On Generalized Fractional Dynamical System With Order Lying in $(0, 2)$: Stability Analysis, Chaotic Behaviour, Control and Synchronization

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On generalized fractional dynamical system with order lying in $(0, 2)$: stability analysis, chaotic behaviour, control and synchronization

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Abstract

The generalized fractional dynamical system with order lying in $(0, 2)$ is investigated. We present the stability analysis of that system using Mittag-Leffler function, the Gronwall-Bellman Lemma and Laplace transform. The bifurcation diagram of generalized fractional-order Chen system is given. We investigate a theorem to control the chaotic generalized fractional-order systems by linear feedback control. Two examples to achieve the theorem of control are given. The synchronization between two different chaotic generalized fractional systems is presented. We give a theorem to calculate the control functions which achieve synchronization. This theorem is applied to achieve the synchronization between different generalized fractional-order systems with order lying in $(0, 1]$. And, also, used to achieve the synchronization between the identical generalized fractional-order Lü systems with order lying in $[1, 2)$. There exist an agreement among analytical results and numerical treatments for stability, control and synchronization theorems.

Keywords: Chaotic systems, Generalized fractional-order, Mittag-Leffler function, Control methods, Synchronization

1. Introduction

In the last few decades, fractional derivatives and fractional differential equations have showed to be valuable tools in physical phenomena modeling in different fields of engineering and science [1]. Many different definitions of fractional derivatives are introduced according to different kernels [2, 3, 4]. These definitions are used in Rieszspace fractional equations [5], timefractional differential equations [6, 7, 8], complex network [9, 10], materials constitutive equations [11, 12], fractional diffusion modeling [13, 14, 15], control theories [15, 16, 17, 18], many kinds of synchronization such as complete synchronization [19], anti synchronization [20], projective synchronization [21], modified projective synchronization [22] and function synchronization [23].

The generalized fractional derivative has many features over the integer derivatives and has potential many applications. Since chaos in fractional-order models is more complicated than the integer cases, then its are

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proposed in image encryption [24, 25, 26]. Recently, Anderson and Ulness are used the generalized fractional derivative in quantum mechanics [27]. Ren and Zhai are investigated the generalized fractional memristor-based impulsive neural network [28]. Chaotic fractional-order models with the two parameters can recover secure communication and information transfer [24]. The generalized fractional derivative will be a novel direction in the fractional calculus because its important applications in many fields [24].

So many physical problems have been mobilised by the help of the Caputo fractional derivative in fractional calculus applications because Caputo derivative is suitable for initial value problems (IVPs) and has many characteristics similar to integer order derivatives. The generalized fractional integral operator [29] is greatly influenced by the value of the parameters α and ρ , thus it gives a valuable tool to control and build mathematical models in fractional calculus applications. Jarad et al. [30] introduced a generalized Caputo-type fractional derivative with properties similar to those of the Caputo derivative. They discussed the relationship between the generalized fractional integral operator and generalized fractional derivative operator. This type of generalized derivative seems closer to ordinary derivatives than other generalized derivatives. Thus, we dedicate this work to investigate generalized fractional-order nonlinear systems governed by the generalized fractional derivative [30] in the following form:

$${}^C\mathcal{D}^{\alpha,\rho}x(t) = Ax(t) + f(x(t)), \quad (1.1)$$

where $0 < \alpha < 2$, $\rho \geq 0$ and ${}^C\mathcal{D}^{\alpha,\rho}$ denotes the left generalized Caputo-type fractional derivative. On the other hand, for the numerical simulation purposes of fractional order models, the predictor-corrector (P-C) techniques are one of the most efficient, stable and accurate methods that was implemented and modified to numerically solve Caputo fractional differential equations [31]. Odibat and Baleanu [32] present an adaptive predictor corrector method for the numerical solution of generalized Caputo-type initial value problems. In order to construct the predictor-corrector method for the IVP (1.1), we will follow the same procedure as in [32].

The pervious papers on the stability of generalized fractional-order nonlinear systems are investigated with fractional-order lying in $(0, 1)$. In this paper, we stated the generalized fractional dynamical system with order in $(0, 2)$. The stability analysis of that system using Mittag-Leffler function, the Gronwall-Bellman Lemma and Laplace transform is illustrated. We show that chaotic solutions for generalized fractional-order models are more complicated than the classical fractional and integer cases. We controlled the chaotic generalized fractional-order systems using the linear feedback control. By this technique of control, we synchronized two different chaotic generalized fractional-order systems. The paper is outlined as follows: In Section 2, we address some important preliminaries. In Section 3, using Mittag-Leffler function, the Gronwall-Bellman Lemma and Laplace transform, we prove the solution of the generalized fractional dynamical system approach to zero at the infinity. We investigate the control of chaotic generalized fractional system by linear feedback control in Section 4. We give two examples to achieve the control theorem. In Section 5, the control functions which achieve synchronized two different chaotic generalized fractional systems is illustrated. The synchronization between the different

generalized fractional-order Chen and Lü systems is presented. And the synchronization between the identical generalized fractional-order Lü systems is introduced. Finally, a conclusion is given to summary our works.

2. Preliminaries

We stated basic definitions of fractional derivatives and some lemmas in this section [2, 33, 34, 35, 36, 37].

2.1. Liouville-Caputo Fractional Calculus

The left-sided and right-sided Riemann-Liouville integrals of order α , when $0 < \alpha < 1$, are defined, respectively, as

$$({}^{RL}\mathcal{I}_a^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(s)ds}{(x-s)^{1-\alpha}}, \quad x > a, \quad (2.1)$$

and

$$({}^{RL}\mathcal{I}_\infty^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(s)ds}{(s-x)^{1-\alpha}}, \quad x < b, \quad (2.2)$$

where Γ represents the Euler Gamma function. The corresponding inverse operators, i.e., the left-sided and right-sided fractional derivatives of order α , are then defined based on (2.1) and (2.2), as

$$({}^{RL}\mathcal{D}_a^\alpha f)(x) = \frac{d}{dx} ({}^{RL}\mathcal{I}_a^{1-\alpha} f)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{f(s)ds}{(x-s)^\alpha}, \quad x > a, \quad (2.3)$$

and

$$({}^{RL}\mathcal{D}_b^\alpha f)(x) = \frac{-d}{dx} ({}^{RL}\mathcal{I}_\infty^{1-\alpha} f)(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{-d}{dx} \right) \int_x^b \frac{f(s)ds}{(s-x)^\alpha}, \quad x < b. \quad (2.4)$$

This allows for the definition of the left and right Riemann-Liouville fractional derivatives of order α ($n-1 < \alpha < n$), $n \in \mathbb{N}$ as

$$({}^{RL}\mathcal{D}_a^\alpha f)(x) = \left(\frac{d}{dx} \right)^n ({}^{RL}\mathcal{I}_a^{n-\alpha} f)(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx} \right)^n \int_a^x \frac{f(s)ds}{(x-s)^{-n+1+\alpha}}, \quad x > a, \quad (2.5)$$

and

$$({}^{RL}\mathcal{D}_b^\alpha f)(x) = \left(\frac{-d}{dx} \right)^n ({}^{RL}\mathcal{I}_\infty^{n-\alpha} f)(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{-d}{dx} \right)^n \int_x^b \frac{f(s)ds}{(s-x)^{-n+1+\alpha}}, \quad x < b. \quad (2.6)$$

Furthermore, the corresponding left-sided and right-sided Caputo derivatives of order α ($n-1 < \alpha < n$) are obtained as

$$({}^C\mathcal{D}_a^\alpha f)(x) = \left({}^{RL}\mathcal{I}_a^{n-\alpha} \frac{d^n f}{dx^n} \right)(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(s)ds}{(x-s)^{-n+1+\alpha}}, \quad x > a, \quad (2.7)$$

and

$$({}^C\mathcal{D}_b^\alpha f)(x) = (-1)^n \left({}^{RL}\mathcal{I}_\infty^{n-\alpha} \frac{d^n f}{dx^n} \right)(x) = \frac{1}{\Gamma(n-\alpha)} \int_x^b \frac{(-1)^n f^{(n)}(s)ds}{(s-x)^{-n+1+\alpha}}, \quad x < b. \quad (2.8)$$

The Caputo operator satisfies the rule

$${}^{RL}\mathcal{I}_a^\alpha {}^C\mathcal{D}_a^\alpha f(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k. \quad (2.9)$$

2.2. Generalized fractional Liouville-Caputo Fractional Calculus

Definition 2.1. (Katugampola [29]) The generalized left-sided Riemann-Liouville integral of f ${}_a^{RL}\mathcal{I}_x^{\alpha,\rho}f(x)$ order α , when $\alpha > 0$, $\rho \geq 0$, is

$$({}_a^{RL}\mathcal{I}_x^{\alpha,\rho}f)(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{s^{\rho-1}f(s)ds}{(x^\rho - s^\rho)^{1-\alpha}}, \quad x > a. \quad (2.10)$$

The generalized left-sided Riemann-Liouville fractional derivatives of order α ($n-1 < \alpha < n$), $n \in \mathbb{N}$ is defined as

$$({}_a^{RL}\mathcal{D}_x^{\alpha,\rho}f)(x) = \left(x^{1-\rho} \frac{d}{dx}\right)^n ({}_a^{RL}\mathcal{I}_x^{n-\alpha,\rho}f)(x) = \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \left(x^{1-\rho} \frac{d}{dx}\right)^n \int_a^x \frac{s^{\rho-1}f(s)ds}{(x^\rho - s^\rho)^{-n+1+\alpha}}, \quad x > a. \quad (2.11)$$

We can observe when $\rho = 1$, we recover the Riemann-Liouville fractional derivative in (2.5). Furthermore, the corresponding generalized left-sided Caputo derivatives of order α ($n-1 < \alpha < n$) are obtained as

$$({}_a^C\mathcal{D}_x^{\alpha,\rho}f)(x) = \left({}_a^{RL}\mathcal{D}_x^{\alpha,\rho} \left[f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (t-a)^k \right] \right)(x), \quad x > a, \quad (2.12)$$

where $n = \lceil \text{Re}(\alpha) \rceil$.

Definition 2.2. (Jarad et al. [30]). The generalized left-sided Caputo derivative of f order α ($n-1 < \alpha < n$), $n \in \mathbb{N}$ is defined by

$$({}_a^C\mathcal{D}_x^{\alpha,\rho}f)(x) = \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \int_a^x (x^\rho - s^\rho)^{-n+1+\alpha} \left(s^{1-\rho} \frac{d}{ds}\right)^n f(s) \frac{ds}{s^{1-\rho}}, \quad x > a, \quad (2.13)$$

further analysis includes

$${}_a^{RL}\mathcal{I}_x^{\alpha,\rho}{}_a^C\mathcal{D}_x^{\alpha,\rho}f(x) = f(x) - \sum_{k=0}^{n-1} \frac{\left(s^{1-\rho} \frac{d}{ds}\right)^k f(s)|_{s=a}}{k!} \left(\frac{t^\rho - a^\rho}{\rho}\right)^k. \quad (2.14)$$

We can observe when $\rho = 1$ in (2.13), we recover the Caputo fractional derivative in (2.7). For the rest of this paper, the Laplace transform will be used to help us in studying stability of the generalized fractional differential equations. The ρ -Laplace transform was recently introduced in the literature [33]. The ρ -Laplace transform of the Caputo generalized fractional derivative is expressed in the following form:

$$\mathcal{L}_\rho\{({}_a^C\mathcal{D}_x^{\alpha,\rho}f)(t)\} = s^\alpha \mathcal{L}_\rho\{f(t)\} - \sum_{j=0}^{n-1} s^{n-j-1} \left((t^{1-\rho} \frac{d}{dt})^j f\right)(0), \quad (2.15)$$

where ${}_a^C\mathcal{D}_x^{\alpha,\rho} = {}_0^C\mathcal{D}_x^{\alpha,\rho}$ (i.e. left generalized Caputo-type fractional derivative of order α).

Besides, the ρ -Laplace transform of a given function f is described in the form:

$$\mathcal{L}_\rho\{f(t)\}(s) = \int_0^\infty e^{-s \frac{t^\rho}{\rho}} f(t) \frac{dt}{t^{1-\rho}}. \quad (2.16)$$

Definition 2.3. [34] The two-parameter Mittag-Leffler function is defined as:

$$E_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + \beta)}, \quad \alpha, \beta > 0, z \in \mathbb{C}. \quad (2.17)$$

Lemma 2.1. [35] Let $F(s) = \mathcal{L}\{f(t)\}$. If all poles of $sF(s)$ are in the open left-half complex plane, then $\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$.

The Laplace transform of the function $z^{\alpha l + \beta - 1} E_{\alpha, \beta}^{(l)}(-az^\alpha)$ is:

$$\mathcal{L}\{z^{\alpha l + \beta - 1} E_{\alpha, \beta}^{(l)}(-az^\alpha)\} = \frac{l! s^{\alpha - \beta}}{(s^\alpha \mp a)^{l+1}}, \quad (R(s) \geq |a|^{\frac{1}{\alpha}}), \quad (2.18)$$

where $E_{\alpha, \beta}^{(l)}(z)$ is

$$E_{\alpha, \beta}^{(l)}(z) = \frac{d^l}{dz^l} \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + \beta)} = \sum_{j=0}^{\infty} \frac{(j+l)! z^j}{j! \Gamma(\alpha j + \alpha l + \beta)}. \quad (2.19)$$

By using definition 2.3, the following relation is obtained as:

$$\frac{dE_{\alpha, \beta}(z)}{dz} = \frac{E_{\alpha, \beta-1}(z) - (\beta-1)E_{\alpha, \beta}(z)}{\alpha z}. \quad (2.20)$$

From (2.20) one gets:

$$E_{\alpha, \beta}^{(k)}(z) = (\alpha z)^{-k} \sum_{m=0}^k d_m E_{\alpha, \beta-m}(z), \quad (2.21)$$

where d_m , $m = 0, 1, \dots, k$ are constants depend on β .

Lemma 2.2. [2] The function $f(x(t))$ satisfies the Lipschitz condition, if

$$\|f(y) - f(x)\| \leq L \|y - x\|, \quad L \in R^+. \quad (2.22)$$

Lemma 2.3. [36]

$$E_{\alpha, \beta}(B) \leq \frac{C}{1 + \|B\|}, \quad 0 < \alpha < 1, B \in R^{n \times n}, C \in R^+, \quad (2.23)$$

where μ satisfies (i) $\pi\alpha/2 \leq \mu \leq \min\{\pi, \pi\alpha\}$ and (ii) $\mu \leq |\arg(\text{eig}(B))| \leq \pi$.

Lemma 2.4. (GronwallBellman lemma[37].) If

$$h(t) \leq g(t) + \int_0^t m(u)h(u)du, \quad (2.24)$$

where $m(t) \geq 0$, $g(t), h(t)$ are nondecreasing continuous functions, $0 \leq t \leq T$ Then $h(t)$ satisfies

$$h(t) \leq g(t) + \int_0^t m(u)g(u)\exp\left[\int_u^t m(v)dv\right]du. \quad (2.25)$$

3. Stability analysis

Theorem 3.1. The zero solution of generalized Caputo fractional-order system (1.1) is stable if:

1. $|\arg(\lambda_i(\frac{A}{\rho^\alpha}))| > \pi\alpha/2$.

$$2. f(0) = 0, \lim_{\|x\| \rightarrow 0} \frac{\|f(x)\|}{\|x\|} = 0.$$

where $x \in \mathbb{R}^{n \times 1}$, $A \in \mathbb{R}^{n \times n}$, $t \in \mathbb{R}^+$ and $\lambda_i(\frac{A}{\rho^\alpha})$ be the eigenvalues of matrix $\frac{A}{\rho^\alpha}$.

Proof: Two cases will be considered separately.

(a) The case $0 < \alpha \leq 1$.

Taking ρ -Laplace transform on (1.1), then taking ρ -Laplace inverse transform by using the inverse Laplace transform formula of the Mittag-Leffler function in two parameters and the integral convolution, it yields

$$x(t) = E_{\alpha,1}[A(\frac{t^\rho}{\rho})^\alpha]x(0) + \int_0^t (\frac{t^\rho-s^\rho}{\rho})^{\alpha-1} E_{\alpha,\alpha}[A(\frac{t^\rho-s^\rho}{\rho})^\alpha] s^{\rho-1} f(x(s)) ds. \quad (3.1)$$

By part 2 of Theorem 3.1, there exists $C > 0$ and δ_0 such that

$$\|x(t)\| < \delta_0 \Rightarrow \|f(x(t))\| < \frac{\|x(t)\|}{C}, \quad t \geq 0. \quad (3.2)$$

Let δ be chosen such that $0 < \delta < \delta_0$ and suppose

$$\|x(0)\| < \delta. \quad (3.3)$$

Using Eqs. (3.2-3.3) and Lemma 2.3, (3.1) gives

$$\|x(t)\| \leq \frac{C}{1+\|A\|(\frac{t^\rho}{\rho})^\alpha} \|x(0)\| + \int_0^t \frac{(\frac{t^\rho-s^\rho}{\rho})^{\alpha-1} s^{\rho-1}}{1+\|A\|(\frac{t^\rho-s^\rho}{\rho})^\alpha} \|x(s)\| ds. \quad (3.4)$$

From Gronwall-Bellman Lemma 2.4 and (3.16), we get

$$\|x(t)\| \leq \frac{C}{1+\|A\|(\frac{t^\rho}{\rho})^\alpha} \|x(0)\| + \int_0^t \frac{C(\frac{t^\rho-s^\rho}{\rho})^{\alpha-1} s^{\rho-1}}{(1+\|A\|(\frac{s^\rho}{\rho})^\alpha)(1+\|A\|(\frac{t^\rho-s^\rho}{\rho})^\alpha)^{1-\frac{1}{\alpha\|A\|}}} \|x(0)\| ds. \quad (3.5)$$

The integral in (3.5) equals

$$\int_0^{t/2} \frac{C(\frac{t^\rho-s^\rho}{\rho})^{\alpha-1} s^{\rho-1}}{(1+\|A\|(\frac{s^\rho}{\rho})^\alpha)(1+\|A\|(\frac{t^\rho-s^\rho}{\rho})^\alpha)^{1-\frac{1}{\alpha\|A\|}}} \|x(0)\| ds + \int_{t/2}^t \frac{C(\frac{t^\rho-s^\rho}{\rho})^{\alpha-1} s^{\rho-1}}{(1+\|A\|(\frac{s^\rho}{\rho})^\alpha)(1+\|A\|(\frac{t^\rho-s^\rho}{\rho})^\alpha)^{1-\frac{1}{\alpha\|A\|}}} \|x(0)\| ds. \quad (3.6)$$

Since $\alpha < 1$, $\rho > 1$ and $t^\rho - s^\rho \geq s^\rho$ for $s \in [0, t/2]$, the first integral of (3.6) can be written as

$$\int_0^{t/2} \frac{C(\frac{t^\rho-s^\rho}{\rho})^{\alpha-1} s^{\rho-1}}{(1+\|A\|(\frac{s^\rho}{\rho})^\alpha)(1+\|A\|(\frac{t^\rho-s^\rho}{\rho})^\alpha)^{1-\frac{1}{\alpha\|A\|}}} \|x(0)\| ds \leq \int_0^{t/2} \frac{C(\frac{s^\rho}{\rho})^{\alpha-1} s^{\rho-1}}{(1+\|A\|(\frac{s^\rho}{\rho})^\alpha)(1+\|A\|(\frac{s^\rho}{\rho})^\alpha)^{1-\frac{1}{\alpha\|A\|}}} \|x(0)\| ds. \quad (3.7)$$

Similarly, the second integral of (3.6) can be written as

$$\int_{t/2}^t \frac{C(\frac{t^\rho-s^\rho}{\rho})^{\alpha-1} s^{\rho-1}}{(1+\|A\|(\frac{s^\rho}{\rho})^\alpha)(1+\|A\|(\frac{t^\rho-s^\rho}{\rho})^\alpha)^{1-\frac{1}{\alpha\|A\|}}} \|x(0)\| ds \leq \int_{t/2}^t \frac{C(\frac{t^\rho-s^\rho}{\rho})^{\alpha-1} s^{\rho-1}}{(1+\|A\|(\frac{t^\rho-s^\rho}{\rho})^\alpha)(1+\|A\|(\frac{t^\rho-s^\rho}{\rho})^\alpha)^{1-\frac{1}{\alpha\|A\|}}} \|x(0)\| ds, \quad (3.8)$$

let $(2^\rho - 1)\tau^\rho = t^\rho - s^\rho$ in (3.8), then

$$\int_{t/2}^t \frac{C(\frac{t^\rho - s^\rho}{\rho})^{\alpha-1} s^{\rho-1}}{(1 + \|A\|(\frac{t^\rho - s^\rho}{\rho})^\alpha)(1 + \|A\|(\frac{t^\rho - s^\rho}{\rho})^\alpha)^{1 - \frac{1}{\alpha\|A\|}}} \|x(0)\| ds = (2^\rho - 1) \int_0^{t/2} \frac{C(\frac{\tau^\rho}{\rho})^{\alpha-1} \tau^{\rho-1}}{(1 + \|A\|(\frac{\tau^\rho}{\rho})^\alpha)(1 + \|A\|(\frac{\tau^\rho}{\rho})^\alpha)^{1 - \frac{1}{\alpha\|A\|}}} \|x(0)\| d\tau. \quad (3.9)$$

From (3.7), (3.9) and $\alpha\varrho(A) > 1$, where $\varrho(A) = \max_{1 \leq i \leq n} |\lambda_i|$ is the spectral radius of A, then (3.5) takes the form

$$\begin{aligned} \|x(t)\| &\leq \frac{C}{1 + \|A\|(\frac{t^\rho}{\rho})^\alpha} \|x(0)\| + \int_0^{t/2} \frac{C(\frac{s^\rho}{\rho})^{\alpha-1} 2^\rho s^{\rho-1}}{(1 + \|A\|(\frac{s^\rho}{\rho})^\alpha)^{2 - \frac{1}{\alpha\|A\|}}} \|x(0)\| ds \\ &= \frac{C\|x(0)\|2^\rho}{\alpha\|A\| - 1} + \frac{C\|x(0)\|}{1 + \|A\|(\frac{t^\rho}{\rho})^\alpha} + \frac{C\|x(0)\|2^\rho}{(-\alpha\|A\| + 1)(1 + \|A\|(\frac{(t/2)^\rho}{\rho})^\alpha)^{1 - \frac{1}{\alpha\|A\|}}}. \end{aligned} \quad (3.10)$$

Using (3.3), then (3.10) gives

$$\|x(t)\| \leq \frac{2^\rho C\delta}{\alpha\|A\| - 1} + \frac{C\delta}{1 + \|A\|(\frac{t^\rho}{\rho})^\alpha} + \frac{2^\rho C\delta}{(-\alpha\|A\| + 1)(1 + \|A\|(\frac{(t/2)^\rho}{\rho})^\alpha)^{1 - \frac{1}{\alpha\|A\|}}}, \quad t \geq 0 \quad (3.11)$$

this means $\|x(t)\| \rightarrow 0$ as $t \rightarrow \infty$ (i.e. the zero solution is stable).

(b) The case $1 < \alpha < 2$

In this case, the initial condition is

$$\left(x^{1-\rho} \frac{d}{dx}\right)^k (0) = x_k, \quad k = 0, 1. \quad (3.12)$$

We can get the solution of (1.1) with the initial condition (3.12) by using the ρ -Laplace transform and ρ -Laplace inverse transform as:

$$x(t) = E_{\alpha,1}[A(\frac{t^\rho}{\rho})^\alpha]x_1 + \frac{t^\rho}{\rho} E_{\alpha,2}[A(\frac{t^\rho}{\rho})^\alpha]x_2 + \int_0^t (\frac{t^\rho - s^\rho}{\rho})^{\alpha-1} E_{\alpha,\alpha}[A(\frac{t^\rho - s^\rho}{\rho})^\alpha] s^{\rho-1} f(x(s)) ds. \quad (3.13)$$

By part 2 of Theorem 3.1, there exists $C > 0$ and δ_0 such that

$$\|x(t)\| < \delta_0 \Rightarrow \|f(x(t))\| < \kappa \frac{(\alpha-1)\|A\|\|x(t)\|}{C}, \quad t \geq 0, \quad 0 < \kappa < 1. \quad (3.14)$$

Using Eqs. (3.3), (3.14) and Lemma 2.3, (3.13) gives

$$\begin{aligned} \|x(t)\| &\leq \frac{c_1\|x_1\|}{1 + \|A\|(\frac{t^\rho}{\rho})^\alpha} + \frac{c_2\|x_2\|t^\rho}{1 + \|A\|(\frac{t^\rho}{\rho})^\alpha} + \kappa \int_0^t \frac{(\frac{t^\rho - s^\rho}{\rho})^{\alpha-1} s^{\rho-1} (\alpha-1)\|A\|}{(1 + \|A\|(\frac{t^\rho - s^\rho}{\rho})^\alpha)} \|x(s)\| ds \\ &\leq \frac{c_1\|x_1\|}{(1 + \|A\|(\frac{t^\rho}{\rho})^\alpha)^\kappa} + \frac{c_2\|x_2\|t^\rho}{1 + \|A\|(\frac{t^\rho}{\rho})^\alpha} + \kappa \int_0^t \frac{(\frac{t^\rho - s^\rho}{\rho})^{\alpha-1} s^{\rho-1} (\alpha-1)\|A\|}{(1 + \|A\|(\frac{t^\rho - s^\rho}{\rho})^\alpha)} \|x(s)\| ds. \end{aligned} \quad (3.15)$$

From Gronwall-Bellman Lemma 2.4 and (3.15), we get

$$\begin{aligned}
\|x(t)\| &\leq \frac{c_1\|x_1\|}{(1+\|A\|(\frac{t^\rho}{\rho})^\alpha)^\kappa} + \frac{c_2\|x_2\|t^\rho}{1+\|A\|(\frac{t^\rho}{\rho})^\alpha} + \kappa \int_0^t \frac{(\frac{t^\rho-s^\rho}{\rho})^{\alpha-1} s^{\rho-1} (\alpha-1) \|A\|}{(1+\|A\|(\frac{t^\rho-s^\rho}{\rho})^\alpha)} \left(\frac{c_1\|x_1\|}{(1+\|A\|(\frac{t^\rho}{\rho})^\alpha)^\kappa} + \frac{c_2\|x_2\|t^\rho}{1+\|A\|(\frac{t^\rho}{\rho})^\alpha} \right) \\
&\quad \exp\left(\int_\tau^t \kappa \frac{(\frac{t^\rho-s^\rho}{\rho})^{\alpha-1} s^{\rho-1} (\alpha-1) \|A\|}{(1+\|A\|(\frac{t^\rho-s^\rho}{\rho})^\alpha)} ds\right) d\tau \\
&= \frac{c_1\|x_1\|}{(1+\|A\|(\frac{t^\rho}{\rho})^\alpha)^\kappa} + \frac{c_2\|x_2\|t^\rho}{1+\|A\|(\frac{t^\rho}{\rho})^\alpha} + \kappa \int_0^t \left(\frac{c_1\|x_1\|}{(1+\|A\|(\frac{t^\rho}{\rho})^\alpha)^\kappa} + \frac{c_2\|x_2\|\tau^\rho}{1+\|A\|(\frac{\tau^\rho}{\rho})^\alpha} \right) \frac{(\frac{t^\rho-\tau^\rho}{\rho})^{\alpha-1} \tau^{\rho-1} (\alpha-1) \|A\|}{(1+\|A\|(\frac{t^\rho-\tau^\rho}{\rho})^\alpha)^{1-\kappa} \frac{\alpha-1}{\alpha}} d\tau \\
&\leq \frac{c_1\|x_1\|}{(1+\|A\|(\frac{t^\rho}{\rho})^\alpha)^\kappa} + \frac{c_2\|x_2\|t^\rho}{1+\|A\|(\frac{t^\rho}{\rho})^\alpha} + \kappa \int_0^t \left(\frac{c_1\|x_1\|}{(1+\|A\|(\frac{t^\rho}{\rho})^\alpha)^\kappa} + \frac{c_2\|x_2\|\tau^\rho}{1+\|A\|(\frac{\tau^\rho}{\rho})^\alpha} \right) \\
&\quad \left(\frac{t^\rho-\tau^\rho}{\rho} \right)^{\kappa(\alpha-1)-1} \tau^{\rho-1} (\alpha-1) \|A\|^{\kappa \frac{\alpha-1}{\alpha}} d\tau \\
&\leq \frac{c_1\|x_1\|}{(1+\|A\|(\frac{t^\rho}{\rho})^\alpha)^\kappa} + \frac{c_2\|x_2\|}{t^{-\rho} + \rho^{-\alpha} \|A\| t^{\rho\alpha-\rho}} + \kappa c_1 (\alpha-1) \|x_1\| \|A\|^{\kappa \frac{\alpha-1}{\alpha} - \kappa} \int_0^t \tau^{\rho-\rho\alpha-1} (t^\rho - \tau^\rho)^{\kappa(\alpha-1)-1} d\tau \\
&\quad + \kappa c_2 \|x_2\| (\alpha-1) \|A\|^{\kappa \frac{\alpha-1}{\alpha} - 1} \int_0^t \tau^{2\rho-\rho\alpha-1} (t^\rho - \tau^\rho)^{\kappa(\alpha-1)-1} d\tau \\
&= \frac{c_1\|x_1\|}{(1+\|A\|(\frac{t^\rho}{\rho})^\alpha)^\kappa} + \frac{c_2\|x_2\|}{t^{-\rho} + \rho^{-\alpha} \|A\| t^{\rho\alpha-\rho}} + \kappa c_1 (\alpha-1) \|x_2\| \|A\|^{\kappa \frac{\alpha-1}{\alpha} - \kappa} \frac{\Gamma(1-\kappa\alpha)\Gamma(\kappa(\alpha-1))}{\rho\Gamma(1-\kappa)} t^{\rho(1-\kappa)-1} \\
&\quad + \kappa c_2 (\alpha-1) \|x_2\| \|A\|^{\kappa \frac{\alpha-1}{\alpha} - 1} \frac{\Gamma(2-\alpha)\Gamma(\kappa(\alpha-1))}{\rho\Gamma(2-\alpha+\kappa(\alpha-1))} t^{\rho(2-\alpha+\kappa(\alpha-1)-1)}.
\end{aligned} \tag{3.16}$$

So, the zero solution of (1.1) is locally asymptotically stable if $\kappa < \frac{\alpha-1}{\alpha}$ and $\rho < \frac{1}{1-\kappa}$. $\square$

Remark 3.1. The proof of Theorem 3.1 is a generalization of the proof of Theorem 1 in [38] for the fractional order α lying in $(0, 1)$ and Theorem 1 in [1] when the fractional order α lying in $(1, 2)$.

4. Control of chaotic generalized fractional-order systems

We introduce a technique of control of solutions of chaotic generalized fractional-order systems by linear feedback control. The generalized fractional-order system (1.1) can be written after adding the vector of control functions $u(t)$ as:

$${}^C\mathcal{D}^{\alpha,\rho}x(t) = Ax(t) + f(x(t)) + u(t). \tag{4.1}$$

We can present the linear feedback control functions as $u(t) = Kx(t)$, where K is $n \times n$ constant matrix. So, the controlled system (4.1) becomes:

$${}^C\mathcal{D}^{\alpha,\rho}x(t) = (A + K)x(t) + f(x(t)). \tag{4.2}$$

We investigate the sufficient conditions to hold that system (4.2) is asymptotically stable in the following theorem.

Theorem 4.1. *The controlled system (4.2) is asymptotically stable for $0 < \alpha < 2$ if:*

1. *we choosing K s.t. the zero solution of ${}^C\mathcal{D}^{\alpha,\rho}x(t) = (A + K)x(t)$ is asymptotically stable.*
2. $\lim_{\|x\| \rightarrow 0} \frac{\|f(x)\|}{\|x\|} = 0.$

Proof: The proof is similar to that of Theorem 3.1. □

We take two examples of chaotic generalized fractional-order systems to achieve Theorem 4.1. The first example for order lying in $(0, 1]$ and the other example for order lying in $[1, 2)$.

4.1. Example 1

In this example, we do a control for chaotic generalized fractional-order Chen system [32] by linear feedback control. The chaotic generalized fractional-order Chen system can be written as:

$$\begin{aligned} {}^C\mathcal{D}^{\alpha,\rho}x_1(t) &= a_1(x_2 - x_1), \\ {}^C\mathcal{D}^{\alpha,\rho}x_2(t) &= (c_1 - a_1)x_1 - x_1x_3 + c_1x_2, \\ {}^C\mathcal{D}^{\alpha,\rho}x_3(t) &= x_1x_2 - b_1x_3, \end{aligned} \tag{4.3}$$

where a_1, b_1 , and c_1 are constant parameters. For the choice $a_1 = 35$, $b_1 = 3$, $c_1 = 28$, $\alpha = 0.98$, $\rho = 2.5$ and the initial values are $x_0 = (0.1, 0.3, 0.5)^T$, system (4.3) has one positive Lyapunov exponent as $\lambda_1 = 1.279 \times 10^4$, $\lambda_2 = -0.135 \times 10^4$, and $\lambda_3 = -1.589 \times 10^4$. This means that system (4.3) has chaotic behavior as shown in Fig. 1. The bifurcation diagram between parameter $b \in (0, 5]$ and solution x of that system is given in Fig. 2, which shows that system has chaotic attractor for the same parameters of Fig. 1. If we take $\rho = 1$ and the same values of other parameters and the initial conditions of Fig. 1, the behaviour of the solution of fractional-order Chen system can be shown in Fig. 3. We can notice that the solution of generalized fractional-order Chen system is more complicated than the solution of usual fractional-order Chen system for the same time ($t=10$).

The chaotic Chen system (4.3) can be written after adding the control functions as:

$$\begin{aligned} {}^C\mathcal{D}^{\alpha,\rho}x_1(t) &= a_1(x_2 - x_1) + u_1, \\ {}^C\mathcal{D}^{\alpha,\rho}x_2(t) &= (c_1 - a_1)x_1 - x_1x_3 + c_1x_2 + u_2, \\ {}^C\mathcal{D}^{\alpha,\rho}x_3(t) &= x_1x_2 - b_1x_3 + u_3. \end{aligned} \tag{4.4}$$

The control functions can be written as:

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} -10 & 0 & 0 \\ 0 & -25 & 0 \\ 0 & 0 & -20 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \tag{4.5}$$

Figure 1: The chaotic behaviour of system (4.3) for $\rho = 2.5$ in (a) (x_1, x_2) space, (b) (x_1, x_3) space.

Figure 2: The bifurcation diagram of chaotic system (4.3) for $b \in (0, 5]$.

Figure 3: The chaotic behaviour of system (4.3) for $\rho = 1$ in (a) (x_1, x_2) space, (b) (x_1, x_3) space.

Using (4.5), the control system (4.4) can be written as

$${}^C\mathcal{D}^{\alpha,\rho} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -45 & 35 & 0 \\ -7 & 3 & 0 \\ 0 & 0 & -23 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ -x_1x_3 \\ x_1x_2 \end{pmatrix}, \quad (4.6)$$

where $A + K = \begin{pmatrix} -45 & 35 & 0 \\ -7 & 3 & 0 \\ 0 & 0 & -23 \end{pmatrix}$ and $f(x) = \begin{pmatrix} 0 \\ -x_1x_3 \\ x_1x_2 \end{pmatrix}$.

System (4.6) holds the sufficient conditions of Theorem 4.1 as:

1. The eigenvalues of $A + K$ are $\lambda_1 = -39.2$, $\lambda_2 = -23$ and $\lambda_3 = -2.8$, then the zero solution of ${}^C\mathcal{D}^{\alpha,\rho}x(t) = (A + K)x(t)$ is asymptotically stable.
2. $\lim_{\|x\| \rightarrow 0} \frac{\sqrt{x_1^2(x_2^2 + x_3^2)}}{\|x\|} \leq \lim_{\|x\| \rightarrow 0} \|x\| = 0$.

In numerical simulation, if we take the same values of the parameters and the initial values of Fig. 1, the solution of the controlled system (4.6) is approach to zero as shown in Fig. 4. This means there exist agreement between Theorem (4.1) and the numerical results.

Figure 4: The behaviour of the controlled system (4.6) in (a) (t, x_1) diagram, (b) (t, x_2) diagram, (c) (t, x_3) diagram.

4.2. Example 2

In this subsection, we use linear feedback control to control the solution of chaotic generalized fractional-order Lü system for order lying in $[1, 2)$. The chaotic generalized fractional-order Lü system takes the form:

$$\begin{aligned} {}^C\mathcal{D}^{\alpha, \rho} x_1(t) &= a(x_2 - x_1), \\ {}^C\mathcal{D}^{\alpha, \rho} x_2(t) &= -x_1 x_3 + c x_2, \\ {}^C\mathcal{D}^{\alpha, \rho} x_3(t) &= x_1 x_2 - b x_3, \end{aligned} \tag{4.7}$$

where a, b , and c are constant parameters. For the choice $a = 36$, $b = 3$, $c = 20$, $\alpha = 1.11$, $\rho = 2.5$ and the initial values are $x_0 = (0.1, 0.5, 0.2)^T$ and $\dot{x}_0 = (1, 2, 3)^T$, system (4.7) has chaotic behavior as shown in Fig. 5.

By adding control functions, system (4.7) can be written as

$$\begin{aligned} {}^C\mathcal{D}^{\alpha, \rho} x_1(t) &= a(x_2 - x_1) + u_1, \\ {}^C\mathcal{D}^{\alpha, \rho} x_2(t) &= -x_1 x_3 + c x_2 + u_2, \\ {}^C\mathcal{D}^{\alpha, \rho} x_3(t) &= x_1 x_2 - b x_3 + u_3, \end{aligned} \tag{4.8}$$

We can write the control functions as

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} -10 & 0 & 0 \\ 0 & -25 & 0 \\ 0 & 0 & -20 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \tag{4.9}$$

Figure 5: The chaotic system (4.7) in (a) (x_1, x_2) space, (b) (x_1, x_3) space.

Using (4.9), the control system (4.8) can be written as

$${}^C\mathcal{D}^{\alpha,\rho} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -46 & 36 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -23 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ -x_1 x_3 \\ x_1 x_2 \end{pmatrix}, \quad (4.10)$$

where $A + K = \begin{pmatrix} -46 & 36 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -23 \end{pmatrix}$ and $f(x) = \begin{pmatrix} 0 \\ -x_1 x_3 \\ x_1 x_2 \end{pmatrix}$.

System (4.10) holds Theorem 4.1 as:

1. The eigenvalues of $A + K$ are $\lambda_1 = -46$, $\lambda_2 = -23$ and $\lambda_3 = -5$, then the zero solution of ${}^C\mathcal{D}^{\alpha,\rho}x(t) = (A + K)x(t)$ is asymptotically stable.
2. $\lim_{\|x\| \rightarrow 0} \frac{\sqrt{x_1^2(x_2^2 + x_3^2)}}{\|x\|} \leq \lim_{\|x\| \rightarrow 0} \|x\| = 0$.

The zero solution of controlled system (4.10) is asymptotically stable as shown in Fig. 6 for the same choice of parameters and initial values of Fig. 5.

5. Synchronization between different and identical chaotic generalized fractional-order system

In this section, we investigate the synchronization between two different chaotic generalized fractional-order systems using linear feedback control method. We present two examples of synchronization between chaotic generalized fractional-order systems. The first one discuss the synchronization between two different chaotic

Figure 6: The behaviour of chaotic system after adding the control (4.8) in (a) (t, x_1) diagram, (b) (t, x_2) diagram, (c) (t, x_3) diagram.

generalized fractional-order Lü and Chen systems with order $\alpha \in (0, 1]$. The second example explains the synchronization between two identical chaotic generalized fractional-order Lü systems with order $\alpha \in (1, 2]$.

Definition 5.1. We can said the drive system (1.1) is synchronized with the following response system

$${}^C\mathcal{D}^{\alpha, \rho}y = By + f(y) + u, \quad (5.1)$$

if $\|e(t)\| = \|y(t) - x(t)\| \rightarrow 0$ as $t \rightarrow \infty$.

From systems (1.1) and (5.1), the error system can be written as:

$${}^C\mathcal{D}^{\alpha, \rho}e = Ae + (B - A)y + f(y) - f(x) + u, \quad (5.2)$$

Theorem 5.1. The solution of the error system (5.2) can be approach to zero if the vector of control functions u takes the form

$$u = (A - B)y - Ke, \quad (5.3)$$

where $K = \text{diag}(k_1, k_2, \dots, k_n)$ is again matrix, $\alpha \in (0, 2]$ and the initial values of the error system (5.2) are $e^{(j)}(0) = e_0^{(j)}$, $j = 0, 1$.

Proof:

Using the control functions (5.3), system (5.2) can be written as:

$${}^C\mathcal{D}^{\alpha, \rho}e = (A - K)e + f(y) - f(x), \quad (5.4)$$

by taking ρ -Laplace transform for system (5.4), then

$$s^\alpha \mathcal{L}_\rho\{e(t)\} - \sum_{j=0}^1 s^{2-j-1}(e^{(j)})(0) = (A - K)\mathcal{L}_\rho\{e(t)\} + \mathcal{L}_\rho\{f(y) - f(x)\}, \quad (5.5)$$

then,

$$\mathcal{L}_\rho\{e(t)\} = \frac{1}{s^\alpha I - A + K} \mathcal{L}_\rho\{f(y) - f(x)\} + \frac{1}{s^\alpha I - A + K} \sum_{j=0}^1 s^{2-j-1}(e^{(j)})(0), \quad (5.6)$$

using Lemma 2.1, we obtain

$$\lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} s \mathcal{L}_\rho\{e(t)\} = \lim_{s \rightarrow 0} \left[\frac{s}{s^\alpha I - A + K} \mathcal{L}_\rho\{f(y) - f(x)\} + \frac{s}{s^\alpha I - A + K} \sum_{j=0}^1 s^{2-j-1}(e^{(j)})(0) \right] = 0, \quad (5.7)$$

then the synchronization between the drive system (1.1) and the response system (5.1) can be achieved. $\square$

5.1. Synchronization between two different chaotic generalized fractional-order Lü and Chen systems

We consider the drive system is chaotic generalized fractional-order Lü system (4.7) and chaotic generalized fractional-order Chen system (4.4) is the response system. The response system after adding the control functions can be written as:

$$\begin{aligned} {}^C\mathcal{D}^{\alpha,\rho}y_1(t) &= a_1(y_2 - y_1) + u_1, \\ {}^C\mathcal{D}^{\alpha,\rho}y_2(t) &= (c_1 - a_1)y_1 - y_1y_3 + c_1y_2 + u_2, \\ {}^C\mathcal{D}^{\alpha,\rho}y_3(t) &= y_1y_2 - b_1y_3 + u_3, \end{aligned} \quad (5.8)$$

Using the drive system (4.7), the response system (5.8) and the control functions (5.3), the error system takes the form

$$\begin{aligned} {}^C\mathcal{D}^{\alpha,\rho}e_1(t) &= a(e_2 - e_1) - k_1e_1, \\ {}^C\mathcal{D}^{\alpha,\rho}e_2(t) &= x_1x_3 - y_1y_3 + (c - k_2)e_2, \\ {}^C\mathcal{D}^{\alpha,\rho}e_3(t) &= y_1y_2 - x_1x_2 - (b + k_3)e_3, \end{aligned} \quad (5.9)$$

For the choice $a = 36, b = 3, c = 20, a_1 = 35, b_1 = 3, c_1 = 28, k_1 = 10, k_2 = 25, k_3 = 20, \alpha = 0.98, \rho = 2.5$ and the initial values of the drive system (4.7) and the response system (5.8), respectively, are $x_0 = (0.1, 0.3, 0.5)^T$ and $y_0 = (0.4, 0.5, 0.6)^T$. The synchronization is achieved and the results are shown in Figs. 7-8. Fig. 7 shows the same chaotic attractor for drive system (4.7) and response system (5.8), while the synchronization errors go to zero as given in Fig. 8.

Figure 7: Chaotic attractors for (a) drive system (4.7) in (x_1, x_3, x_2) space, (b) response system (5.8) in (y_1, y_3, y_2) space.

Figure 8: The synchronization errors of drive system (4.7) and response system (5.8) in (a) (t, e_1) diagram, (b) (t, e_2) diagram, (c) (t, e_3) diagram.

Figure 9: Synchronization between drive system (4.7) (solid curves) and response system (5.10) (dashed curves): (a) x_1 and y_1 versus t , (b) x_2 and y_2 versus t and (c) x_3 and y_3 versus t .

5.2. Synchronization between two identical chaotic generalized fractional-order Lü systems

In this subsection, we present the identical synchronization between two generalized fractional Lü systems with order $\alpha \in (1, 2]$. We consider the chaotic Lü system (4.7) the drive system and the response takes the form:

$$\begin{aligned} {}^C\mathcal{D}^{\alpha,\rho}y_1(t) &= a(y_2 - y_1) + u_1, \\ {}^C\mathcal{D}^{\alpha,\rho}y_2(t) &= -y_1y_3 + cy_2 + u_2, \\ {}^C\mathcal{D}^{\alpha,\rho}y_3(t) &= y_1y_2 - by_3 + u_3. \end{aligned} \tag{5.10}$$

Using Theorem 5.1, the error system is

$$\begin{aligned} {}^C\mathcal{D}^{\alpha,\rho}e_1(t) &= a(e_2 - e_1) - k_1e_1, \\ {}^C\mathcal{D}^{\alpha,\rho}e_2(t) &= x_1x_3 - y_1y_3 + (c - k_2)e_2, \\ {}^C\mathcal{D}^{\alpha,\rho}e_3(t) &= y_1y_2 - x_1x_2 - (b + k_3)e_3, \end{aligned} \tag{5.11}$$

For the choice $a = 36, b = 3, c = 20, k_1 = 10, k_2 = 25, k_3 = 20, \alpha = 1.11, \rho = 2.5$ and the initial values of the drive system (4.7) and the response system (5.8), respectively, are $x_0 = (0.1, 0.3, 0.5)^T, \dot{x}_0 = (0.1, 0.3, 0.5)^T$ and $y_0 = (0.4, 0.5, 0.6)^T, \dot{y}_0 = (0.5, 0.6, 0.7)^T$. The synchronization is achieved and the results are shown in Figs. 9-10. Fig. 9 shows the state variables of drive system (4.7) and response system (5.10) versus t . The synchronization errors approach to zero as shown in Fig. 10. The generalized fractional dynamical system was simulated using Adams-Bashforth-Moulton method in this paper.

Figure 10: The synchronization errors of drive system (4.7) and response system (5.8) in (a) (t, e_1) diagram, (b) (t, e_2) diagram, (c) (t, e_3) diagram.

6. Conclusion

We introduced the generalized fractional dynamical system with order in $(0,2)$. In Theorem 3.1, the stability analysis of that system is investigated using Mittag-Leffler function, the Gronwall-Bellman Lemma and Laplace transform. The chaotic generalized fractional-order Chen and Lü systems and the bifurcation diagram of Chen system are presented. Using linear feedback control, we illustrated the control of chaotic generalized fractional-order system in general and two examples are given to achieve the control Theorem 4.1. We investigated the synchronization between two different chaotic generalized fractional systems. The control functions (5.3) which achieve synchronization are given in Theorem 5.1. The synchronization between the different generalized fractional-order Chen and Lü systems and between the identical generalized fractional-order Lü systems are achieved. Other examples of generalized fractional-order systems can be similarly studied.

Compliance with ethical standards

Conict of Interest:

We have no conict of interest.

Declarations:

Not applicable.

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Figures

Figure 1

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