details here.

Corollary 7.1. Let U be a bounded open p -convex subset of a p -(semi)norm space $(E, \|\cdot\|_p)$ ($0 < p \leq 1$) the zero $0 \in U$. Assume $F : \overline{U} \rightarrow 2^E$ is a 1-set contractive and upper semi-continuous mapping with non-empty closed p -convex values, satisfying the condition (H) or (H1) above. Then F has at least one fixed point if one of the the following conditions holds for $x \in \overline{\partial U}$ and $y \in F(x)$:

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- (i) $\|y\|_p \leq \|x\|_p$,
- (ii) $\|y\|_p \leq \|y - x\|_p$,
- (iii) $\|y + x\|_p \leq \|y\|_p$,
- (iv) $\|y + x\|_p \leq \|x\|_p$,
- (v) $\|y + x\|_p \leq \|y - x\|_p$,
- (vi) $\|y\|_p \cdot \|y + x\|_p \leq \|x\|_p^2$,
- (vii) $\|y\|_p \cdot \|y + x\|_p \leq \|y - x\|_p \cdot \|x\|_p$.

If the p -seminorm space E is a uniformly convex Banach space $(E, \|\cdot\|)$ (for p -norm space with $p = 1$), then we have the following general existence result (which actually is true for non-expansive set-valued mappings).

Theorem 7.6. Let U be a bounded open convex subset of a uniformly convex Banach space $(E, \|\cdot\|)$ (with $p = 1$) with zero $0 \in U$. Assume $F : \overline{U} \rightarrow E$ is a semi-contractive and continuous single-valued mapping with non-empty values. In addition, for any $x \in \overline{\partial U}$, we have $\lambda x \neq F(x)$ for any $\lambda > 1$ (i.e., the ‘‘Leray-Schauder boundary condition’’). Then F has at least one fixed point.

Proof. By the assumption that F is a semi-contractive and continuous single-valued mapping with non-empty values, it follows by Lemma 3.2 in p.338 of Petryshyn [92], f is a 1-set contractive single-valued mapping. Moreover, by the assumption that E is a uniformly convex Banach, indeed $(I - F)$ is closed at zero, i.e., F is semiclosed (see Browder [15], or Goebel and Kirk [41]). Thus all assumptions of Theorem 7.1 are satisfied with the (H1) condition. The conclusion follows by Theorem 7.1, and the proof is completes. \square

Like Lemma 7.1 shows that a single-valued nonexpansive mapping defined in a uniformly convex Banach space (see also Theorem 7.6) satisfied the (H1) condition. Actually, the nonexpansive set-valued mappings defined on a special class of Banach spaces with the so-called the “Opial’s condition” do not only satisfy the condition (H1), but also belong to the classes of semiclosed 1-set contractive mappings as shown below.

The notion of the so-called “Opial’s condition” first given by Opial [78], which says that a Banach space X is said to satisfy Opial’s condition if $\liminf_{n \rightarrow \infty} \|w_n - w\| < \liminf_{n \rightarrow \infty} \|w_n - p\|$ whenever (w_n) is a sequence in X weakly convergent to w and $p \neq w$, we know that Opial’s condition plays an important role in the fixed point theory, e.g., see Lami Dozo [65], Goebel and Kirk [42], Xu [124] and references where. The following result shows that there nonexpansive set-valued mappings in Banach spaces with Opial’s condition (see Lami Dozo [65]) satisfying the condition (H1).

Lemma 7.2. Let C be a convex weakly compact of a Banach space X which satisfies Opial’s condition. Let $T : C \rightarrow K(C)$ be a non-expansive set-valued mapping with non-empty compact values. Then the graph of $(I - T)$ is closed in $(X, \sigma(X, X^*) \times (X, \|\cdot\|))$, thus T satisfies the “(H1) condition”, where, I denotes the identity on X , $\sigma(X, X^*)$ the weak topology, and $\|\cdot\|$ the norm (or strong) topology.

Proof. By following Theorem 3.1 of Lami Dozo [65], it follows that the mapping T is demiclosed, thus T satisfies the “(H1) condition”. The proof is complete. \square

As an application of Lemma 7.2, we have the following results for non-expansive mappings.

Theorem 7.7. Let C is a nonempty convex weakly compact subset of a Banach space X which satisfies Opial’s condition and $0 \in \text{int}C$. Let $T : C \rightarrow K(X)$ be a nonexpansive set-valued mapping with non-empty compact convex values. In addition, for any $x \in \partial\bar{C}$, we have $\lambda x \neq F(x)$ for any $\lambda > 1$ (i.e., the “Leray-Schauder boundary condition”). Then F has at least one fixed point.

Proof. As T is nonexpansive, it is 1-set contractive, By Lemma 7.1, it is then semi-contractive and continuous. Then the (H1) condition of Theorem 7.1 is satisfied. The conclusion follows by Theorem 7.1, and the proof is complete. \square .

Before the end of this section, by considering the p -seminorm space $(E, \|\cdot\|)$ is a seminorm space with $p = 1$, the following result is a special case of corresponding results from Theorem 7.2 to Theorem 7.5, and thus we omit its proof.

Corollary 7.2. Let U be a bounded open convex subset of a norm space $(E, \|\cdot\|)$. Assume $F : \bar{U} \rightarrow 2^E$ is a 1-set contractive and upper semi-continuous mapping with non-empty closed p -convex values, satisfying the condition (H) or (H1) above. Then F has at least one fixed point if there exist $\alpha > 1$, $\beta \geq 0$, such that any one of the following conditions satisfied

- (i) for each $x \in \partial\bar{U}$, and any $y \in F(x)$, $\|y - x\|^\alpha \geq \|y\|^{(\alpha+\beta)}\|x\|^{-\beta} - \|x\|^\alpha$,
- (ii) for each $x \in \partial\bar{U}$, and any $y \in F(x)$, $\|y + x\|^{(\alpha+\beta)} \leq \|y\|^\alpha\|x\|^\beta + \|x\|^{(\alpha+\beta)}$,
- (iii) for each $x \in \partial\bar{U}$, and any $y \in F(x)$, $\|y - x\|^\alpha\|x\|^\beta \geq \|y\|^\alpha\|y + x\|^\beta - \|x\|^{(\alpha+\beta)}$,
- (iv) for each $x \in \partial\bar{U}$, and any $y \in F(x)$, $\|y + x\|^{(\alpha+\beta)} \leq \|y - x\|^\alpha\|x\|^\beta + \|y\|^\beta\|x\|^\alpha$.

Remark 7.2. As discussed by Lemma 7.1 and the proof of Theorem 7.6, when the p -vector space is a uniformly convex Banach space, the semi-contractive or nonexpansive mappings automatically satisfy the condition (H) or (H1). Moreover, our results from Theorem 7.1 to Theorem 7.6, Corollary 7.1 and Corollary 7.2 also improve or unify corresponding results given by Browder [15], Li [66], Li et al.[67], Goebel and Kirk [41], Petryshyn [91]-[92], Reich [98], Tan and Yuan [115], Xu [123], Xu [126], Xu et al.[127], and results from the reference therein by extending the non-self mappings to the classes of 1-set contractive set-valued mappings in p -seminorm spaces with $p \in (0,1]$ (including the normed space or Banach space when $p = 1$, and for p -seminorm spaces).

8. Fixed Points for classes of Semiclosed 1-Set Contractive Mappings in p -Seminorm Spaces

In order to study the fixed point theory for a class of semiclosed 1-set contractive mappings in p -seminorm spaces, we first introduce the following definition which is a set-valued generalization of single-value semiclosed 1-set mappings first discussed by Li [66], Xu [126] (see also Li et al.[67], Xu et al.[127] and references therein).

Definition 8.1. Let D be a non-empty (bounded) closed subset of p -vector spaces $(E, \|\cdot\|_p)$ with p -seminorm for p -vector spaces, where $p \in (0,1]$ (which include norm space, or Banach spaces as special classes), and suppose $T : D \rightarrow X$ is a set-valued mapping. Then T is said to be a semiclosed 1-set contraction mapping if T is 1-set contraction, and $(I - T)$ is closed, which means that for a given net $\{x_n\}_{i \in I}$, for each $i \in I$, there exists $y_i \in T(x_i)$ with $\lim_{i \in I}(x_i - y_i) = 0$, then $0 \in (I - T)(\overline{D})$, i.e., there exists $x_0 \in \overline{D}$ such that $x_0 \in T(x_0)$.

Remark 8.1. By Lemma 7.1 and Lemma 7.2 above, it follows that each non-expansive (single-valued) mapping defined on a subset of uniformly convex Banach spaces, and nonexpansive set-valued mappings defined on a subset of Banach spaces satisfying Opial's condition are semiclosed 1-set contractive mapping (see also Goebel [40], Goebel and Kirk [41], Petrusel et al.[93], Xu [124], Yangai [128] for related discussion by the reference therein). In particular, under the setting of metric spaces or Banach spaces with certain property, it is clear that each semiclosed 1-set contractive mapping satisfies the condition (H1) above.

Though we know that compared to the single-valued case, based on the study in the literature about the approximation of fixed points for multi-valued mappings, a well-known counterexample due to Pietramala [94] (see also Muglia and Marino [74]) proved in 1991 that Browder approximation Theorem 1 given by Browder [13] cannot be extended to the genuine multivalued case even on a finite dimensional space \mathbb{R}^2 . Moreover, if a Banach space X satisfies Opial's property (see Opial [78]) that is, if x_n weakly converges to x , then we have that, $\limsup \|x_n - x\| < \limsup \|x_n - y\|$ for all $x \in X$ and $y \neq x$, then $I - f$ is demiclosed at 0 (see Lami Dozo [65], Yanagi [128] and related references therein) provided $f : C \rightarrow K(C)$ is non-expansive (here $K(C)$ denotes the family of nonempty compact subsets of C). We know that all Hilbert spaces and L^p spaces $p \in (1, \infty)$ have Opial's property, but it seems that whether $I - f$ is demiclosed at zero 0 if f is a nonexpansive set-valued mapping defined on the space X which is uniformly convex (e.g., $L[0,1]$, $1 < p < \infty$, $\neq 2$) and $f : C \rightarrow K(C)$ is nonexpansive. Here we remark that for a single-valued nonexpansive mapping f is yes, which is the famous theorem of Browder [12]. A remarkable fixed point theorem for multi-valued mappings is Lim's result in [69] which says that: If C is a

nonempty closed bounded convex subset of a uniformly convex Banach space X and $f : C \rightarrow K(C)$ is nonexpansive, then f has a fixed point.

Now based on the concept for the semiclosed 1-set contractive mappings, we give the existence results for their best approximation, fixed points and related nonlinear alternative under the framework of p -seminorm spaces for $p \in (0, 1]$.

Theorem 8.1 (Schauder Fixed Point Theorem for semiclosed 1-set contractive mappings). Let U be a non-empty bounded open subset of a (Hausdorff) locally p -convex space E and its zero $0 \in U$, and $C \subset E$ be a closed p -convex subset of E such that $0 \in C$, with $0 < p \leq 1$. If $F : C \cap \bar{U} \rightarrow 2^{C \cap \bar{U}}$ is an upper semi-continuous and semiclosed 1-set contractive set-valued mappings with non-empty p -convex closed values. Then T has at least one fixed point in $C \cap \bar{U}$.

Proof. As the mapping T is 1-set contractive, taking an increasing sequence $\{\lambda_n\}$ such that $0 < \lambda_n < 1$ and $\lim_{n \rightarrow \infty} \lambda_n = 1$, where $n \in \mathbb{N}$. Now we define a mapping $F_n : C \rightarrow 2^C$ by $F_n(x) := \lambda_n F(x)$ for each $x \in C$ and $n \in \mathbb{N}$. Then it follows that F_n is a λ_n -set-contractive mapping with $0 < \lambda_n < 1$. By Theorem 4.5 on the condensing mapping F_n in p -vector space with p -seminorm P_U for each $n \in \mathbb{N}$, there exists $x_n \in C$ such that $x_n \in F_n(x_n) = \lambda_n F(x_n)$. Thus there exists $y_n \in F(x_n)$ such that $x_n = \lambda_n y_n$. Let P_U is the Minkowski p -functional of U in E , it follows that P_U is continuous as $0 \in \text{int}(U) = U$. Note that for each $n \in \mathbb{N}$, $\lambda_n x_n \in \bar{U} \cap C$, which imply that $x_n = r(\lambda_n y_n) = \lambda_n y_n$, thus $P_U(\lambda_n y_n) \leq 1$ by Lemma 2.2. Note that

$$P_U(y_n - x_n) = P_U(y_n - \lambda_n y_n) = P_U(y_n - \lambda_n y_n) = P_U\left(\frac{(1 - \lambda_n)\lambda_n y_n}{\lambda_n}\right) \leq \left(\frac{1 - \lambda_n}{\lambda_n}\right)^p P_U(\lambda_n y_n) \leq \left(\frac{1 - \lambda_n}{\lambda_n}\right)^p,$$

which implies that $\lim_{n \rightarrow \infty} P_U(y_n - x_n) = 0$. Now by the assumption that F is semiclosed, which means that $(I - F)$ is closed at zero, thus there exists one point $x_0 \in \bar{C}$ such that $0 \in (I - F)(\bar{C})$, thus we have $x_0 \in F(x_0)$.

Indeed, without loss of the generality, we assume that $\lim_{n \rightarrow \infty} x_n = x_0$, here $y_n \in F(x_n)$ is with $x_n = \lambda_n y_n$, and $\lim_{n \rightarrow \infty} \lambda_n = 1$, it implies that $x_0 = \lim_{n \rightarrow \infty} (\lambda_n y_n)$, which means $y_0 := \lim_{n \rightarrow \infty} y_n = x_0$. There exists $y_0 (= x_0) \in F(x_0)$. We complete the proof. \square

Theorem 8.2 (Best approximation for semiclosed 1-set contractive mappings). Let U be a bounded open p -convex subset of a locally p -convex space E ($0 \leq p \leq 1$) the zero $0 \in U$, and C a (bounded) closed p -convex subset of E with also zero $0 \in C$. Assume $F : \bar{U} \cap C \rightarrow 2^C$ is a semiclosed 1-set contractive and upper semi-continuous mapping with non-empty closed p -convex values, and for each $x \in \partial_C U$ with $y \in F(x) \cap (C \setminus \bar{U})$, $(P_U^{\frac{1}{p}}(y) - 1)^p \leq P_U(y - x)$ for $0 < p \leq 1$ (this is trivial when $p = 1$). Then we have that there exist $x_0 \in C \cap \bar{U}$ and $y_0 \in F(x_0)$ such that $P_U(y_0 - x_0) = d_P(y_0, \bar{U} \cap C) = d_p(y_0, \overline{I_U^p(x_0)} \cap C)$, where P_U is the Minkowski p -functional of U . More precisely, we have the following either (I) or (II) holding:

- (I) F has a fixed point $x_0 \in U \cap C$, i.e., $0 = P_U(y_0 - x_0) = d_P(y_0, \bar{U} \cap C) = d_p(y_0, \overline{I_U^p(x_0)} \cap C)$,
- (II) there exists $x_0 \in \partial_C(U)$ and $y_0 \in F(x_0) \setminus \bar{U}$ with

$$P_U(y_0 - x_0) = d_P(y_0, \bar{U} \cap C) = d_p(y_0, \overline{I_U^p(x_0)} \cap C) = (P_U^{\frac{1}{p}}(y_0) - 1)^p > 0.$$

Proof. Let $r : E \rightarrow U$ be a retraction mapping defined by $r(x) := \frac{x}{\max\{1, (P_U(x))^{\frac{1}{p}}\}}$ for each $x \in E$, where P_U is the Minkowski p -functional of U . Since the space E 's zero $0 \in U (= \text{int}U$ as U is open), it follows that r is continuous by Lemma 2.2. As the mapping F is 1-set contractive, taking an increasing sequence $\{\lambda_n\}$ such that $0 < \lambda_n < 1$ and $\lim_{n \rightarrow \infty} \lambda_n = 1$, where $n \in \mathbb{N}$. Now we define a mapping $F_n : C \cap \bar{U} \rightarrow 2^C$ by $F_n(x) := \lambda_n F \circ r(x)$ for each $x \in C \cap \bar{U}$ and $n \in \mathbb{N}$. Then it follows that F_n is a λ_n -set-contractive mapping with $0 < \lambda_n < 1$ for each $n \in \mathbb{N}$. As C and \bar{U} are p -convex, we have $r(C) \subset C$ and $r(\bar{U}) \subset \bar{U}$, so $r(C \cap \bar{U}) \subset C \cap \bar{U}$. thus F_n is a self-mapping defined on $C \cap \bar{U}$. By Theorem 4.5 for condensing mapping F_n , for each $n \in \mathbb{N}$, there exists $z_n \in C \cap \bar{U}$ such that $z_n \in F_n(z_n) = \lambda_n F \circ r(z_n)$. Let $x_n = r(z_n)$, then we have $x_n \in C \cap \bar{U}$. and there exists $y_n \in F(x_n)$ with $x_n = r(\lambda_n y_n)$ such that the following (1) or (2) holding for each $n \in \mathbb{N}$:

(1): $\lambda_n y_n \in C \cap \bar{U}$; or (2): $\lambda_n y_n \in C \setminus \bar{U}$.

Now we prove the conclusion by considering the following two cases:

Case (I): For each $n \in \mathbb{N}$, $\lambda_n y_n \in C \cap \bar{U}$; or

Case (II): There there exists a positive integer n such that $\lambda_n y_n \in C \setminus \bar{U}$.

First, by the case (I), for each $n \in \mathbb{N}$, $\lambda_n y_n \in \bar{U} \cap C$, which imply that $x_n = r(\lambda_n y_n) = \lambda_n y_n$, thus $P_U(\lambda_n y_n) \leq 1$ by Lemma 2.2. Note that

$$P_U(y_n - x_n) = P_U(y_n - \lambda_n y_n) = P_U(y_n - \lambda_n y_n) = P_U\left(\frac{(1 - \lambda_n)\lambda_n y_n}{\lambda_n}\right) \leq \left(\frac{1 - \lambda_n}{\lambda_n}\right)^p P_U(\lambda_n y_n) \leq \left(\frac{1 - \lambda_n}{\lambda_n}\right)^p,$$

which implies that $\lim_{n \rightarrow \infty} P_U(y_n - x_n) = 0$. Now by the facet that F is semiclosed, it implies that there exists a point $x_0 \in \bar{U}$ (i.e., the consequence $\{x_n\}_{n \in \mathbb{N}}$ has a convergent subsequence with the limit x_0) such that $x_0 \in F(x_0)$. Indeed, without the loss of the generality, we assume that $\lim_{n \rightarrow \infty} x_n = x_0$, where $y_n \in F(x_n)$ is with $x_n = \lambda_n y_n$, and $\lim_{n \rightarrow \infty} \lambda_n = 1$, and as $x_0 = \lim_{n \rightarrow \infty} (\lambda_n y_n)$, which implies that $y_0 = \lim_{n \rightarrow \infty} y_n = x_0$. Thus there exists $y_0 (= x_0) \in F(x_0)$, we have $0 = d_p(x_0, F(x_0)) = d(y_0, \bar{U} \cap C) = d_p(y_0, \overline{I_U^p(x_0)} \cap C)$ as indeed $x_0 = y_0 \in F(x_0) \in \bar{U} \cap C \subset \overline{I_U^p(x_0)} \cap C$.

Second, by the case (II) there exists a positive integer n such that $\lambda_n y_n \in C \setminus \bar{U}$. Then we have that $P_U(\lambda_n y_n) > 1$, and also $P_U(y_n) > 1$ as $\lambda_n < 1$. As $x_n = r(\lambda_n y_n) = \frac{\lambda_n y_n}{(P_U(\lambda_n y_n))^{\frac{1}{p}}}$, which implies that $P_U(x_n) = 1$, thus $x_n \in \partial_C(U)$. Note that

$$P_U(y_n - x_n) = P_U\left(\frac{(P_U(y_n))^{\frac{1}{p}} - 1}{P_U(y_n)^{\frac{1}{p}}}\right) = (P_U^{\frac{1}{p}}(y_n) - 1)^p.$$

By the assumption, we have $(P_U^{\frac{1}{p}}(y_n) - 1)^p \leq P_U(y_n - x)$ for $x \in C \cap \partial \bar{U}$, it follows that

$$P_U(y_n) - 1 \leq P_U(y_n) - \sup\{P_U(z) : z \in C \cap \bar{U}\} \leq \inf\{P_U(y_n - z) : z \in C \cap \bar{U}\} = d_p(y_n, C \cap \bar{U}).$$

Thus we have the best approximation: $P_U(y_n - x_n) = d_P(y_n, \bar{U} \cap C) = (P_U^{\frac{1}{p}}(y_n) - 1)^p > 0$.

Now we want to show that $P_U(y_n - x_n) = d_P(y_n, \bar{U} \cap C) = d_p(y_n, \overline{I_U^p(x_0)} \cap C) > 0$.

By the fact that $(\bar{U} \cap C) \subset \overline{I_U^p(x_n)} \cap C$, let $z \in \overline{I_U^p(x_n)} \cap C \setminus (\bar{U} \cap C)$, we first claim that $P_U(y_n - x_n) \leq P_U(y_n - z)$. If not, we have $P_U(y_n - x_n) > P_U(y_n - z)$. As $z \in \overline{I_U^p(x_n)} \cap C \setminus (\bar{U} \cap C)$, there exists $y \in \bar{U}$ and a non-negative number c (actually $c \geq 1$ as shown soon below) with $z = x_n + c(y - x_n)$. Since $z \in C$, but $z \notin \bar{U} \cap C$, it implies that $z \notin \bar{U}$. By the fact that $x_n \in \bar{U}$ and $y \in \bar{U}$, we must have the constant $c \geq 1$; otherwise, it implies that

$z(= (1-c)x_n + cy) \in \bar{U}$, this is impossible by our assumption, i.e., $z \notin \bar{U}$. Thus we have that $c \geq 1$, which implies that $y = \frac{1}{c}z + (1 - \frac{1}{c})x_n \in C$ (as both $x_n \in C$ and $z \in C$). On the other hand, as $z \in I_{\bar{U}}^p(x_n) \cap C \setminus (\bar{U} \cap C)$, and $c \geq 1$ with $(\frac{1}{c})^p + (1 - \frac{1}{c})^p = 1$, combing with our assumption that for each $x \in \partial_C \bar{U}$ and $y \in F(x_n) \setminus \bar{U}$, $P_U^{\frac{1}{c}}(y) - 1 \leq P_U^{\frac{1}{c}}(y-x)$ for $0 < p \leq 1$, it then follows that

$$P_U(y_n - y) = P_U[\frac{1}{c}(y_n - z) + (1 - \frac{1}{c})(y_n - x_n)] \leq [(\frac{1}{c})^p P_U(y_n - z) + (1 - \frac{1}{c})^p P_U(y_n - x_n)] < P_U(y_n - x_n),$$

which contradicts that $P_U(y_n - x_n) = d_P(y_n, \bar{U} \cap C)$ as shown above we know that $y \in \bar{U} \cap C$, we should have $P_U(y_n - x_n) \leq P_U(y_n - y)$! This helps us to complete the claim: $P_U(y_n - x_n) \leq P_U(y_n - z)$ for any $z \in I_{\bar{U}}^p(x_n) \cap C \setminus (\bar{U} \cap C)$, which means that the following best approximation of Fan's type (see [35]-[36]) holding:

$$0 < d_P(y_n, \bar{U} \cap C) = P_U(y_n - x_n) = d_p(y_n, I_{\bar{U}}^p(x_n) \cap C).$$

Now by the continuity of P_U , it follows that the following best approximation of Fan type is also true:

$$0 < P_U(y_n - x_n) = d_P(y_n, \bar{U} \cap C) = d_p(y_n, I_{\bar{U}}^p(x_n) \cap C) = d_p(y_n, \overline{I_{\bar{U}}^p(x_n)} \cap C).$$

The proof is complete. \square

For a p -vector space when $p = 1$, it is a (Hausdorff) topological vector space E , we have the following best approximation for the outward set $\overline{O_{\bar{U}}(x_0)}$ based on the point $\{x_0\}$ respect the convex subset U in E .

Theorem 8.3 (Best approximation for outward sets). Let U be a bounded open convex subset of a locally convex space E (i.e., $p = 1$) with zero $0 \in \text{int}U = U$ (the interior $\text{int}U = U$ as U is open), and C a closed p -convex subset of E with also zero $0 \in C$. Assume that $F : \bar{U} \cap C \rightarrow C$ is a semiclosed 1-set-contractive continuous mapping. Then there exist $x_0 \in \bar{U} \cap X$ and $y_0 \in F(x_0)$ such that $P_U(y_0 - x_0) = d_P(y_0, \bar{U} \cap C) = d_p(y_0, \overline{O_{\bar{U}}(x_0)} \cap C)$, where P_U is the Minkowski p -functional of U . More precisely, we have the following either (I) or (II) holding:

- (I) F has a fixed point $x_0 \in U \cap C$, i.e., $P_U(y_0 - x_0) = P_U(y_0 - x_0) = d_P(y_0, \bar{U} \cap C) = d_p(y_0, \overline{O_{\bar{U}}(x_0)} \cap C) = 0$,
- (II) there exists $x_0 \in \partial_C(U)$ and $y_0 \in F(x_0) \setminus \bar{U}$ with

$$P_U(y_0 - x_0) = d_P(y_0, \bar{U} \cap C) = d_p(y_0, \overline{O_{\bar{U}}(x_0)} \cap C) > 0.$$

Proof. We define a new mapping $F_1 : \bar{U} \cap C \rightarrow 2^C$ by $F_1(x) := \{2x\} - F(x)$ for each $x \in \bar{U} \cap C$, then F_1 is also compact and upper semi-continuous mapping with non-empty closed convex values, and F_1 satisfies all hypothesis of Theorem 5.2 wit $p = 1$. It follows by Theorem 8.2 that there exists $x_0 \in \bar{U} \cap X$ and $y_1 \in F_1(x_0)$ such that $P_U(y_1 - x_0) = d_P(y_1, \bar{U} \cap C) = d_p(y_1, \overline{I_{\bar{U}}^p(x_0)} \cap C)$. More precisely, we have the following either (I) or (II) holding:

- (I) F_1 has a fixed point $x_0 \in U \cap C$ (so $0 = P_U(y_1 - x_0) = P_U(y_1 - x_0) = d_P(y_1, \bar{U} \cap C) = d_p(y_1, \overline{I_{\bar{U}}^p(x_0)} \cap C)$);
- (II) there exists $x_0 \in \partial_C(U)$ and $y_1 \in F_1(x_0) \setminus \bar{U}$ with

$$P_U(y_1 - x_0) = d_P(y_1, \bar{U} \cap C) = d_p(y_1, \overline{O_{\bar{U}}(x_0)} \cap C) > 0.$$

Now for any $x \in O_{\bar{U}}(x_0)$, there exist $r < 0, u \in \bar{U}$ such that $x = x_0 + r(u - x_0)$. Let $x_1 = 2x_0 - x$, then $x_1 = 2x_0 - x_0 - r(u - x_0) = x_0 + (-r)(u - x_0) \in I_{\bar{U}}(x_0)$. Let $y_1 = 2x_0 - y_0$, for some $y_0 \in F(x_0)$. As we have

$P_U(y_1 - x_0) = d_P(y_1, \overline{U} \cap C) = d_p(y_1, \overline{I_U(x_0)} \cap C)$, it follows that $P_U(y_1 - x_0) \leq P_U(y_1 - x_1)$, which implies that

$$P_U(x_0 - y_0) = P_U(y_1 - x_0) \leq P_U(y_1 - x_1) = P_U(2x_0 - y_0 - (2x_0 - x)) = P_U(y_0 - x)$$

for all $x \in O_{\overline{U}}(x_0)$. Thus we have $P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, O_{\overline{U}}(x_0) \cap C)$ and by the continuity of P_U , it follows that

$$P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{O_{\overline{U}}(x_0)} \cap C) (P_U^{\frac{1}{p}}(y_0) - 1)^p > 0.$$

This completes the proof. \square

Now by the application of Theorem 8.2 and Theorem 8.3, we have the the following general principle for the existence of solutions for Birkhoff-Kellogg Problems in p -seminorm spaces, where $(0 < p \leq 1)$.

Theorem 8.4 (Principle of Birkhoff-Kellogg alternative). Let U be a bounded open p -convex subset of a locally p -convex space E ($0 \leq p \leq 1$) with zero $0 \in \text{int}U = (U)$ (the interior $\text{int}U$ as U is open), and C a closed p -convex subset of E with also zero $0 \in C$. Assume that $F : \overline{U} \cap C \rightarrow C$ is a semiclosed 1-set-contractive continuous mapping. Then F has at least one of the following two properties:

(I) F has a fixed point $x_0 \in U \cap C$ such that $x_0 \in F(x_0)$,

(II) there exist $x_0 \in \partial_C(U)$, $y_0 \in F(x_0) \setminus \overline{U}$, and $\lambda = \frac{1}{(P_U(y_0))^{\frac{1}{p}}} \in (0, 1)$ such that $x_0 = \lambda y_0 \in \lambda F(x_0)$; In addition

if for each $x \in \partial_C U$, $P_U^{\frac{1}{p}}(y) - 1 \leq P_U^{\frac{1}{p}}(y - x)$ for $0 < p \leq 1$ (this is trivial when $p = 1$), then the best approximation between x_0 and y_0 given by

$$P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{I_U^p(x_0)} \cap C) = (P_U^{\frac{1}{p}}(y_0) - 1)^p > 0.$$

Proof. If (I) is not the case, then (II) is proved by the Remark 5.2 and by following the proof in Theorem 8.2 for the case (ii): $y_0 \in C \setminus \overline{U}$, with $y_0 = f(x_0) \in f(x_0)$. Indeed, as $y_0 \notin \overline{U}$, it follows that $P_U(y_0) > 1$, and $x_0 = f(y_0) = y_0 \frac{1}{(P_U(y_0))^{\frac{1}{p}}}$. Now let $\lambda = \frac{1}{(P_U(y_0))^{\frac{1}{p}}}$, we have $\lambda < 1$ and $x_0 = \lambda y_0$ with $y_0 \in F(x_0)$. Finally, the additionally assumption in (II) allows us to have the best approximation between x_0 and y_0 obtained by following the proof of Theorem 8.2 as $P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{I_U^p(x_0)} \cap C) > 0$. This completes the proof. \square

As an application of Theorem 8.2 for the non-self set-valued mappings discussed in Theorem 8.3 with outward set condition, we have the following general principle of Birkhoff-Kellogg alternative in topological vector spaces.

Theorem 8.5 (Principle of Birkhoff-Kellogg alternative in TVS). Let U be a bounded open p -convex subset of the topological vector space E with the zero $0 \in U$, and C a closed convex subset of E with also zero $0 \in C$. Assume the $F : \overline{U} \cap C \rightarrow 2^C$ is a semiclosed 1-set contractive and upper semi-continuous mapping with non-empty closed convex values. Then it has at least one of the following two properties:

(I) F has a fixed point $x_0 \in U \cap C$ such that $x_0 \in F(x_0)$,

(II) there exists $x_0 \in \partial_C(U)$ and $y_0 \in F(x_0) \setminus \overline{U}$ and $\lambda \in (0, 1)$ such that $x_0 = \lambda y_0$, and the best approximation

between x_0 and y_0 is given by $P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{I_U^p(x_0)} \cap C) > 0$.

On the other hand, by the Proof of Theorems 8.2, we note that for case (II) of Theorem 8.2, the assumption “each $x \in \partial_C U$ with $y \in F(x)$, $P_U^{\frac{1}{p}}(y) - 1 \leq P_U^{\frac{1}{p}}(y - x)$ ” is only used to guarantee the best approximation

" $P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{I_U^p(x_0)} \cap C) > 0$ ", thus we have the following Leray-Schauder alternative in p -vector spaces, which, of course, includes the corresponding results in locally convex spaces as special cases.

Theorem 8.6 (The Leray-Schauder Nonlinear Alternative). Let C a closed p -convex subset of p -seminorm space E with $0 \leq p \leq 1$ and the zero $0 \in C$. Assume the $F : C \rightarrow 2^C$ is a semiclosed 1-set contractive and upper semi-continuous mapping with non-empty closed p -convex values. Let $\varepsilon(F) := \{x \in C : x \in \lambda F(x), \text{ for some } 0 < \lambda < 1\}$. Then either F has a fixed point in C or the set $\varepsilon(F)$ is unbounded.

Proof. By assuming the case (I) is not true, i.e., F has no fixed point, then we claim that the set $\varepsilon(F)$ is unbounded. Otherwise, assume the set $\varepsilon(F)$ is bounded. and assume P is the continuous p -seminorm for E , then there exists $r > 0$ such that the set $B(0, r) := \{x \in E : P(x) < r\}$, which contains the set $\varepsilon(F)$, i.e., $\varepsilon(F) \subset B(0, r)$, which means for any $x \in \varepsilon(F)$, $P(x) < r$. Then $B(0, r)$ is an open p -convex subset of E and the zero $0 \in B(0, r)$ by Lemma 2.2 and Remark 2.4. Now let $U := B(0, r)$ in Theorem 8.4, it follows that for the mapping $F : B(0, r) \cap C \rightarrow 2^C$ satisfies all general conditions of Theorem 8.4, and we have that any $x_0 \in \partial_C B(0, r)$, no any $\lambda \in (0, 1)$ such that $x_0 = \lambda y_0$, where $y_0 \in F(x_0)$. Indeed, for any $x \in \varepsilon(F)$, it follows that $P(x) < r$ as $\varepsilon(F) \subset B(0, r)$, but for any $x_0 \in \partial_C B(0, r)$, we have $P(x_0) = r$, thus the conclusion (II) of Theorem 8.4 does not have hold. By Theorem 8.4 again, F must have a fixed point, but this contradicts with our assumption that F is fixed point free. This completes the proof. \square

Now assume a given p -vector space E equipped with the P -seminorm (by assuming it is continuous at zero) for $0 < p \leq 1$, then we know that $P : E \rightarrow \mathbb{R}^+$, $P^{-1}(0) = 0$, $P(\lambda x) = |\lambda|^p P(x)$ for any $x \in E$ and $\lambda \in \mathbb{R}$. Then we have the following useful result for fixed points due to Rothe and Altman types in p -vector spaces, which plays important roles for optimization problem, variational inequality, complementarity problems.

Corollary 8.1. Let U be a bounded open p -convex subset of a locally p -convex space E and zero $0 \in U$, plus C is a closed p -convex subset of E with $U \subset C$, where $0 < p \leq 1$. Assume that $F : \overline{U} \rightarrow 2^C$ is a semiclosed 1-set contractive upper semicontinuous mapping with non-empty closed p -convex values. f one of the following is satisfied,

- (1) (Rothe type condition): $P_U(y) \leq P_U(x)$ for $y \in F(x)$, where $x \in \partial U$,
- (2) (Petryshyn type condition): $P_U(y) \leq P_U(y - x)$ for $y \in F(x)$, where $x \in \partial U$,
- (3) (Altman type condition): $|P_U(y)|^{\frac{2}{p}} \leq [P_U(y - x)]^{\frac{2}{p}} + [P_U(x)]^{\frac{2}{p}}$ for $y \in F(x)$, where $x \in \partial U$,

then F has at least one fixed point.

Proof. By the conditions (1), (2) and (3), it follows that the conclusion of (II) in Theorem 8.4 "there exist $x_0 \in \partial_C(U)$ and $\lambda \in (0, 1)$ such that $x_0 \notin \lambda F(x_0)$ " does not hold, thus by the alternative of Theorem 8.4, F has a fixed point. This completes the proof. \square .

1300 By the fact that when $p = 1$, each p -vector space is a topological vector space, and thus we have the following classical Fan's best approximation (see [35]) as a powerful tool for the study in the optimization, mathematical programming, games theory, and mathematical economics, and others related topics in applied mathematics.

Corollary 8.2 (Fan's best approximation). Let U be a bounded open convex subset of a topological vector

space E with the zero $0 \in U$, and C a closed convex subset of E with also zero $0 \in C$, and assume $F : \bar{U} \cap C \rightarrow 2^C$ is a semiclosed 1-set contractive and upper semi-continuous mapping with non-empty closed convex values. Then there exist $x_0 \in \bar{U} \cap C$ and $y_0 \in F(x_0)$ such that $P_U(y_0 - x_0) = d_P(y_0, \bar{U} \cap C) = d_p(y_0, \overline{I_{\bar{U}}(x_0)} \cap C)$, where P_U being the Minkowski p -functional of U in E . More precisely, we have the following either (I) or (II) holding, where $W_{\bar{U}}(x_0)$ is either inward set $I_{\bar{U}}(x_0)$, or the outward set $O_{\bar{U}}(x_0)$:

- (I) F has a fixed point $x_0 \in U \cap C$, $0 = P_U(y_0 - x_0) = P_U(y_0 - x_0) = d_P(y_0, \bar{U} \cap C) = d_p(y_0, \overline{W_{\bar{U}}(x_0)} \cap C)$,
- (II) there exists $x_0 \in \partial_C(U)$ and $y_0 \in F(x_0) \setminus \bar{U}$ with

$$P_U(y_0 - x_0) = d_P(y_0, \bar{U} \cap C) = d_p(y_0, \overline{W_{\bar{U}}(x_0)} \cap C) = P_U(y_0) - 1 > 0.$$

Proof. When $p = 1$, then it automatically satisfies that the inequality: $P_U^{\frac{1}{p}}(y) - 1 \leq P_U^{\frac{1}{p}}(y - x)$, and indeed we have that for $x_0 \in \partial_C(U)$, with $y_0 \in F(x_0)$, we have $P_U(y_0 - x_0) = d_P(y_0, \bar{U} \cap C) = d_p(y_0, \overline{W_{\bar{U}}(x_0)} \cap C) = P_U(y_0) - 1$. The conclusions are given by Theorem 8.2 (or Theorem 8.3). The proof is complete. \square

We like to point out the similar results on Rothe and Leray-Schauder alternative have been developed by Isac [52], Park [84], Potter [96], Shahzad [105]-[107], Xiao and Zhu [121], and related references therein as tools of nonlinear analysis in topological vector spaces. As mentioned above, when $p = 1$ and take F as a continuous mapping, then we obtain the version of Leray-Schauder in TVS and this we omit its statement in details.

9. Principle of Nonlinear Alternatives for Semiclosed 1-Set Contractive Mappings

As applications of results in Section 8 above, we now establish general results for the existence of solutions for Birkhoff-Kellogg problem, and the principle of Leray-Schauder alternatives for semiclosed 1-set contractive mappings in p -vector spaces for $0 < p \leq 1$.

Theorem 9.1 (Birkhoff-Kellogg alternative in p -vector spaces). Let U be a bounded open p -convex subset of a locally p -convex space E (where, $0 \leq p \leq 1$) with the zero $0 \in U$, and C a closed p -convex subset of E with also zero $0 \in C$, and assume $F : \bar{U} \cap C \rightarrow 2^C$ is a semiclosed 1-set contractive and upper semi-continuous mapping with non-empty closed p -convex values. In addition, for each $x \in \partial_C(U)$ with $y \in F(x)$, $P_U^{\frac{1}{p}}(y) - 1 \leq P_U^{\frac{1}{p}}(y - x)$ for $0 < p \leq 1$ (this is trivial when $p = 1$), where P_U is the Minkowski p -functional of U . Then we have that either (I) or (II) holding below:

- (I) there exists $x_0 \in \bar{U} \cap C$ such that $x_0 \in F(x_0)$,
- (II) there exists $x_0 \in \partial_C(U)$ with $y_0 \in F(x_0) \setminus \bar{U}$ and $\lambda > 1$ such that $\lambda x_0 = y_0 \in F(x_0)$, i.e., $F(x_0) \cap \{\lambda x_0 : \lambda > 1\} \neq \emptyset$.

Proof. By following the argument and notations used by Theorem 8.2, we have that either

- (1) F has a fixed point $x_0 \in U \cap C$; or
- (2) there exists $x_0 \in \partial_C(U)$ and $y_0 \in F(x_0)$ with $x_0 = f(y_0)$ such that

$$P_U(y_0 - x_0) = d_P(y_0, \bar{U} \cap C) = d_p(y_0, \overline{I_{\bar{U}}(x_0)} \cap C) = P_U(y_0) - 1 > 0,$$

where $\partial_C(U)$ denotes the boundary of U relative to C in E , and f is the restriction of the continuous retraction r respect to the set U in E .

If F has no fixed point, then above (2) holds and $x_0 \notin F(x_0)$. As given by the proof of Theorem 8.2, we have that $y_0 \in F(x_0)$ and $y_0 \notin \bar{U}$, thus $P_U(y_0) > 1$ and $x_0 = f(y_0) = \frac{y_0}{(P_U(y_0))^{\frac{1}{p}}}$, which means $y_0 = (P_U(y_0))^{\frac{1}{p}}x_0$. Let $\lambda = (P_U(y_0))^{\frac{1}{p}}$, then $\lambda > 1$ and we have $\lambda x_0 = y_0 \in F(x_0)$. This completes the proof. \square

Theorem 9.2 (Birkhoff-Kellogg alternative in TVS). Let U be a bounded open convex subset of a topological space E with the zero $0 \in U$, and C a closed convex subset of E with also zero $0 \in C$, and assume $F : \bar{U} \cap C \rightarrow 2^C$ is a semiclosed 1-set contractive and upper semi-continuous mapping with non-empty closed convex values. Then we have the following either (I) or (II) holding, where $W_{\bar{U}}(x_0)$ is either inward set $I_{\bar{U}}(x_0)$, or the outward set $O_{\bar{U}}(x_0)$:

(I) there exists $x_0 \in \bar{U} \cap C$ such that $x_0 \in F(x_0)$,

(II) there exists $x_0 \in \partial_C(U)$ with $y_0 \in F(x_0) \setminus \bar{U}$ and $\lambda > 1$ such that $\lambda x_0 = y_0 \in F(x_0)$, i.e., $F(x_0) \cap \{\lambda x_0 : \lambda > 1\} \neq \emptyset$.

Proof. When $p = 1$, then it automatically satisfies that the inequality: $P_U^{\frac{1}{p}}(y) - 1 \leq P_U^{\frac{1}{p}}(y - x)$, and indeed we have that for $x_0 \in \partial_C(U)$, with $y_0 \in F(x_0)$, we have $P_U(y_0 - x_0) = d_p(y_0, \bar{U} \cap C) = d_p(y_0, \overline{W_{\bar{U}}(x_0)} \cap C) = P_U(y_0) - 1$. The conclusions are given by Theorems 8.3 and 8.4. The proof is complete. \square

Indeed, we have the following fixed points for non-self mappings in p -vector spaces for $0 < p \leq 1$ under different boundary conditions.

Theorem 9.3 (Fixed Points of non-self mappings). Let U be a bounded open p -convex subset of a locally p -convex space E (where, $0 \leq p \leq 1$) with the zero $0 \in U$, and C a closed p -convex subset of E with also zero $0 \in C$, and assume $F : \bar{U} \cap C \rightarrow 2^C$ is a semiclosed 1-set contractive and upper semi-continuous mapping with non-empty closed p -convex values. In addition, for each $x \in \partial_C(U)$ with $y \in F(x)$, $P_U^{\frac{1}{p}}(y) - 1 \leq P_U^{\frac{1}{p}}(y - x)$ for $0 < p \leq 1$ (this is trivial when $p = 1$), where P_U is the Minkowski p -functional of U . If F satisfies any one of the following conditions for any $x \in \partial_C(U) \setminus F(x)$

(i) For each $y \in F(x)$, $P_U(y - z) < P_U(y - x)$ for some $z \in \overline{I_{\bar{U}}(x)} \cap C$;

(ii) For each $y \in F(x)$, there exists λ with $|\lambda| < 1$ such that $\lambda x + (1 - \lambda)y \in \overline{I_{\bar{U}}(x)} \cap C$;

(iii) $F(x) \subset \overline{I_{\bar{U}}(x)} \cap C$;

(iv) $F(x) \cap \{\lambda x : \lambda > 1\} = \emptyset$;

(v) $F(\partial U) \subset \bar{U} \cap C$;

(vi) For each $y \in F(x)$, $P_U(y - x) \neq ((P_U(y))^{\frac{1}{p}} - 1)^p$;

then F must has a fixed point.

Proof. By following the argument and symbols used in the proof of Theorem 8.2 (see also Theorem 8.4), we have that either

(1) F has a fixed point $x_0 \in U \cap C$; or

(2) there exists $x_0 \in \partial_C(U)$ and $y_0 \in F(x_0)$ with $x_0 = f(y_0)$ such that

$$P_U(y_0 - x_0) = d_p(y_0, \bar{U} \cap C) = d_p(y_0, \overline{I_{\bar{U}}(x_0)} \cap C) = P_U(y_0) - 1 > 0,$$

where $\partial_C(U)$ denotes the boundary of U relative to C in E , and f is the restriction of the continuous retraction r respect to the set U in E .

First, suppose that F satisfies the condition (i), if F has no fixed point, then above (2) holds and $x_0 \notin F(x_0)$. Then by the condition (i), it follows that $P_U(y_0 - z) < P_U(y_0 - x_0)$ for some $z \in \overline{I_{\overline{U}}(x)} \cap C$, this contradicts with the best approximation equations given by (2) above, thus F must have a fixed point.

Second, suppose that F satisfies the condition (ii), if F has no fixed point, then above (2) holds and $x_0 \notin F(x_0)$. Then by the condition (ii), there exists $\lambda > 1$ such that $\lambda x_0 + (1 - \lambda)y_0 \in \overline{I_{\overline{U}}(x)} \cap C$. It follows that

$$P_U(y_0 - x_0) \leq P_U(y_0 - (\lambda x_0 + (1 - \lambda)y_0)) = P_U(\lambda(y_0 - x_0)) = |\lambda|^p P_U(y_0 - x_0) < P_U(y_0 - x_0)$$

this is impossible and thus F must have a fixed point in $\overline{U} \cap C$.

Third, suppose that F satisfies the condition (iii), i.e., $F(x) \subset \overline{I_{\overline{U}}(x)} \cap C$; then the (2), we have that $P_U(y_0 - x_0)$ and thus $x_0 = y_0 \in F(x_0)$, which means F has a fixed point.

Forth, suppose that F satisfies the condition (iv), if F has no fixed point, then above (2) holds and $x_0 \notin F(x_0)$. As given by the proof of Theorem 8.2, we have that $y_0 \notin \overline{U}$, thus $P_U(y_0) > 1$ and $x_0 = f(y_0) = \frac{y_0}{(P_U(y_0))^{\frac{1}{p}}}$, which means $y_0 = (P_U(y_0))^{\frac{1}{p}} x_0$, where $(P_U(y_0))^{\frac{1}{p}} > 1$, this contradicts with the assumption (iv), thus F must have a fixed point in $\overline{U} \cap C$.

Fifth, suppose that F satisfies the condition (v), then $x_0 \notin F(x_0)$. As $x_0 \in \partial_C U$, now by the condition (v), we have that $F(\partial U) \subset \overline{U} \cap C$, it follows that for any $y_0 \in F(x_0)$, we have $y_0 \in \overline{U} \cap C$, thus $y \notin \overline{U} \setminus \cap C$, which implies that $0 < P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = 0$, this is impossible, thus F must have a fixed point. Here, as pointed out by Remark 5.2, we know that based on the condition (v), the mapping F has a fixed point by applying $F(\partial U) \subset \overline{U} \cap C$ is enough, not needing the general hypothesis: “for each $x \in \partial_C(U)$ with $y \in F(x)$, $P_U^{\frac{1}{p}}(y) - 1 \leq P_U^{\frac{1}{p}}(y - x)$ for $0 < p \leq 1$ ”.

Finally, suppose that F satisfies the condition (vi), if F has no fixed point, then above (2) holds and $x_0 \notin F(x_0)$. Then the condition (v) implies that $P_U(y_0 - x_0) \neq ((P_U(y_0))^{\frac{1}{p}} - 1)^p$, but the our proof in Theorem shows that $P_U(y_0 - x_0) = ((P_U(y_0))^{\frac{1}{p}} - 1)^p$, this is impossible, thus F must have a fixed point. Then the proof is complete. \square

Now by taking the set C in Theorem 8.1 as the whole p -vector space E itself, we have the following general results for non-self upper semi-continuous set-valued mappings which include results of Rothe, Petryshyn, Altman and Leray-Schauder types' fixed points as special cases.

Taking $p = 1$ and $C = E$ in Theorem 9.3, we have the following fixed points for non-self upper semi-continuous set-valued mappings associated with inward or outward sets in topological vector spaces (TVS) as follows.

Theorem 9.4 (Fixed Points of non-self mappings with boundary conditions). Let U be a bounded open convex subset of the topological vector spaces E with the zero $0 \in U$, and assume $F : \overline{U} \rightarrow 2^E$ is a semiclosed 1-set contractive and upper semi-continuous mapping with non-empty closed convex values. If F satisfies any one of the following conditions for any $x \in \partial(U) \setminus F(x)$

- (i) For each $y \in F(x)$, $P_U(y - z) < P_U(y - x)$ for some $z \in \overline{I_{\overline{U}}(x)}$ (or $z \in \overline{O_{\overline{U}}(x)}$)

(ii) For each $y \in F(x)$, there exists λ with $|\lambda| < 1$ such that $\lambda x + (1 - \lambda)y \in \overline{I_U(x)}$ (or, $\overline{O_U(x)}$).

(iii) $F(x) \subset \overline{I_U(x)}$ (or $\overline{O_U(x)}$)

(iv) $F(x) \cap \{\lambda x : \lambda > 1\} = \emptyset$;

(v) $F(\partial(U)) \subset \overline{U}$;

(vi) For each $y \in F(x)$, $P_U(y - x) \neq P_U(y) - 1$;

then F must has a fixed point.

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In what follow, based on the best approximation theorem in p -seminorm space, we will also give some fixed point theorems for non-self set-valued mappings with various boundary conditions which are related to the study for the existence of solutions for PDE and differential equations with boundary problems (see, Browder [15], Petryshyn [91]-[92], Reich [98]), which would play roles in nonlinear analysis for p -seminorm space as shown below.

First, as discussed by Remark 5.2, the proof of Theorem 9.2, with the strongly boundary condition " $F(\partial(U)) \subset \overline{U} \cap C$ " only, we can prove that F has a fixed point, thus we have the following fixed point theorem of Rothe type in p -vector spaces.

Theorem 9.5 (Rothe Type). Let U be a bounded open p -convex subset of a locally p -convex space E (where, $0 \leq p \leq 1$) with the zero $0 \in U$. Assume $F : \overline{U} \rightarrow 2^E$ is a semi 1-set contractive and upper semi-continuous mapping with non-empty closed p -convex values, and such that $F(\partial(U)) \subset \overline{U}$, then F must has a fixed point.

Now as applications of Theorem 9.5, we give the following Leray-Schauder Alternative in p -vector spaces for non-self set-valued mappings associated with the boundary condition which often appear in the applications (see Isac [52] and references therein for the study of complementary problems and related topics in optimization).

Theorem 9.6 (Leray-Schauder Alternative in p -Vector Spaces). Let E be a locally p -convex space E , where $0 < p \leq 1$, $B \subset E$ a bounded closed p -convex such that $0 \in \text{int}B$. Let $F : [0, 1] \times B \rightarrow E$ be semiclosed 1-set contractive and upper semi-continuous set-valued with non-empty closed p -convex values, and such that the set $F([0, 1] \times B)$ be relatively compact in E . If the following assumptions are satisfied:

(1) $x \notin F(t, x)$ for all $x \notin \partial B$ and $t \in [0, 1]$,

(2) $F(\{0\} \times \partial B) \subset B$,

then there is an element $x^* \in B$ such that $x^* \in F(1, x^*)$.

Proof. For any $n \in N$, we consider the mapping

$$F_n(x) = \begin{cases} F\left(\frac{1 - P_B(x)}{\epsilon_n}, \frac{x}{P_B(x)}\right), & \text{if } 1 - \epsilon \leq P_B(x) \leq 1, \\ F\left(1, \frac{x}{1 - \epsilon_n}\right), & \text{if } P_B(x) < 1 - \epsilon_n, \end{cases} \quad (4)$$

where P_B is the Minkowski p -functional of B and $\{\epsilon_n\}_{n \in N}$ is a sequence of real numbers such that $\lim_{n \rightarrow \infty} \epsilon_n = 0$ and $0 < \epsilon_n < \frac{1}{2}$ for any $n \in N$. We observe that for each $n \in N$, the mapping F_n is 1-set contractive upper semi-continuous with non-empty closed p -convex values on B . From assumption (2), we have that $F_n(\partial B) \subset B$, and the assumptions of Theorem 9.5 are satisfied, then for each $n \in N$, there exists an element $u_n \in B$ such that $u_n \in F_n(u_n)$.

We first prove the following statement: “It is impossible to have an infinite number of the elements u_n satisfy the following inequality: $1 - \epsilon_n \leq P_B(u_n) \leq 1$. ”

If not, we assume to have an infinite number of the elements u_n satisfy the following inequality:

$$1 - \epsilon_n \leq P_B(u_n) \leq 1.$$

As $F_n(B)$ is relatively compact and by the definition of mappings F_n , we have that $\{u_n\}_{n \in N}$ is contained in a compact set in E . Without loss of the generality (indeed, each compact set is also countably compact), we define the sequence $\{t_n\}_{n \in N}$ by $t_n := \frac{1 - P_B(u_n)}{\epsilon}$ for each $n \in N$. Then we have that $\{t_n\}_{n \in N} \subset [0, 1]$ and we may assume that $\lim_{n \rightarrow \infty} t_n = t \in [0, 1]$. The corresponding subsequence of $\{u_n\}_{n \in N}$ is denoted again by $\{u_n\}_{n \in N}$ and it also satisfies the inequality: $1 - \epsilon_n \leq P_B(u_n) \leq 1$, which implies that $\lim_{n \rightarrow \infty} P_B(u_n) = 1$.

Now let u^* be an accumulation point of $\{u_n\}_{n \in N}$, thus have $\lim_{n \rightarrow \infty} (t_n, \frac{u_n}{P_B(u_n)}, u_n) = (t, u^*, u^*)$. By the fact that F is compact, we have assume that $u_n \in F(t_n, \frac{u_n}{P_B(u_n)})$ for each $n \in N$, it follows that $u^* \in F(t, u^*)$, this contradicts with the assumption (1) as we have $\lim_{n \rightarrow \infty} P_B(u_n) = 1$ (which means that $u^* \in \partial B$, this is impossible).

Thus it is impossible to have that “to have an infinite number of elements u_n satisfy the inequality: $1 - \epsilon_n \leq P_B(u_n) \leq 1$ ”, which means that there is only a finite number of elements of sequence $\{u_n\}_{n \in N}$ satisfying the inequality: $1 - \epsilon_n \leq P_B(u_n) \leq 1$. Now, without loss of the generality, for $n \in N$, we have the following inequality:

$$P_B(u_n) < 1 - \epsilon_n.$$

By the fact that $\lim_{n \rightarrow \infty} (1 - \epsilon_n) = 1$, $u_n \in F(1, \frac{u_n}{1 - \epsilon_n})$ for all $n \in N$ and assume that $\lim_{n \rightarrow \infty} u_n = u^*$, then the upper semi-continuity of F with non-empty closed values implies that the graph of F is closed, and by the fact $u_n \in F(1, \frac{u_n}{1 - \epsilon_n})$, it implies that $u^* \in F(1, u^*)$. This completes the proof. \square

As a special case of Theorem 9.6, we have the following principle for the implicit form of Leray-Schauder type alternative for set-valued mappings in p -vector spaces for $0 < p \leq 1$.

Corollary 9.1 (The Implicit Leray-Schauder Alternative). Let E be a locally p -convex space E , where $0 < p \leq 1$, $B \subset E$ a bounded closed p -convex such that $0 \in \text{int}B$. Let $F : [0, 1] \times B \rightarrow E$ be semiclosed 1-set contractive and continuous with non-empty closed p -convex values, and the set $F([0, 1] \times B)$ be relatively compact in E . If the following assumptions are satisfied:

- (1) $F(\{0\} \times \partial B) \subset B$,
- (2) $x \notin F(0, x)$ for all $x \in \partial B$,

then at least one of the following properties is satisfied:

- (i) there exists $x^* \in B$ such that $x^* \in F(1, x^*)$; or
- (ii) there exists $(\lambda^*, x^*) \in (0, 1) \times \partial B$ such that $x^* \in F(\lambda^*, x^*)$.

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Proof. The result is an immediate consequence of Theorem 9.6, this completes the proof. \square

We like to point out the similar results on Rothe and Leray-Schauder alternative have been developed by Furi and Pera [37], Granas and Dugundji [46], Górniewicz [44], Górniewicz et al.[45], Isac [52], Li et al.[67], Liu [70], Park [84], Potter [96], Shahzad [105]-[107], Xu [126], Xu et al.[127], and related references therein as tools of nonlinear

analysis in the Banach space setting and applications to the boundary value problems for ordinary differential equations in noncompact problems, a general class of mappings for nonlinear alternative of Leray-Schauder type in normal topological spaces, and some Birkhoff-Kellogg type theorems for general class mappings in topological vector spaces are also established by Agarwal et al.[1], Agarwal and O'Regan [2]-[3], Park [86], and references therein for more in detail; and in particular, recently O'Regan [80] uses the Leray-Schauder type coincidence theory to establish some Birkhoff-Kellogg problem, Furi-Pera type results for a general class of mappings.

Before closing this section, we like to share that as the application of the best approximation result for 1-set contractive mappings, we can establish the fixed point theorems and general principle of Leray-Schauder alternative for non-self mappings, which seem would play important roles for the nonlinear analysis under the framework of p -seminorm spaces, as the achievement of nonlinear analysis for the underling being locally topological vector spaces, normed spaces, or in Banach spaces.

10. Fixed Points for Semiclosed 1-Set Contractive Mappings

In this section, based on the best approximation Theorem 8.2 established for the 1-set contractive mappings in Section 8, we will show how it is used as a useful tool for us to develop fixed point theorems for semiclosed 1-set contractive non-self upper semi-continuous mappings in p -seminorm spaces (for $p \in (0, 1]$, by including seminorm, norm spaces, and uniformly convex Banach spaces as special cases).

By following the Definition 7.1 above, we first observe that if f is a continuous demicompact mapping, then $(I - f)$ is closed, where I is the identity mapping on X . it is also clear from definitions that every demicompact map is hemicompact in seminorm spaces, but the converse is not true in general (e.g., see the example in p.380 by Tan and Yuan [115]). It is evident that if f is demicompact, then $I - f$ is demiclosed. It is know that for each condensing mapping f , when D or $f(D)$ is bounded, then f is hemicompact; and also f is demicompact in metric spaces by Lemma 2.1 and Lemma 2.2 of Tan and Yuan [115], respectively. In addition, it is known that every nonexpansive map is a 1-set-contractive map; and also if f is a hemicompact 1-set-contractive mapping, then f is a 1-set-contractive mapping satisfying the following “**Condition (H1)**” (the same as (H1), and slightly different from the condition (H) used in Section 5):

(H1) Condition: Let D be a nonempty bounded subset of a space E and assume $F : \overline{D} \rightarrow 2^E$ a set-valued mapping. If $\{x_n\}_{n \in \mathbb{N}}$ is any sequence in D such that for each x_n , there exists $y_n \in F(x_n)$ with $\lim_{n \rightarrow \infty} (x_n - y_n) = 0$, then there exists a point $x \in \overline{D}$ such that $x \in F(x)$.

We first note that the “(H1) Condition” above is actually the “Condition (C)” used by Theorem 1 of Petryshyn [92]. Indeed, by following Goebel and Kirk [42] (see also Xu [124] and reference therein), Browder [15] (see also [16], p.103) proved that if K is a closed and convex subset of a uniformly convex Banach space X , and if $T : K \rightarrow X$ is nonexpansive, then the mapping $f := I - T$ is demiclosed on X . This result, known as Browder’s demiclosedness principle (Browder’s proof, which was inspired by the technique of Göhde in [43]), is one of the fundamental results in the theory of nonexpansive mappings, which satisfies the “(H1) condition”.

The following is the Browder’s demiclosedness principle proved by Browder [15] that says that a nonexpansive mapping in a uniformly convex Banach X enjoys the condition (H1) as shown below.

Lemma 10.1. Let D be a nonempty bonded convex subset of a uniformly convex Banach space E . Assume $F : \overline{D} \rightarrow E$ is a non-expansive single-valued mapping, then the mapping $P := I - F$ defined by $P(x) := (x - F(x))$ for each $x \in \overline{D}$ is demiclosed, and in particular, the “(H1) Condition” holds.

Proof. By following the argument given in p.329 (see also the proof of Theorem 2.2 and Corollary 2.1) by Petryshyn [92], by the Browder demiclosedness principle (see Goebel and Kirk [42] or Xu [124]), $P = (I - F)$ is closed at zero, thus there exists $x_0 \in \overline{D}$ such $0 \in (I - F)x_0$, which means that $x_0 \in F(x_0)$. The proof is complete. \square .

On the other hand, by following the notion called “Opial’s condition” given by Opial [78], which says that a Banach space X is said to satisfy Opial’s condition if $\liminf_{n \rightarrow \infty} \|w_n - w\| < \liminf_{n \rightarrow \infty} \|w_n - p\|$ whenever (w_n) is a sequence in X weakly convergent to w and $p \neq w$, we know that Opial’s condition plays an important role in the fixed point theory, e.g., see Lami Dozo [65], Goebel and Kirk [42], Xu [124] and references where. Actually, the following result shows that there exists a class of non-expansive set-valued mappings in Banach spaces with Opial’s condition (see Lami Dozo [65] satisfying the “(H1) Condition”.

Lemma 10.2. Let C is a nonempty convex weakly compact subset of a Banach space X which satisfies Opial’s condition. Let $T : C \rightarrow K(C)$ be a non-expansive set-valued mapping with non-empty compact-values. Then the graph of $(I - T)$ is closed $(X, \sigma(X, X^*) \times (X, \|\cdot\|))$, thus T satisfies the “(H1) condition”, where, I denotes the identity on X , $\sigma(X, X^*)$ the weak topology, and $\|\cdot\|$ the norm (or strong) topology.

Proof. By following Theorem 3.1 of Lami Dozo [65], it follows that the mapping T is demi-closed, thus T satisfies the “(H1) condition”. The proof is complete. \square

By Theorem 3.1 of Lami Dozo [65], indeed we have the following statement which is an another version by using the term of “distance convergence” for Lemma 10.2.

Lemma 10.3. Let C be a nonempty closed convex subset of a Banach space (X, d) which satisfies the Opial condition. Let $T : C \rightarrow K(C)$ be a multi-valued nonexpansive mapping (with the fixed points). Let $(y_n)_{n \in \mathbb{N}}$ be a bounded sequence, such that $\lim_{n \rightarrow \infty} d(y, T(y_n)) = 0$, then the weak cluster points of (y_n) , $n \in \mathbb{N}$ is a fixed point of T .

Proof. It is Theorem 3.1 of Lami Dozo [65] (see also Lemma 3.2 of Xu and Muglia [125]). \square

We note that another class of set-valued mappings, called “*-nonexpansive mappings in Banach spaces (introduced by Husain and Tarafdar [50], see also Husain and Latif [49]) which was proved to hold the demiclosedness principle in reflexive Banach spaces satisfying Opial’s condition by Muglia and Marino (i.e., Lemma 3.4 in [74], thus the demiclosedness principle also holds in reflexive Banach spaces with duality mapping that is weakly sequentially continuous since these satisfy Opial’s condition.

Let E denote a Hausdroff locally convex topological vector space, and \mathfrak{F} to denote the family of oontinuous scminorms generating the topology of E . Also $C(E)$ will denote the family of nonempty compact subsets of E .

For each $p \in \mathfrak{F}$ and $A, B \in C(E)$, we can define $\delta(A, B) := \sup\{p(a - b) : a \in A, b \in B\}$. and $D_p(A, B) := \max\{\sup_{a \in A} \inf_{b \in B} P(a - b), \sup_{b \in B} \inf_{a \in A} P(a - b)\}$. Though p is only a seminorm, D_p is a Hausdorff metric on $C(E)$ (e.g., see Ko and Tsai [61]).

Definition 10.1. Let K be a nonempty subset of E . A mapping $T : K \rightarrow C(E)$ is said to be a multi-valued contraction if there exists a constant $k_p \in (0, 1)$ such that $D_p(T(x), T(y)) \leq k_p P(x - y)$. T is said to be non-expansive if for any $x, y \in K$, we have $P_p(T(x), T(y)) \leq P(x - y)$.

By Chen and Singh [27], we now have the following definition of Opial's condition in locally convex spaces.

Definition 10.2. The locally convex space E is said to satisfy the Opial's condition if for each $x \in E$ and every net (x_α) converging weakly to x , then for each $P \in \mathfrak{F}$, we have $\liminf P(x_\alpha - y) > \liminf P(x_\alpha - x)$ for any $y \neq x$.

Now we have the following demiclosedness principle for nonexpansive set-valued mappings in (Hausdorff) local convex spaces E , which is indeed Theorem 1 of Chen and Singh [27]).

Lemma 10.4. Let K be a nonempty, weakly, compact and convex subset of E . Let $T : K \rightarrow C(E)$ be non-expansive. If E satisfies the Opial's condition, then graph $(I - G)$ is closed in $E_w \times E$, where E_w is E with its weak topology and I is the identity mapping.

Proof. The conclusion follows by Theorem 1 of Chen and Singh [27]. \square

Remark 10.1. When a p -vector space E is with a p -norm, then both (H1) and (H) conditions for their convergence can be described by the convergence weakly, and strongly by the weak topology and strong topology induced by p -norm for $p \in (0, 1]$. Secondly, if a given p -vector space E has a non-empty open p -convex subset U containing zero, then any mapping satisfying the "(H) condition" is a hemicompact mapping (with respect P_U for a given bounded open p -convex subset U containing zero of p -vector space E), thus satisfying the "(H) condition" used in Theorem 5.1.

By the fact that each semiclosed 1-set mappings satisfy the "(H1) condition", we have the existence of fixed points for the class of semiclosed 1-set mappings. First as an application of Theorem 8.2, we have the following result for non-self mappings in p -seminorm spaces for $p \in (0, 1]$.

Theorem 10.1. Let U be a bounded open p -convex subset of a p -seminorm space E ($0 < p \leq 1$) the zero $0 \in U$. Assume $F : \bar{U} \rightarrow 2^E$ is a semiclosed 1-set contractive and upper semi-continuous mapping with non-empty closed p -convex values. In addition, for any $x \in \partial \bar{U}$ and $y \in F(x)$, we have $\lambda x \neq y$ for any $\lambda > 1$ (i.e., the "Leray-Schauder boundary condition"). Then F has at least one fixed point.

Proof. By the proof of Theorem 8.2 with $C = E$, we actually have the following either (I) or (II) holding:

(I) F has a fixed point $x_0 \in U$, i.e., $P_U(y_0 - x_0) = 0$,

(II) there exists $x_0 \in \partial(U)$ and $y_0 \in F(x_0)$ with $P_U(y_0 - x_0) = (P_U^{\frac{1}{p}}(y_0) - 1)^p > 0$.

If F has no fixed point, then above (II) holds and $x_0 \notin F(x_0)$. By the proof of Theorem 8.2, we have that $x_0 = f(y_0)$ and $y_0 \notin \bar{U}$. Thus $P_U(y_0) > 1$ and $x_0 = f(y_0) = \frac{y_0}{(P_U(y_0))^{\frac{1}{p}}}$, which means $y_0 = (P_U(y_0))^{\frac{1}{p}} x_0$, where

$(P_U(y_0))^{\frac{1}{p}} > 1$, this contradicts with the assumption. Thus F must have a fixed point. The proof is complete. \square

By following the idea used and developed by Browder [15], Li [66], Li et al.[67], Goebel and Kirk [41], Petryshyn [91]-[92], Tan and Yuan [115], Xu [126], Xu et al.[127] and references therein, we have the following existence theorems for the principle of Leray-Schauder type alternatives in p -seminorm spaces $(E, \|\cdot\|_p)$ for $p \in (0, 1]$.

Theorem 10.2. Let U be a bounded open p -convex subset of a p -seminorm space $(E, \|\cdot\|_p)$ ($0 < p \leq 1$) the zero $0 \in U$. Assume $F : \bar{U} \rightarrow 2^E$ is a semiclosed 1-set contractive and upper semi-continuous mapping with non-empty closed p -convex values. In addition, there exist $\alpha > 1$, $\beta \geq 0$, such that for each $x \in \partial\bar{U}$, we have that for any $y \in F(x)$, $\|y - x\|_p^{\alpha/p} \geq \|y\|_p^{(\alpha+\beta)/p} \|x\|_p^{-\beta/p} - \|x\|_p^{\alpha/p}$. Then F has at least one fixed point.

Proof. By assuming F has no fixed point, we prove the conclusion by showing the Leray-Schauder boundary condition in Theorem 10.1 does not hold. If we assume F has no fixed point, by the boundary condition of Theorem 10.1, there exist $x_0 \in \partial\bar{U}$, $y_0 \in F(x_0)$ and $\lambda_0 > 1$ such that $y_0 = \lambda_0 x_0$.

Now, consider the function f defined by $f(t) := (t-1)^\alpha - t^{\alpha+\beta} + 1$ for $t \geq 1$. We observe that f is a strictly decreasing function for $t \in [1, \infty)$ as the derivative of $f'(t) = \alpha(t-1)^{\alpha-1} - (\alpha+\beta)t^{\alpha+\beta-1} < 0$ by the differentiation, thus we have $t^{\alpha+\beta} - 1 > (t-1)^\alpha$ for $t \in (1, \infty)$. By combining the boundary condition, we have that $\|y_0 - x_0\|_p^{\alpha/p} = \|\lambda_0 x_0 - x_0\|_p^{\alpha/p} = (\lambda_0 - 1)^\alpha \|x_0\|_p^{\alpha/p} < (\lambda_0^{\alpha+\beta} - 1) \|x_0\|_p^{(\alpha+\beta)/p} \|x_0\|_p^{-\beta/p} = \|y_0\|_p^{(\alpha+\beta)/p} \|x_0\|_p^{-\beta/p} - \|x_0\|_p^{\alpha/p}$, which contradicts the boundary condition given by Theorem 10.2. Thus, the conclusion follows and the proof is complete.

\square

Theorem 10.3. Let U be a bounded open p -convex subset of a p -seminorm space $(E, \|\cdot\|_p)$ ($0 < p \leq 1$) the zero $0 \in U$. Assume $F : \bar{U} \rightarrow 2^E$ is a semiclosed 1-set contractive and upper semi-continuous mapping with non-empty closed p -convex values. In addition, there exist $\alpha > 1$, $\beta \geq 0$, such that for each $x \in \partial\bar{U}$, we have that for any $y \in F(x)$, $\|y + x\|_p^{(\alpha+\beta)/p} \leq \|y\|_p^{\alpha/p} \|x\|_p^{\beta/p} + \|x\|_p^{(\alpha+\beta)/p}$. Then F has at least one fixed point.

Proof. We prove the conclusion by showing the Leray-Schauder boundary condition in Theorem 10.1 does not hold. If we assume F has no fixed point, by the boundary condition of Theorem 10.1, there exist $x_0 \in \partial\bar{U}$, $y_0 \in F(x_0)$ and $\lambda_0 > 1$ such that $y_0 = \lambda_0 x_0$.

Now, consider the function f defined by $f(t) := (t+1)^{\alpha+\beta} - t^\alpha - 1$ for $t \geq 1$. We then can show that f is a strictly increasing function for $t \in [1, \infty)$, thus we have $t^\alpha + 1 < (t+1)^{\alpha+\beta}$ for $t \in (1, \infty)$. By the boundary condition given in Theorem 7.3, we have that

$$\|y_0 + x_0\|_p^{(\alpha+\beta)/p} = (\lambda_0 + 1)^{\alpha+\beta} \|x_0\|_p^{(\alpha+\beta)/p} > (\lambda_0^\alpha + 1) \|x_0\|_p^{(\alpha+\beta)/p} = \|y_0\|_p^{\alpha/p} \|x_0\|_p^{\beta/p} + \|x_0\|_p^{(\alpha+\beta)/p},$$

which contradicts the boundary condition given by Theorem 10.3. Thus, the conclusion follows and the proof is complete. \square

Theorem 10.4. Let U be a bounded open p -convex subset of a p -seminorm space $(E, \|\cdot\|_p)$ ($0 < p \leq 1$) the zero $0 \in U$. Assume $F : \bar{U} \rightarrow 2^E$ is a semiclosed 1-set contractive and upper semi-continuous mapping with non-empty closed p -convex values. In addition, there exist $\alpha > 1$, $\beta \geq 0$ (or alternatively, $\alpha > 1$, $\beta \geq 0$) such that for each $x \in \partial\bar{U}$, we have that for any $y \in F(x)$, $\|y - x\|_p^{\alpha/p} \|x\|_p^{\beta/p} \geq \|y\|_p^{\alpha/p} \|y + x\|_p^{\beta/p} - \|x\|_p^{(\alpha+\beta)/p}$. Then F has at least

one fixed point.

Proof. The same as above, we prove the conclusion by showing the Leray-Schauder boundary condition in Theorem 10.1 does not hold. If we assume F has no fixed point, by the boundary condition of Theorem 10.1, there exist $x_0 \in \partial\bar{U}$, $y_0 \in F(x_0)$ and $\lambda_0 > 1$ such that $y_0 = \lambda_0 x_0$.

Now, consider the function f defined by $f(t) := (t-1)^\alpha - t^\alpha(t-1)^\beta + 1$ for $t \geq 1$. We then can show that f is a strictly decreasing function for $t \in [1, \infty)$, thus we have $(t-1)^\alpha < t^\alpha(t+1)^\beta - 1$ for $t \in (1, \infty)$.

By the boundary condition given in Theorem 10.3, we have that

$$\|y_0 - x_0\|_p^{\alpha/p} \|x_0\|_p^{\beta/p} = (\lambda_0 - 1)^\alpha \|x_0\|_p^{(\alpha+\beta)/p} < (\lambda_0^\alpha (\lambda_0 + 1)^\beta - 1) \|x_0\|_p^{(\alpha+\beta)/p} = \|y_0\|_p^{\alpha/p} \|y_0 + x_0\|_p^{\beta/p} - \|x_0\|_p^{(\alpha+\beta)/p},$$

which contradicts the boundary condition given by Theorem 10.4. Thus, the conclusion follows and the proof is complete. \square

Theorem 10.5. Let U be a bounded open p -convex subset of a p -seminorm space $(E, \|\cdot\|_p)$ ($0 < p \leq 1$) the zero $0 \in U$. Assume $F : \bar{U} \rightarrow 2^E$ is a semiclosed 1-set contractive and upper semi-continuous mapping with non-empty closed p -convex values. In addition, there exist $\alpha > 1$, $\beta \geq 0$, we have that for any $y \in F(x)$, $\|y + x\|_p^{(\alpha+\beta)/p} \leq \|y - x\|_p^{\alpha/p} \|x\|_p^{\beta/p} + \|y\|_p^{\beta/p} \|x\|_p^{\alpha/p}$. Then F has at least one fixed point.

Proof. The same as above, we prove the conclusion by showing the Leray-Schauder boundary condition in Theorem 7.1 does not hold. If we assume F has no fixed point, by the boundary condition of Theorem 10.1, there exist $x_0 \in \partial\bar{U}$, $y_0 \in F(x_0)$ and $\lambda_0 > 1$ such that $y_0 = \lambda_0 x_0$.

Now, consider the function f defined by $f(t) := (t+1)^{\alpha+\beta} - (t-1)^\alpha - t^\beta$ for $t \geq 1$. We then can show that f is a strictly increasing function for $t \in [1, \infty)$, thus we have $(t+1)^{\alpha+\beta} > (t-1)^\alpha + t^\beta$ for $t \in (1, \infty)$.

By the boundary condition given in Theorem 10.3, we have that $\|y_0 + x_0\|_p^{(\alpha+\beta)/p} = (\lambda_0 + 1)^{\alpha+\beta} \|x_0\|_p^{(\alpha+\beta)/p} > ((\lambda_0 - 1)^\alpha + \lambda_0^\beta) \|x_0\|_p^{(\alpha+\beta)/p} = \|\lambda_0 x_0 - x_0\|_p^{\alpha/p} \|x_0\|_p^{\beta/p} + \|\lambda_0 x_0\|_p^{\beta/p} \|x_0\|_p^{\alpha/p} = \|y_0 - x_0\|_p^{\beta/p} \|x_0\|_p^{\alpha/p} + \|y_0\|_p^{\beta/p} \|x_0\|_p^{\alpha/p}$, which implies that

$$\|y_0 + x_0\|_p^{(\alpha+\beta)/p} > \|y_0 - x_0\|_p^{\beta/p} \|x_0\|_p^{\alpha/p} + \|y_0\|_p^{\beta/p} \|x_0\|_p^{\alpha/p},$$

this contradicts the boundary condition given by Theorem 10.5. Thus, the conclusion follows and the proof is complete. \square

As an application of Theorems 10.1 by testing the Leray-Schauder boundary condition, we have the following conclusion for each special case, and thus we omit their proofs in details here.

Corollary 10.1. Let U be a bounded open p -convex subset of a p -seminorm space $(E, \|\cdot\|_p)$ ($0 < p \leq 1$) the zero $0 \in U$. Assume $F : \bar{U} \rightarrow 2^E$ is a semiclosed 1-set contractive and upper semi-continuous mapping with non-empty closed p -convex values. Then F has at least one fixed point if one of the the following (strong) conditions holds for $x \in \partial\bar{U}$ and $y \in F(x)$:

- (i) $\|y\|_p \leq \|x\|_p$,
- (ii) $\|y\|_p \leq \|y - x\|_p$,
- (iii) $\|y + x\|_p \leq \|y\|_p$,

- (iv) $\|y + x\|_p \leq \|x\|_p$,
- (v) $\|y + x\|_p \leq \|y - x\|_p$,
- (vi) $\|y\|_p \cdot \|y + x\|_p \leq \|x\|_p^2$,
- (vii) $\|y\|_p \cdot \|y + x\|_p \leq \|y - x\|_p \cdot \|x\|_p$.

If the p -(semi)norm space E is a uniformly convex Banach space $(E, \|\cdot\|)$ (for p -norm space with $p = 1$), then we have the following general existence result which can apply to general non-expansive (single-valued) mappings, too.

Theorem 10.6. Let U be a bounded open convex subset of a uniformly convex Banach space $(E, \|\cdot\|)$ (with $p = 1$) with zero $0 \in U$. Assume $F : \overline{U} \rightarrow E$ is a semi-contractive and continuous (single-valued) mapping. In addition, for any $x \in \partial\overline{U}$, we have $\lambda x \neq F(x)$ for any $\lambda > 1$ (i.e., the ‘‘Leray-Schauder boundary condition’’). Then F has at least one fixed point.

Proof. By Lemma 10.1, F is a semiclosed 1-set contractive mapping. Moreover, by the assumption that E is a uniformly convex Banach, the mapping $(I - F)$ is closed at zero, and thus F is semiclosed at zero (see Browder [15], or Goebel and Kirk [41]). Thus all assumptions of Theorem 10.2 are satisfied. The conclusion follows by Theorem 10.2. The proof is completes. \square

Now we have the following results for nonexpansive set-valued mappings in a Banach space X with Opial’s condition.

Theorem 10.7. Let C is a nonempty convex weakly compact subset of a Banach space X which satisfies Opial’s condition and $0 \in \text{int}C$. Let $T : C \rightarrow K(X)$ be a nonexpansive set-valued mapping with non-empty compact convex values. In addition, for any $x \in \partial\overline{C}$, we have $\lambda x \neq F(x)$ for any $\lambda > 1$ (i.e., the ‘‘Leray-Schauder boundary condition’’). Then F has at least one fixed point.

Proof. As T is nonexpansive, it is 1-set contractive, By Lemma 10.2, it is then semi-contractive and continuous. Then all conditions of Theorem 10.1 are satisfied, the conclusion follows by Theorem 10.1, and the proof is complete. \square .

By using Lemma 10.4, we have the following result in local convex spaces for nonexpansive set-valued mappings.

Theorem 10.8. Let C is a nonempty convex weakly compact subset of a local convex space X which satisfies Opial’s condition and $0 \in \text{int}C$. Let $T : C \rightarrow K(X)$ be a nonexpansive set-valued mapping with non-empty compact convex values. In addition, for any $x \in \partial\overline{C}$, we have $\lambda x \neq F(x)$ for any $\lambda > 1$ (i.e., the ‘‘Leray-Schauder boundary condition’’). Then F has at least one fixed point.

Proof. As T is nonexpansive, it is 1-set contractive, By Lemma 10.4, it is then semi-contractive and continuous. Then all conditions of Theorem 10.1 are satisfied, then the conclusion follows by Theorem 10.1, and the proof is complete. \square .

By considering p -seminorm space $(E, \|\cdot\|)$ with a seminorm for $p = 1$, the following corollary is a special case of corresponding results from Theorem 10.2 to Theorem 10.5, and thus we omit its proof.

Corollary 10.2. Let U be a bounded open convex subset of a norm space $(E, \|\cdot\|)$. Assume $F : \bar{U} \rightarrow 2^E$ is a semiclosed 1-set contractive and upper semi-continuous mapping with non-empty closed p -convex values. Then F has at least one fixed point if there exist $\alpha > 1$, $\beta \geq 0$, such that any one of the following conditions satisfied

- (i) for each $x \in \partial\bar{U}$, and any $y \in F(x)$, $\|y - x\|^\alpha \geq \|y\|^{(\alpha+\beta)}\|x\|^{-\beta} - \|x\|^\alpha$,
- (ii) for each $x \in \partial\bar{U}$, and any $y \in F(x)$, $\|y + x\|^{(\alpha+\beta)} \leq \|y\|^\alpha\|x\|^\beta + \|x\|^{(\alpha+\beta)}$,
- (iii) for each $x \in \partial\bar{U}$, and any $y \in F(x)$, $\|y - x\|^\alpha\|x\|^\beta \geq \|y\|^\alpha\|y + x\|^\beta - \|x\|^{(\alpha+\beta)}$,
- (iv) for each $x \in \partial\bar{U}$, and any $y \in F(x)$, $\|y + x\|^{(\alpha+\beta)} \leq \|y - x\|^\alpha\|x\|^\beta + \|y\|^\beta\|x\|^\alpha$.

Remark 10.2. As discussed by Lemma 10.1 and the proof of Theorem 10.6, when the p -vector space is a uniformly convex Banach space, the semi-contractive or nonexpansive mappings automatically satisfy the conditions (see (H1)) required by Theorem 10.1, that is, the mappings are indeed semiclosed. Moreover, our results from Theorem 10.1 to Theorem 10.6, Corollary 10.1 and Corollary 10.2 also improve or unify corresponding results given by Browder [15], Li [66], Li et al.[67], Goebel and Kirk [41], Petryshyn [91]-[92], Reich [98], Tan and Yuan [115], Xu [123], Xu [126], Xu et al.[127], and results from the reference therein by extending the non-self mappings to the classes of semiclosed 1-set contractive set-valued mappings in p -seminorm spaces with $p \in (0,1]$ (including the norm space or Banach space when $p = 1$ for p -seminorm spaces).

Before the ending of this paper, we like to share with readers that the main goal of this paper is to develop some new results and tools in the nature way for the category of nonlinear analysis for 1-set contractive mappings under the general framework of p -vector spaces (where, $(0 < p \leq 1)$), and we expect that they are expected to become useful tools for the study on optimization, nonlinear programming, variational inequality, complementarity, game theory, mathematical economics, and related other social science area. In particular, we first establish one best approximation, acting as a tool to establish the principle of nonlinear alternative, which then allows us to give general principle of nonlinear alternative under for 1-set contractive mappings.

Finally, like what mentioned in the beginning of this paper, we do expect that nonlinear results and principles of the best approximation theorem established in this paper would play a very important role for the nonlinear analysis under the general framework of p -vector spaces for $(0 < p \leq 1)$, as shown by those results give in Section 5 and Section 6 above for the fixed point theorems of non-self mappings, principle of nonlinear alternative, Rothe type, Leray-Schauder alternative which do not include corresponding results in the existing literature as special cases, but also would be important tools for the study of optimization, nonlinear programming, variational inequality, complementarity, game theory, mathematical economics and related topics and area forthcoming; and finally in Section 7, by considering p -(semi)norm spaces for $p \in (0,1]$, as application of best approximation, we unify and improve the corresponding results in the existing literature under the general framework of p -vector spaces.

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Compliance with Ethical Standards

The author declares that he has no conflict of interest.

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