Centers and limit cycles for a generalized kukles differential systems of degree 8

Loubna Damene

Universite de Bordj Bou Arreridj

Rebiha Benterki (✉ r.benterki@univ-bba.dz)

Universite de Bordj Bou Arreridj <https://orcid.org/0000-0001-6745-2747>

Original Research

Keywords: Phase portrait, Cyclicity, Limit cycle, Averaging method

Posted Date: February 5th, 2021

DOI: <https://doi.org/10.21203/rs.3.rs-170528/v1>

License: This work is licensed under a Creative Commons Attribution 4.0 International License.

[Read Full License](#)

Centers and limit cycles for a generalized kukles differential systems of degree 8

Damene Loubna · Rebiha Benterki

Received: date / Accepted: date

Abstract In this paper we provide all the global phase portraits of the generalized kukles differential systems

$$\dot{x} = -y, \quad \dot{y} = x + ax^8 + bx^6y^2 + cx^4y^4 + dx^2y^6 + ey^8,$$

symmetric with respect to the x -axis, with $a^2 + b^2 + c^2 + d^2 + e^2 \neq 0$, and by using the averaging theory up to seven order, we give the upper bounds of limit cycles which can bifurcate from its center when we perturb it inside the class of all polynomial differential systems of degree 8.

The main tool used for proving these results is based in the first integrals of the systems which form the discontinuous piecewise differential systems.

Keywords Phase portrait · Cyclicity · Limit cycle · Averaging method

Mathematics Subject Classification (2010) Primary 34A36 · 34C25 · 34A26

1 Introduction

In our work we are interesting in planar differential systems of the form

$$\dot{x} = -y, \quad \dot{y} = x + Q_n(x, y), \tag{1}$$

where Q_n is a real homogeneous polynomial of degree n . These kind of systems are called *Kukles homogeneous differential systems*. In the literature we find many works studying kukles differential systems.

Damene Loubna
Département de Mathématiques, Université Mohamed El Bachir El Ibrahimi, Bordj Bou Arréridj 34000, El Anasser, Algeria
E-mail: loubna.damene@univ-bba.dz

Rebiha Benterki (Corresponding author)
Département de Mathématiques, Université Mohamed El Bachir El Ibrahimi, Bordj Bou Arréridj 34000, El Anasser, Algeria
E-mail: r.benterki@univ-bba.dz

Volokitin and Ivanov [14] in 1999 were the first who studied the problem of center-focus for Kukles differential systems (1) for $n \geq 2$, where they conjectured that the origin of these systems is a center if and only if systems (1) are symmetric with respect to one of the coordinates axes, and they proved this conjecture for $n = 2$ and $n = 3$. In 2002 Giné [7] proved this conjecture for $n = 4$ and $n = 5$. In 2015 Giné et al [8] proved that the conjecture is true for all $n \geq 2$.

For the global phase portraits of Kukles differential systems (1) we can cite the following works. Vulpe [15] studied the global phase portraits for all center quadratic differential systems and since systems (1) for $n = 2$ is a particular case of these systems, we know that their phase portraits are studied. Buzzi et al. [3], Malkin [13], Vulpe and Sibirskii [16] and Żołądek [17, 18] classified the global phase portraits of cubic polynomial differential systems with a symmetry with respect to a straight line. Benterki and Llibre [1] provided the global phase portraits of systems (1) for $n = 4$.

Llibre and Silva [11, 12] classified the phase portraits of the systems (1) with $n = 5, 6$, and the global phase portraits for the case $n = 7$ it was studied by Benterki and Llibre [2].

The first objective of this paper is to classify all the global phase portraits of the generalized Kukles differential systems of degree 8 symmetric with respect to the x -axis

$$\dot{x} = -y, \quad \dot{y} = x + ax^8 + bx^6y^2 + cx^4y^4 + dx^2y^6 + ey^8. \quad (2)$$

The second objective is to solve the second part of the Hilbert 16th problem for systems (2), when we perturb them inside the class of all polynomial differential systems of degree 8.

Remark 1 The Kukles differential systems (2) are invariant under the transformation $(x, y, t, a, b, c, d, e) \rightarrow (-x, y, -t, -a, -b, -c, -d, -e)$, then we only need to study the system for $e > 0$, $e = 0$ and $d > 0$, $e = d = 0$ and $c > 0$, $e = d = c = 0$ and $b > 0$, or $e = d = c = b = 0$ and $a \neq 0$.

We know that to study the global phase portraits of such differential system, we have to compute its equilibrium points, for this reason we need to know the real roots of the polynomial $P(x, y) = ax^8 + bx^6y^2 + cx^4y^4 + dx^2y^6 + ey^8$. By doing the following change of variables $X = x^2$ and $Y = y^2$ the polynomial $P(x, y)$ becomes

$$P(X, Y) = aX^4 + bX^2Y^3 + cX^2Y^2 + dXY^3 + eY^4,$$

which is a polynomial of degree four, so it has many different kind of roots which are summarized in the following 69 cases.

If the polynomial $P(x, y)$ has four simple real roots, so $P(x, y) = e(y^2 - r_1x^2)(y^2 - r_2x^2)(y^2 - r_3x^2)(y^2 - r_4x^2)$, and we distinguish the 9 subcases

- (1) $0 < r_1 < r_2 < r_3 < r_4$,
- (2) $r_1 = 0 < r_2 < r_3 < r_4$,
- (3) $r_1 < 0 < r_2 < r_3 < r_4$,
- (4) $r_1 < r_2 = 0 < r_3 < r_4$,
- (5) $r_1 < r_2 < 0 < r_3 < r_4$,
- (6) $r_1 < r_2 < r_3 = 0 < r_4$,
- (7) $r_1 < r_2 < r_3 < 0 < r_4$,
- (8) $0 < r_1 < r_2 < r_3 < r_4 = 0$,

(9) $r_1 < r_2 < r_3 < r_4 < 0$,

If the polynomial $P(x, y)$ has three real roots two simples and one double, so $P(x, y) = e(y^2 - r_1x^2)(y^2 - r_2x^2)(y^2 - r_3x^2)^2$, and we distinguish the 7 subcases

(10) $0 < r_1 < r_2 < r_3$,

(11) $r_1 = 0 < r_2 < r_3$,

(12) $r_1 < 0 < r_2 < r_3$,

(13) $r_1 < r_2 = 0 < r_3$,

(14) $r_1 < r_2 < 0 < r_3$,

(15) $r_1 < r_2 < r_3 = 0$,

(16) $r_1 < r_2 < r_3 < 0$,

If the polynomial $P(x, y)$ has two double real roots, so $P(x, y) = e(y^2 - r_1x^2)^2(y^2 - r_2x^2)^2$, and we distinguish the 5 subcases

(17) $0 < r_1 < r_2$,

(18) $r_1 = 0 < r_2$,

(19) $r_1 < 0 < r_2$,

(20) $r_1 < r_2 = 0$,

(21) $r_1 < r_2 < 0$,

If the polynomial $P(x, y)$ has one simple and one triple real roots, so $P(x, y) = e(y^2 - r_1x^2)(y^2 - r_2x^2)^3$, and we distinguish the 5 subcases

(22) $0 < r_1 < r_2$,

(23) $r_1 = 0 < r_2$,

(24) $r_1 < 0 < r_2$,

(25) $r_1 < r_2 = 0$,

(26) $r_1 < r_2 < 0$,

If the polynomial $P(x, y)$ has one real root order four, so $P(x, y) = e(y^2 - r_1x^2)^4$, and we distinguish the 3 subcases

(27) $0 < r_1$,

(28) $r_1 = 0$,

(29) $r_1 < 0$,

If the polynomial $P(x, y)$ has two simple real roots and two complexes, so $P(x, y) = e(y^2 - r_1x^2)(y^2 - r_2x^2)(y^4 - 2\alpha x^2y^2 + (\alpha^2 + \beta^2)x^4)$, and we distinguish the 5 subcases

(30) $0 < r_1 < r_2$,

(31) $r_1 = 0 < r_2$,

(32) $r_1 < 0 < r_2$,

(33) $r_1 < r_2 = 0$,

(34) $r_1 < r_2 < 0$,

If the polynomial $P(x, y)$ has one double real root and two complexes, so $P(x, y) = e(y^2 - r_1x^2)^2(y^4 - 2\alpha x^2y^2 + (\alpha^2 + \beta^2)x^4)$, and we distinguish the 3 subcases

(35) $0 < r_1$,

(36) $r_1 = 0$,

(37) $r_1 < 0$,

If the polynomial $P(x, y)$ has no real root, so $P(x, y) = e(y^4 - 2\alpha x^2y^2 + (\alpha^2 + \beta^2)x^4)^2$, and we have to study the case

(38) $e > 0$,

If $e = 0$ and the polynomial $P(x, y)$ has three simple real roots, so $P(x, y) = dx^2(y^2 - r_1x^2)(y^2 - r_2x^2)(y^2 - r_3x^2)$, and we distinguish the 7 subcase

(39) $0 < r_1 < r_2 < r_3$,

(40) $r_1 = 0 < r_2 < r_3$,

$$(41) \ r_1 < 0 < r_2 < r_3,$$

$$(42) \ r_1 < r_2 = 0 < r_3,$$

$$(43) \ r_1 < r_2 < 0 < r_3,$$

$$(44) \ r_1 < r_2 < r_3 = 0,$$

$$(45) \ r_1 < r_2 < r_3 < 0,$$

If $e = 0$ and the polynomial $P(x, y)$ has two real roots one simple and one double, so $P(x, y) = dx^2(y^2 - r_1x^2)(y^2 - r_2x^2)^2$, and we distinguish the 5 subcase

$$(46) \ 0 < r_1 < r_2,$$

$$(47) \ r_1 = 0 < r_2,$$

$$(48) \ r_1 < 0 < r_2,$$

$$(49) \ r_1 < r_2 = 0,$$

$$(50) \ r_1 < r_2 < 0,$$

If $e = 0$ and the polynomial $P(x, y)$ has one triple real root, so $P(x, y) = dx^2(y^2 - r_1x^2)^3$, and we distinguish the 3 subcase

$$(51) \ 0 < r_1,$$

$$(52) \ r_1 = 0,$$

$$(53) \ r_1 < 0,$$

If $e = 0$ and the polynomial $P(x, y)$ has one simple real root and two complexes, so $P(x, y) = dx^2(y^2 - r_1x^2)(y^4 - 2\alpha x^2y^2 + (\alpha^2 + \beta^2)x^4)$, and we distinguish the 3 subcase

$$(54) \ 0 < r_1,$$

$$(55) \ r_1 = 0,$$

$$(56) \ r_1 < 0,$$

If $e = 0, d = 0$ and the polynomial $P(x, y)$ has two simple real roots, so $P(x, y) = cx^4(y^2 - r_1x^2)(y^2 - r_2x^2)$, and we distinguish the 5 subcase

$$(57) \ 0 < r_1 < r_2,$$

$$(58) \ r_1 = 0 < r_2,$$

$$(59) \ r_1 < 0 < r_2,$$

$$(60) \ r_1 < r_2 = 0,$$

$$(61) \ r_1 < r_2 < 0,$$

If $e = 0, d = 0$ and the polynomial $P(x, y)$ has one double real root, so $P(x, y) = cx^4(y^2 - r_1x^2)^2$, and we distinguish the 3 subcase

$$(62) \ 0 < r_1,$$

$$(63) \ r_1 = 0,$$

$$(64) \ r_1 < 0,$$

If $e = 0, d = 0$ and the polynomial $P(x, y)$ has two complexes roots, so $P(x, y) = cx^4(y^4 - 2\alpha x^2y^2 + (\alpha^2 + \beta^2)x^4)$, then we have the following subcase

$$(65) \ c > 0,$$

If $e = 0, d = 0$ and $c = 0$, the polynomial $P(x, y)$ can have one simple real root, so $P(x, y) = bx^6(y^2 - r_1x^2)$, then we have the following 3 subcases

$$(66) \ 0 < r_1,$$

$$(67) \ r_1 = 0,$$

$$(68) \ r_1 < 0,$$

If $e = 0, d = 0, c = 0$ and $b = 0$, so $P(x, y) = ax^8$, we have the following subcase

$$(69) \ a \neq 0.$$

Remark 2 Some cases of the previous ones has more then one configuration, according to the position of their separatrices:

- the case (7) has two different configurations (7.1) and (7.2);

- the case (11) has two different configurations (11.1) and (11.2);
- the case (12) has two different configurations (12.1) and (12.2);
- the case (13) has two different configurations (13.1) and (13.2);
- the case (17) has two different configurations (17.1) and (17.2);
- the case (22) has two different configurations (22.1) and (22.2);
- the case (30) has two different configurations (30.1) and (30.2);
- the case (31) has two different configurations (31.1) and (31.2);
- the case (32) has two different configurations (32.1) and (32.2);
- the case (39) has two different configurations (39.1) and (39.2);
- the case (40) has two different configurations (40.1) and (40.2);
- the case (41) has two different configurations (41.1) and (41.2);
- the case (43) has three different configurations (43.1), (43.2) and (43.3);
- the case (51) has four different configurations (51.1), (51.2), (51.3) and (51.4);
- the case (57) has two different configurations (57.1) and (57.2);
- the case (58) has two different configurations (58.1) and (58.2);
- the case (59) has two different configurations (59.1) and (59.2);
- the case (60) has two different configurations (60.1) and (60.2);
- the case (62) has two different configurations (62.1) and (62.2).

Our first main result is the following

Theorem 1 *Suppose that $a^2 + b^2 + c^2 + d^2 + e^2 \neq 0$. Then the polynomial differential systems (2) have 38 topologically non-equivalent phase portraits in the Poincaré disc. More precisely, the phase portrait in the Poincaré disc of Figure*

- 1 is realizable by the case (1);
- 2 is realizable by the case (2);
- 3 is realizable by the cases (3) and (39.1);
- 4 is realizable by the case (4) and (40.1);
- 5 is realizable by the cases (5), (30.1), (41.1) and (57.1);
- 6 is realizable by the cases (6), (31.1) and (58.2);
- 7 is realizable by the cases (7.1), (32.2), (43.2), (51.2) and (59.1);
- 8 is realizable by the cases (7.2), (32.1), (43.1), (54) and (59.2);
- 9 is realizable by the case (8), (15), (20), (44) and (60.1);
- 10 is realizable by the case (9), (16), (21), (26), (29), (34), (37), (38), (45), (50), (53), (56), (61), (64), (65), (68) and (69);
- 11 is realizable by the cases (10);
- 12 is realizable by the cases (11.1);
- 13 is realizable by the cases (11.2);
- 14 is realizable by the cases (12.1);
- 15 is realizable by the cases (12.2) and (46);
- 16 is realizable by the cases (13.1), (18) and (47);
- 17 is realizable by the cases (13.2);
- 18 is realizable by the cases (14), (19), (27), (35), (48) and (62.2);
- 19 is realizable by the case (17.1);
- 20 is realizable by the case (17.2);
- 21 is realizable by the case (22.1);
- 22 is realizable by the case (22.2);
- 23 is realizable by the case (23);
- 24 is realizable by the case (24), and (51.1);

25 is realizable by the case (25) and (49);
 26 is realizable by the cases (28), (33), (36), (52), (55), (60.2), (63) and (67);
 27 is realizable by the case (30.2);
 28 is realizable by the case (31.2);
 29 is realizable by the cases (39.2);
 30 is realizable by the cases (40.2);
 31 is realizable by the cases (40.3);
 32 is realizable by the case (41.2);
 33 is realizable by the cases (42) and (58.1);
 34 is realizable by the cases (51.3);
 35 is realizable by the cases (51.4) and (43.3);
 36 is realizable by the cases (57.2);
 37 is realizable by the cases (62.1);
 38 is realizable by the cases (66).

Theorem 1 is proved in section 3.

Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6

In subsection 3 we are interesting to provide the number of limit cycles of a differential system (2) when we perturb it inside the class of all polynomial differential systems of degree 8. More precisely we will use the averaging theory up to seven order for studying the number of limit cycles of the following systems

$$\begin{aligned}
 \dot{x} &= -y + \sum_{s=1}^7 \varepsilon^s \sum_{0 \leq i+j \leq 8} a_{ij}^s x^i y^j, \\
 \dot{y} &= x + ax^8 + bx^6 y^2 + cx^4 y^4 + dx^2 y^6 + ey^8 + \sum_{s=1}^7 \varepsilon^s \sum_{0 \leq i+j \leq 8} b_{ij}^s x^i y^j,
 \end{aligned} \tag{3}$$

Fig. 7

Fig. 8

Fig. 9

Fig. 10

Fig. 11

Fig. 12

Fig. 13

Fig. 14

Fig. 15

Fig. 16

Fig. 17

Fig. 18

where a_{ij}^s and b_{ij}^s are real parameters, for $0 \leq i, j \leq 8$ and $1 \leq s \leq 7$.

We define the average function $f_j(r)$ of order j for $j = 1, \dots, 7$ of the differential system (3) in subsection 2.2.

Theorem 2 Assume that $f_j = 0$ for $j = 1, \dots, k-1$ and $f_k \neq 0$. Then if $\bar{r}$ is a simple zero of f_k , the small amplitude limit cycle $(x(t, \varepsilon), y(t, \varepsilon))$ associated to this zero is of the form $(x(t, \varepsilon), y(t, \varepsilon)) = \varepsilon(\bar{r} \cos t, \bar{r} \sin t) + O(\varepsilon^2)$. Moreover, for $|\varepsilon| \neq 0$ sufficiently small the maximum number of small amplitude limit cycles of

Fig. 19

Fig. 20

Fig. 21

Fig. 22

Fig. 23

Fig. 24

Fig. 25

Fig. 26

Fig. 27

Fig. 28

Fig. 29

Fig. 30

the differential system (3) bifurcating from the periodic solutions of the center (2), is

- (a) 0 if the first order average function f_1 is non-zero,
- (b) 0 if $f_1 = 0$ and the second order average function f_2 is non-zero;
- (c) 1 if $f_1 = f_2 = 0$ and the third order average function f_3 is non-zero, where $\bar{r}_1 = \sqrt{-M_0/M_2}$ with $M_0 M_2 < 0$;
- (d) 1 if $f_1 = f_2 = f_3 = 0$ and the fourth order average function f_4 is non-zero, where $\bar{r}_1 = \sqrt{-N_0/N_2}$ with $N_0 N_2 < 0$;

Fig. 31

Fig. 32

Fig. 33

Fig. 34

Fig. 35

Fig. 36

Fig. 37

Fig. 38

- (e) 2 if $f_1 = f_2 = f_3 = f_4 = 0$ and the fifth order average function f_5 is non-zero, when the equation $H_4r^4 + H_2r^2 + H_0 = 0$ has four simple solutions;
- (f) 2 if $f_1 = f_2 = f_3 = f_4 = f_5 = 0$ and the sixth order average function f_6 is non-zero, when the equation $T_4r^4 + T_2r^2 + T_0 = 0$ has four simple solutions.
- (g) 3 if $f_1 = f_2 = f_3 = f_4 = f_5 = f_6 = 0$ and the seventh order average function f_7 is non-zero, when the equation $G_6r^6 + G_4r^4 + G_2r^2 + G_0 = 0$ has six simple solutions and this result is reached by an Example with exactly 3 limit cycles.

The values of the coefficients $M_0, M_2, N_0, N_2, H_0, H_2, H_4, T_0, T_2, T_4$ are given in section 4.

Theorem 2 is proved in section 4.

2 Preliminaries and basic results

In this section we present some basic results and notations which are necessary for proving our results.

2.1 Singular points

As usual we classify the singular points of a planar differential system in *hyperbolic*, *semi-hyperbolic*, *nilpotent* and *linearly zero*. The *hyperbolic* ones are the singular points such that its linear part of the differential system has eigenvalues with nonzero real part see for instance Theorem 2.15 of [4] for the classification of their local phase portrait. The *semi-hyperbolic* are the ones having a unique eigenvalue equal to zero, their phase portraits are characterized in Theorem 2.19 of [4]. The *nilpotent* singular points have both eigenvalues zero but their linear part is not identically zero. Finally the *linearly zero* singular points are the ones such that their linear part is identically zero, and their local phase portraits must be studied using the change of variables called Blow-ups, see for instance chapter 2 and 3 of [4].

2.2 The averaging theory up to order 7

To study the limit cycles that can bifurcate from the periodic orbits of center, we can use one of the three following techniques; Abelian integral; averaging theory; and Melnikov function, which produce the same results in the plane. In this paper we use the averaging theory up to seven order to study the number of limit cycles that can have the differential systems (3).

We consider the differential system

$$\dot{x}(t) = \sum_{i=0}^k \varepsilon^i F_i(t, x) + \varepsilon^{k+1} R(t, x, \varepsilon), \quad (4)$$

where the functions $F_i : \mathbb{R} \times D \rightarrow \mathbb{R}$ for $i = 0, 1, \dots, k$ and $R : \mathbb{R} \times D \times (-\varepsilon_0, \varepsilon_0) \rightarrow \mathbb{R}$ are T -periodic in the first variable and continuous functions, D is an open interval of $\mathbb{R}^2$, and ε a small parameter. Following the results of [10] we define the functions $y_j(t, z)$ for $j = 1, 2, 3, 4, 5$ related to system (12).

In [?,?] it was proved that these averaged functions are

$$f_i(z) = \frac{y_i(2\pi, z)}{i!}, \quad (5)$$

where $y_i : \mathbb{R} \times D \rightarrow \mathbb{R}$ for $i = 1, 2, \dots, k$, are defined recurrently by the following integral equations

$$\begin{aligned} y_1(t, z) &= \int_0^t F_1(s, \varphi(s, z)) ds, \\ y_i(t, z) &= i! \int_0^t \left(F_i(s, \varphi(s, z)) + \sum_{l=1}^i \sum_{S_l} \frac{1}{b_1! b_2! 2!^{b_2} \dots b_l! l!^{b_l}} \right. \\ &\quad \left. \cdot \partial^L F_{i-l}(s, \varphi(s, z)) \prod_{j=1}^l y_j(s, z)^{b_j} \right) ds, \quad \text{for } i = 2, \dots, k. \end{aligned}$$

Here $\partial^L G(\phi, z)$ denotes the derivative order L of a function G with respect to the variable z , and S_l is the set of all l -tuples of non-negative integers $(b_1, b_2, \dots, b_l)$

satisfying $b_1 + 2b_2 + \dots + lb_l = l$, and $L = b_1 + b_2 + \dots + b_l$. The averaging theory for studying the periodic solutions of a non-autonomous differential system (12) works as follows, see [10] for more details. Suppose that the average functions $f_j(z) = 0$ for $j = 1, \dots, k-1$ and $f_k(z) \neq 0$ for some $k \geq 1$, we assume that $f_0(z) = 0$. By [10] if $\bar{z}$ is a simple zero of $f_k(z)$, then there is a limit cycle $r(\theta, \varepsilon)$ of system (12) such that $r(0, \varepsilon) = \bar{z} + O(\varepsilon)$.

3 Phase portraits in the Poincaré disc

In this section we are interesting to study the phase portraits in the Poincaré disc for the polynomial differential systems (2) when $(a, b, c, d, e) \neq (0, 0, 0, 0, 0)$.

We will characterize all the finite and infinite singular points with their local phase portraits, then we use the symmetry of the solutions with respect to the x -axis, we get their phase portraits in the Poincaré disc.

3.1 Finite equilibrium points

For a planar polynomial differential system (2) its finite equilibrium points are characterized in the following proposition.

Proposition 1 *The differential system (2) has a center at the origin of coordinates. In addition when $a < 0$ it has a hyperbolic saddle at $(-a^{-1/7}, 0)$.*

Proof The eigenvalues of the linear part of system (2) at the origin are $\pm i$ so the origin is either a focus or a center, but due to the fact that the system is symmetric with respect to x -axis, it is a center. When $a < 0$, the system has in addition to the origin the equilibrium point $(-a^{-1/7}, 0)$ with eigenvalues $\pm\sqrt{7}$. By using the Theorem 2.15 of [4] it is a saddle.

3.2 Infinite equilibrium points

In this subsection and by using the definition of subsection 2.2 we perform the analysis of the local phase portraits at the infinite singular points in the Poincaré disc of system (2)

In the local chart U_1 . The differential system (2) in the local chart U_1 is

$$\dot{u} = a + bu^2 + cu^4 + du^6 + eu^8 + v^7 + u^2v^7, \quad \dot{v} = uv^8. \quad (6)$$

Any arbitrary infinite singular point of system (6) takes the form $(u_0, 0)$ with $u_0 = \pm\sqrt{r_j}/r_j > 0$ is a root of the polynomial $p(u) = (a + bu + cu^2 + du^3 + eu^4)$ and r_j 's are the ones described in section 1. If $p(u_0) = 0$ then the Jacobian matrix of system (6) evaluated at u_0

$$\begin{pmatrix} p'(u_0) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Which means that all infinite singularities are either sem-hyperbolic or linearly zero.

Due to the symmetry of (6) with respect to the x -axis, in the next proposition we only provide the local phase portrait of the infinite equilibrium point $(\sqrt{r_j}, 0)$ with $r_j \geq 0$, because due to the symmetry $(x, y, t) \rightarrow (x, -y, -t)$ of system (2) with respect to x -axis the local phase portrait at the infinite equilibrium point $(-\sqrt{r_j}, 0)$ is the same at the equilibrium point $(\sqrt{r_j}, 0)$.

Proposition 2

The local phase portraits at the infinite equilibrium points $(\sqrt{r_j}, 0)$ with $r_j \geq 0$ of the local chart U_1 are

- (a) *four semi-hyperbolic saddle-nodes in case (1);*
- (b) *a linearly zero at the origin of coordinates with four hyperbolic sectors, and three semi-hyperbolic saddle-node in case (2);*
- (c) *Three semi-hyperbolic saddle-nodes in cases (3) and (39);*
- (d) *a linearly zero at the origin of coordinates with four hyperbolic sectors, and two semi-hyperbolic saddle-nodes in cases (4) and (40);*
- (e) *two semi-hyperbolic saddle-nodes in cases (5), (41) and (57);*
- (f) *a linearly zero at the origin of coordinates with four hyperbolic sectors, and one semi-hyperbolic saddle-node in cases (6), (42) and (58);*
- (g) *two semi-hyperbolic saddle-nodes in cases (7), (43), (54), (59) and (66);*
- (h) *a linearly zero at the origin of coordinates with one hyperbolic sector, two parabolic sectors and one elliptic sector in cases (8), (15), (20), (44) and (60);*
- (i) *no infinite equilibria in the local chart U_1 in cases (9), (16), (21), (26), (29), (37), (38), (45), (50), (53), (55), (61), (64), (65), (68) and (69);*
- (j) *a linearly zero with two hyperbolic sectors and one parabolic sector, two semi-hyperbolic saddle-nodes in cases (10);*
- (k) *a semi-hyperbolic saddle-node and a linearly zero with two hyperbolic sectors and one parabolic sector, a linearly zero at the origin of coordinates with four hyperbolic sectors in case (11);*
- (l) *a linearly zero with two hyperbolic sectors and one parabolic sector, a semi-hyperbolic saddle-node in cases (12), (22) and (46);*
- (m) *a linearly zero with two hyperbolic sectors and one parabolic sector, another linearly zero at the origin of coordinates with four hyperbolic sectors in cases (13), (18), (23) and (47);*
- (n) *a linearly zero with two hyperbolic sectors and one parabolic sector in cases (14), (19), (24), (48) and (51);*
- (o) *two linearly zero singularities with two hyperbolic sectors and one parabolic sector in cases (17), (27) and (35);*
- (p) *linearly zero at the origin of coordinates with two hyperbolic sectors and one parabolic sector in cases (25) and (47);*
- (q) *a linearly zero at the origin of coordinates with four hyperbolic sectors in cases (28), (36), (52), (55), (63) and (67);*
- (r) *a linearly zero with four hyperbolic sectors in case (62)*

Proof We will prove with all details the statements (a) and (b), and by the similar way we prove the other statements.

In statements (a) and (b) system (1) writes as

$$\dot{x} = -y; \quad \dot{y} = x + e(y^2 - r_1x^2)(y^2 - r_2x^2)(y^2 - r_3x^2)(y^2 - r_4x^2), \quad (7)$$

with $e > 0$. This system in the local chart U_1 becomes

$$\begin{aligned}\dot{u} &= er_1r_2r_3r_4 - e(r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4)u^2 + e(r_1r_2 + r_1r_3 \\ &\quad + r_2r_3 + r_1r_4 + r_2r_4 + r_3r_4)u^4 - e(r_1 + r_2 + r_3 + r_4)u^6 + eu^8 \\ &\quad + v^7 + u^2v^7, \\ \dot{v} &= uv^8.\end{aligned}\tag{8}$$

Assume $0 < r_1 < r_2 < r_3 < r_4$. Then the eigenvalues at the infinite equilibrium point $(\sqrt{r_j}, 0)$ for $j = 1, 2, 3, 4$ are 0 and $h'(\sqrt{r_j}) = e\sqrt{r_j}(-6r_j^2(r_1 + r_2 + r_3 + r_4) + 4r_j(r_1(r_2 + r_3 + r_4) + r_2(r_3 + r_4) + r_3r_4) - 2r_1r_2r_3 - 2r_1r_2r_4 - 2r_1r_3r_4 - 2r_2r_3r_4 + 8r_j^3) \neq 0$, because the four roots $\sqrt{r_j}$ for $j = 1, 2, 3, 4$ of the polynomial $h(u) = \dot{u}|_{v=0} = er_1r_2r_3r_4 - e(r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4)u^2 + e(r_1r_2 + r_1r_3 + r_2r_3 + r_1r_4 + r_2r_4 + r_3r_4)u^4 - e(r_1 + r_2 + r_3 + r_4)u^6 + eu^8$ are simple. The points $(\sqrt{r_j}, 0)$ for $j = 1, 2, 3, 4$ are semi-hyperbolic equilibria. In order to know the local phase portraits of the infinite equilibria $(\sqrt{r_j}, 0)$ for $j = 1, 2, 3, 4$ we apply Theorem 2.19 of [4]. First we translate the equilibrium point $(\sqrt{r_j}, 0)$ to the origin of coordinates doing the change $(u, v) = (Y + \sqrt{r_j}, X)$. Thus we obtain the differential system

$$\begin{aligned}\dot{X} &= A(X, Y) = \text{sign}(h'(\sqrt{r_j}))(\sqrt{r_j}X^8 + X^8Y), \\ \dot{Y} &= \text{sign}(h'(\sqrt{r_j}))Y + B(X, Y),\end{aligned}\tag{9}$$

where

$$\begin{aligned}B(X, Y) &= +8e\sqrt{r_j}Y^7 + eY^8 - e(r_1 + r_2 + r_3 + r_4 - 28r_j)Y^6 - 2e(3r_1 + 3r_2 + 3r_3 \\ &\quad + 3r_4 - 28r_j)\sqrt{r_j}Y^5 + (-e(r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4 - 6r_1r_2r_j \\ &\quad - 6r_1r_3r_j - 6r_2r_3r_j - 6r_1r_4r_j - 6r_2r_4r_j - 6r_3r_4r_j + 15r_1r_j^2 + 15r_2r_j^2 \\ &\quad + 15r_3r_j^2 + 15r_4r_j^2 - 28r_j^3) + X^7)Y^2 + 4e\sqrt{r_j}(r_1r_2 + r_1r_3 + r_2r_3 + r_1r_4 \\ &\quad + r_2r_4 + r_3r_4 - 5r_1r_j - 5r_2r_j - 5r_3r_j - 5r_4r_j + 14r_j^2)Y^3 + e(r_1r_2 + r_1r_3 \\ &\quad + r_2r_3 + r_1r_4 + r_2r_4 + r_3r_4 - 15r_1r_j - 15r_2r_j - 15r_3r_j - 15r_4r_j + 70r_j^2)Y^4 \\ &\quad + (1 + r_j)X^7 + 2\sqrt{r_j}X^7Y.\end{aligned}$$

Applying Theorem 2.19. we get the following expressions of the functions $f(X)$ and $g(X)$

$$\begin{aligned}f(X) &= -\frac{1 + r_j}{h'(\sqrt{r_j})}X^7 + \text{h.o.t.}, \\ g(X) &= \text{sign}(h'(\sqrt{r_j}))\sqrt{r_j}X^8 + \text{h.o.t.},\end{aligned}$$

where h.o.t. denotes higher order terms. So using the notation of Theorem 2.19 we have that $a_m = \text{sign}(h'(\sqrt{r_j}))\sqrt{r_j}$ and $m = 8$ is even. Then, by using Theorem 2.19 we get that the equilibria $(\sqrt{r_j}, 0)$ for $j = 1, 4$ are saddle-nodes. This completes the proof of statement (a).

Assume $r_1 = 0 < r_2 < r_3 < r_4$. To prove that the equilibria $(\sqrt{r_2}, 0), (\sqrt{r_3}, 0)$ and $(\sqrt{r_4}, 0)$ are semi-hyperbolic saddle-node we use the same steps as the proof of the statement (a). Then we shall show that the local phase portrait of the equilibrium $(\sqrt{r_1}, 0) = (0, 0)$ is constituted by four hyperbolic sectors. After doing the blow-up change of variables $v = wu$ to system (8) and by eliminating the

common factor u of $\dot{u}$ and $\dot{w}$ by doing the rescaling $d\tau = udt$ in the independent variable, we get the differential system

$$\begin{aligned} u' &= -er_2r_3r_4u + er_2r_3u^3 + er_2r_4u^3 + er_3r_4u^3 - er_2u^5 \\ &\quad - er_3u^5 - er_4u^5 + eu^7 + u^6w^7 + u^8w^7, \\ w' &= er_2r_3r_4w - er_2r_3u^2w - er_2r_4u^2w - er_3r_4u^2w + er_2u^4w \\ &\quad + er_3u^4w + er_4u^4w - eu^6w - u^5w^8. \end{aligned} \quad (10)$$

The unique equilibrium on the w -axis of system (10) is the origin of coordinates, which is a saddle whose four separatrices are contained in the two axes. Going back through the two changes of variables, first the change $d\tau = udt$ and after the change $v = wu$, and taking into account that in system (8) we have that $\dot{u}|_{u=0} = v^7$, we obtain that the local phase portrait at the origin of system (8) is formed by two hyperbolic sectors. This completes the proof of statement (b).

Proposition 3 *The local phase portrait at the origin of the local chart U_2 is a hyperbolic stable node in all the cases*

Proof We prove the Proposition (3) for the statement (a) and the other statements are proved in a similar way.

In the case (1) system (1) becomes system (7). In the local chart U_2 it becomes

$$\begin{aligned} \dot{u} &= -eu + (er_1 + er_2 + er_3 + er_4)u^3 + (-er_1r_2 - er_1r_3 \\ &\quad - er_2r_3 - er_1r_4 - er_2r_4 - er_3r_4)u^5 + (er_1r_2r_3 + er_1r_2r_4 \\ &\quad + er_1r_3r_4 + er_2r_3r_4)u^7 - er_1r_2r_3r_4u^9 - v^7 - u^2v^7, \\ \dot{v} &= -ev + (er_1 + er_2 + er_3 + er_4)u^2v + (-er_1r_2 - er_1r_3 \\ &\quad - er_2r_3 - er_1r_4 - er_2r_4 - er_3r_4)u^4v + (er_1r_2r_3 + er_1r_2r_4 \\ &\quad + er_1r_3r_4 + er_2r_3r_4)u^6v - er_1r_2r_3r_4u^8v - uv. \end{aligned} \quad (11)$$

The origin is a singular point of system (11). The eigenvalues of its associated Jacobian matrix are $(-e)$ and $(-e)$. Then the origin is a stable node.

4 Proof of Theorem 2

For studying the limit cycles which bifurcate in a Hopf bifurcation from the center of the differential system (2) when it is perturbed inside the class of all polynomial differential systems of degree 8, see (3), we work as follows. First doing the scaling $x = \varepsilon X$, $y = \varepsilon Y$ we introduce a small parameter ε . Thus we obtain the differential system $(\dot{X}, \dot{Y})$. Now performing the polar change of coordinates $X = r \cos \theta$, $Y = r \sin \theta$, the differential system $(\dot{X}, \dot{Y})$ written in polar coordinates becomes a differential system $(\dot{r}, \dot{\theta})$. Taking as independent variable the angle θ the differential system $(\dot{r}, \dot{\theta})$ produces the differential equation $dr/d\theta$. Finally doing a Taylor expansion in the variable r at $r = 0$ and truncating at 7-th order in ε we obtain the differential equation

$$r' = \frac{dr}{d\theta} = \sum_{i=0}^7 \varepsilon^i F_i(\theta, r) + O(\varepsilon^8). \quad (12)$$

The functions $F_i(\theta, r)$ $i = 1, \dots, 7$ of the differential system (12) are analytic, and since the independent variable θ appears through the sinus and cosinus of θ , they are 2π -periodic. Hence the assumptions for applying the averaging theory described in subsection 2.3 are satisfied. Now we shall study the limit cycles bifurcating from the center of system (2) when it is perturbed as in (3) following the steps described in subsection 2.3. We give only the expressions of functions $F_1(r, \theta)$ and $F_2(r, \theta)$. The explicit expressions of $F_i(r, \theta)$ for $i = 3, \dots, 7$ are very long, therefore we shall omit them here. Thus we have

$$F_1(r, \theta) = a_{00}^2 \cos \theta + b_{00}^2 \sin \theta + \frac{1}{2}r(b_{01}^1 + a_{10}^1 - b_{01}^1 \cos 2\theta + a_{10}^1 \cos 2\theta + b_{10}^1 \sin 2\theta + a_{01}^1 \sin 2\theta)$$

$$F_2(r, \theta) = \frac{1}{r}(a_{00}^2 \cos \theta + a_{10}^1 r \cos^2 \theta + b_{00}^2 \sin \theta + b_{10}^1 r \cos \theta \sin \theta + a_{01}^1 r \cos \theta \sin \theta + b_{01}^1 r \sin^2 \theta)(-b_{00}^2 \cos \theta - b_{10}^1 r \cos^2 \theta + a_{00}^2 \sin \theta - b_{01}^1 r \cos \theta \sin \theta + a_{10}^1 r \cos \theta \sin \theta + a_{01}^1 r \sin^2 \theta) + (a_{00}^3 \cos \theta + a_{10}^2 r \cos^2 \theta + a_{20}^1 r^2 \cos^3 \theta + b_{00}^3 \sin \theta + b_{10}^2 r \cos \theta \sin \theta + a_{01}^2 r \cos \theta \sin \theta + b_{20}^1 r^2 \cos^2 \theta \sin \theta + a_{11}^1 r^2 \cos^2 \theta \sin \theta + b_{00}^2 1r \sin^2 \theta + b_{11}^1 r^2 \cos \theta \sin^2 \theta + a_{02}^1 r^2 \cos \theta \sin^2 \theta + b_{02}^1 r^2 \sin^3 \theta).$$

and

Using the formulas given in subsection 2.3 the averaged function of first order is

$$f_1(r) = (b_{01}^1 + a_{10}^1)r$$

Clearly the polynomial $f_1(r)$ no positive root, so the first average function does not provide any information on the limit cycles that bifurcate from the center when we perturb it as in system (3). So, from the proof of the theorem follows for $k = 1$. Taking $b_{01}^1 = -a_{10}^1$, we obtain $f_1(r) \equiv 0$. We apply the averaging theory of second order and we get the averaged function of second order is

$$f_2(r) = (b_{01}^2 + a_{10}^2)r$$

Again the second averaged function does not provide any limit cycles. Then the proof of statement (b) holds. Now taking $b_{01}^2 = -a_{10}^2$, we get $f_2(r) \equiv 0$, and applying the averaging theory of third order, we obtain the third averaged function $f_3(r) = r(M_2 r^2 + M_0)$ where

$$M_0 = -(b_{11}^1 b_{00}^2 - b_{01}^3 + 2b_{00}^2 a_{20}^1 - 2b_{02}^1 a_{00}^2 - a_{11}^1 a_{00}^2 - a_{10}^3),$$

$$M_2 = \frac{1}{4}(3b_{03}^1 + b_{21}^1 + a_{12}^1 + 3a_{30}^1).$$

Clearly the coefficients M_0 and M_2 are independent, so the polynomial $f_3(r)$ can have at most one positive real root, so statement (c) of the theorem is proved.

In order to apply the averaging theory of fourth order we need that $f_3(r) \equiv 0$. So we take

$$a_{10}^3 = b_{11}^1 b_{00}^2 - b_{01}^3 + 2b_{00}^2 a_{20}^1 - 2b_{02}^1 a_{00}^2 - a_{11}^1 a_{00}^2, \quad a_{12}^1 = -3b_{03}^1 - b_{21}^1 - 3a_{30}^1.$$

Computing the function $f_4(r)$, we obtain $f_4(r) = r(N_2 r^2 + N_0)$, where

$$N_2 = -\frac{1}{4}(b_{02}^1 b_{11}^1 + b_{11}^1 b_{20}^1 + b_{10}^1 b_{21}^1 - 3b_{03}^2 - b_{21}^2 + b_{21}^1 a_{01}^1 - 2b_{02}^1 a_{02}^1 - 2b_{12}^1 a_{10}^1 - a_{02}^1 a_{11}^1 + 2b_{20}^1 a_{20}^1 - a_{11}^1 a_{20}^1 - 2a_{10}^1 a_{21}^1 + 3b_{10}^1 a_{30}^1 + 3a_{01}^1 a_{30}^1 - a_{12}^2 - 3a_{30}^2),$$

$$\begin{aligned}
N_0 = & -(-b_{10}^1 b_{11}^1 b_{00}^2 + b_{00}^2 b_{11}^2 + b_{11}^1 b_{00}^3 - b_{01}^4 + 2b_{02}^1 b_{00}^2 a_{10}^1 + b_{00}^2 a_{10}^1 a_{11}^1 - 2b_{10}^1 b_{00}^2 a_{20}^1 \\
& + 2b_{00}^3 a_{20}^1 - 2b_{02}^2 a_{00}^2 - 2b_{02}^1 a_{01}^1 a_{00}^2 - b_{11}^1 a_{10}^1 a_{00}^2 - a_{01}^1 a_{11}^1 a_{00}^2 - 2a_{10}^1 a_{20}^1 a_{00}^2 \\
& + 2b_{00}^2 a_{20}^2 - 2b_{02}^1 a_{00}^3 - a_{00}^2 a_{11}^2 - a_{11}^1 a_{00}^3 - a_{10}^4).
\end{aligned}$$

From the expression of the polynomial $f_4(r)$ we get that it has at most one positive real root. Hence statement (d) of the theorem is proved. Doing $N_2 = 0$ and $N_0 = 0$ and we extract the value of a_{12}^2 from $N_2 = 0$ and the value of a_{10}^4 from $N_0 = 0$, we have that $f_4(r) \equiv 0$, and we can apply the averaging theory of order five, and the fifth averaged function is $f_5(r) = r(H_4 r^4 + H_2 r^2 + H_0)$, where

$$\begin{aligned}
H_0 = & -((b_{10}^1)^2 b_{11}^1 b_{00}^2 - b_{21}^1 (b_{00}^2)^2 - b_{11}^1 b_{00}^2 b_{10}^2 - b_{10}^1 b_{00}^2 b_{11}^2 - b_{10}^1 b_{11}^1 b_{00}^3 + b_{11}^2 b_{00}^3 + b_{00}^2 b_{11}^3 \\
& + 2b_{00}^2 b_{02}^2 a_{10}^1 + 2b_{02}^1 b_{00}^3 a_{10}^1 + 2b_{02}^1 b_{00}^2 a_{01}^1 a_{10}^1 + b_{11}^1 b_{00}^2 (a_{10}^1)^2 - 2b_{02}^1 b_{10}^1 b_{00}^2 a_{10}^1 \\
& - b_{10}^1 b_{00}^2 a_{10}^1 a_{11}^1 + b_{00}^3 a_{10}^1 a_{11}^1 + b_{00}^2 a_{01}^1 a_{10}^1 a_{11}^1 + 2(b_{10}^1)^2 b_{00}^2 a_{20}^1 - 2b_{00}^2 b_{10}^2 a_{20}^1 \\
& + 2b_{00}^4 a_{20}^1 + 2b_{00}^2 (a_{10}^1)^2 a_{20}^1 - 3(b_{00}^2)^2 a_{30}^1 + b_{11}^1 b_{00}^4 - 2b_{10}^1 b_{00}^3 a_{20}^1 + 2b_{12}^1 b_{00}^2 a_{00}^2 \\
& - 2b_{00}^3 2a_{00}^2 - 2b_{02}^2 a_{01}^1 a_{00}^2 - 2b_{02}^1 (a_{01}^1)^2 a_{00}^2 + b_{10}^1 b_{11}^1 a_{10}^1 a_{00}^2 - b_{11}^2 a_{10}^1 a_{00}^2 - a_{10}^5 \\
& - b_{11}^1 a_{01}^1 a_{10}^1 a_{00}^2 - 2b_{02}^1 (a_{10}^1)^2 a_{00}^2 + b_{21}^1 (a_{00}^2)^2 + 3a_{30}^1 (a_{00}^2)^2 - 2b_{02}^1 a_{00}^2 a_{01}^1 - b_{01}^5 \\
& - (a_{01}^1)^2 a_{11}^1 a_{00}^2 - (a_{10}^1)^2 a_{11}^1 a_{00}^2 + 2b_{10}^1 a_{10}^1 a_{20}^1 a_{00}^2 - 2a_{01}^1 a_{10}^1 a_{20}^1 a_{00}^2 + 2b_{00}^2 a_{21}^1 a_{00}^2 \\
& + b_{00}^2 a_{11}^1 a_{20}^1 - b_{11}^1 a_{00}^2 a_{10}^2 - 2a_{20}^1 a_{00}^2 a_{10}^2 + b_{00}^2 a_{10}^1 a_{11}^2 - a_{01}^1 a_{00}^2 a_{11}^2 - a_{11}^1 a_{00}^2 a_{01}^2 \\
& - 2b_{10}^1 b_{00}^2 a_{20}^2 + 2b_{00}^3 a_{20}^2 - 2a_{10}^1 a_{00}^2 a_{20}^2 - 2b_{02}^2 a_{00}^3 - 2b_{02}^1 a_{01}^1 a_{00}^3 + 2b_{02}^1 b_{00}^2 a_{10}^2 \\
& - b_{11}^1 a_{10}^1 a_{00}^3 - a_{01}^1 a_{11}^1 a_{00}^3 - 2a_{10}^1 a_{20}^1 a_{00}^3 - a_{11}^2 a_{00}^3 - a_{00}^2 a_{11}^3 - 2b_{02}^1 a_{00}^4 - a_{11}^1 a_{00}^4 \\
& + 2b_{00}^2 a_{20}^3), \\
H_2 = & -\frac{1}{4}(-b_{02}^1 b_{10}^1 b_{11}^1 - 2b_{10}^1 b_{11}^1 b_{20}^1 - (b_{10}^1)^2 b_{21}^1 + 3b_{13}^1 b_{00}^2 + 3b_{31}^1 b_{00}^2 + b_{11}^1 b_{02}^2 + b_{21}^1 b_{10}^2 \\
& + b_{02}^2 b_{11}^2 + b_{20}^1 b_{11}^2 + b_{11}^1 b_{20}^2 + b_{10}^1 b_{21}^2 - 3b_{03}^3 - b_{11}^3 - b_{11}^1 b_{20}^1 a_{01}^1 - b_{10}^1 b_{21}^1 a_{01}^1 \\
& - 2b_{02}^2 a_{02}^1 - 2b_{02}^1 a_{01}^1 a_{02}^1 + 2(b_{02}^1)^2 a_{10}^1 + (b_{11}^1)^2 a_{10}^1 + 2b_{10}^1 b_{12}^1 a_{10}^1 + b_{21}^1 a_{01}^1 \\
& + 2b_{02}^1 b_{20}^1 a_{10}^1 - 2b_{12}^2 a_{10}^1 - b_{11}^1 a_{02}^1 a_{10}^1 - a_{01}^1 a_{02}^1 a_{11}^1 - b_{02}^1 a_{10}^1 a_{11}^1 + 3b_{10}^2 a_{30}^1 \\
& + b_{20}^1 a_{10}^1 a_{11}^1 - a_{10}^1 (a_{11}^1)^2 - 4b_{10}^1 b_{20}^1 a_{20}^1 + 2b_{20}^2 a_{20}^1 - 2b_{20}^1 a_{01}^1 a_{20}^1 - 3(b_{10}^1)^2 a_{30}^1 \\
& + b_{11}^1 a_{10}^1 a_{20}^1 - 2a_{02}^1 a_{10}^1 a_{20}^1 + b_{10}^1 a_{11}^1 a_{20}^1 - 2a_{10}^1 (a_{20}^1)^2 + 2b_{10}^1 a_{10}^1 a_{21}^1 + 2b_{00}^2 a_{22}^1 \\
& - 12b_{04}^1 a_{00}^2 - 2b_{22}^1 a_{00}^2 - 3a_{13}^1 a_{00}^2 - 3a_{31}^1 a_{00}^2 + b_{21}^1 a_{01}^2 - 3b_{10}^1 a_{01}^1 a_{30}^1 + 12b_{00}^2 a_{40}^1 \\
& + 3a_{30}^1 a_{01}^2 - 2b_{02}^1 a_{02}^2 - a_{11}^2 a_{00}^2 - 2b_{12}^2 a_{10}^2 - 2a_{21}^1 a_{10}^2 - a_{02}^2 a_{11}^2 - a_{20}^1 a_{11}^2 \\
& + 2b_{20}^1 a_{20}^2 - a_{11}^2 a_{20}^2 - 2a_{10}^1 a_{21}^2 + 3b_{10}^1 a_{30}^2 + 3a_{01}^1 a_{30}^2 - p_{12}^3 - 3a_{30}^3), \\
H_4 = & \frac{1}{8}(5b_{05}^1 + b_{23}^1 + b_{41}^1 + a_{14}^1 + a_{32}^1 + 5a_{50}^1).
\end{aligned}$$

The rank of the Jacobian matrix of the function $\mathcal{H} = (H_0, H_2, H_4)$ with respect to the coefficient a_{ij}^s and b_{ij}^s which appear in their expressions is maximal, i.e. it is 3. Then the coefficients H_i for $i = 0, 2, 4$ which appear in the expression of $f_5(r)$ are linearly independent. By the roots of a quadratic polynomial in the variable r^2 it follows that $f_5(r)$ can have at most two positive real roots. Therefore statement

(e) of the theorem is proved.

Imposing that $H_0 = 0$, $H_2 = 0$ and $H_4 = 0$ we obtain that $f_5(r) \equiv 0$, and applying the averaging theory of order six we obtain the sixth average function $f_6(r) = r(T_4 r^4 + T_2 r^2 + T_0)$, where

$$\begin{aligned}
T_0 = & (b_{10}^1)^3 b_{11}^1 b_{00}^2 - 2b_{02}^1 b_{11}^1 (b_{00}^2)^2 - b_{11}^1 b_{20}^1 (b_{00}^2)^2 - 2b_{10}^1 b_{21}^1 (b_{00}^2)^2 - b_{10}^1 (a_{10}^1)^2 a_{11}^1 a_{00}^2 \\
& - (b_{10}^1)^2 b_{00}^2 b_{11}^2 + (b_{00}^2)^2 b_{21}^2 - (b_{10}^1)^2 b_{11}^1 b_{00}^3 - 2b_{10}^1 b_{11}^1 b_{00}^2 b_{10}^2 + 2a_{10}^1 a_{11}^1 a_{00}^2 a_{10}^2 \\
& + b_{10}^1 b_{11}^1 b_{00}^3 + b_{10}^1 b_{00}^2 b_{11}^3 - b_{00}^3 b_{11}^3 + b_{10}^1 b_{11}^1 b_{00}^4 - b_{11}^2 b_{00}^4 - b_{00}^2 b_{11}^4 + b_{11}^1 b_{10}^2 b_{00}^3 \\
& + b_{00}^8 - 2b_{02}^1 (b_{10}^1)^2 b_{00}^2 a_{10}^1 + 2b_{12}^1 (b_{00}^2)^2 a_{10}^1 + 2b_{10}^1 b_{00}^2 b_{02}^2 a_{10}^1 + 2b_{02}^1 b_{00}^2 b_{01}^3 \\
& + 2b_{02}^1 b_{00}^2 b_{10}^1 a_{10}^1 + 2b_{02}^1 b_{10}^1 b_{00}^3 a_{10}^1 - 2b_{02}^2 b_{00}^3 a_{10}^1 - 2b_{00}^2 b_{02}^3 a_{10}^1 - 4b_{00}^2 a_{10}^1 a_{20}^1 a_{10}^2 \\
& - 2b_{02}^2 b_{00}^4 a_{10}^1 + 2b_{02}^2 b_{10}^1 b_{00}^2 a_{01}^1 a_{10}^1 - 2b_{00}^2 b_{02}^2 a_{01}^1 a_{10}^1 - 2b_{02}^1 b_{00}^3 a_{01}^1 a_{10}^1 + b_{11}^1 b_{00}^2 b_{10}^3 \\
& - 2b_{02}^1 b_{00}^2 (a_{01}^1)^2 a_{10}^1 + 2b_{10}^1 b_{11}^1 b_{00}^2 (a_{10}^1)^2 - b_{00}^2 b_{11}^2 (a_{10}^1)^2 - b_{11}^1 b_{00}^3 (a_{10}^1)^2 - b_{00}^3 a_{10}^1 a_{11}^2 \\
& - b_{11}^1 b_{00}^2 a_{01}^1 (a_{10}^1)^2 - 2b_{02}^1 b_{00}^2 (a_{10}^1)^3 - (b_{11}^1 b_{00}^2)^2 a_{11}^1 - b_{00}^4 a_{10}^1 a_{11}^1 + b_{00}^2 b_{00}^3 1 a_{11}^1 \\
& - (b_{10}^1)^2 b_{00}^2 a_{10}^1 a_{11}^1 + b_{00}^2 b_{10}^1 a_{10}^1 a_{11}^1 + b_{10}^1 b_{00}^3 a_{10}^1 a_{11}^1 - b_{00}^2 (a_{01}^1)^2 a_{10}^1 a_{11}^1 - b_{00}^2 a_{01}^1 a_{11}^1 a_{10}^2 \\
& + b_{10}^1 b_{00}^2 a_{01}^1 a_{10}^1 a_{11}^1 - b_{00}^3 a_{01}^1 a_{10}^1 a_{11}^1 - b_{00}^2 (a_{10}^1)^3 a_{11}^1 - 4b_{10}^1 b_{00}^2 b_{10}^2 a_{20}^1 + 2b_{11}^1 b_{00}^2 a_{20}^1 a_{00}^2 \\
& - 2(b_{10}^1)^2 b_{00}^3 a_{20}^1 + 2b_{02}^1 b_{00}^3 a_{20}^1 + 2b_{00}^2 b_{10}^3 a_{20}^1 + 2b_{10}^1 b_{00}^4 a_{20}^1 + 4b_{10}^1 b_{00}^2 (a_{10}^1)^2 a_{20}^1 \\
& - 2b_{00}^3 (a_{10}^1)^2 a_{20}^1 - 2b_{00}^2 a_{01}^1 (a_{10}^1)^2 a_{20}^1 - (b_{00}^2)^2 a_{11}^1 a_{20}^1 + 2(b_{00}^2)^2 a_{10}^1 a_{21}^1 + 2(b_{10}^1)^3 b_{00}^2 a_{10}^1 \\
& - 6b_{10}^1 (b_{00}^2)^2 a_{10}^1 a_{30}^1 + 6b_{00}^2 b_{00}^3 a_{10}^1 a_{30}^1 + 4(b_{02}^1)^2 b_{00}^2 a_{00}^2 + (b_{11}^1)^2 b_{00}^2 a_{00}^2 + (b_{10}^1)^2 b_{11}^1 a_{10}^1 a_{00}^2 \\
& + 2b_{10}^1 b_{12}^1 b_{00}^2 a_{00}^2 - 2b_{00}^2 b_{12}^2 a_{00}^2 - 2b_{12}^1 b_{00}^3 a_{00}^2 - b_{11}^1 b_{00}^3 1 a_{00}^2 + 2b_{02}^4 a_{00}^2 - 2b_{02}^1 (b_{00}^2)^2 a_{20}^1 \\
& - 2b_{12}^1 b_{00}^3 a_{01}^1 a_{00}^2 + 2b_{00}^3 2 a_{01}^1 a_{00}^2 + 2b_{02}^1 (a_{01}^1)^2 a_{00}^2 + 2b_{02}^1 (a_{01}^1)^3 a_{00}^2 - 2b_{20}^1 (b_{00}^2)^2 a_{20}^1 \\
& - b_{11}^1 b_{10}^2 a_{10}^1 a_{00}^2 - b_{10}^1 b_{11}^2 a_{10}^1 a_{00}^2 + b_{11}^3 a_{10}^1 a_{00}^2 - b_{10}^1 b_{11}^1 a_{01}^1 a_{10}^1 a_{00}^2 + 2(a_{01}^1 a_{10}^1)^2 a_{11}^1 a_{00}^2 \\
& + b_{11}^2 a_{01}^1 a_{10}^1 a_{00}^2 + b_{11}^1 (a_{01}^1)^2 a_{10}^1 a_{00}^2 - 2b_{02}^1 b_{10}^1 (a_{10}^1)^2 a_{00}^2 + 2b_{02}^2 (a_{10}^1)^2 a_{00}^2 - 2b_{01}^1 a_{20}^1 a_{00}^2 \\
& + 4b_{02}^1 a_{01}^1 (a_{10}^1)^2 a_{00}^2 + b_{11}^1 (a_{10}^1)^3 a_{00}^2 + 2b_{02}^1 b_{00}^2 a_{11}^1 a_{00}^2 + (a_{01}^1)^3 a_{11}^1 a_{00}^2 + 2b_{21}^1 b_{00}^2 b_{00}^3 \\
& + 2(b_{10}^1)^2 a_{10}^1 a_{20}^1 a_{00}^2 - 2b_{10}^2 a_{10}^1 a_{20}^1 a_{00}^2 - 2b_{10}^1 a_{01}^1 a_{10}^1 a_{20}^1 a_{00}^2 + 2(a_{01}^1)^2 a_{10}^1 a_{20}^1 a_{00}^2 \\
& + 2(a_{10}^1)^3 a_{20}^1 a_{00}^2 + 2b_{10}^1 b_{00}^2 a_{21}^1 a_{00}^2 - 2b_{00}^3 a_{21}^1 a_{00}^2 - 2b_{00}^2 a_{01}^1 a_{21}^1 a_{00}^2 + 4b_{02}^1 a_{10}^1 a_{00}^2 a_{10}^2 \\
& + b_{11}^1 b_{20}^1 (a_{00}^2)^2 + b_{10}^1 b_{21}^1 (a_{00}^2)^2 - b_{21}^2 (a_{00}^2)^2 - b_{21}^1 a_{01}^1 (a_{00}^2)^2 + b_{11}^1 a_{01}^1 a_{00}^2 a_{10}^2 - b_{00}^3 a_{11}^1 a_{10}^2 \\
& - 2b_{02}^1 a_{20}^1 (a_{00}^2)^2 + 2b_{20}^1 a_{20}^1 (a_{00}^2)^2 - a_{11}^1 a_{20}^1 (a_{00}^2)^2 + 3b_{10}^1 a_{30}^1 (a_{00}^2)^2 - 3a_{01}^1 a_{30}^1 (a_{00}^2)^2 \\
& - 2b_{02}^1 b_{00}^2 a_{10}^1 a_{01}^1 - b_{00}^2 a_{10}^1 a_{11}^1 a_{01}^1 + 2b_{02}^2 a_{00}^2 a_{01}^1 + 4b_{02}^1 a_{01}^1 a_{00}^2 a_{01}^1 + b_{10}^1 b_{00}^2 a_{11}^1 a_{10}^2 \\
& + b_{11}^1 a_{10}^1 a_{00}^2 a_{01}^1 + 2a_{01}^1 a_{11}^1 a_{00}^2 a_{01}^1 + 2a_{10}^1 a_{20}^1 a_{00}^2 a_{01}^1 + 2b_{02}^1 b_{10}^1 b_{00}^2 a_{10}^2 + b_{11}^2 a_{00}^2 a_{10}^2 \\
& - 2b_{00}^2 b_{02}^2 a_{10}^2 - 2b_{02}^1 b_{00}^3 a_{10}^2 - 2b_{02}^1 b_{00}^2 a_{01}^1 a_{10}^2 - 2b_{11}^1 b_{00}^2 a_{10}^1 a_{10}^2 - b_{10}^1 b_{11}^1 a_{00}^2 a_{10}^2 \\
& + b_{00}^2 b_{10}^2 b_{11}^1 - 2b_{10}^1 a_{20}^1 a_{00}^2 a_{10}^2 + 2a_{01}^1 a_{20}^1 a_{00}^2 a_{10}^2 + b_{10}^1 b_{00}^2 a_{10}^1 a_{11}^2 - b_{00}^2 a_{01}^1 a_{10}^1 a_{11}^2 \\
& + a_{00}^2 a_{01}^2 a_{11}^2 - b_{00}^2 a_{10}^2 a_{11}^2 - 2(b_{10}^1)^2 b_{00}^2 a_{20}^2 + 2b_{00}^2 b_{10}^2 a_{20}^2 + 2b_{10}^1 b_{00}^3 a_{20}^2 + 2b_{00}^3 2 a_{00}^3 \\
& - 2b_{00}^4 a_{20}^2 - 2b_{00}^2 (a_{10}^1)^2 a_{20}^2 - 2b_{10}^1 a_{10}^1 a_{00}^2 a_{20}^2 + 2a_{01}^1 a_{10}^1 a_{00}^2 a_{20}^2 + 2a_{00}^2 a_{10}^2 a_{20}^2 + a_{10}^6 \\
& - 2b_{00}^2 a_{00}^2 a_{21}^2 - 3(a_{00}^2)^2 a_{30}^2 - 2b_{12}^1 b_{00}^2 a_{00}^3 + (a_{01}^1)^2 a_{00}^2 a_{11}^2 + (a_{10}^1)^2 a_{00}^2 a_{11}^2 - 2b_{00}^2 a_{20}^4
\end{aligned}$$

$$\begin{aligned}
& +2b_{02}^2a_{01}^1a_{00}^3 + 2b_{02}^1(a_{01}^1)^2a_{00}^3 - b_{10}^1b_{11}^1a_{10}^1a_{00}^3 + b_{11}^2a_{10}^1a_{00}^3 + 3(b_{00}^2)^2a_{30}^2 - b_{11}^1b_{00}^5 \\
& +2b_{02}^1(a_{10}^1)^2a_{00}^3 + (a_{01}^1)^2a_{11}^1a_{00}^3 + (a_{10}^1)^2a_{11}^1a_{00}^3 - 2b_{10}^1a_{10}^1a_{20}^1a_{00}^3 + b_{11}^1a_{01}^1a_{10}^1a_{00}^3 \\
& +2a_{01}^1a_{10}^1a_{20}^1a_{00}^3 - 2b_{00}^2a_{21}^1a_{00}^3 - 2b_{21}^1a_{00}^2a_{00}^3 - 6a_{30}^1a_{00}^2a_{00}^3 - 2a_{20}^1b_{00}^5 + a_{11}^2a_{00}^4 \\
& +2b_{02}^1a_{01}^2a_{00}^3 + a_{11}^1a_{01}^2a_{00}^3 + b_{11}^1a_{10}^2a_{00}^3 + 2a_{20}^1a_{10}^2a_{00}^3 + a_{01}^1a_{11}^2a_{00}^3 + 2a_{10}^1a_{20}^1a_{00}^4 \\
& +2a_{10}^1a_{20}^2a_{00}^3 + 2b_{02}^1a_{01}^2a_{00}^3 + a_{11}^1a_{01}^2a_{00}^3 - b_{00}^2a_{10}^1a_{11}^3 + a_{01}^1a_{00}^2a_{11}^3 + a_{00}^2a_{11}^4 \\
& +a_{00}^3a_{11}^3 + 2b_{10}^1b_{00}^2a_{20}^3 - 2b_{00}^3a_{20}^3 + 2a_{10}^1a_{00}^2a_{20}^3 + 2b_{02}^1a_{00}^5 + a_{11}^1a_{00}^5 + a_{01}^1a_{11}^1a_{00}^4 \\
& +2b_{02}^2a_{00}^4 + 2b_{02}^1a_{01}^1a_{00}^4 + b_{11}^1a_{10}^1a_{00}^4.
\end{aligned}$$

$$\begin{aligned}
T_2 = & \frac{1}{4}(-b_{02}^1(b_{10}^1)^2b_{11}^1 - 3(b_{10}^1)^2b_{11}^1b_{20}^1 - (b_{10}^1)^3b_{21}^1 + 6b_{02}^1b_{03}^1b_{00}^2 + 3b_{11}^1b_{12}^1b_{00}^2 - 3b_{00}^2b_{31}^2 \\
& +4b_{02}^1b_{21}^1b_{00}^2 + 4b_{20}^1b_{21}^1b_{00}^2 + 3b_{11}^1b_{30}^1b_{00}^2 + 6b_{10}^1b_{31}^1b_{00}^2 + b_{10}^1b_{11}^1b_{02}^2 + b_{02}^1b_{11}^1b_{10}^2 \\
& +2b_{11}^1b_{20}^1b_{10}^2 + 2b_{10}^1b_{21}^1b_{10}^2 + b_{02}^1b_{10}^1b_{11}^2 + b_{02}^2a_{10}^1a_{11}^1 - b_{20}^2a_{10}^1a_{11}^1 + b_{02}^1a_{01}^1a_{10}^1a_{11}^1 \\
& +2b_{10}^1b_{20}^1b_{11}^2 - b_{02}^2b_{11}^2 - 3b_{00}^2b_{13}^2 + 2b_{10}^1b_{11}^1b_{20}^2 - b_{11}^1b_{20}^2 + a_{02}^1(a_{10}^1)^2a_{11}^1 - b_{10}^2b_{21}^2 \\
& -2b_{12}^1b_{01}^3 - b_{11}^1b_{02}^3 - b_{21}^1b_{10}^3 - b_{02}^1b_{11}^3 - b_{20}^1b_{11}^3 - b_{11}^1b_{20}^3 + (b_{10}^1)^2b_{21}^2 + 3b_{10}^1b_{13}^1b_{00}^2 \\
& +3b_{03}^1 + b_{21}^4 - 2b_{10}^1b_{11}^1b_{20}^1a_{01}^1 - (b_{10}^1)^2b_{21}^1a_{01}^1 - 3b_{13}^1b_{00}^3 - 3b_{31}^1b_{00}^3 + 2b_{21}^1b_{00}^2a_{11}^1 \\
& +b_{10}^1b_{21}^2a_{01}^1 - 2b_{12}^1b_{00}^2a_{02}^1 + 2b_{00}^32a_{02}^1 + 2b_{02}^2a_{01}^1a_{02}^1 + 3b_{31}^1b_{00}^2a_{01}^1 + (a_{01}^1)^2a_{02}^1a_{11}^1 \\
& +2b_{02}^1(a_{01}^1)^2a_{02}^1 + 2(b_{02}^1)^2b_{10}^1a_{10}^1 + 2b_{10}^1(b_{11}^1)^2a_{10}^1 + 2(b_{10}^1)^2b_{12}^1a_{10}^1 + b_{20}^1b_{11}^2a_{01}^1 \\
& +4b_{02}^1b_{10}^1b_{20}^1a_{10}^1 - 12b_{04}^1b_{00}^2a_{10}^1 - 6b_{22}^1b_{00}^2a_{10}^1 - 4b_{02}^1b_{02}^2a_{10}^1 + b_{21}^1b_{10}^2a_{01}^1 - b_{21}^3a_{01}^1 \\
& -2b_{20}^1b_{02}^2a_{10}^1 - 2b_{12}^1b_{10}^2a_{10}^1 - 2b_{11}^1b_{11}^2a_{10}^1 - 2b_{10}^1b_{12}^2a_{10}^1 - b_{02}^1b_{10}^1a_{10}^1a_{11}^1 + 15a_{50}^2 \\
& +2b_{12}^3a_{10}^1 - 2(b_{02}^1)^2a_{01}^1a_{10}^1 - b_{10}^1b_{11}^1a_{02}^1a_{10}^1 + b_{11}^2a_{02}^1a_{10}^1 + b_{11}^1b_{20}^2a_{01}^1 - b_{10}^1b_{21}^2 \\
& +b_{11}^1a_{01}^1a_{02}^1a_{10}^1 - 3b_{02}^1b_{11}^1(a_{10}^1)^2 - b_{11}^1b_{20}^1(a_{10}^1)^2 + 2b_{02}^1a_{02}^1(a_{10}^1)^2 + 3b_{03}^1b_{00}^2a_{11}^1 \\
& -b_{10}^1a_{10}^1(a_{11}^1)^2 + a_{01}^1a_{10}^1(a_{11}^1)^2 - 3b_{00}^2a_{10}^1a_{13}^1 - 6(b_{10}^1)^2b_{20}^1a_{20}^1 + 2b_{10}^1b_{20}^1a_{10}^1a_{11}^1 \\
& +6b_{30}^1b_{00}^2a_{20}^1 + 4b_{20}^1b_{10}^2a_{20}^1 + 4b_{10}^1b_{20}^2a_{20}^1 - 2b_{20}^3a_{20}^1 - 4b_{10}^1b_{20}^1a_{01}^1a_{20}^1 - 2b_{02}^1b_{20}^2a_{10}^1 \\
& +2b_{20}^2a_{01}^1a_{20}^1 + 2b_{10}^1b_{11}^1a_{10}^1a_{20}^1 - b_{11}^2a_{10}^1a_{20}^1 - 2b_{10}^1a_{02}^1a_{10}^1a_{20}^1 - 2b_{02}^1b_{20}^2a_{10}^1 \\
& +2a_{01}^1a_{02}^1a_{10}^1a_{20}^1 - 2(b_{20}^1a_{10}^1)^2a_{20}^1 + (b_{10}^1)^2a_{11}^1a_{20}^1 - b_{10}^2a_{11}^1a_{20}^1 + 3(a_{10}^1)^2a_{11}^1a_{20}^1 \\
& -4b_{10}^1a_{10}^1(a_{20}^1)^2 + 2b_{11}^1b_{00}^2a_{21}^1 - 2b_{00}^31a_{21}^1 - 2b_{00}^2a_{02}^1a_{21}^1 + 2(b_{10}^1)^2a_{10}^1a_{21}^1 \\
& -2b_{10}^2a_{10}^1a_{21}^1 - 2b_{00}^2a_{20}^1a_{21}^1 + 2b_{10}^1b_{00}^2a_{22}^1 - 2b_{00}^3a_{22}^1 - 3(b_{10}^1)^3a_{30}^1 - b_{11}^1a_{20}^1a_{10}^2 \\
& +6b_{02}^1b_{00}^2a_{30}^1 + 12b_{20}^1b_{00}^2a_{30}^1 + 6b_{10}^1b_{10}^2a_{30}^1 - 3b_{10}^3a_{30}^1 - 3(b_{10}^1)^2a_{01}^1a_{30}^1 + a_{20}^1a_{11}^3 \\
& +3b_{10}^2a_{01}^1a_{30}^1 + 3b_{00}^2a_{11}^1a_{30}^1 - 9b_{00}^2a_{10}^1a_{31}^1 + 24b_{10}^1b_{00}^2a_{40}^1 - 12b_{00}^3a_{40}^1 + a_{02}^1a_{11}^3 \\
& +12b_{00}^2a_{01}^1a_{40}^1 - 3b_{03}^1b_{11}^1a_{00}^2 - 6b_{02}^1b_{12}^1a_{00}^2 - 2b_{12}^1b_{20}^1a_{00}^2 - 2b_{11}^1b_{21}^1a_{00}^2 - 3a_{30}^1a_{01}^3 \\
& -2b_{10}^1b_{22}^1a_{00}^2 + 12b_{00}^24a_{00}^2 + 2b_{22}^2a_{00}^2 + 12b_{04}^1a_{01}^1a_{00}^2 - 4b_{21}^1a_{02}^1a_{00}^2 + (a_{11}^1)^2a_{10}^2 \\
& +6b_{02}^1a_{03}^1a_{00}^2 + 9b_{13}^1a_{10}^1a_{00}^2 + 3b_{31}^1a_{10}^1a_{00}^2 + 3a_{01}^1a_{13}^1a_{00}^2 + 3a_{03}^1a_{11}^1a_{00}^2 + a_{11}^1a_{02}^2 \\
& -6b_{03}^1a_{20}^1a_{00}^2 - 4b_{21}^1a_{20}^1a_{00}^2 - 4b_{02}^1a_{21}^1a_{00}^2 - 2b_{20}^1a_{21}^1a_{00}^2 + a_{11}^1a_{21}^1a_{00}^2 + 2b_{02}^1a_{02}^3
\end{aligned}$$

$$\begin{aligned}
& +6a_{10}^1a_{22}^2a_{00}^2 - 3b_{11}^1a_{30}^1a_{00}^2 - 12a_{02}^1a_{30}^1a_{00}^2 - 6a_{20}^1a_{30}^1a_{00}^2 - 3b_{10}^1a_{31}^1a_{00}^2 - b_{21}^2a_{01}^2 \\
& +12a_{10}^1a_{40}^1a_{00}^2 + b_{11}^1b_{20}^1a_{01}^2 + b_{10}^1b_{21}^1a_{01}^2 + 2b_{02}^1a_{02}^1a_{01}^2 + a_{02}^1a_{11}^1a_{01}^2 + 3(b_{10}^1)^2a_{30}^2 \\
& +3b_{10}^1a_{30}^1a_{01}^2 + 2b_{02}^2a_{02}^2 + 2b_{02}^1a_{01}^1a_{02}^2 - 2b_{20}^1a_{20}^3 + a_{11}^1a_{20}^3 + 2a_{10}^1a_{21}^3 - 3b_{10}^1a_{30}^3 \\
& +b_{11}^1a_{10}^1a_{02}^2 + a_{01}^1a_{11}^1a_{02}^2 + 2a_{10}^1a_{20}^1a_{02}^2 - 2(b_{02}^1)^2a_{10}^2 - (b_{11}^1)^2a_{10}^2 - 2b_{10}^1b_{12}^1a_{10}^2 \\
& -2b_{02}^1b_{20}^1a_{10}^2 + 2b_{12}^2a_{10}^2 + b_{11}^1a_{02}^1a_{10}^2 + b_{02}^1a_{11}^1a_{10}^2 - b_{20}^1a_{11}^1a_{10}^2 + a_{12}^4 + 3a_{30}^4 \\
& +2a_{02}^1a_{20}^1a_{10}^2 + 2(a_{20}^1)^2a_{10}^2 + a_{11}^2a_{20}^2 - 2b_{10}^1a_{10}^1a_{21}^2 + 2a_{10}^2a_{21}^2 - 2b_{00}^2n_{22}^2 + a_{00}^2a_{21}^2 \\
& -2b_{10}^1a_{21}^1a_{10}^2 + a_{01}^1a_{02}^1a_{11}^2 + b_{02}^1a_{10}^1a_{11}^2 - b_{20}^1a_{10}^1a_{11}^2 + 2a_{10}^1a_{11}^1a_{11}^2 - 2b_{20}^2a_{20}^2 \\
& +2b_{20}^1a_{01}^1a_{20}^2 - b_{11}^1a_{10}^1a_{20}^2 + 2a_{02}^1a_{10}^1a_{20}^2 - b_{10}^1a_{11}^1a_{20}^2 + 4a_{10}^1a_{20}^1a_{20}^2 - b_{10}^1a_{20}^1a_{11}^2 \\
& -3b_{10}^2a_{30}^3 + 3b_{10}^1a_{01}^1a_{30}^2 - 3a_{02}^1a_{30}^2 + 3a_{00}^2a_{31}^2 - 12b_{00}^2a_{40}^2 + 3a_{00}^2a_{13}^2 + 4b_{10}^1b_{20}^1a_{20}^2 \\
& +12b_{04}^1a_{00}^3 + 2b_{22}^1a_{00}^3 + 3a_{13}^1a_{00}^3 + 3a_{31}^1a_{00}^3 - b_{21}^1a_{01}^3 + 2b_{20}^1a_{20}^1a_{01}^2 - 3a_{01}^1a_{30}^3). \\
T_4 = & \frac{1}{24}(15b_{05}^1b_{10}^1 + 5b_{04}^1b_{11}^1 - 6b_{03}^1b_{12}^1 - 9b_{02}^1b_{13}^1 - 3b_{13}^1b_{20}^1 - 3b_{12}^1b_{21}^1 - b_{11}^1b_{22}^1 - 3b_{21}^1b_{30}^1 \\
& -4b_{02}^1a_{22}^1 - 2b_{20}^1a_{22}^1 + 4b_{22}^1a_{20}^1 - 6b_{40}^1a_{20}^1 + 7a_{13}^1a_{20}^1 - 6b_{03}^1a_{21}^1 - 3b_{21}^1a_{21}^1 + 3a_{32}^2 \\
& +a_{11}^1a_{22}^1 + 6a_{10}^1a_{23}^1 - 9b_{30}^1a_{30}^1 - 9a_{03}^1a_{30}^1 - 3a_{21}^1a_{30}^1 + 3a_{02}^1a_{31}^1 + 9a_{20}^1a_{31}^1 + 3a_{14}^2 \\
& -3b_{12}^1a_{30}^1 - 22b_{02}^1a_{40}^1 - 20b_{20}^1a_{40}^1 - 5a_{11}^1a_{40}^1 + 12a_{10}^1a_{41}^1 - 15b_{10}^1a_{50}^1 - 15a_{01}^1a_{50}^1 \\
& -7b_{02}^1b_{31}^1 - 5b_{20}^1b_{31}^1 - 3b_{11}^1b_{40}^1 - 3b_{10}^1b_{41}^1 + 15b_{05}^2 + 3b_{23}^2 + 3b_{41}^2 + 15b_{05}^1a_{01}^1 \\
& +6b_{02}^1a_{04}^1 + 12b_{14}^1a_{10}^1 + 6b_{32}^1a_{10}^1 - 3b_{41}^1a_{01}^1 + 20b_{04}^1a_{02}^1 + 2b_{22}^1a_{02}^1 - 3b_{21}^1a_{03}^1 \\
& -2b_{31}^1a_{11}^1 + 3a_{04}^1a_{11}^1 + 2b_{11}^1a_{13}^1 + 5a_{02}^1a_{13}^1 + 3b_{10}^1a_{14}^1 + 3a_{01}^1a_{14}^1 + 22b_{04}^1a_{20}^1).
\end{aligned}$$

Taking $T_0 = 0, T_2 = 0$ and $T_4 = 0$ we obtain that $f_6(r) \equiv 0$. Computing the seventh average function we obtain $f_7(r) = r(G_6r^6 + G_4r^4 + G_2r^2 + G_0)$. We do not give the big expressions of the coefficients G_i for $i = 0, 2, 4, 6$. According to the expression of the polynomial $f_7(r)$ it follows easily that $f_7(r)$ can have at most three positive real roots. Now we are going to reach our result by giving an example with exactly three limit cycles.

Example with three limit cycles. We consider the kukles differential systems of degree 8 symmetric with respect to the x -axis, perturbed inside a class of polynomial differential systems of degree 8.

We consider the system (13) defined by

$$\begin{aligned}
\dot{x} = & -y + \epsilon^7 (r_0 + r_2x^2) + \epsilon^6 (u_2x^2 - 1) + \epsilon^4 \left(\frac{815x^3}{36} - 8xy^2 - 5xy + \frac{17x}{2} \right) + \epsilon^5 (x^3 + x^2 \\
& -xy) + \epsilon^3 \left(-\frac{851x^5}{30} - x^3y + x^3 - 7x^2y - \frac{37xy^2}{2} - \frac{3x}{2} \right) + \epsilon^2 \left(-\frac{14x^5}{3} + 2x^3y^2 - 9x^2y^3 \right. \\
& + x^2 - 2xy^2 + xy - x - 2y^3 + y^2 + 1) + \epsilon \left(-2x^7 - 4x^4y^2 - \frac{2x^4y}{3} + 2x^3y^4 - x^3y^3 - 7x^3y^2 \right. \\
& \left. + x^3y - 7x^2y^4 + x^2y^2 + \frac{73xy^6}{5} - 3xy^5 + 2xy^2 - 2x - 2y^4 + 4y^3 + 2y^2 - 2y \right),
\end{aligned}$$

$$\begin{aligned} \dot{y} = & x + ax^8 + bx^6y^2 + cx^4y^4 + dx^2y^6 + ey^8 + \epsilon^6(g_1x + g_2x^2) + \epsilon^7(k_0 + k_1x) + \epsilon^5\left(\frac{387x^2y}{4}\right. \\ & - \frac{1743xy}{4} - 2y^3 + 9y^2 - 32y - \frac{122}{3}) + \epsilon^4\left(-5x^2y - 6xy^2 + xy - \frac{y^3}{4}\right) + \epsilon^2(-x^2y^2 \\ & - 2xy^4 + 4x + y^5 + y - \frac{1}{2}) + \epsilon^3\left(4x^4y - 2x^3y - 4x^2 - \frac{xy^3}{2} - \frac{xy^2}{3} - 3y^5\right) + \epsilon(5x^4y^2 \\ & + 2x^4 + x^3y^3 + x^3y + x^3 + 4x^2y^5 + 2x^2y^3 - 2x^2y + 3xy + 3x + y^7 + y^5 + 2y), \end{aligned}$$

an exhausting computation shows that for the polynomial differential systems (??), we obtain $f_1(r) \equiv f_2(r) \equiv f_3(r) \equiv f_4(r) \equiv f_5(r) \equiv f_6(r) \equiv 0$, and

$$f_7(r) = r(r-3)(r-2)(r-1)(r+1)(r+2)(r+3).$$

Then for this systems we have three limit cycles bifurcate from the periodic orbits of the center $\dot{x} = y, \dot{y} = x + ax^8 + bx^6y^2 + cx^4y^4 + dx^2y^6 + ey^8$. Moreover, in polar coordinates (r, θ) the periodic orbits that bifurcate are $r = 1, 2, 3$. This completes the proof of the theorem when $k = 7$.

Conclusion

In many of my works, I am interesting to solve the problems of cyclicity and the global phase portraits of many classes of differential equations. In [1] of my manuscript I solved these problems with J.Llibre for a class of differential equations of degree 4, this class is called Kukles differential systems, in the same year I studied with J.Llibre another work of kukles differential systems of degree seven and we found a different results we put it in reference [2]. Recently, and in this manuscript, I and Loubna Damene we solved these problems for a new generalized class of kukles differential systems of degree eight.

Funding

The authors received no specific funding for this work.

Data Availability Statement

The datasets analysed during the current study are available from the corresponding author on reasonable request.

Conflict of interest

The authors declare no conflict of interest.

Author Contributions

Conceptualization: Rebiha Benterki; Methodology: Loubna Damene; Formal analysis and investigation: Loubna Damene and Rebiha Benterki; Writing - original draft preparation: Loubna Damene; Writing - review and editing: Rebiha Benterki; Resources: Rebiha Benterki; Supervision: Rebiha Benterki.

References

1. Benterki, R., Llibre, J.: Centers and limit cycles of polynomial differential systems of degree 4 via averaging theory. *J. Computational and Appl. Math.* **313**, 273–283 (2017)
2. Benterki, R., Llibre, J.: The centers and their cyclicity for a class of polynomial differential systems of degree 7. *J. Computational and Appl. Math.* **368**, 112456 (2020)
3. Buzzi, C. A., Llibre, J., Medrado, J.C.: Phase portraits of reversible linear differential systems with cubic homogeneous polynomial nonlinearities having a non-degenerate center at the origin. *Qual. Theory Dyn. Syst.* **7**, 369–403 (2009)
4. Dumortier, F., Llibre, J., Artés, J.C.: *Qualitative theory of planar differential systems*. Universitext, Springer-Verlag, (2006)
5. Du, C., Huang, W. Center-focus problem and limit cycles bifurcations for a class of cubic Kolmogorov model. *Nonlinear Dyn* **72**, 197206 (2013). <https://doi.org/10.1007/s11071-012-0703-9>
6. Ferragut, A., Valls, C.: Phase portraits of Abel quadratic differential systems of the second kind. *J. Dynamical Systems* (2017)
7. Giné, J.: Conditions for the existence of a center for the Kukles homogenous systems. *Comput. Math. Appl.* **43**, 1261–1269 (2002)
8. Giné, J., Llibre, J., Valls, C.: Centers for the Kukles homogeneous systems with odd degree. *Bull. London Math. Soc.* **47**, 315–324 (2015)
9. Liang, H., Huang, J. Zhao, Y. Classification of global phase portraits of planar quartic quasi-homogeneous polynomial differential systems. *Nonlinear Dyn* **78**, 16591681 (2014). <https://doi.org/10.1007/s11071-014-1541-8>
10. Llibre, J., Novaes, D.D., Teixeira, M.A.: Higher order averaging theory for finding periodic solutions via Brouwer degree. *Nonlinearity* **27**, 563–583 (2014)
11. Llibre, J., da Silva, M.F.: Global phase portraits of Kukles differential systems with homogenous polynomial nonlinearities of degree 6 having a center and their small limit cycles. *Int. J. of Bifurcation and Chaos* **26**, 1650044, 25 pp (2016)
12. Llibre, J., da Silva, M.F.: Global phase portraits of Kukles differential systems with homogenous polynomial nonlinearities of degree 5 having a center. *Topol. Methods in Nonlinear Analysis* **48**, 257–282 (2016)
13. Malkin, K.E.: Criteria for the center for a certain differential equation. (Russian) *Volz. Mat. Sb. Vyp.* **2**, 87–91 (1964)
14. Volokitin, E.P., Ivanov, V.V.: Isochronicity and Commutation of polynomial vector fields. *Siberian Mathematical Journal* **40**, 22–37 (1999)
15. Vulpe N.I.: Affine-invariant conditions for the topological discrimination of quadratic systems with a center. *Differential Equations* **19**, 273–280 (1983)
16. Vulpe, N.I., Sibirskii, K.S.: Centro-affine invariant conditions for the existence of a center of a differential system with cubic nonlinearities. (Russian) *Dokl. Akad. Nauk SSSR* **301**, 1297–1301 (1988); translation in *Soviet Math. Dokl.* **38**, 198–201 (1989)
17. Żoładek, H.: The classification of reversible cubic systems with center. *Topol. Methods Nonlinear Anal.* **4**, 79–136 (1994)
18. Żoładek, H.: Remarks on: The classification of reversible cubic systems with center. *Topol. Methods Nonlinear Anal.* **4** (1994), 79–136; *Topol. Methods Nonlinear Anal.* **8**, 335–342 (1996)