Analogical Reasoning as an Inference Scheme

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ANALOGICAL REASONING AS AN INFERENCE SCHEME

Abstract: Analogy plays an important role in science as well as in non-scientific domains such as taxonomy or learning. We make explicit the difference and complementarity between the concept of analogical statement, which merely states that two objects have a relevant similarity, and the concept of analogical inference, which relies on the former in order to draw a conclusion from some premises. For the first, we show that it is not possible to give an absolute definition of what it means for two objects to be analogous; a relative definition of analogy is introduced, only relevant from some point of view. For the second, we argue that it is necessary to introduce a background over-hypothesis relating two sets of properties; the belief strength of the conclusion is then directly related to the belief strength of the over-hypothesis. Moreover, we assert the syntactical identity between analogical inference and single case induction despite important pragmatic differences.

Keywords: analogy, induction, reasoning mode, similarity, taxonomy

1. Introduction

Reasoning by analogy is a very usual mode of reasoning, explicit or implicit in epistemic practice. Many examples punctuate the history of science, for instance the parallelism of structure between sound and light or between hydraulic and electric systems. Some typical applications are at work in everyday life, for instance when a child learns a new language or when a lawyer compares different normative situations. Various analogies are used to exemplify some ideas, for instance when comparing natural and artificial selection. Finally, analogies help to form perceptions, to build taxonomies, to infer relations or to solve problems.

More precisely, three functions of analogies and analogical reasoning may be considered according to their aim in the reasoning process. The *heuristic function* consists in suggesting ex ante the possible existence of some new property possessed by an object when it is similar to another one in other respects. The *argumentative function* intends to sustain ex post the belief in a new property attributed to some object on the basis of its similarity to another object. The *didactic function* aims at providing a simple, compact and evocating image, either realist or poetic, of some complex phenomenon, in order to fulfill a communicative or a pedagogic aim.

Of course, any analogical reasoning can be judged as more or less relevant. Hesse (1966) asserts that some deep analogical reasoning forms the core of scientific research. Conversely, Bouveresse (1999) observes that several analogies in sociology result from a fanciful mode of reasoning. A majority of observers think that analogies and analogical reasoning have to be accepted or discarded through a one by one judgment. More constructively, some epistemological works try to link and even to reduce analogical reasoning to more classical ones, while others view it as a specific reasoning mode.

This paper relies on several previous philosophical or logical works, although it departs clearly from them on some important points. The analysis of Quine (1969) is a starting point in order to show the

difficulties of a logical definition of absolute analogy, or similarity. However our proposition of defining analogy by a relative concept departs from Quine's.

The most current intuitive theory of analogical inference states that a good analogical inference relies simply on a good analogy, associated to the fact that the two objects share many common properties. The structure of an analogical inference may then be associated to a general kind of enthymematic reasoning, relying on a general principle of analogy stating that objects sharing many properties generally share other properties. It is obvious that in many situations this way of reasoning leads to unwanted conclusions: some additional conditions are required. Several authors tried to add structure in the analogical inference in order to list the properties which make it more robust and relevant.

Hesse (1966) proposed a tabular representation of an analogical argument, which separates the source domain and the target domain, each domain including a set of objects, properties and relations. She defines the "vertical relations" as the relations between within each domain (such as causal relations between properties) and the "horizontal relations" as the relations between the domains (such as similarities between objects). Then, she formulates several qualitative requirements in order for an analogical argument to be acceptable. Bartha (2010, 2013) proposed a synthesis and a criticism of her requirements and built a more complete analysis of analogy. He listed eight "commonsense guidelines" of analogical reasoning and criticized their vagueness and internal contradictions. He proposed a complex "articulation model", focused on the different types of vertical relations in the source and the target domains, in order to give a more formal account of different kinds of analogical reasoning. Our paper follows the same goal of clarification and shares the importance of the vertical relations between the properties of the objects but departs largely from Bartha's paper: the paper assumes that reasoning by analogy obeys the same syntactical principles whatever its field of application or its function. These principles will be expressed in a formal way by avoiding as much as possible all ill-defined or ambiguous concepts (e.g. "essential", "causal", "relevant", "critical") on which Bartha's work relies. The importance of avoiding redundancy in analogical reasoning is also a key point, which misses clarity in Bartha's analysis.

The idea of non-redundancy is the starting point of Davies & Russel (1987), who proposed an analysis of analogical reasoning based on "determination rules", linking the shared properties and the projected properties of the analogous objects. It results in a deductive account of analogical reasoning. In the same spirit, our paper considers these determination rules as a central concept but compares and proposes different expressions of those rules and concentrates on a more general inference scheme pattern, which can be compatible either with a deductive or with a non-deductive framework.

The paper proposes an original unified view of analogical statements: it shows how different kinds of analogical statements, including those where the properties of the analogous objects are *prima facie* different, can be casted into a same definition. An "analogical inference" or an "analogical argument" relies on an analogical statement in order to transfer some additional property from one object to the other, by taking into account an explicit background hypothesis. These two steps are strongly linked since an analogical statement may prepare an analogical argument, contrary to a mere similarity. The complete procedure allows to evaluate the rational belief in its conclusion with regard to its assumptions.

Our main original thesis is that analogical reasoning should neither be reduced to any such and such existing or new logic, nor associated to a specific relation of entailment. It is an inference scheme general pattern, which can be validated by different kinds of relations of entailment, according to the belief status of a background over hypothesis.

The rest of the paper is composed of four parts. In the first part a general framework is introduced and different forms of analogical statements are considered, an absolute analogy between two objects is first defined and soon discarded by a relative one (§2). The second part deals with analogical inference: a general inference scheme pattern is sketched linking target and source properties through a background hypothesis, a deductive specification of this pattern using a specific over-hypothesis as the relevant background hypothesis is detailed, and a probabilistic account of this inference scheme is given (§3). The third part compares analogical reasoning to single case induction (§4). In a fourth part, a comparison with some logical and AI works is proposed (§5). Finally, some conclusions about the specificity of our approach are sketched and some insights for future analysis are suggested (§6).

2. Analogical statement

2.1. General framework

We adopt the framework of the first order logic. We assume the existence of a universe **X** of objects denoted *X*, *Y*, etc., which are the constants of the language. Objects can be material (people, cars, trees) as well as symbolic (numbers, propositions, values). Objects can be specific (John, my car, the Eiffel tower) or generic (a man, a car, a monument). Note that a specific object is just a given entity while a generic object is already a class of specific ones.

We suppose moreover that a set **P** of properties, defined on the universe **X** of objects and denoted *P*, *Q*, etc. is given. Properties are predicates (which can be one-place or many-place). They can concern either material aspects (to be red, to be heavy) or symbolic ones (to be greater than 3, to be nice). For a given object *X*, a property *P* is said to be relevant or not whether it applies or not to that object. For instance, a ball is red or not, but this is irrelevant for a number: so the property “to be red” is undefined for a number. These predicates may be two-place or *n*-place predicates, corresponding to relations between two or *n* objects, for instance a child may be related to his father by a relation of “sonness”.

An analogy always relates in an oriented way two or more objects belonging respectively to two different domains, the “source domain” and the “target domain”. Two types of relations are introduced when linking these domains (Hesse, 1966). The horizontal relations link similar properties present in the two domains. The vertical relations link different properties present in the same domain. An analogical statement can be of different types.

The simplest analogical statement is called “notional analogy”. It expresses that “*A* is like *B*”, written “ $A \sim B$ ”, and just states that there is a specific kind of similarity between two specific objects (*John is like Ophelia*) or between two generic ones (*an airplane is like a bird*) or even between mixed ones (*this building is like a ship*). The two objects are compared as concern their relevant properties. For a given property, the two objects are covalent if both satisfy it or do not satisfy it, and contravalent in the other case.

A more elaborate form is called “relational analogy”. It expresses that “*A* is to *B* what *C* is to *D*”, written “ $A:B :: C:D$ ”, and actually points to a similarity between two couples of objects. It can concern specific objects (*Dante is to Italy what Shakespeare is to England*), generic ones (*a hoof is to a horse what a foot is to a man*) or even mixed ones (*beer is to Belgium what wine is to France*).

This form is in fact not logically different from the previous one since it can be re-written as a notional analogy between two couples: “ $(A,B) \sim (C,D)$ ”. Relational analogy has often been assimilated

with analogy in general, since it gave its name to the concept in Aristotle's work, *αναλογία* applying to an identity in proportion. That is why it is often called "proportional analogy" in the literature, though it does not rely exclusively on the numerical concept of proportion.

An analogical statement can be extended with the same method to a t-uple of objects, and stays always reducible to the simplest notional form. Hence, an analogy may concern two formal models, where a model represents a multidimensional generic object. It is called "structural analogy" and is written $[A] \sim [B]$ where $[X]$ is a model. Such a structural analogy can be defined at two levels. It is "formal" when it concerns only the respective structure of related variables and equations; Hempel (1965a) speaks of a nomic isomorphism. It is "substantial" when some common interpretation is involved. For instance, consider the "Newton law" (the force between two bodies is proportional to their masses and inversely proportional to the square of their distance). It has a substantial analogy with the "Coulomb law" (the force between two electric charges is proportional to their charges and inversely proportional to their distance) since they both reflect attractive forces between masses or charges. It has only a formal analogy with the "traffic law" (the traffic between two towns is proportional to their population and inversely proportional to their distance) since the underlying phenomena have no common interpretation.

Likewise, the analogy between the "Fechner's law" in psychology and the law of decreasing utility in economics is substantial since both are interpreted as a marginal effect of a material stimulus on a subjective state of mind. Conversely, the analogy between the Lotka-Volterra model in biology and the Goodwin model in economics is formal since no common phenomenon is at work despite their similar structure.

2.2. Absolute analogy

Tracing back to Joyce ("Principles of logic", 1936), quoted by Norton (2018), an analogy has often been considered as an absolute judgement such as " S_2 resembles S_1 in being M". It may differ from mere similarity on the basis that it is a judgement intended to prepare an analogical reasoning: " S_1 is P, S_2 resembles S_1 in being M, therefore S_2 is P". Hence for instance, it may involve the use of more criteria such as multiple similarities or few counter-similarities.

However, following Quine (1969), it can be shown that it is not possible to propose any logical definition of absolute analogy, since it is not even possible to give a logical definition of the weaker notion of similarity. The simplest intuitive definition is:

- (1) $A \sim B$ iff there exists some property P such that $P(A) \ \& \ P(B)$

Analogy is defined by the existence of a property commonly shared by both objects. Of course, it is an equivalence relation but this definition is far too lax and is even trivial: it is always possible to find a common property between any two objects. For example: *a tiger is like a zebra, they have stripes*.

A more restrictive attempt is the following:

- (2) $A \sim B$ iff for any property P, $P(A) \ \& \ P(B)$

But this covalence of properties leads to an extreme situation which reduces analogy to identity.

Quine (1969) proposes to try an intermediary definition of similarity:

- (3) $A \sim B$ iff A and B have "many" common properties

But, as he highlights, this notion is too vague because one cannot tell how many properties are required. In fact, the question is to determine what counts as a property. If any set of objects counts as a property, then any two objects will be members of an arbitrary number of sets and will share “many” properties. If one restricts the type of sets to the properties collecting similar objects, we are led to a circular definition.

An attempt to escape the problem faced by those general definitions may be to restrict the admissible properties to a unique subset of externally defined properties that are relevant for any object, say W . Hence, amended formulations of (1), (2), (3) may be:

(1a) $A \sim B$ iff there exists a property P belonging to W such that $P(A) \ \& \ P(B)$

(2a) $A \sim B$ iff for any property P belonging to W , $P(A) \ \& \ P(B)$

(3a) $A \sim B$ iff A and B have “many” common properties P belonging to W

Clearly, (1a) may prevent the triviality of (1), (2a) may prevent the reducibility of (2) to identity, (3a) may prevent the arbitrary nature of (3). The question is then to be able to give a relevant definition for a universal W since universality is required for a definition of absolute analogy.

An intuitive way to do this would be to identify W with the set of “natural kinds”, also considered by Quine (1969), who pointed the intuitive relationship between this concept and the notion of similarity: a natural kind is a collection of similar objects and reversely similar objects seem to be those very objects which are instances of the same natural kind. However, this approach leads to many philosophical issues and is highly controversial, unless one accepts a very specific essentialist position (see Bird and Tobin, 2015, for a critical presentation of this position).

Firstly, the properties which define natural kinds are supposed to be the properties which are “really important” for classifying objects in “genuinely natural ways”. But critics of that position deny that any of our classifications is natural. Classifications are mere human tools built within current language and science for practical purposes, not hard categories in the world as for Plato. Dupré (1993) argues for instance in favour of a “ubiquitous realism,” stressing that there are always a large number of ways to build taxonomies and kinds, depending on the theoretical interests pursued. In fact, what is called “natural kinds” does not correspond to essences or necessities existing in nature but to evolving categories that are established according to complex pragmatic optimizations.

Secondly, Quine’s argumentation leads to the conclusion that defining natural kinds relies again on too vague notions or to obvious circularities with the notion of similarity. Hence there is neither a philosophical nor a logical way to define “natural kinds”, and to identify which set of properties could be a proper W .

But the impossibility to find a formal definition of absolute analogy comes moreover from the debates necessarily open when an analogy is refuted by a counter-analogy. The counter-analogy suggests a better partner than the one proposed, for the source as well as for the target.

For instance, for a notional analogy:

- *Bruges is the Venice of the North; it is a city built on canals.*

- *No, Bruges is not the Venice of the North; it is a city that never had any major economic influence.*

-It is Antwerp which is the Venice of the North, since it was a European economic capital like Venice.

Likewise, for a relational analogy:

- Freud is to psychoanalysis what Piaget is to cognitive developmental psychology, its most well-known inventor;

- No, Freud is not to psychoanalysis what Piaget is to cognitive developmental psychology, he was not its first inventor.

- It is Joseph Breuer who is to psychoanalysis what Piaget is to cognitive developmental psychology, since he is the first inventor of psychoanalysis according to Freud himself.

These debates underline the vacuity of vindicating any absolute analogy: even if A and B were similar with respect to one point of view, they would usually differ from another point of view. No pair of non-identical objects are similar from the standpoint of all possible properties, even if we limit these properties to current categories: is an apple similar to a pear, because it is a fruit, or to a tennis ball, because it is round?

2.3. Relative analogy

So we are led to consider that any analogy has always to be expressed with respect to one property or to a domain of properties. This is obviously an appropriate answer to the debates on analogical statements: there is no more any assertion stated from a universal standpoint but only relative points of views on the similarity between two objects.

The simplest way to represent relative analogy for notional analogy is again:

(1b) $A \sim_P B$ iff there is some property P such that $P(A) \ \& \ P(B)$ ¹

For instance, *an apple is like a pear, relatively to "fruitness", they are fruits.*

For relational analogy, the condition states:

(1b') $A:B ::_R C:D$ iff there is a relation R (which is a two-place predicate), such that $R(A,B) \ \& \ R(C,D)$

For instance, *Paul is to Ana what Bob is to Julia, relatively to "sonness", he is her son.*

But on second thoughts, this seems to be a very restrictive way to express things. Indeed, an analogy is significant because it spotlights that two objects share one particular property among a list of other possible properties all pertaining to a certain way of describing the objects. These two cars are analogous relatively to the colour if they are both blue, these two animals are analogous relatively to the species if they are both dogs. But it would be rather odd to say that these two cars are analogous relatively to their "blueness" or that these two animals are analogous relatively to their "dogness".

What is meant is the fact that these objects are analogous relatively to some "point of view" which can be expressed by a set of related properties (for example colours or animal species). They share the same property in this set while they could have two different ones (one car could be blue and the other one red, one animal could be a dog and the other one a cat). What is stressed by the analogy is that it

¹Be careful not to confuse definition (1b) with definition (1) whose form is very near. The latter was intended to define analogy in an absolute form. The former uses the same condition (the existence of a common property) but states that the two objects are similar only relatively to this property. That's why it is denoted $\sim_P$.

is not the case that they have two different properties inside the set which is considered. So these cars are analogous relatively to their colour, these animals are analogous relatively to their specie.

Let's define a domain Z as a set of possible disjoint properties² (or disjoint values of one property) which are associated with the same point of view. The relativization of any analogy to a given domain expresses the speaker's intention to choose a specific point of view and her intention to speak only with regard to this aspect of the world. A "point of view" is a mental attitude consisting of applying a filter on the properties of things or events. This notion is represented by the set Z which is not any set of properties, but a set of disjoint properties which are then correlated through this disjunction. The combination of two points of view by considering the product of two set Z and Z' is indeed also a point of view.

Relative notional analogy can then be defined by:

(1c) $(A \sim_Z B)$ iff there exists a property P within the domain Z, such as $P(A) \ \& \ P(B)$

For instance, *an apple is like a pear, with regard to vegetal kinds, they are fruits*

Likewise, relative relational analogy is defined by:

(1c') $A:B ::_Z C:D$ iff there is a relation R within the domain Z, such as $R(A,B) \ \& \ R(C,D)$

For instance, *Paul is to Ana what Bob is to Julia, with regard to family relationship, he is her son.*

One can check that relative notional analogy expressed by (1c) or by (1c') satisfies all the minimal principles required for a relevant definition of analogy. It can be used for any kind of object and satisfies the usual theoretical properties:

- it is neither reduced to identity nor to triviality.
- it is not circular since Z is chosen by the agent with respect to the point of view she wants to stress and does not need analogy to be defined.

Moreover, it is easily checked that notional relative analogy is an equivalence relation satisfying the following principles:

- Reflexivity: $A \sim A$ (*an airplane is like an airplane*)
- Symmetry: if $A \sim B$, then $B \sim A$ (*if an airplane is like a bird, then a bird is like an airplane*)
- Transitivity: If $A \sim B$ and $B \sim C$, then $A \sim C$ (*if an airplane is like a bird and a bird is like a bee, then an airplane is like a bee*).

For a relational analogy, these principles become:

- Horizontal Reflexivity: $A:B :: A:B$
- Horizontal Symmetry: if $A:B :: C:D$ then $C:D :: A:B$
- Horizontal Transitivity: if $A:B :: C:D$ and $C:D :: E:F$ then $A:B :: E:F$

These principles are not pre-requisites for our definition of relative analogy but result logically from our definition which has its own justification. Hence we believe that it would rather need strong justifications to reject them. But these principles can all be considered at least to be compatible with a relevant logical analysis of analogical statements. This needs to be analyzed independently of any pragmatic considerations of assertability and intentionality.

² The fact that these properties are disjoint means that no object can satisfy simultaneously two properties.

Indeed, Reflexivity is obvious: an object is at least analogous to itself, even if asserting this analogy may be strange and useless in the conversation. It is generally accepted that analogy, although it logically relies on a similarity *ex post* in a symmetrical way is not symmetrical *ex ante*, contrary to the general concept of similarity. Analogy is associated with an illocutionary intention that distinguishes a part which inherits a property from a part for which the property is well known. When one says “*your eyes are blue like the sky*”, the well-known property of the intense blue of the sky is attributed to someone for making her a compliment. However, this remark concerns the intentional aspect contrary to a purely formal syntactic point of view. Indeed, if one will generally not assert it or use it for building an analogical reasoning, the sentence “the sky is blue like your eyes” is true if the former sentence is true. Transitivity comes from the fact that Z is defined as a list of disjoint properties. It is noticeable that transitivity is not respected with the usual definition of absolute analogy since the properties shared by A and B are not necessary the same than those shared by B and C. This cannot be the case here. It is a useful principle for reasoning on analogical statements, chaining one to another. Since analogy is based on similarity of properties, this principle is quite intuitive, and it has to be clearly distinguished from transitivity between “close measures” which is not respected.

In a vast philosophical litterature, contextualistic positions have been vindicated in order to explain that knowledge, belief, or truth are related to a « context ». It is a relativist conception which refers to the idea that the truth or belief conditions of sentences « vary in certain ways according to the context in which they are uttered » (Keith DeRose, 1999). But even if it looks to have the same flavor, the notion of domain used in relative analogy is not contextualistic: it does not express the fact that the absolute analogical assertion is only true in a certain context but that there is no such logical thing as an absolute analogical assertion. Speaking about analogies in a coherent way is always speaking about relative analogies, and those are more or less acceptable independently of any context.

It could be possible to argue that the situation can be a little bit more complex if one considers several common properties or relations involved in the relative analogy. For instance:

- *An apple is like a pear, with regard to vegetal kinds and colours, they are yellow fruits.*
- *Paul is to Ana what Bob is to Julia, with regard to family relationship and social relationship, he is her son and he doesn't care about her.*

But these situations may be reduced to the simplest one by using a conjunction of predicates. For instance: to be yellow and to be a fruit for the first example and to be a son and to not care about his mother, for the second. The unique domain Z is then the product set of disjoint conjunctions formed by using these two predicates or relations. For instance in the two previous examples and even if it may appear semantically heterogeneous: the domain of colours and kind of botanic species, the domain of kind of family relationships and attitudes toward someone else.

The situation may even be more complex when the common properties involved in analogical statements are different, though appearing to “correspond”, as several authors (Hesse (1966), Juthe (2005), Bartha (2010)) pointed out. The problem is then to define rigorously this intuitive but vague notion of “correspondence” between properties, and to ask if this should induce a simple refinement or a radical change in our general definition of analogical statements.

We can start with a simple example. While lungs (A) make it possible to breathe in the air (P_1), gills (B) make it possible to breathe in the water (P_2). The lungs therefore seem to be analogous to the gills, although their properties are indeed different: the associated chemical transformations are not the same. This is because they share a common property, “making extraction of oxygen possible”, from water or from air. Each of those properties is an application of this general property to animals that are aerial or aquatic. It is in this precise meaning that they “correspond”. This idea may be made more precise.

The property to breathe in the air and the property to breathe in the water both imply the property to breath. Hence lungs and gills share a same general property: to make possible to breath, which is one of the alternative functions of vital organs (others being for instance digestion, blood circulation...) whose set is the domain from the point of view of which the analogy is expressed. Let’s make it more general:

Let Z be a domain of properties over a domain of objects D , Z_1 be the restriction of Z on $D_1 \subset D$, Z_2 be the restriction of Z on $D_2 \subset D$. Then: if $P_1 \in Z_1$ and $P_2 \in Z_2$ it will be said that “ P_1 corresponds to P_2 in Z ” if there is a property P within Z , such that $\forall X P_1(X) \rightarrow P(X)$ and $\forall X P_2(X) \rightarrow P(X)$. Then, we see that: If $[P_1(A), P_2(B), (P_1 \text{ corresponds to } P_2 \text{ in } Z)]$, then $(A \sim B)_Z$ because $P(A)$ and $P(B)$.

Thus the cases of analogies with corresponding but different properties may be easily casted in the general definition of analogies. This reduction to the simpler general case can easily be extended to the correspondence between properties Q_1 and Q_2 transferred by analogical inferences.

This form of analogy is frequent for scientific models, for instance when the equations of different domains express the same mathematical relations between obviously different but “corresponding” measures, for example those of electric and hydraulic networks. One can say they share a common point of view, that of “constrained flows”. The electric intensity is like the hydraulic debit because they imply both a quantity of fluids. The tension is like the pressure variation because they imply both a potential of movement. Moreover, the law of nodes (for intensities as well as debits) and the law of loops (for tensions as well as pressure variations) apply to both of them. Finally, the Ohm law linking linearly intensity to tension is analogous to the law linking linearly debit to pressure variation.

3. Analogical inference

3.1. The general inference scheme

Typically, analogical inference consists in using a similarity between two objects as a premise for inferring new similarities as conclusions. Then, in a way, it is considered that an analogical statement is a kind of similarity judgement which justifies its extension to other properties: it is this point which differentiates “mere similarities” from “analogies” or “relevant similarities”. As expressed by Bartha (2010): “An analogical argument is an explicit representation of analogical reasoning that cites accepted similarities between two systems in support of the conclusion that some further similarities exist”.

Analogical inference uses an analogical statement to transfer the properties of an object to another object. The analogical statement may be explicit or not, and represented by facts which imply it, as illustrated for instance by the “violinist argument” (Thomson, 1971). Let Q be a new one-place predicate or S a new two-place predicate in $\mathbf{P}$. In the two basic forms (notional analogy and relational analogy), an analogical inference states that:

$$(4) [A \sim_z B, Q(B)] \rightarrow Q(A)$$

or

$$(4') [A:B ::_Z C:D, S(C,D)] \rightarrow S(A,B)$$

Analogical inference inherits the intentional asymmetric nature of analogical statement: it is based on the property of the “source” transferred to the “target”. The symbol $\rightarrow$ denotes a relation of entailment between premises and conclusion, which has to be further characterized. . The scheme can be easily extended to structural analogy.

For instance, from the fact that *a pear is like an apple relatively to fruitness* and that *an apple can be eaten*, one could infer that *a pear can be eaten*. Likewise, from the fact that *a pear is to a pear tree what an apple is to an apple tree relatively to production* and that *an apple tree grows by sowing apple seeds*, one could infer that *a pear tree grows by sowing pear seeds*. For the very same reasons that led us to introduce a domain Z listing the properties considered for defining the relative analogy which is the premise, we introduce a domain Z' listing the properties that will be considered for the conclusion. The analogical reasoning (4) can then be developed into:

$$(5) [P \in Z, Q \in Z', P(A), P(B), Q(B)] \rightarrow Q(A)$$

Why should we accept this inference scheme? Our answer does not rely on the construction of a new consequence relation but on an external supplementary hypothesis HE, used by the person who argues in favour of the conclusion.

Our approach is that this inference scheme has to be completed by an external hypothesis HE that will be integrated in the reasoning as follows (for notional analogies):

$$(6) [(A \sim_Z B), Q(B), HE] \Rightarrow Q(A)$$

which can be developed into:

$$(7) [P \in Z, Q \in Z', P(A), P(B), Q(B), HE] \Rightarrow Q(A)$$

The symbol $\Rightarrow$ is used instead of $\rightarrow$ in order to acknowledge the fact that including HE in the premises is intended to give a better epistemic status to the inference scheme. The extension to relational analogies or to analogies between t-uples is obvious, replacing P and Q by two-place or t-place predicates.

The formulas (6) and (7) are the general inference scheme patterns that can be transformed into a formal inference scheme of a precise logic as soon as the arrow $\Rightarrow$ is replaced by a precise relation of entailment.

We will first identify the content of HE, then consider the relationships between HE and two kinds of relations of entailments which can validate it, deduction or probabilistic entailment. These two cases are to be considered only as typical examples of the several ways by which this general inference scheme could be activated, our general thesis being independent of which one is actually chosen by the agent.

Other logical frameworks such as non-monotonic logic (Krauss, Lehman and Magidor, 1990) or belief revision (Alchourrón, Gärdenfors & Makinson, 1985), could be considered in further works. Our point is that analogical reasoning should not be reduced to any such and such existing or new logic, associated to a specific relation of entailment, but to an inference scheme general pattern, which can be validated by different kinds of relations of entailment.

We have then to identify the structure of the background hypothesis relevant for validating this inference scheme according to a specific logic or relation of entailment.

3.2. Structure of the background hypothesis in a deductive framework

In order for HE to be relevant for explaining how analogical inference runs, we impose that it obeys the two following additional principles:

- HE must not lead to “trivialize” the reasoning in a way which would make unnecessary the consultation of one of the other premises. This is the “non-redundancy” condition.
- The empirical protocol for believing in HE must be coherent, and one must be able to describe it.

We consider now more and more sophisticated expressions of HE. Let’s examine a first candidate.

$$(8) \text{ HE1 : } \forall X [P(X) \rightarrow Q(X)]$$

Using HE = HE1 within (7) makes the analogical inference scheme valid in deductive logic. But trivialization is obviously at work in this extreme case. Due to the redundancy of the premises, the reasoning becomes both trivial and deductive. Analogical inference becomes a “focusing” operation from a prior generic belief on a specific case: knowing that P(A), the conclusion Q(A) is acquired without any need to refer to P(B).

To avoid redundancy, Davies & Russell (1987) proposed an interesting candidate for HE, called the “determination clause”. It is initially written as follows:

$$(9) \text{ HE2: } [\forall X (P(X) \rightarrow Q(X))] \text{ or } [\forall X (P(X) \rightarrow \neg Q(X))]$$

Like for HE1, using HE = HE2 within (7) makes the analogical inference scheme valid in deductive logic but it involves all premises: it is necessary to use Q(B) to infer Q(A). For instance, if HE2 states that: *any golden object is insensible to acid or any golden object is sensible to acid*, the fact that my watch is golden and insensible to acid implies that yours which is also golden is insensible to acid.

How is it possible to acquire the knowledge of a hypothesis like HE2? If all X such as P(X) have been observed, the fact that Q(X) or -Q(X) for all these X is already known and there is no need for the analogical reasoning to know that Q(A). Of course, if -Q(x) is the case, HE2 is irrelevant for this analogical reasoning which is false. But if the fact that all X are such that Q(X) or if the fact that all X are such that -Q(X) is already known, the belief in HE2 is reduced to the belief in one of the two possibilities, either to HE1 or to its contrary and we are back to redundancy. If only some X such as P(X) have been observed and if all associated are such that Q(X) or all such that -Q(X), the belief in HE2 may stem from an inductive process leading possibly to a probabilistic belief but again, the belief will concern only one of the two possibilities mentioned in HE2. Every empirical observation which does not refute HE2 will lead to believe either in its first part or in its second part. Then there is no way to learn a hypothesis such as HE2 as it is.

The problem comes from the fact that H2 is not the right over-hypothesis that one needs in order to complete the analogical reasoning. This is again a question of the right level of properties to express things. The domains Z and Z’ whose important role has been noticed have a role to play in the over-hypothesis. A more relevant over-hypothesis, whose major difference with HE2 is not mentioned in Davies & Russel (1987), is then the following:

$$(10) \text{ HE3: For any property P in Z, for any property Q in Z', } [\forall X (P(X) \rightarrow Q(X))] \text{ or } [\forall X (P(X) \rightarrow \neg Q(X))]$$

Again, using HE = HE3 within (7) makes the analogical inference scheme valid in deductive logic.

Using HE1, HE2 or HE3 as over-hypotheses HE makes analogical reasoning look like an enthymeme, i.e. an inference to which a premise lacks (according to the modern meaning of a concept created by

Aristotle with the broader meaning of “deductions from likelihoods and indices”, see Boyer, 1995). Musgrave (1989) was one of the first to suggest the transformation of inductive inferences into deductive enthymemes, and in the present context, analogical inferences are of the same kind. But we will see in §3.3 that it is not the most general situation.

Considering the formalization of HE3, the apparent use of second order logic does not raise specific problems here as long as in this framework sets of predicates within Z and Z' are supposed to be finite. Indeed, in this case, quantization on predicates can be considered as a scheme of axioms that can be developed into a finite list of axioms of first order logic, each explicitly mentioning one of the predicates in each domain of quantization.

Considering its meaning, HE3 is a meta-level “over hypothesis”, which does not rely on the direct relationship between P and Q , but on the relationship between the set of properties Z and the set of properties Z' . Due to the quantification over Z and Z' , HE3 is actually a set of hypotheses for each couple of predicates, one in Z and one in Z' . Z and Z' are lists of disjoint properties. Then each object X can satisfy at most one property in Z and at most one in Z' . HE3 means that all the objects satisfying one given property in Z must satisfy one same other property in Z' . In a way, HE3 links each property in Z with (at most) one property in Z' .

For instance, consider that this car is like mine, it's a Chevrolet Silverado, and I want to infer that it costs nearly the same price than mine. Davies & Russel's HE2 hypothesis would state: every Chevrolet Silverado costs between 28 000€ and 32 000 € or no Chevrolet Silverado costs between 28 000 € and 32 000 €; my Chevrolet Silverado costs 30 000 €; hence this other Chevrolet Silverado should cost between 28 000 € and 32 000 €. The proposed HE3 over-hypothesis states instead: the cost of any car of a given type is situated in a range of prices; my Chevrolet Silverado costs between 28000€ and 32000 €; hence this other Chevrolet Silverado should cost between 28000 € and 32000 €. Ranges of price are defined exogenously by the brand of cars.

It is exactly the situation which was suggested by Goodman (1947) in his “prospects for a theory of projection”. Suppose that we are interested in the colors k of the marbles drawn from a bag h which belongs to a stack of bags. What he calls “over-hypothesis” H of a hypothesis G such as “all the marbles in the bag B are red” is a hypothesis H such as “every bagful of the stack is uniform in color”. Goodman considers the situation where many bags of the stack have been observed (but not the bag B itself) and where this observation leads to confirm H . Having H in mind, observing a red marble from the bag B will support G . In this case, Z is the set of predicates “to belong to the bag number h ” and Z' is the set of predicates “to be of color k ”. The over-hypothesis is the fact that each bag is associated with only one color.

Let's come back to the example of the car. HE2 states that every Chevrolet Silverado costs between 28.000€ and 32.000 € or no Chevrolet Silverado costs between 28.000 € and 32.000 €. As noticed, knowing that would mean having examined all (or a very large number) of Chevrolet Silverado and noted that all cost between 28.000 € and 32.000 € (otherwise this would be incompatible with the premise that mine costs between 28.000 € and 32.000 €). But in this case, it is not HE2 that we would believe in but in “Every Chevrolet Silverado costs between 28.000€ and 32.000 €”. Imagine now that we have observed a large number of different models of cars and noted that all cars of the same type cost approximately the same price. This observation, which is really plausible, would lead to the general hypothesis: For any car of a given type, the cost is in the same price range.

So, if Z is a list of types of cars and Z' a list of mutually exclusive price ranges, this would be expressed in HE3 form as:

[For any car A, for any price range T, [$\forall \text{car}, (\text{car of model A} \rightarrow \text{price of the car is in range T})$] or [$\forall \text{car}, (\text{car of model A} \rightarrow \text{price of the car is not in range T})$]]

We immediately see that, contrary to HE2, the process allowing to acquire the belief in HE3 is realist and it does not lead to a mutilation of the hypothesis in only one part of the alternative.

Moreover, it appears that HE2 is very strange. In absence of the premise “my Chevrolet Silverado costs 30.000 €”, one does not see where the hypothesis “every Chevrolet Silverado costs between 28.000€ and 32.000 € or no Chevrolet Silverado costs between 28.000 € and 32.000 €” could come from. Why using this particular range? The only answer seems to be because many Chevrolet Silverado for which price lies in this range have been observed. But in this case, why stating the whole HE2 and not only “every Chevrolet Silverado costs between 28.000 € and 32.000 €”? On the contrary, HE3 does not mention any particular range. It just states that, having made a partition of all possible prices in some arbitrary ranges for any car, each model of car is associated with only one range. This is not only more reasonable but intuitively, this is exactly the way we think in various subjects.

Indeed, such an over-hypothesis frequently states a regularity between classes of objects already defined:

- nationality determines the mother tongue;
- age and skills determines the class followed by pupils;
- species, gender and age determine what animals weight;

Each one of these examples is an example of HE3: A partition of a set Z of possible properties is set in a one to one relation with a partition of another set Z' of possible properties.

This type of belief is usually acquired in a very natural way: having seen many objects satisfying one property of a list of properties of the same type, it appears that each of them also satisfies one property of another list and that to each one first property, the same second property is always attached. This process is typically inductive in that we infer a general law from a limited (but possibly large) number of cases. But the over-hypothesis could also result from an abductive process since the conclusions may be the generic best explanation of the antecedents.

One may worry about the fact that this way of conceiving an analogical inference is leading to an infinite regression, since it relies on an over-hypothesis which has itself to be justified. But it is not the case since foundationalism is not the goal here. The way to acquire HE3 is not itself a part of analogical reasoning: what is required is only the fact that it is possible to attribute to HE3 a degree of belief by a clear empirical protocol, without redundancy with the analogical reasoning that it supports. But this degree of belief is exogenous to the present analysis.

3.3 Probabilistic over-hypothesis

One could be willing to try avoiding complex meta-level hypothesis such as HE3, simply by weakening the too strong HE1 and giving it a probabilistic status such as:

$$(11) \text{ HE4: } \Pr(Q(X) / P(X)) = \alpha$$

Using HE = HE4 within our general inference scheme (7) leads to the conclusion that $\Pr(Q(A)) = \alpha$, according to a simple probabilistic inference rule. It validates that the inference scheme (7) if $\Rightarrow$ is an entailment relation associated to a degree of probabilistic belief α , assumed to be part of the background belief of the agent. This inference is no more deductive since it is now defeasible: a further fact concerning A or B may change the probability of Q(A) (as stressed for instance by Hempel, 1965a). Another way to putting it is to realize that there may be several competing HE4, based upon other

properties $P'(X)$, such that the belief in the conclusion changes when $P'(X)$ occurs. But this formula still leads to redundancy: it is not necessary to have any information about B in order to get a probability degree over $Q(A)$. So it does not do the job better than HE1.

The only way to escape redundancy is to use a meta-level hypothesis such as HE3 (HE2 having been shown to be irrelevant). But in main - if not all - empirical situations of science or real life, we do not have such a hypothesis in our background belief but only uncertain versions of it: we do not believe for sure that species determines what an animal eats and we may only guess that such and such list of physical properties of planets may determine the existence of life (example drawn from Bartha, 2010).

Probabilistic degrees of belief is one way of expressing this general uncertainty. Other ways, such as non-monotonic beliefs, could be explored but do not change the nature of our general thesis, which concerns the inference scheme pattern.

We can formulate this reasoning by assuming that the agent uses the following kind of hypothesis:

$$(12) \text{ HE5: } \Pr(\text{HE3}) = \alpha$$

Using $\text{HE} = \text{HE5}$ within (7) leads to the conclusion that $\Pr(Q(A)) = \alpha$. Again, this inference scheme is not deductive, and is not an enthymeme, since it is defeasible by any new information concerning A or B. This feature is the basis which opens to analogical debates.

Four typical different situations which are worth to be analysed can then arise.

- Situation 1: $\Pr(\text{HE3})$ is so high that HE3 is “accepted” (meaning for instance that its probability is close to 1 as formalized by Adam’s semantics) (Pearl, 1988). Then the analogical reasoning will get a real strength and the conclusion will be accepted with the same strong degree of belief: *my car is a Twingo like yours. Its price is well below 10000 €. Hence, yours must also cost less than 10000 € because the model of a car determines its price.* The fact that the model of a car determines its price is almost certain so I can be almost sure of the conclusion.
- Situation 2: $\Pr(\text{HE3})$ can be simply higher than $\Pr(Q(A))$. In this case, the degree of belief in $Q(A)$ increases to the level of the degree of belief in HE3. This corresponds to a relative confirmation (Zwirn & Zwirn, 1996) in which the belief in the conclusion is strengthened while being not large enough to lead to accept it (absolute confirmation). The strength of the analogical reasoning in situation 2 is lower than in situation 1: *Bjorn and Anna are Swedish students. Bjorn speaks English fluently. So I can believe that Anna speaks also English very well* (because the nationality and the level of studies determine fairly well the general linguistic competences). A priori, I cannot know if Anna who is Swedish speaks English fluently and my degree of belief in it is low. On the other hand, I have a fairly good confidence in the fact that the nationality and the level of studies determine the linguistic competence. Then, when I learn that Bjorn, who is a Swedish student like Anna is, speaks English fluently, I increase my belief in the fact that so does Anna.
- Situation 3: $\Pr(\text{HE3})$ can be totally unknown : no degree of belief is attached to HE3. In this case, the analogical reasoning will have no proof value but may have a purely heuristic interest to help noticing a new possibility worth to be explored: *Mars and the Earth rotate around the sun not too far from it, they have similar gravity and surface temperature. There is life on Earth. Perhaps there is life on Mars* (because rotating not too far around the sun, having gravity and a surface temperature similar to those of the Earth are conditions in which life appears). It is clear that nobody today knows the real conditions that are necessary and sufficient for life to appear. So, the over-hypothesis here is very risky and even probably false. Nonetheless, it seems worthier exploring if Mars can harbour life than exploring if Pluto can, since the last is totally different from Earth.

- Situation 4: $\Pr(\text{HE3})$ is known to be very low or close to 0. In this case, the conclusion of the analogical reasoning is not taken seriously and is considered even as silly. This corresponds to cases where the analogical reasoning is considered as a bad reasoning mode: *the tiger has a tail as well as my cat. My cat is kind hence the tiger is kind* (because the tail induces the behaviour).

There may be many intermediate situations but the general principle is the same: the degree of belief in $Q(A)$ is determined by the degree of belief in HE3. Of course, HE3 is not always made explicit by the agent in her analogical reasoning. But even if not made explicit, a hypothesis of type HE3 has always to be in the background belief of the agent when following such a reasoning scheme with some consistency. It can also happen that an adversary makes this hypothesis explicit just in order to show the weakness of the reasoning: "to infer that, you need this background over-hypothesis, yet it is clearly absurd or has a very low probability".

Of course, because it always relies on relative analogies, reasoning by analogy is open to revision. Especially, the over-hypothesis can be modified in its structure as well as in its degree of belief. For instance, a counter-analogy, true for A and B in a domain Z^* different from Z, may relativize the conclusion of a first analogical inference, even if the belief in the over-hypothesis is high, insofar as this counter-analogy could be associated with another over-hypothesis which leads to an exception to the first one and lessens the belief in it. This is the case with the over-hypothesis "all bodies fall on the floor" which is refuted by balloons. The reason here is that some relevant causes ("hidden factors") necessary to explain the fall of bodies were ignored.

Ideally, a rational agent will consider the total evidence available to her to form her belief, which will lead directly or by discussion with other rational agents to attribute to the over-hypothesis a belief that already anticipates the possible counter-analogies.

One may also use analogical inference as a *reductio ad absurdum* reasoning. Let's consider a situation where the premises are: $P \in Z$, $Q \in Z'$, $P(A)$, $P(B)$, $Q(B)$, and where one has a degree of belief x that $Q(A)$ is false. Then one ought to have the same degree of belief that the properties P within the domain Z do not determine the properties Q within the domain Z' . Indeed, in this case, if HE3 was true then $Q(A)$ could not be false (because it is indeed deductive). If we have some degree of belief that $Q(A)$ is false, we therefore must have the same degree of belief that HE3 is false.

Hence, the syntactic expression of determination rules takes in charge many intricate intuitions of some philosophical works on analogical reasoning, and stresses the important notion of non-redundancy. However, as has been shown, Davies & Russel (1987) pointed this idea but they miss to analyze precisely why there is an important difference between an over-hypothesis like HE2 and an over-hypothesis like HE3 for expressing those rules without redundancy and to analyze the empirical process explaining how these over-hypothesis can be learnt. Their definition of a determination rule (*), on page 2 of their paper, is clearly equivalent to HE2 and not satisfying.

Moreover, Davies & Russel stick only to the deductive version of the determination rules, making analogical reasoning a purely deductive process. But they miss the fact that in most if not all empirical contexts, HE3 is uncertain and that one has to use a probabilistic version of the over-hypothesis such HE5 (or any other representation of uncertainty relevant in the context). This leads to make analogical reasoning a defeasible inference scheme. More generally, their analysis presents analogical reasoning as one single deductive inference scheme, while ours presents it as a general form of inference scheme, which may be validated within several logical framework.

Bartha (2010) criticizes the Davies and Russel's determination with the following argument: "*Scientific analogies are commonly applied to problems where we do not possess useful determination rules. In many (perhaps most) cases, researchers are not aware of all relevant factors*". But his argument leads to a confusion about the role played by these rules: even when an agent does not know them, or is uncertain about them, they always play a normative role for evaluating the strength of the analogical argument, by attributing degrees of belief to them or by showing that one cannot attribute any degree of belief to them as a counter-argument

4. Comparison with single-case induction

We first formalize the main reasoning modes in our general framework. Let H be a universal hypothesis such that $\forall X, [P(X) \rightarrow Q(X)]$. The following schemes hold for $A, B, C \in X$:

- single-case induction: $P(B), Q(B) \rightarrow H$
- deduction: $H, P(A) \rightarrow Q(A)$
- abduction: $H, Q(C) \rightarrow P(C)$

In single-case induction (the expression is used by Bartha, 2010), the hypothesis is confirmed by a single positive confirmation instance. By combination of single-case induction and deduction, one obtains: for $A, B \in X$, $[P(A), P(B), Q(B)] \rightarrow Q(A)$, which is precisely the raw form of reasoning by analogy. Syntactically, reasoning by analogy is then equivalent to single induction, where a positive confirmation instance of a hypothesis is confirmed by another positive confirmation instance. Abduction is a different scheme, when the inference goes in the reverse way.

Analogical reasoning and single case induction transfer a property from one object to another in exactly the same way, relying on the fact that both objects share already another property. A property Q observed for an object of "type P " is transferred on another object of "type P ", like in the standard example: *this raven in front of me is black, hence the next raven I'll see should be black (an instance of "all ravens are black")*. That does not mean that reasoning by analogy is reducible to a kind of induction but rather that each individual step of any enumerative induction consists in a reasoning by analogy. Single case induction is nothing else than a kind of analogical reasoning, but there are different ways to express such reasoning in the usual language since it can be set in different pragmatic contexts.

In single case induction, objects of the analogical statement are usually directly designed by the property they share according to this statement, preceded by a demonstrative pronoun of time, place, ownership etc.:

- This P is Q , then this other P is also Q
- These P are Q , then these other P are also Q

For instance, induction implies that *since my canary is yellow, yours should be yellow too*.

In other kinds of reasoning by analogy, which are the kinds which are the most popular because they seem different from classical examples of single case induction, objects are primarily designed by their name, or by any designator independent of the properties they are supposed to share according to the analogical statement, which are attributed to them in a sentence:

- A is P , B is P , B is Q , then A is Q
- A 's are P , B 's are P , B 's are Q , then A 's are Q

For instance, *a canary has wings and is able to fly, then since an airplane has wings too it should be able to fly*.

In single case induction, the property P shared by two objects is used as the “principal designator” (to be a canary). In reasoning by analogy, the shared property is used only as a secondary qualifier; it is explicitly mentioned as the shared property of two objects which are designated through another qualifier (to have wings for a canary). Common properties used in single case induction are pre-existing: they correspond often to a class inside a current taxonomy. These taxonomies and these classes have been built inside the language precisely because they are well suited for maximising causal effects with other properties, hence to build over-hypotheses of type HE. Common properties used in other analogical reasoning are usually selected at the moment when the reasoning is made. They are not used as current principal designators of any class in a usual taxonomy, and can be fanciful.

This helps understanding why single case induction is often considered more reliable than other analogical reasoning. The basic idea is that current categories are defined in a way which exactly takes into account the relation of determination between many properties that define each category. For example, the fact to be a canary has a lot of other consequences. This is why this category is useful. Hypotheses of type HE3 or HE5 linked to this kind of category are enough to help drawing inferences from the fact of knowing that something is a canary.

In a nutshell, single case induction is nothing else than a reasoning by analogy but in a normalized context where it relies on over-hypotheses that are linked to the properties which are used as principal designators for the objects which are considered. These over-hypotheses are well entrenched because they are linked to categories currently used inside the language. On the other hand, over-hypotheses used in other cases of analogical reasoning are linked to properties that are less general and hence less entrenched. Both reasoning modes are essentially of the same nature but used in different pragmatic contexts.

Of course, that does not mean that single case induction cannot be fanciful too: *since this stone is small, this other stone is small too*. The fact that the designator is a current category does not imply that all over-hypotheses that could be linked to it are relevant. The fact to be a stone has no effect on its size.

Finally, the role of principal designator can be contextual. Consider, for example, as principal designator “being a New-Yorker”, to whom we can attribute some secondary properties such as “wearing purple shorts”. In general, the first will serve as a basis for classical inductive reasoning (this New-Yorker is a runner, then this other too ...) and interest for the second will be found only in reasoning by analogy in which one will designate the individuals concerned by their names (Paul is wearing a purple short). But if in a basketball competition one sees individuals wearing either purple shorts or white shorts, this property may become contextually a principal designator: as it is learned that this purple short holder belongs to the team A, it will be inferred that this other purple short wearer also belongs to team A. These contextual situations do not contradict the preceding remarks: the particular context adds data on the situation, which makes properties usually subsidiary become relevant as principal designators; these properties may then be used in this context as meta-hypotheses of type HE3 or HE5 (e.g. the team to which belongs an athlete in a competition determines the color of the shorts that he is wearing).

5. Comparison with some logical and AI works

In an AI field, several works have given accounts of analogical reasoning. Those accounts have a different goal than ours since they are all oriented toward implementation of effective reasoning or machine learning programs, while the present paper is purely focused on a philosophical analysis of the rationality of analogical reasoning. Some of those works bring however fruitful insights to this philosophical analysis and may be compared directly to our's, by proposing explicit logical definitions of analogical reasoning, while some others are more specific to AI and far away from the approach of this paper.

Davies & Russel (1987) paper, which has an objective of implementation in a logic programming system, has already been largely commented.

In an important number of articles (e.g. Prade & Richard, 2011, 2012a, 2012b, 2014, Amgoud, Ouannani & Prade, 2014), Prade and al. develop a very different approach, which intends to provide a logical definition of analogy in a propositional framework, always guided by logic programming goals. This approach focuses on the definition of 4-term formulas such as $a:b :: c:d$ (where a, b, c, d are propositions) which are the basic structure of relational analogies.

On one side, they propose a list of principles that such formulas could satisfy, not limited to those who constitute an equivalence relation. On the other side, they consider several criteria of "logical proportions", whose components are analyzed in terms of "similarity indicators" ($a \& b$; $-a \& -b$) and "dissimilarity indicators" ($a \& -b$; $-a \& b$). The preferred formula for analogy is what they called "Analogical Proportion", which is expressed by:

(AP) $a:b :: c:d$ iff $((a \& -b) \equiv c \& -d) \& ((-a \& b) \equiv (-c \& d))$

This is supposed to mean that "a is to b what c is to d", and the authors argue that this definition satisfies the more relevant principles for representing a relational analogy in this propositional framework.

This conclusion seems to conflict with the thesis of the present paper since it may imply that, despite our previous arguments, it is possible to give a relevant logical definition of absolute analogy: no restriction to a domain is mentioned in definition (AP). However, the work of Prade & al. concerns only Boolean propositions as very specific objects of analogy, and cannot be considered as a general theory of analogy. It is assumed to be a transposition in a Boolean framework of the standard "proportional analogy":

(PA) $x:y :: z:t$ iff $x/y = z/t$, where x, y, z, t are real numbers.

As relevant as this classical example of analogy may be for numbers, it cannot be taken seriously as a basis for a definition of the philosophical concept of analogy, since it cannot be applied as such to any other objects, e.g. ravens, apples or human beings.... The same applies to (AP) which concerns only propositions. The misleading aspect of Prade & al. suggestion is that it seems that, contrary to numbers, propositions may express our beliefs "in general". But in fact, even if analogical statements are themselves propositions, they are propositions expressing beliefs which stand between objects, not between propositions. It is not possible to consider the definition (AP) in a propositional way in order to represent this general relation between objects. Hence, these attempts to define analogy in a propositional framework are at best interesting Boolean exercises but they are of no help for the philosophical understanding neither of the very concept of analogy nor of reasoning by analogy.

Polya (1945) identified some patterns of plausible reasoning among which this one:

(PR) *If a and b are analogous and a is true, then the truth of b is more plausible.*

Prade & Richard (2011) on their side suggested the following definition in a non-monotonic framework:

(NM) $a \sim b$ iff $\neg a \equiv b$

It is easy to check that (NM) validates (PR) if the inference relation $\sim$ in (NM) is for instance a preferential non monotonic consequence, as defined in Kraus, Lehmann and Magidor (1990) and if the conclusion of a non-monotonic inference is interpreted as a “plausible conclusion”.

However this definition raises three problems which manifest the fact that their analysis is of little help for a philosophical analysis of analogy:

- It concerns again only propositions and it is unclear how it may apply to the very general notion of analogy between objects, such as *“an apple is like a pear”*.
- It introduces an absolute definition of analogy (NM) which is too restrictive: a and b are analogous if and only if they are true in exactly the same “normal” possible worlds. This almost reduces analogy to identity, with the only exception of the most implausible worlds
- It does not explain how an object A could be similar to an object B from a point of view and not from another, since the normal worlds are a fixed subset of the possible worlds.

Other AI works are more difficult to compare to this paper, since they do not rely on the same kind of logical framework and philosophical discussion. However, the problems considered can be directly or indirectly related to analogical reasoning. The above examples are not intended to be exhaustive.

Firstly, in a Machine-Learning context, Gentner proposed in several papers (Gentner, 1983, Gentner & Maravilla, 2018) to conceptualize analogical reasoning in his “structure mapping theory”. His analysis makes a relevant distinction between analogies based on simple common properties, and “structural analogies” based on higher level relations. As usual, he considers first analogical assessment called “structural alignment” as the systematic comparison of properties and relations of source and target objects. He further considers analogical judgment called “abstraction and inference projection” as the inference of new properties through an abstract common structure. He discusses moreover different properties satisfied in each reasoning step.

The common aspect of all AI works on analogy is then to stress the existence of a meta-structure generalizing the basic properties or relations of objects, in accordance with the prefix “ana” (i.e. bottom up) in the word “analogy”. Called “meta-structure” by Gentner, (1983), it is also known as “determining structure” (Weitzenfeld, 1984), “generative model” (Tenenbaum et alii, 2011) or even “class of universality”. It can be expressed by different mathematical tools: function, geometrical figures, graphs, matrices.

Secondly, Case-based reasoning is an approach to problem solving which represents a whole field of AI work in the recent years. Among others, Voskoglou & Salem (2014) argues that analogical and case-based reasoning are two similar problems « solving processes based on the adaptation of the solution of past problems for use with a new analogous problem ».

Indeed, in a typical case-based reasoning situation, an expert has to make an expectation of some outcome or to reach a decision about it. In order to do so, he gathers a set of past situations which are similar to the new situation and observes for each of them what conditions were realized and what outcome was obtained or what decision was implemented. A first example concerns the effect of some innovation on a given system. Different systems are considered, differing by the characteristics of the innovation, the influence mechanisms of the innovation and the distribution of the population involved. The output is the efficiency of the innovation measured by an aggregate index. This index is a positive one, the idea being that similar situations should lead to similar outcomes. A second example concerns the judgment associated to a judicial trial. Different cases are examined, differing by various social circumstances, the operation mode of the suspect and the personality of the suspect. The output is the penalty imposed to the suspect by the judge. This penalty is a normative one, the idea being that in similar situations, the same verdict should be applied.

In fact, case-based reasoning is a quantitative generalization of an analogical process. Objects become cases, old ones and a new one. Properties become indicators, concerning the conditions observed and the privileged effect. As concerns analogical assessment, a quantitative similarity index is defined between each old case and the new one by combining all conditions. As concerns analogical inference, the selected property in the new case is assumed to be a combination of the old selected properties weighted by their similarity to the new case.

6. Conclusion

We develop an analytic scheme of reasoning by analogy which is decomposed into two steps. Analogical assessment states that some objects are similar as concerns a fixed set of properties. The analogical inference states that a new property possessed by these similar objects can be transferred to another object already similar to the considered ones. But our conceptual scheme differs from the existing works in several respects.

Our analysis considers that an analogical assessment is not true or false, good or bad in an absolute way, but is relative to a point of view expressed by a domain of properties. If some debates are raised about it and make it defeasible, they concern rival analogical inferences and not analogical assertions by themselves.

Our analysis expresses that analogy is an inference such that the degree of belief in the conclusion is defined coherently with the degree of belief in a background over-hypothesis supporting it. The value of this kind of reasoning cannot be established on a purely syntactical basis but is linked to externally defined beliefs, which should be made explicit within analogical reasoning disputes.

Hence, our analysis reconciles the opposite ideas that analogical reasoning is a useful method, especially in science, and a fanciful method, especially in current reasoning. It may be both, in all circumstances, while using always the same inference scheme: the plausibility of the conclusion depends only of the plausibility of the background over-hypothesis used in this scheme. Our analysis departs from those which state that performing good or bad analogical inferences depends of the fact to fulfil or not a good reasoning method, or to rely or not on good analogies.

The background over-hypothesis used in an analogical reasoning is a meta-level hypothesis in that it concerns upper level belief about properties. Consideration of this upper level avoids to trivialise the

reasoning into a focusing redundant one and explains in a coherent way the possible – though multiple – origins of those background belief, especially in an inductive way.

Our analysis departs finally from the usual analysis which considers analogical reasoning as a specific reasoning mode different from deduction, induction and abduction. On the contrary, it relies on the idea that it is a category which includes single case induction, but with two specific pragmatic contexts, one of them corresponding to the more traditional examples of single case induction.

Further work can be done in some theoretical directions. First, one could study the relevance of quantitative types of similarity indices which are used in AI for establishing a similarity assessment. Second, one could analyse more closely the different ways to empirically build the kind of belief inherent in the over-hypothesis, and to propose a non-monotonic framework for expressing the analogical inference scheme. Third, one could examine how the usual paradoxes of induction such as Hempel's and Goodman's apply to general analogical reasoning.

Further investigations can also be done in empirical domains. On the one hand, one could precise on historical examples how analogy is used in the process of science, in combination with other reasoning modes, especially in the "context of discovery" as opposed to the "context of justification". On the other hand, one could examine how scientific or popular analogies are revised through time, under the pressure of new information and hot debates, and become widely accepted or definitely discarded.

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