Noname manuscript No.
(will be inserted by the editor)

L-concave derived internal relations and L-convex derived enclosed relations

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Abstract Derived operator plays an important role in describing algebraic properties of many mathematical structures such as topology, matroid, convergence and convex space. In this paper, we present the notions of L-concave derived internal relation space and L-convex derived enclosed relation space by which we characterize L-concave space and L-convex space. Based on this, we further introduce some other structures such as L-concave derived hull space and L-convex derived hull space. We find that these spaces are isomorphic to L-concave space and L-convex space.

Keywords L-concave derived internal relation space · L-convex derived enclosed relation space · L-concave derived hull space · L-convex derived hull space

Mathematics Subject Classification (2010) 54A40 · 52A01

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1 Introduction

In an abstract convex space, a convex structure on a nonempty set is a family of subsets containing the empty set and the largest set, and is closed under arbitrary intersections and nested unions. Its theory is called the abstract convex theory which involves many mathematical structures such as lattice, graph, median algebra, metric space, poset and vector space [25].

Convex structure has been extended into fuzzy settings by many ways. Rosa introduced the notion of fuzzy convex structure which was further extended by Maruyama who introduced L-fuzzy convex structure [16,7]. In the framework of L-convex spaces, many theories has been studied [11–13,31,38]. Later, Shi and Xiu introduced M-fuzzifying convex structures [19]. Many subsequent studies have been done [3,15,20,27,30,35–37]. Further, Shi and Xiu introduced (L, M)-fuzzy convex structure which is a unified form of L-convex structure and M-fuzzifying convex structure [21]. Its characterizations have been studied recently [14,28,32,33]. Now, these fuzzy forms of convex structures have being applied to many fuzzy mathematical structures such as fuzzy topology [6,24,26,33,39], fuzzy convergence [4,5,9,10,15,38] and fuzzy matroid [22,23,29].

Derived operator was originally introduced on the real line by Cantor in 1872 [2]. Shi extended this concept into fuzzy setting by which he characterized fuzzy topologies [18]. Also, derived operator has been applied to M-fuzzifying matroid [34,40] and M-fuzzifying convex space [1,17]. In the viewpoint of convex relations, Liao et al introduced L-convex enclosed relations and L-concave internal relations by which they characterize L-convex spaces and L-concave spaces [6,31]. Then a natural question arises: can derived operator be applied to L-convex enclosed relation or L-concave internal relation? Specifically, is there any L-concave derived internal relation or L-convex derived enclosed relation such that the following diagrams communicate?

Fig. 1 Problem 1.

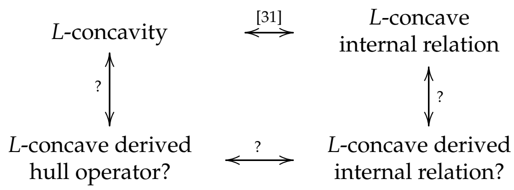
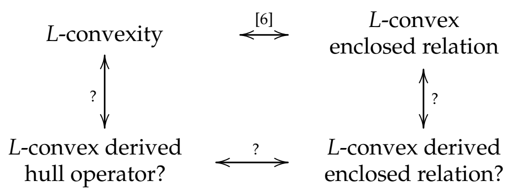


Fig. 2 Problem 2.



To solve the above problems, we present this paper. The arrangement of this paper is as follows. In Section 2, we recall some basic concepts, denotations and results of L-convex space and L-concave space. In Section 3, we introduce L-concave derived internal relation space and L-concave derived hull space by which we characterize L-concave space. In Section 4, we introduce L-convex derived enclosed relation space and L-convex derived hull space by which we characterize L-convex space.

2 preliminaries

In this paper, X and Y are nonempty sets. The power set of X is denoted by 2^X. For any $A \in 2^X$, 2^A_{fin} is the set of all finite subsets of A. L is a completely distributive lattice. The smallest element and the largest element in L are respectively denoted by $\perp$ and $\top$. An element $a \in L$ is called a co-prime, if for all $b, c \in L$, $a \leq b \vee c$ implies $a \leq b$ or $a \leq c$. The set of all co-primes in $L\backslash\{\perp\}$ is denoted by $J(L)$. For any $a \in L$, there is an $L_1 \subseteq J(L)$ such that $a = \bigvee_{b \in L_1} b$. A binary relation $\prec$ on L is defined by $a \prec b$ iff for any $L_1 \subseteq L$, $b \leq \bigvee L_1$ implies some $d \in L_1$ such that $a \leq d$. The mapping $\beta : L \to 2^L$, defined by $\beta(a) = \{b : b \prec a\}$, satisfies $\beta(\bigvee_{i \in I} a_i) = \bigcup_{i \in I} \beta(a_i)$ for $\{a_i\}_{i \in I} \subseteq L$. For any $a \in L$, $\beta(a)$ and $\beta^*(a) = \beta(a) \cap J(L)$ satisfies $a = \bigvee \beta(a) = \bigvee \beta^*(a)$ [24].

An L-fuzzy set on X is a mapping $A : X \to L$. The set of all L-fuzzy sets on X is denoted by L^X. The smallest element and the largest element in L^X are respectively denoted by $\underline{\perp}$ and $\underline{\top}$. A subset $\{A_i\}_{i \in I} \subseteq L^X$ is said to be up-directed (or, down-directed), denoted by $\{A_i\}_{i \in I} \overset{dir}{\subseteq} L^X$ (or $\bigvee^{dir}_{i \in I} A_i$), if for all $i, j \in I$, there is a $k \in I$ such that $A_i \vee A_j \leq A_k$ (or, $A_k \leq A_i \wedge A_j$). In this case, we denote $\bigvee_{i \in I} A_i$ (or, $\bigwedge_{i \in I} A_i$) by $\bigvee^{dir}_{i \in I} A_i$ (or, $\bigwedge^{ddir}_{i \in I} A_i$).

For any $A \in L^X$, we denote $\beta^*(A) = \{x_\lambda \in L^X : \lambda \in \beta^*(A(x))\}$ and $\mathfrak{F}(A) = \{F \in L^X : \exists \varphi \in 2^{\beta^*(A)}_{fin}, F = \bigvee \varphi\}$. In particular, $\mathfrak{F}(\underline{\top})$ is written by $\mathfrak{F}(L^X)$. Then (1) $B \leq A$ iff $\mathfrak{F}(B) \subseteq \mathfrak{F}(A)$; (2) $\beta^*(A) \subseteq \mathfrak{F}(A)$ and $\bigvee \mathfrak{F}(A) = A$; (3) $\mathfrak{F}(\bigvee \varphi) = \bigcup_{A \in \varphi} \mathfrak{F}(A)$ for any up-directly subset $\varphi \subseteq L^X$ [32].

For a mapping $f : X \to Y$, the L-fuzzy mapping $f^{\to}_L : L^X \to L^Y$ is defined by $f^{\to}_L(A)(y) = \bigvee\{A(x) : f(x) = y\}$ for $A \in L^X$ and $y \in Y$, and the mapping $f^{\leftarrow}_L : L^Y \to L^X$ is defined by $f^{\leftarrow}_L(B)(x) = B(f(x))$ for $B \in L^Y$ and $x \in X$ [24].

Next, we recall some results and denotations of L-convex space and L-concave space.

Definition 1 ([7]) A subset $C \subseteq L^X$ is called an L-convex structure on L^X and the pair (X, C) is called an L-convex space if

(LC1) $\underline{\top}, \underline{\bot} \in \mathcal{C}$;
(LC2) $\bigwedge_{i\in I} A_i \in \mathcal{C}$ for any subset $\{A_i\}_{i\in I} \subseteq \mathcal{C}$;
(LC3) $\bigvee^{dir}_{i\in I} A_i \in \mathcal{C}$ for any $\{A_i\}_{i\in I} \overset{dir}{\subseteq} \mathcal{C}$.

Theorem 1 ([11]) *The L-convex hull operator $co_{\mathcal{C}} : L^X \to L^X$ of an L-convex space $(X,\mathcal{C})$ is defined by $co_{\mathcal{C}}(A) = \bigwedge\{B \in \mathcal{C} : A \leq B\}$ for any $A \in L^X$. It satisfies*

(LCO1) $co_{\mathcal{C}}(\underline{\bot}) = \underline{\bot}$;
(LCO2) $A \leq co_{\mathcal{C}}(A)$;
(LCO3) $co_{\mathcal{C}}(co_{\mathcal{C}}(A)) = co_{\mathcal{C}}(A)$;
(LCO4) $co_{\mathcal{C}}(\bigvee^{dir}_{i\in I} A_i) = \bigvee_{i\in I} co_{\mathcal{C}}(A_i)$ for any $\{A_i\}_{i\in I} \overset{dir}{\subseteq} L^X$.

Conversely, if an operator $co : L^X \to L^X$ satisfies (LCO1)–(LCO4), then the set $\mathcal{C}_{co} = \{A \in L^X : co(A) = A\}$ is an L-convex structure satisfying $co_{\mathcal{C}_{co}} = co$.

Let $(X,\mathcal{C}_X)$ and $(Y,\mathcal{C}_Y)$ be L-convex spaces. A mapping $f : X \to Y$ is called an L-convexity preserving mapping, if $f_L^{\leftarrow}(A) \in \mathcal{C}_X$ for any $A \in \mathcal{C}_Y$. It is proved that a mapping $f : X \to Y$ is an L-convex preserving mapping if and only if $f_L^{\rightarrow}(co_{\mathcal{C}_X}(A)) \leq co_{\mathcal{C}_Y}(f_L^{\rightarrow}(A))$ for any $A \in L^X$. The category of L-convex spaces and L-convex preserving mappings is denoted by L-**CS** [11].

Definition 2 ([11]) A subset $\mathcal{A} \subseteq L^X$ is called an L-concave structure on L^X and the pair $(X,\mathcal{A})$ is called an L-concave space if

(LCA1) $\underline{\top}, \underline{\bot} \in \mathcal{A}$;
(LCA2) $\bigvee_{i\in I} A_i \in \mathcal{A}$ for any $\{A_i\}_{i\in I} \subseteq \mathcal{A}$;
(LCA3) $\bigwedge^{ddir}_{i\in I} A_i \in \mathcal{A}$ for any $\{A_i\}_{i\in I} \overset{ddir}{\subseteq} \mathcal{A}$.

Theorem 2 ([11]) *The L-concave hull operator $ca_{\mathcal{A}} : L^X \to L^X$ of an L-concave space $(X,\mathcal{A})$ is defined by $ca_{\mathcal{A}}(A) = \bigvee\{B \in \mathcal{A} : B \leq A\}$ for any $A \in L^X$. It satisfies*

(LCAH1) $ca_{\mathcal{A}}(\underline{\top}) = \underline{\top}$;
(LCAH2) $ca_{\mathcal{A}}(A) \leq A$;
(LCAH3) $ca_{\mathcal{A}}(ca_{\mathcal{A}}(A)) = ca_{\mathcal{A}}(A)$;
(LCAH4) $ca_{\mathcal{A}}(\bigwedge^{ddir}_{i\in I} A_i) = \bigwedge_{i\in I} ca_{\mathcal{A}}(A_i)$ for any $\{A_i\}_{i\in I} \overset{ddir}{\subseteq} L^X$.

Conversely, if an operator $ca : L^X \to L^X$ satisfies (LCAH1)–(LCAH4), then the set $\mathcal{A}_{ca} = \{A \in L^X : ca(A) = A\}$ is an L-concave structure satisfying $ca_{\mathcal{A}_{ca}} = ca$.

Let $(X,\mathcal{A}_X)$ and $(Y,\mathcal{A}_Y)$ be L-concave spaces. A mapping $f : X \to Y$ is called an L-concavity preserving mapping, if $f_L^{\leftarrow}(A) \in \mathcal{A}_X$ for any $A \in \mathcal{A}_Y$. It is proved that a mapping $f : X \to Y$ is an L-concavity preserving mapping if and only if $f_L^{\leftarrow}(ca_{\mathcal{A}_Y}(B)) \leq ca_{\mathcal{A}_X}(f_L^{\leftarrow}(B))$ for any $B \in L^Y$. The category of L-concave spaces and L-concavity preserving mappings is denoted by L-**CAS** [11].

Definition 3 ([6]) A binary relation $\preccurlyeq$ on L^X is called an L-convex enclosed relation and the pair $(X,\preccurlyeq)$ is called an L-convex enclosed relation space, if $\preccurlyeq$ satisfies

(LCER1) $\underline{\bot} \preccurlyeq \underline{\bot}$;
(LCER2) $A \preccurlyeq B$ implies $A \leq B$;
(LCER3) $A \preccurlyeq \bigwedge_{i\in I} B_i$ iff $A \preccurlyeq B_i$ for all $i \in I$;
(LCER4) $A \preccurlyeq B$ implies some $C \in L^X$ with $A \preccurlyeq C \preccurlyeq B$;
(LCER5) $\bigvee^{dir}_{i\in I} A_i \preccurlyeq B$ iff $A_i \preccurlyeq B$ for any $i \in I$.

Let $(X,\preccurlyeq_X)$ and $(Y,\preccurlyeq_Y)$ be L-convex enclosed relation spaces. A mapping $f : X \to Y$ is called an L-convex enclosed relation preserving mapping, if $f_L^{\leftarrow}(A) \preccurlyeq_X f_L^{\leftarrow}(B)$ for all $A, B \in L^X$ with $A \preccurlyeq_Y B$.

The category of L-convex enclosed relation spaces and L-convex enclosed relation preserving mappings is denoted by L-**CERS**. It has been proved that L-**CERS** and L-**CS** are isomorphic [6].

Theorem 3 ([6]) *(1) Let $(X,\mathcal{C})$ be an L-convex space. Define a binary relation $\preccurlyeq_{\mathcal{C}}$ by*

$$\forall A, B \in L^X, \;\; A \preccurlyeq_{\mathcal{C}} B \;\Leftrightarrow\; co_{\mathcal{C}}(A) \leq B.$$

Then $\preccurlyeq_{\mathcal{C}}$ is an L-convex enclosed relation.

(2) Let $(X,\preccurlyeq)$ be an L-convex enclosed relation space. Define an operator $co_{\preccurlyeq} : L^X \to L^X$ by

$$\forall A \in L^X, \;\; co_{\preccurlyeq}(A) = \bigwedge\{B \in L^X : A \preccurlyeq B\}.$$

Then $co_{\preccurlyeq}$ is an L-convex hull operator of an L-convex structure $\mathcal{C}_{\preccurlyeq}$.

*(3) L-**CS** is isomorphic to L-**CERS**.*

Definition 4 ([31]) A binary relation $\preccurlyeq$ on L^X is called an L-concave internal relation and the pair $(X,\preccurlyeq)$ is called an L-concave internal relation space, if $\preccurlyeq$ satisfies

(LCIR1) $\underline{\top} \preccurlyeq \underline{\top}$;
(LCIR2) $A \preccurlyeq B$ implies $A \leq B$;
(LCIR3) $\bigvee_{i\in I} A_i \preccurlyeq B$ iff $A_i \preccurlyeq B$ for all $i \in I$;
(LCIR4) $A \preccurlyeq B$ implies some $C \in L^X$ with $A \preccurlyeq C \preccurlyeq B$;
(LCIR5) $A \preccurlyeq \bigwedge^{ddir}_{i\in I} B_i$ iff $A \preccurlyeq B_i$ for any $i \in I$.

Let $(X, \preccurlyeq_X)$ and $(Y, \preccurlyeq_Y)$ be L-concave internal relation spaces. A mapping $f : X \to Y$ is called an L-concave internal relation preserving mapping, if $f_L^{\leftarrow}(A) \preccurlyeq_X f_L^{\leftarrow}(B)$ for all $A, B \in L^X$ with $A \preccurlyeq_Y B$.

The category of L-concave internal relation spaces and L-concave internal relation preserving mappings is denoted by L-**CIRS**. It has been proved that L-**CIRS** and L-**CAS** are isomorphic [31].

Theorem 4 ([31]) *(1) Let $(X, \mathcal{A})$ be an L-concave space. Define a binary relation $\preccurlyeq_{\mathcal{A}}$ by*

$$\forall A, B \in L^X, \ \ A \preccurlyeq_{\mathcal{A}} B \ \ \Leftrightarrow \ \ A \leq ca_{\mathcal{A}}(B).$$

Then $\preccurlyeq_{\mathcal{A}}$ is an L-concave internal relation.

(2) Let $(X, \preccurlyeq)$ be an L-concave internal relation space. Define an operator $ca_{\preccurlyeq} : L^X \to L^X$ by

$$\forall A \in L^X, \ \ ca_{\preccurlyeq}(A) = \bigvee \{B \in L^X : B \preccurlyeq A\}.$$

Then $ca_{\preccurlyeq}$ is an L-concave hull operator of an L-concave structure $\mathcal{A}_{\preccurlyeq}$.

*(3) L-**CAS** is isomorphic to L-**CIRS**.*

3 *L*-concave derived internal relation spaces

In this section, we introduce the notion of L-concave derived internal relation space which can be used to characterize L-concave internal relation space and L-concave space. Also, we introduce the notion of L-concave derived hull operator by which we can obtain a simple characterization of L-concave derived internal relation space.

Definition 5 A binary relation $\preccurlyeq$ on L^X is called an L-concave derived internal relation and the pair $(X, \preccurlyeq)$ is called an L-concave derived internal relation space, if for all $A, B, C \in L^X$ and $x_\lambda \in \beta^*(\underline{\top})$,

(LCDIR1) $\underline{\top} \preccurlyeq \underline{\top}$;
(LCDIR2) $A \preccurlyeq B$ iff $x_\lambda \preccurlyeq B \vee x_\lambda$ for any $x_\lambda \in \beta^*(A)$;
(LCDIR3) $\bigvee_{i\in I} A_i \preccurlyeq B$ iff $A_i \preccurlyeq B$ for any $i \in I$;
(LCDIR4) $A \preccurlyeq B$ implies some $C \in L^X$ such that $A \wedge B \preccurlyeq C \preccurlyeq B$ and $A \wedge B \leq C \leq B$;
(LCDIR5) $A \preccurlyeq \bigwedge^{ddir}_{i\in I} B_i$ iff $A \preccurlyeq B_i$ for any $i \in I$.

Let $(X, \preccurlyeq_X)$ and $(Y, \preccurlyeq_Y)$ be L-concave derived internal relation spaces. A mapping $f : X \to Y$ is called an L-concave derived internal relation preserving mapping, if for all $f_L^{\leftarrow}(A \wedge B) \preccurlyeq_X f_L^{\leftarrow}(B)$ for all $A, B \in L^Y$ with $A \preccurlyeq_Y B$.

The category of L-concave derived internal relation spaces and L-concave derived internal relation preserving mappings is denoted by L-**CDIRS**.

Theorem 5 *Let $(X, \preccurlyeq)$ be an L-concave derived internal relation space. Define a binary relation $\preccurlyeq_{\preccurlyeq}$ on L^X by: for all $A, B \in L^X$,*

$$A \preccurlyeq_{\preccurlyeq} B \ \ \Leftrightarrow \ \ \exists C \in L^X,\ C \preccurlyeq B \text{ and } A = B \wedge C.$$

Then $\preccurlyeq_{\preccurlyeq}$ is an L-concave internal relation.

Proof (LCIR1). We have $\underline{\top} \preccurlyeq \underline{\top}$ and $\underline{\top} \wedge \underline{\top} = \underline{\top}$ by (LCDIR1). Thus $\underline{\top} \preccurlyeq_{\preccurlyeq} \underline{\top}$.

(LCIR2). It directly follows from the definition.

(LCIR3). Let $\bigvee_{i\in I} A_i \preccurlyeq_{\preccurlyeq} B$. Then there is a $C \in L^X$ such that $C \preccurlyeq B$ and $\bigvee_{i\in I} A_i = B \wedge C$. For any $i \in I$, we have $A_i \leq \bigvee_{i\in I} A_i \preccurlyeq B$. Thus $A_i \preccurlyeq B$ and $A_i = B \wedge A_i$. Hence $A_i \preccurlyeq_{\preccurlyeq} B$.

Conversely, assume that $A_i \preccurlyeq_{\preccurlyeq} B$ for any $i \in I$. For any $i \in I$, there is a $C_i \in L^X$ such that $C_i \preccurlyeq B$ and $A_i = B \wedge C_i$. Thus $\bigvee_{i\in I} C_i \preccurlyeq B$ by (LCDIR3). Further, we have

$$\bigvee_{i\in I} A_i = \bigvee_{i\in I} (B \wedge C_i) = B \wedge \bigvee_{i\in I} C_i.$$

Hence $\bigvee_{i\in I} A_i \preccurlyeq_{\preccurlyeq} B$.

(LCIR4). Let $A \preccurlyeq_{\preccurlyeq} B$. There is a $D \in L^X$ such that $D \preccurlyeq B$ and $A = B \wedge D$. By $D \preccurlyeq B$ and (LCDIR4), there is $C \in L^X$ such that $A = D \wedge B \preccurlyeq C \preccurlyeq B$ and $A \leq C \leq B$. Thus $A \preccurlyeq_{\preccurlyeq} C \preccurlyeq_{\preccurlyeq} B$.

(LCIR5). If $A \preccurlyeq_{\preccurlyeq} \bigwedge_{i\in I}^{ddir} B_i$, then there is a $D \in L^X$ such that $D \preccurlyeq \bigwedge_{i\in I}^{ddir} B_i$ and $A = \bigwedge_{i\in I}^{ddir}(B_i \wedge D)$. For any $j \in I$, we have $A \leq D \preccurlyeq \bigwedge_{i\in I}^{ddir} B_i \leq B_j$. Thus $A \preccurlyeq B_j$ and $A \leq B_j$. Hence $A \preccurlyeq_{\preccurlyeq} B_j$.

Conversely, assume that $\{B_i\}_{i\in I} \overset{ddir}{\subseteq} L^X$ with $A \preccurlyeq_{\preccurlyeq} B_i$ for any $i \in I$. Then there is a $D_i \in L^X$ and $D_i \preccurlyeq B_i$ such that $A = D_i \wedge B_i$. Let $D = \bigwedge_{i\in I} D_i$. We have $D \leq D_i \preccurlyeq B_i$ for any $i \in I$. Thus $D \preccurlyeq B_i$ for any $i \in I$. Hence $D \preccurlyeq \bigwedge_{i\in I}^{ddir} B_i$ by (LCDIR5). Further, by

$$A = \bigwedge_{i\in I}(D_i \wedge B_i) = D \wedge \bigwedge_{i\in I}^{ddir} B_i,$$

we conclude that $A \preccurlyeq_{\preccurlyeq} \bigwedge_{i\in I}^{ddir} B_i$.

Theorem 6 *Let $(X, \preccurlyeq_X)$ and $(Y, \preccurlyeq_Y)$ be L-concave derived internal relation spaces. If $f : X \to Y$ is an L-concave derived internal relation preserving mapping, then $f : (X, \preccurlyeq_{\preccurlyeq_X}) \to (Y, \preccurlyeq_{\preccurlyeq_Y})$ is an L-concave internal relation preserving mapping.*

Proof If $A \preccurlyeq_{\preccurlyeq_Y} B$, then there is a $C \in L^Y$ such that $C \preccurlyeq_Y B$ and $A = B \wedge C$. Thus $f_L^{\leftarrow}(C \wedge B) \preccurlyeq_X f_L^{\leftarrow}(B)$ and $f_L^{\leftarrow}(A) = f_L^{\leftarrow}(C \wedge B) \wedge f_L^{\leftarrow}(B)$. Hence $f_L^{\leftarrow}(A) \preccurlyeq_{\preccurlyeq_X} f_L^{\leftarrow}(B)$. Therefore f is an L-concave internal relation preserving mapping.

Theorem 7 *Let $(X, \preccurlyeq)$ be an L-concave internal relation space. Define a binary relation $\preccurlyeq_{\preccurlyeq}$ on L^X by: for all $A, B \in L^X$,*

$$A \preccurlyeq_{\preccurlyeq} B \quad \Leftrightarrow \quad \forall x_\lambda \in \beta^*(A),\ x_\lambda \preccurlyeq B \vee x_\lambda.$$

Then $\preccurlyeq_{\preccurlyeq}$ is an L-concave internal relation.

Proof It is clear that $A \preccurlyeq_{\preccurlyeq} B$ for any $A, B, C, D \in L^X$ with $A \leq C \preccurlyeq_{\preccurlyeq} D \leq B$.

(LCDIR1). If $x_\lambda \in \beta^*(\underline{\top})$, then $x_\lambda \leq \underline{\top} \preccurlyeq \underline{\top} = \underline{\top} \vee x_\lambda$. Thus $x_\lambda \preccurlyeq \underline{\top}$. Hence $\underline{\top} \preccurlyeq_{\preccurlyeq} \underline{\top}$.

(LCDIR2). Let $A \preccurlyeq_{\preccurlyeq} B$ and let $x_\lambda \in \beta^*(A)$. To prove that $x_\lambda \preccurlyeq_{\preccurlyeq} B \vee x_\lambda$, let $x_\eta \in \beta^*(x_\lambda)$. Then $x_\eta \in \beta^*(A)$. By $A \preccurlyeq_{\preccurlyeq} B$, we have $x_\eta \preccurlyeq B \vee x_\eta \leq (B \vee x_\lambda) \vee x_\eta$. Thus $x_\eta \preccurlyeq (B \vee x_\lambda) \vee x_\eta$. Hence $x_\lambda \preccurlyeq_{\preccurlyeq} B \vee x_\lambda$.

Conversely, assume that $x_\lambda \preccurlyeq_{\preccurlyeq} B \vee x_\lambda$ for any $x_\lambda \in \beta^*(A)$. To prove that $A \preccurlyeq_{\preccurlyeq} B$, let $x_\lambda \in \beta^*(A)$. By $x_\lambda \preccurlyeq_{\preccurlyeq} B \vee x_\lambda$, we have $x_\eta \preccurlyeq B \vee x_\lambda \vee x_\eta = B \vee x_\lambda$ for any $x_\eta \in \beta^*(x_\lambda)$. Thus $x_\lambda = \bigvee_{x_\eta \in \beta^*(x_\lambda)} \preccurlyeq B \vee x_\lambda$ by (LCIR3). Hence $A \preccurlyeq_{\preccurlyeq} B$.

(LCDIR3). If $\bigvee_{i\in I} A_i \preccurlyeq_{\preccurlyeq} B$, then it is clear that $A_i \preccurlyeq_{\preccurlyeq} B$ for any $i \in I$. Conversely, assume that $A_i \preccurlyeq_{\preccurlyeq} B$ for any $i \in I$. To prove that $\bigvee_{i\in I} A_i \preccurlyeq_{\preccurlyeq} B$, let $x_\lambda \in \beta^*(\bigvee_{i\in I} A_i)$. Then there is an $i \in I$ such that $x_\lambda \in \beta^*(A_i)$. By $A_i \preccurlyeq_{\preccurlyeq} B$, we have $x_\lambda \preccurlyeq B \vee x_\lambda$. Therefore $\bigvee_{i\in I} A_i \preccurlyeq_{\preccurlyeq} B$.

(LCDIR4). Let $A \preccurlyeq_{\preccurlyeq} B$. Let

$$D = \bigvee\{F \in L^X : F \leq B,\ F \preccurlyeq_{\preccurlyeq} B\}.$$

We have $A \wedge B \leq D \leq B$. In addition, $D \preccurlyeq_{\preccurlyeq} B$ by (LCDIR3). To prove that $A \wedge B \preccurlyeq_{\preccurlyeq} D$, we check that $y_\eta \preccurlyeq D \vee y_\eta$ for any $y_\eta \in \beta^*(A \wedge B)$.

Let $y_\eta \in \beta^*(A \wedge B)$. By $A \preccurlyeq_{\preccurlyeq} B$, we have $y_\eta \preccurlyeq B \vee y_\eta = B$. By (LCIR4), there is a $C \in L^X$ such that $y_\eta \preccurlyeq C \preccurlyeq B$. Thus $y_\eta \leq C \leq B$ by (LCIR2). For any $z_\theta \in \beta^*(C)$, we have $z_\theta \leq C \preccurlyeq B \leq B \vee z_\theta$ which implies that $z_\theta \preccurlyeq B \vee z_\theta$. Hence $C \preccurlyeq_{\preccurlyeq} B$ followed by $C \leq D$. Further, we have $y_\eta \preccurlyeq D \vee y_\eta$ by $y_\eta \preccurlyeq C \leq D \vee y_\eta$. Therefore $A \wedge B \preccurlyeq_{\preccurlyeq} D \preccurlyeq_{\preccurlyeq} B$ and $A \wedge B \leq D \leq B$ as desired.

(LCDIR5). If $A \preccurlyeq_{\preccurlyeq} \bigwedge_{i\in I}^{ddir} B_i$, then $A \preccurlyeq_{\preccurlyeq} B_i$ for any $i \in I$. Conversely, assume that $\{B_i\}_{i\in I} \overset{ddir}{\subseteq} L^X$ with $A \preccurlyeq_{\preccurlyeq} B_i$ for any $i \in I$. Then $x_\lambda \preccurlyeq B_i \vee x_\lambda$ for any $x_\lambda \in \beta^*(A)$ and any $i \in I$. Thus, by (LCIR5),

$$x_\lambda \preccurlyeq \bigwedge_{i\in I}^{ddir}(B_i \vee x_\lambda) = (\bigwedge_{i\in I}^{ddir} B_i) \vee x_\lambda.$$

Therefore $A \preccurlyeq_{\preccurlyeq} \bigwedge_{i\in I}^{ddir} B_i$.

Theorem 8 *Let $(X, \preccurlyeq_X)$ and $(Y, \preccurlyeq_Y)$ be L-concave internal relation spaces. If $f : X \to Y$ is an L-concave internal relation preserving mapping, then $f : (X, \preccurlyeq_{\preccurlyeq_X}) \to (Y, \preccurlyeq_{\preccurlyeq_Y})$ is an L-concave derived internal relation preserving mapping.*

Proof Let $A \preccurlyeq_{\preccurlyeq_Y} B$. If $f_L^{\leftarrow}(A \wedge B) = \underline{\bot}$, then $f_L^{\leftarrow}(A \wedge B) \preccurlyeq_{\preccurlyeq_X} f_L^{\leftarrow}(B)$ is trivial. Let $f_L^{\leftarrow}(A \wedge B) \neq \underline{\bot}$. If $x_\lambda \in \beta^*(f_L^{\leftarrow}(A \wedge B))$, then $f_L^{\rightarrow}(x_\lambda) \in \beta^*(A \wedge B)$. Thus $f_L^{\rightarrow}(x_\lambda) \preccurlyeq_Y B \vee f_L^{\rightarrow}(x_\lambda)$ and

$$f_L^{\leftarrow}(f_L^{\rightarrow}(x_\lambda)) \preccurlyeq_X f_L^{\leftarrow}(B) \vee f_L^{\leftarrow}(f_L^{\rightarrow}(x_\lambda)) = f_L^{\leftarrow}(B).$$

Further, since $x_\lambda \leq f_L^{\leftarrow}(f_L^{\rightarrow}(x_\lambda))$, we have $x_\lambda \preccurlyeq_X f_L^{\leftarrow}(B)$. Hence $x_\lambda \preccurlyeq_X f_L^{\leftarrow}(B)$ and so $f_L^{\leftarrow}(A \wedge B) \preccurlyeq_X f_L^{\leftarrow}(B)$ by (LCIR3). Therefore f is an L-concave derived internal relation preserving mapping.

Theorem 9 *We have $\preccurlyeq_{\preccurlyeq_\preccurlyeq} = \preccurlyeq$ for any L-concave interval relation space $(X, \preccurlyeq)$ and $\preccurlyeq_{\preccurlyeq_\preccurlyeq} = \preccurlyeq$ for any L-concave derived internal relation space $(X, \preccurlyeq)$.*

Proof Let $(X, \preccurlyeq)$ be an L-concave internal relation space. If $A \preccurlyeq_{\preccurlyeq_\preccurlyeq} B$, then $A \leq B$ by (LCIR2). In addition, there is a $C \in L^X$ such that $C \preccurlyeq_\preccurlyeq B$ and $A = B \wedge C$. Thus $A \preccurlyeq_\preccurlyeq B$ which implies that $x_\lambda \preccurlyeq B \vee x_\lambda = B$ for any $x_\lambda \in \beta^*(A)$. Hence $A = \bigvee_{x_\lambda \in \beta^*(A)} x_\lambda \preccurlyeq B$ by (LCIR3).

Conversely, if $A \preccurlyeq B$ then $A \leq B$ by (LCIR2). For any $x_\lambda \in \beta^*(A)$, we have $x_\lambda \leq A \preccurlyeq B$ and so $x_\lambda \preccurlyeq B = B \vee x_\lambda$. Hence $A \preccurlyeq_\preccurlyeq B$. Since $A \wedge B = A$, we have $A \preccurlyeq_{\preccurlyeq_\preccurlyeq} B$.

In conclusion, for all $A, B \in L^X$, we have $A \preccurlyeq_{\preccurlyeq_\preccurlyeq} B$ iff $A \preccurlyeq B$. That is, $\preccurlyeq_{\preccurlyeq_\preccurlyeq} = \preccurlyeq$.

Let $(X, \preccurlyeq)$ be an L-concave derived internal relation space.

If $A \preccurlyeq_{\preccurlyeq_\preccurlyeq} B$ and $x_\lambda \in \beta^*(A)$, then $x_\lambda \preccurlyeq_\preccurlyeq B \vee x_\lambda$. Thus there is a $C \in L^X$ such that $C \preccurlyeq B$ and $x_\lambda = (B \vee x_\lambda) \wedge C$. Hence $x_\lambda \leq C \preccurlyeq B \vee x_\lambda$ and so $x_\lambda \preccurlyeq B \vee x_\lambda$. Therefore $A \preccurlyeq B$ by (LCDIR2).

Conversely, assume that $A \preccurlyeq B$. If $x_\lambda \in \beta^*(A)$, then $x_\lambda \leq A \preccurlyeq B \leq B \vee x_\lambda$. Thus $x_\lambda \preccurlyeq B \vee x_\lambda$. By this result and $x_\lambda = x_\lambda \wedge (B \vee x_\lambda)$, we have $x_\lambda \preccurlyeq_\preccurlyeq B \vee x_\lambda$. Hence $A \preccurlyeq_{\preccurlyeq_\preccurlyeq} B$.

In conclusion, for all $A, B \in L^X$, we have $A \preccurlyeq_{\preccurlyeq_\preccurlyeq} B$ iff $A \preccurlyeq B$. That is, $\preccurlyeq_{\preccurlyeq_\preccurlyeq} = \preccurlyeq$.

Based on Theorems 5 and 6, we obtain a factor $\mathbb{U} : L$-**CDIRS**$\to L$-**CIRS** defined by

$$\mathbb{U}((X, \preccurlyeq)) = (X, \preccurlyeq_\preccurlyeq), \quad \mathbb{U}(f) = f.$$

Based on Theorems 5–9, $\mathbb{U}$ is an isomorphic factor. Thus we have the following conclusion.

Theorem 10 *The category L-**CDIRS** is isomorphic to the category L-**CIRS**.*

To simply characterize L-**CDIRS**, we introduce L-concave derived hull space as follows.

Definition 6 An operator $\mathcal{I} : L^X \to L^X$ is called an L-concave derived hull operator on L^X and the pair $(X, \mathcal{I})$ is called an L-concave derived hull space if for all $A, B \in L^X$,

(LCADH1) $\mathcal{I}(\underline{\top}) = \underline{\top}$;

(LCADH2) $A \leq \mathcal{I}(B)$ iff $x_\lambda \leq \mathcal{I}(B \vee x_\lambda)$ for any $x_\lambda \in \beta^*(A)$;

(LCADH3) $A \wedge \mathcal{I}(A) \leq \mathcal{I}(A \wedge \mathcal{I}(A))$;

(LCADH4) $\mathcal{I}(\bigwedge^{ddir}_{i \in I} A_i) = \bigwedge_{i \in I} \mathcal{I}(A_i)$.

Let $(X, \mathcal{I}_X)$ and $(Y, \mathcal{I}_Y)$ be L-concave derived hull spaces. A mapping $f : X \to Y$ is called an L-concave derived hull preserving mapping, if $f_L^{\leftarrow}(\mathcal{I}_Y(B) \wedge B) \leq \mathcal{I}_X(f_L^{\leftarrow}(B))$ for all $B \in L^Y$. The category of L-concave derived hull spaces and L-concave derived hull preserving mappings is denoted by L-**CADHS**.

Theorem 11 *Let $(X, \mathcal{I})$ be an L-concave derived hull space. Define a binary relation $\preccurlyeq_\mathcal{I}$ on L^X by*

$$\forall A, B \in L^X, \;\; A \preccurlyeq_\mathcal{I} B \;\; \Leftrightarrow \;\; A \leq \mathcal{I}(B).$$

Then $(X, \preccurlyeq_\mathcal{I})$ is an L-concave derived internal relation space.

Proof We check that $\preccurlyeq_\mathcal{I}$ satisfies (LCDIR1)–(LCDIR5).

(LCDIR1). We have $\mathcal{I}(\underline{\top}) = \underline{\top}$ by (LCADH1). Thus $\underline{\top} \preccurlyeq_\mathcal{I} \underline{\top}$.

(LCDIR2). It directly follows from (LCADH2).

(LCDIR3). If $\bigvee_{i \in I} A_i \preccurlyeq_\mathcal{I} B$ then $A_j \leq \bigvee_{i \in I} A_i \leq \mathcal{I}(B)$ for any $j \in I$. Thus $A_j \preccurlyeq_\mathcal{I} B$ for any $j \in I$. Conversely, assume that $A_i \preccurlyeq_\mathcal{I} B$ for any $i \in I$. Then $A_i \leq \mathcal{I}(B)$ for any $i \in I$. Hence $\bigvee_{i \in I} A_i \leq \mathcal{I}(B)$. Therefore $\bigvee_{i \in I} A_i \preccurlyeq_\mathcal{I} B$.

(LCDIR4). Let $A \preccurlyeq_\mathcal{I} B$ and let $E = B \wedge \mathcal{I}(B)$. By $A \preccurlyeq_\mathcal{I} B$, we have $A \leq \mathcal{I}(B)$. Thus $A \wedge B \leq E \leq B$. Since $E \leq \mathcal{I}(B)$, we have $E \preccurlyeq_\mathcal{I} B$. In addition, by (LCADH3), we have

$$A \wedge B \leq E = B \wedge \mathcal{I}(B) \leq \mathcal{I}(B \wedge \mathcal{I}(B)) = \mathcal{I}(E).$$

Thus $A \wedge B \preccurlyeq_\mathcal{I} E$. Therefore $A \wedge B \preccurlyeq_\mathcal{I} E \preccurlyeq_\mathcal{I} B$ and $A \wedge B \leq E \leq B$ as desired.

(LCDIR5). By (LCADH4), $\mathcal{I}$ is monotonic. So the desired result follows from (LCADH4).

Theorem 12 *Let $(X, \mathcal{I}_X)$ and $(Y, \mathcal{I}_Y)$ be L-concave derived hull spaces. If $f : X \to Y$ is an L-concave derived hull preserving mapping, then $f : (X, \preccurlyeq_{\mathcal{I}_X}) \to (Y, \preccurlyeq_{\mathcal{I}_Y})$ is an L-concave derived internal relation preserving mapping.*

Proof If $A \preccurlyeq_{\mathcal{I}_Y} B$ then $A \leq \mathcal{I}_Y(B)$. Thus $f_L^{\leftarrow}(A) \leq f_L^{\leftarrow}(\mathcal{I}_Y(B))$ and

$$\begin{aligned} f_L^{\leftarrow}(A \wedge B) &\leq f_L^{\leftarrow}(\mathcal{I}_Y(B)) \wedge f_L^{\leftarrow}(B) \\ &= f_L^{\leftarrow}(\mathcal{I}_Y(B) \wedge B) \\ &\leq \mathcal{I}_X(f_L^{\leftarrow}(B)). \end{aligned}$$

Hence $f_L^{\leftarrow}(A \wedge B) \preccurlyeq_{\mathcal{I}_X} f_L^{\leftarrow}(B)$. Therefore f is an L-concave derived internal relation preserving mapping.

Theorem 13 *Let $(X, \preccurlyeq)$ be an L-concave derived internal relation space. Define an operator $\mathcal{I}_{\preccurlyeq} : L^X \to L^X$ by*

$$\forall A \in L^X,\ \ \mathcal{I}_{\preccurlyeq}(A) = \bigvee\{B \in L^X : B \preccurlyeq A\}.$$

Then $(X, \mathcal{I}_{\preccurlyeq})$ is an L-concave derived hull space.

Proof (LCADH1). We have $\underline{\top} \leq \mathcal{I}_{\preccurlyeq}(\underline{\top})$ by (LCDIR1). Thus $\mathcal{I}_{\preccurlyeq}(\underline{\top}) = \underline{\top}$.

(LCADH2). Let $A \leq \mathcal{I}_{\preccurlyeq}(B)$. If $x_\lambda \in \beta^*(A)$, then $x_\lambda \prec \mathcal{I}_{\preccurlyeq}(B)$. Thus there is a $D \in L^X$ such that $x_\lambda \leq D$ and $D \preccurlyeq B$. Hence $x_\lambda \leq D \preccurlyeq B \leq B \vee x_\lambda$ followed by $x_\lambda \preccurlyeq B \vee x_\lambda$. Therefore $x_\lambda \leq \mathcal{I}_{\preccurlyeq}(B \vee x_\lambda)$.

Conversely, assume that $x_\lambda \leq \mathcal{I}_{\preccurlyeq}(B \vee x_\lambda)$ for any $x_\lambda \in \beta^*(A)$. To prove that $A \leq \mathcal{I}_{\preccurlyeq}(B)$, let $x_\lambda \in \beta^*(A)$. Then $x_\lambda \leq \mathcal{I}_{\preccurlyeq}(B \vee x_\lambda)$. For any $x_\eta \in \beta^*(x_\lambda)$, we have $x_\eta \prec \mathcal{I}_{\preccurlyeq}(B \vee x_\lambda)$. Thus there is a $D \in L^X$ such that $x_\eta \leq D \preccurlyeq B \vee x_\lambda$. Hence $x_\eta \preccurlyeq B \vee x_\lambda$. So $x_\lambda \preccurlyeq B \vee x_\lambda$ by (LCDIR3). Thus $A \preccurlyeq B$ by (LCDIR2). Therefore $A \leq \mathcal{I}_{\preccurlyeq}(B)$.

(LCADH3). By (LCDIR3), we have $\mathcal{I}_{\preccurlyeq}(A) \preccurlyeq A$. For any $x_\lambda \in \beta^*(A \wedge \mathcal{I}_{\preccurlyeq}(A))$, there is a $D \in L^X$ such that $x_\lambda \leq D \preccurlyeq A$. By $D \preccurlyeq A$ and (LCDIR4), there is a $C \in L^X$ such that $D \wedge A \preccurlyeq C \preccurlyeq A$ and

$$x_\lambda \leq D \wedge A \leq C \leq A$$

Thus $C \leq \mathcal{I}_{\preccurlyeq}(A) \wedge A$. Hence $D \wedge A \preccurlyeq \mathcal{I}_{\preccurlyeq}(A) \wedge A$ which implies that

$$x_\lambda \leq D \wedge A \leq \mathcal{I}_{\preccurlyeq}(\mathcal{I}_{\preccurlyeq}(A) \wedge A).$$

Therefore $A \wedge \mathcal{I}_{\preccurlyeq}(A) \leq \mathcal{I}_{\preccurlyeq}(\mathcal{I}_{\preccurlyeq}(A) \wedge A)$.

(LCADH4). If $\{A_i\}_{i\in I} \overset{ddir}{\subseteq} L^X$, then it is cleat that $\mathcal{I}_{\preccurlyeq}(\bigwedge^{ddir}_{i\in I} A_i) \leq \bigwedge_{i\in I} \mathcal{I}_{\preccurlyeq}(A_i)$. Conversely, let $x_\lambda \in \beta^*(\bigwedge_{i\in I} \mathcal{I}_{\preccurlyeq}(A_i))$. Since $x_\lambda \in \beta^*(\mathcal{I}_{\preccurlyeq}(A_i))$ for any $i \in I$, there is a $C_i \in L^X$ such that $x_\lambda \leq C_i \preccurlyeq A_i$. Let $C = \bigwedge_{i\in I} C_i$. We have $x_\lambda \leq C \preccurlyeq A_i$ for any $i \in I$. Thus $x_\lambda \leq C \preccurlyeq \bigwedge^{ddir}_{i\in I} A_i$ which implies that $x_\lambda \leq \mathcal{I}_{\preccurlyeq}(\bigwedge^{ddir}_{i\in I} A_i)$. Therefore $\bigwedge_{i\in I} \mathcal{I}_{\preccurlyeq}(A_i) \leq \mathcal{I}_{\preccurlyeq}(\bigwedge^{ddir}_{i\in I} A_i)$.

Theorem 14 *Let $(X, \preccurlyeq_X)$ and $(Y, \preccurlyeq_Y)$ be L-concave derived internal relation spaces. If $f : X \to Y$ is an L-concave derived internal relation preserving mapping, then $f : (X, \mathcal{I}_{\preccurlyeq_X}) \to (Y, \mathcal{I}_{\preccurlyeq_Y})$ is an L-concave derived hull preserving mapping.*

Proof Let $B \in L^Y$. To prove that $f_L^{\leftarrow}(\mathcal{I}_{\preccurlyeq_Y}(B) \wedge B) \leq \mathcal{I}_{\preccurlyeq_X}(f_L^{\leftarrow}(B))$, let $x_\lambda \in \beta^*(f_L^{\leftarrow}(\mathcal{I}_{\preccurlyeq_Y}(B) \wedge B))$. Then $f_L^{\rightarrow}(x_\lambda) \prec \mathcal{I}_{\preccurlyeq_Y}(B) \wedge B$. By $f_L^{\rightarrow}(x_\lambda) \prec \mathcal{I}_{\preccurlyeq_Y}(B)$, there is a $D \in L^Y$ such that $f_L^{\rightarrow}(x_\lambda) \leq D \preccurlyeq_Y B$. Thus

$$x_\lambda \leq f_L^{\leftarrow}(D) \wedge f_L^{\leftarrow}(B) = f_L^{\leftarrow}(D \wedge B) \preccurlyeq_X f_L^{\leftarrow}(B).$$

Hence $x_\lambda \leq \mathcal{I}_{\preccurlyeq_X}(f_L^{\leftarrow}(B))$ followed by $f_L^{\leftarrow}(\mathcal{I}_{\preccurlyeq_Y}(B) \wedge B) \leq \mathcal{I}_{\preccurlyeq_X}(f_L^{\leftarrow}(B))$. Therefore f is an L-concave derived hull preserving mapping.

Theorem 15 *We have $\mathcal{I}_{\preccurlyeq_{\mathcal{I}}} = \mathcal{I}$ for any L-concave derived hull space $(X, \mathcal{I})$ and $\preccurlyeq_{\mathcal{I}_{\preccurlyeq}} = \preccurlyeq$ for any L-concave derived internal relation space $(X, \preccurlyeq)$.*

Proof Let $(X, \mathcal{I})$ be an L-concave derived hull space and $A \in L^X$. It is clear that

$$\mathcal{I}_{\preccurlyeq_{\mathcal{I}}}(A) = \bigvee\{D \in L^X : D \preccurlyeq_{\mathcal{I}} A\} \leq \mathcal{I}(A).$$

Conversely, we have $\mathcal{I}(A) \preccurlyeq_{\mathcal{I}} A$ by $\mathcal{I}(A) \leq \mathcal{I}(A)$. Thus $\mathcal{I}(A) \leq \mathcal{I}_{\preccurlyeq_{\mathcal{I}}}(A)$. Therefore $\mathcal{I}_{\preccurlyeq_{\mathcal{I}}} = \mathcal{I}$.

Let $(X, \preccurlyeq)$ be an L-concave derived internal relation space. If $A \preccurlyeq B$ then

$$A \leq \bigvee\{E \in L^X : E \preccurlyeq B\} = \mathcal{I}_{\preccurlyeq}(B).$$

Thus $A \preccurlyeq_{\mathcal{I}_{\preccurlyeq}} B$. Conversely, if $A \preccurlyeq_{\mathcal{I}_{\preccurlyeq}} B$, then $A \leq \mathcal{I}_{\preccurlyeq}(B)$. For any $x_\lambda \in \beta^*(A)$, we have $x_\lambda \prec \mathcal{I}_{\preccurlyeq}(B)$. Thus there is an $E \in L^X$ such that $x_\lambda \leq E \preccurlyeq B$. Hence $x_\lambda \preccurlyeq B \vee x_\lambda$. By (LCDIR2), we have $A \preccurlyeq B$.

In conclusion, for any $A, B \in L^X$, we have $A \preccurlyeq B$ iff $A \preccurlyeq_{\mathcal{I}_{\preccurlyeq}} B$. That is, $\preccurlyeq_{\mathcal{I}_{\preccurlyeq}} = \preccurlyeq$.

Based on Theorems 13 and 14, we obtain a factor $\mathbb{W} : L$-**CDIRS**→ L-**CADHS** by

$$\mathbb{W}((X, \preccurlyeq)) = (X, \mathcal{I}_{\preccurlyeq}), \quad \mathbb{W}(f) = f.$$

Based on Theorems 11–15, $\mathbb{W}$ is an isomorphic factor. Thus we have the following result.

Theorem 16 *The category L-**CDIRS** is isomorphic to the category L-**CADHS**.*

Based on Theorem 4 and Theorems 5–10, relations between L-concave space and L-concave derived internal relation space can be presented as follows.

Corollary 1 *(1) Let $(X, \mathcal{A})$ be an L-concave space. Define a binary relation $\preccurlyeq_{\mathcal{A}}$ on L^X by: for all $A, B \in L^X$,*

$$A \preccurlyeq_{\mathcal{A}} B \quad \Leftrightarrow \quad \forall x_\lambda \in \beta^*(A),\ x_\lambda \leq ca_{\mathcal{A}}(B \vee x_\lambda).$$

Then $\preccurlyeq_{\mathcal{A}}$ is an L-concave derived internal relation.

(2) Let $(X, \preccurlyeq)$ be an L-concave derived internal relation space. Define an operator $ca_{\preccurlyeq} : L^X \to L^X$ by:

$$\forall A \in L^X,\ \ ca_{\preccurlyeq}(A) = A \wedge \bigvee \{B \in L^X : B \preccurlyeq A\}.$$

Then $ca_{\preccurlyeq}$ is an L-concave hull operator which induces an L-concave structure denoted by $\mathcal{A}_{\preccurlyeq}$.

*(3) The category L-**CDIRS** is isomorphic to the category L-**CAS**.*

Based on Corollary 1 and Theorems 11–16, relations between L-concave derived hull spaces and L-concave spaces can be presented as follows.

Corollary 2 *(1) Let $(X, \mathcal{I})$ be an L-concave derived hull space. Define an operator $ca_{\mathcal{I}} : L^X \to L^X$ by:*

$$\forall A \in L^X,\ \ ca_{\mathcal{I}}(A) = A \wedge \mathcal{I}(A).$$

Then $ca_{\mathcal{I}}$ is an L-concave hull operator of an L-concave space $(X, \mathcal{A}_{\mathcal{I}})$;

(2) Let $(X, \mathcal{A})$ be an L-concave space. Define an operator $\mathcal{I}_{\mathcal{A}} : L^X \to L^X$ by

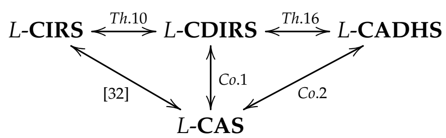
$$\forall A \in L^X,\ \ \mathcal{I}_{\mathcal{A}}(A) = \bigvee \{B \in L^X : \forall x_\lambda \in \beta^*(B),\ x_\lambda \leq ca_{\mathcal{A}}(A \vee x_\lambda)\}.$$

Then $\mathcal{I}_{\mathcal{A}}$ is an L-concave derived hull operator;

*(3) The category L-**CADHS** is isomorphic to the category L-**CAS**.*

Relations among categories mentioned in this section can be showed by the following diagram.

Fig. 3 Relations 1.



4 *L*-convex derived enclosed relation spaces

In this section, we introduce the notion of L-convex derived enclosed relations by which we characterize L-convex enclosed relation spaces and L-convex spaces. Further, we introduce the notion of L-convex derived hull space and get a brief characterization of L-convex derived enclosed relation space. For these purposes, we introduce the following notions.

For $A \in L^X$ and $x_\lambda \in \beta^*(\underline{\top})$, we denote $\beta^*_\lambda(L) = \{\mu \in \beta^*(\top) : \lambda \in \beta^*(\mu)\}$ and

$$A_{x_\lambda} = \bigvee \{y_\mu \in \beta^*(A) : x_\lambda \nleq y_\mu\}.$$

For convenience, we also denote $y_\eta \nleq^* A$ for $y_\eta \in \beta^*(\underline{\top})$ and $y_\eta \nleq A$.

Proposition 1 *For all $x_\lambda, y_\eta \in \beta^*(\underline{\top})$, $A \in L^X$ and $\{A_i\}_{i\in I} \subseteq L^X$, we have*

(1) $x_\lambda \nleq^* A$ implies $A_{x_\lambda} = A$;

(2) $A \leq B$ implies $A_{x_\lambda} \leq B_{x_\lambda}$;

(3) $(A_{x_\lambda})_{x_\lambda} = A_{x_\lambda}$;

(4) $\mu \in \beta^*_\lambda(L)$ implies $A_{x_\lambda} \leq A_{x_\mu}$ and $(A_{x_\mu})_{x_\lambda} = (A_{x_\lambda})_{x_\mu} = A_{x_\lambda}$;

(5) $y_\eta \nleq \underline{\top}_{x_\lambda}$ iff $x = y$ and $\eta \in \beta^*_\lambda(L)$;

(6) $A = \bigwedge_{x_\lambda \nleq^* A} \underline{\top}_{x_\lambda}$;

(7) $(\bigvee_{i\in I} A_i)_{x_\lambda} = \bigvee_{i\in I} (A_i)_{x_\lambda}$.

Proof (1) and (2) are direct.

(3). We have $(A_{x_\lambda})_{x_\lambda} \leq A_{x_\lambda}$ by (2). Conversely, for any $z_\nu \in \beta^*(\underline{\top})$ with $z_\nu \prec A_{x_\lambda}$, there is a $z_\mu \in \beta^*(A)$ such that $x_\lambda \nleq z_\mu$ and $z_\nu \prec z_\mu$. Thus $z_\nu \prec z_\mu \leq A_{x_\lambda}$ which implies that $z_\nu \in \beta^*(A_{x_\lambda})$. Further, since $x_\lambda \nleq z_\nu$, we have $z_\nu \leq (A_{x_\lambda})_{x_\lambda}$. Hence $A_{x_\lambda} \leq (A_{x_\lambda})_{x_\lambda}$. Therefore $(A_{x_\lambda})_{x_\lambda} = A_{x_\lambda}$.

(4). For any $\mu \in \beta^*_\lambda(L)$, it is clear that $A_{x_\lambda} \leq A_{x_\mu}$. Further, by (2) and (3), we have

$$A_{x_\lambda} = (A_{x_\lambda})_{x_\lambda} \leq (A_{x_\lambda})_{x_\mu} \leq A_{x_\lambda}.$$

Thus $(A_{x_\lambda})_{x_\mu} = A_{x_\lambda}$. Similarly, $A_{x_\lambda} = (A_{x_\lambda})_{x_\lambda} \leq (A_{x_\mu})_{x_\lambda} \leq A_{x_\lambda}$. Therefore $(A_{x_\mu})_{x_\lambda} = A_{x_\lambda}$.

(5). Assume that $y_\eta \nleq \underline{\top}_{x_\lambda}$. Then there is a $\nu \in \beta^*(\eta)$ such that $y_\nu \nleq \underline{\top}_{x_\lambda}$. Thus $x_\lambda \leq y_\nu$. Hence $x = y$ and $\lambda \leq \nu \prec \eta$. Therefore $\eta \in \beta^*_\lambda(L)$. Conversely, assume that $\eta \in \beta^*_\lambda(L)$ and $x = y$. Suppose that $y_\eta \leq \underline{\top}_{x_\lambda}$. Then $x_\lambda \prec \underline{\top}_{x_\lambda}$. Thus there is an $x_\theta \in \beta^*(\underline{\top})$ such that $x_\lambda \nleq x_\theta$

and $x_\lambda \leq x_\theta$. It is a contradiction. Therefore $y_\eta \not\leq \underline{\top}_{x_\lambda}$.

(6). For any $z_\mu \in \beta^*(\underline{\top})$ with $z_\mu \not\leq \bigwedge_{x_\lambda \not\leq^* A} \underline{\top}_{x_\lambda}$, we have $z_\mu \not\leq \underline{\top}_{x_\lambda}$ for some $x_\lambda \not\leq^* A$. Since $z_\mu \not\leq \underline{\top}_{x_\lambda}$, we have $z = x$ and $\mu \in \beta^*_\lambda(L)$. Thus $z_\mu \not\leq A$. Hence $A \leq \bigwedge_{x_\lambda \not\leq^* A} \underline{\top}_{x_\lambda}$.

Conversely, suppose that $\bigwedge_{x_\lambda \not\leq^* A} \underline{\top}_{x_\lambda} \not\leq A$. Then there is a $z_\nu \in \beta^*(\underline{\top})$ such that a $z_\nu \not\leq^* A$ and $z_\nu \leq \bigwedge_{x_\lambda \not\leq^* A} \underline{\top}_{x_\lambda}$. By $z_\nu \not\leq A$, there is a $\theta \in \beta^*(\nu)$ such that $z_\theta \not\leq^* A$. Thus $z_\theta \prec z_\nu \leq \bigwedge_{x_\lambda \not\leq^* A} \underline{\top}_{x_\lambda} \leq \underline{\top}_{z_\theta}$. It is a contradiction. Therefore $\bigwedge_{x_\lambda \not\leq^* A} \underline{\top}_{x_\lambda} \leq A$.

(7). $(\bigvee_{i\in I} A_i)_{x_\lambda} = \bigvee\{y_\mu \in \bigcup_{i\in I} \beta^*(A_i) : x_\lambda \not\leq y_\mu\} = \bigvee_{i\in I}(A_i)_{x_\lambda}$.

Definition 7 A binary relation $\preccurlyeq$ on L^X is called an L-convex derived enclosed relation and the pair $(X, \preccurlyeq)$ is called an L-convex derived enclosed relation space, if for all $A, B, C \in L^X$ and $x_\lambda \in \beta^*(\underline{\top})$,

(LCDER1) $\underline{\bot} \preccurlyeq \underline{\bot}$;

(LCDER2) $A \preccurlyeq B$ iff $A_{x_\mu} \preccurlyeq \underline{\top}_{x_\lambda}$ and $A_{x_\mu} \leq \underline{\top}_{x_\lambda}$ for any $x_\lambda \not\leq^* B$ and any $\mu \in \beta^*_\lambda(L)$;

(LCDER3) $A \preccurlyeq \bigwedge_{i\in I} B_i$ iff $A \preccurlyeq B_i$ for any $i \in I$;

(LCDER4) $A \preccurlyeq B$ implies some $C \in L^X$ such that $A \preccurlyeq C \preccurlyeq A \vee B$ and $A \leq C \leq A \vee B$;

(LCDER5) $\bigvee^{dir}_{i\in I} A_i \preccurlyeq B$ iff $A_i \preccurlyeq B$ for any $i \in I$.

Let $(X, \preccurlyeq_X)$ and $(Y, \preccurlyeq_Y)$ be L-convex derived enclosed relation spaces. A mapping $f : X \to Y$ is called an L-convex derived enclosed relation preserving mapping if $f^\leftarrow_L(A) \preccurlyeq_X f^\leftarrow_L(B) \vee f^\leftarrow_L(A)$ for all $A, B \in L^Y$ with $A \preccurlyeq_Y B$.

The category of L-convex derived enclosed relation spaces and L-convex derived enclosed relation preserving mappings is denoted by L-**CDERS**.

Now, we discuss the relations between L-convex derived enclosed relation space and L-convex enclosed relation space.

Theorem 17 *Let $(X, \preccurlyeq)$ be an L-convex derived enclosed relation space. Define a binary relation $\preccurlyeq_\preccurlyeq$ on L^X by: for all $A, B \in L^X$,*

$$A \preccurlyeq_\preccurlyeq B \quad \Leftrightarrow \quad \exists C \in L^X,\ A \preccurlyeq C \text{ and } A \vee C = B.$$

Then $\preccurlyeq_\preccurlyeq$ is an L-convex enclosed relation space.

Proof (LCER1). We have $\underline{\bot} \preccurlyeq \underline{\bot}$ by (LCDER1). In addition, by $\underline{\bot} = \underline{\bot} \vee \underline{\bot}$, we have $\underline{\bot} \preccurlyeq_\preccurlyeq \underline{\bot}$.

(LCER2). It directly follows from the definition.

(LCER3). If $A \preccurlyeq_\preccurlyeq \bigwedge_{i\in I} B_i$, then there is a $C \in L^X$ such that $A \preccurlyeq C$ and $A \vee C = \bigwedge_{i\in I} B_i$. Then $A \vee C \leq B_i$ for any $i \in I$. Thus $A \preccurlyeq B_i$ and $A \vee B_i = B_i$. That is, $A \preccurlyeq_\preccurlyeq B_i$ for any $i \in I$. Conversely, assume that $A \preccurlyeq_\preccurlyeq B_i$ for any $i \in I$. Then there is a $C_i \in L^X$ such that $A \preccurlyeq C_i$ and $A \vee C_i = B_i$ for any $i \in I$. Thus $A \preccurlyeq \bigwedge_{i\in I} B_i$ by (LCDER3) and

$$A \vee \bigwedge_{i\in I} C_i = \bigwedge_{i\in I}(A \vee C_i) = \bigwedge_{i\in I} B_i.$$

Thus $A \preccurlyeq_\preccurlyeq \bigwedge_{i\in I} B_i$.

(LCER4). Let $A \preccurlyeq_\preccurlyeq B$. Then there is a $D \in L^X$ such that $A \preccurlyeq D$ and $A \vee D = B$. By (LCDER4), there is a $C \in L^X$ such that $A \preccurlyeq C \preccurlyeq B$ and $A \leq C \leq B$. Hence $A \preccurlyeq_\preccurlyeq C \preccurlyeq_\preccurlyeq B$. Therefore C satisfies the requirement.

(LCER5). Let $\bigvee^{dir}_{i\in I} A_i \preccurlyeq_\preccurlyeq B$. Then there is a $D \in L^X$ such that $\bigvee^{dir}_{i\in I} A_i \preccurlyeq D$ and $\bigvee^{dir}_{i\in I} A_i \vee D = B$. Thus $A_i \preccurlyeq B$ and $A_i \vee B = B$ for any $i \in I$. That is $A_i \preccurlyeq_\preccurlyeq B$ for any $i \in I$. Conversely, assume that $A_i \preccurlyeq_\preccurlyeq B$ for any $i \in I$. Then there is a $D_i \in L^X$ such that $A_i \preccurlyeq D_i$ and $A_i \vee D_i = B$ for any $i \in I$. Let $D = \bigvee_{i\in I} D_i$. We have $A_i \preccurlyeq D$ for any $i \in I$. Thus $\bigvee^{dir}_{i\in I} A_i \preccurlyeq D$ by (LCDER5). Further, since $\bigvee^{dir}_{i\in I} A_i \vee D = B$, we have $\bigvee^{dir}_{i\in I} A_i \preccurlyeq_\preccurlyeq B$.

Theorem 18 *Let $(X, \preccurlyeq_X)$ and $(Y, \preccurlyeq_Y)$ be L-convex derived enclosed relation spaces. If $f : X \to Y$ is an L-convex derived enclosed relation preserving mapping, then $f : (X, \preccurlyeq_{\preccurlyeq_X}) \to (Y, \preccurlyeq_{\preccurlyeq_Y})$ is an L-convex enclosed relation preserving mapping.*

Proof Let $A \preccurlyeq_{\preccurlyeq_Y} B$. Then there is a $C \in L^Y$ such that $A \preccurlyeq_Y C$ and $A \vee C = B$. Thus $f^\leftarrow_L(A) \preccurlyeq_X f^\leftarrow_L(A \vee C)$ and

$$f^\leftarrow_L(A) \vee f^\leftarrow_L(A \vee C) = f^\leftarrow_L(A \vee C) = f^\leftarrow_L(B).$$

Hence $f^\leftarrow_L(A) \preccurlyeq_{\preccurlyeq_X} f^\leftarrow_L(B)$. Therefore f is an L-convex enclosed relation preserving mapping.

Theorem 19 *Let $(X, \preccurlyeq)$ be an L-convex enclosed relation space. Define a binary relation $\preccurlyeq_\preccurlyeq$ on L^X by: for all $A, B \in L^X$,*

$$A \preccurlyeq_\preccurlyeq B \quad \Leftrightarrow \quad \forall x_\lambda \not\leq^* B, \forall \mu \in \beta^*_\lambda(L),\ A_{x_\mu} \prec \underline{\top}_{x_\lambda}.$$

Then $(X, \lessdot_{\leqslant})$ *is an L-convex derived enclosed relation space.*

Proof Clearly, $A \lessdot_{\leqslant} B$ for any $A, B, C, D \in L^X$ with $A \leq C \lessdot_{\leqslant} D \leq B$.

(LCDER1). It directly follows from (LCER1) of $\leqslant$.

(LCDER2). Let $A \lessdot_{\leqslant} B$. Let $x_\lambda \nleq^* B$ and $\mu \in \beta^*_\lambda(L)$. We have $B \leq \underline{\top}_{x_\lambda}$ by $x_\lambda \nleq B$. Thus $A_{x_\mu} \leq A \lessdot_{\leqslant} B \leq \underline{\top}_{x_\lambda}$ which implies that $A_{x_\mu} \lessdot_{\leqslant} \underline{\top}_{x_\lambda}$. Further, by $A \lessdot_{\leqslant} B$, we have $A_{x_\mu} \leqslant \underline{\top}_{x_\lambda}$. Hence $A_{x_\mu} \leq \underline{\top}_{x_\lambda}$ by (LCER2).

Conversely, assume that $A_{x_\mu} \lessdot_{\leqslant} \underline{\top}_{x_\lambda}$ and $A_{x_\mu} \leq \underline{\top}_{x_\lambda}$ for any $x_\lambda \nleq^* B$ and any $\mu \in \beta^*_\lambda(L)$. Suppose that $A \not\lessdot_{\leqslant} B$. Then there are $x_\lambda \nleq^* B$ and $\mu \in \beta^*_\lambda(L)$ such that $A_{x_\mu} \not\leqslant \underline{\top}_{x_\lambda}$. Since $\mu \in \beta^*_\lambda(L)$, we have $x_\mu \nleq \underline{\top}_{x_\lambda}$. Further, by $A_{x_\mu} \lessdot_{\leqslant} \underline{\top}_{x_\lambda}$, we have

$$A_{x_\mu} = (A_{x_\mu})_{x_\mu} \leqslant (\underline{\top}_{x_\lambda})_{x_\mu} = \underline{\top}_{x_\lambda}.$$

It is a contradiction. Therefore $A \lessdot_{\leqslant} B$.

(LCDER3). If $A \lessdot_{\leqslant} \bigwedge_{i\in I} B_i$, then it is clear that $A \lessdot_{\leqslant} B_i$ for any $i \in I$. Conversely, assume that $A \lessdot_{\leqslant} B_i$ for any $i \in I$. For any $x_\lambda \nleq^* \bigwedge_{i\in I} B_i$, there is an $i \in I$ such that $x_\lambda \nleq^* B_i$. By $A \lessdot_{\leqslant} B_i$, we have $A_{x_\mu} \leqslant \underline{\top}_{x_\lambda}$ for any $\mu \in \beta^*_\lambda(L)$. Therefore $A \lessdot_{\leqslant} \bigwedge_{i\in I} B_i$.

(LCDER4). Let $A \lessdot_{\leqslant} B$ and let

$$D = \bigwedge\{F \in L^X : A \lessdot_{\leqslant} F\}.$$

Then $A \leq A \vee D \leq A \vee B$. Next, we verify that $E = A \vee D$ satisfies that $A \lessdot_{\leqslant} E \lessdot_{\leqslant} A \vee B$.

Indeed, we have $A \lessdot_{\leqslant} D$ by (LCDER3). Thus $A \lessdot_{\leqslant} E$. To prove that $E \lessdot_{\leqslant} A \vee B$, let $x_\lambda \nleq^* A \vee B$. We prove that $E_{x_\mu} \leqslant \underline{\top}_{x_\lambda}$ for any $\mu \in \beta^*_\lambda(L)$.

By $A \lessdot_{\leqslant} B$, we have $A = A_{x_\mu} \leqslant \underline{\top}_{x_\lambda}$. By (LCER4), there is a $C \in L^X$ such that $A \leqslant C \leqslant \underline{\top}_{x_\lambda}$. Thus $A \leq C \leq \underline{\top}_{x_\lambda}$ by (LCER2). For any $z_\eta \nleq^* C$ and any $\theta \in \beta^*_\eta(L)$, we have

$$A_{z_\theta} = A \leqslant C = C_{z_\eta} \leq \underline{\top}_{z_\eta}.$$

Hence $A \lessdot_{\leqslant} C$ which implies that $D \leq C \leqslant \underline{\top}_{x_\lambda}$. So $E \leq C \leqslant \underline{\top}_{x_\lambda}$ followed by $E_{x_\mu} \leqslant \underline{\top}_{x_\lambda}$. Therefore $E \lessdot_{\leqslant} A \vee B$. In conclusion, E satisfies the requirement.

(LCDER5). If $\bigvee^{dir}_{i\in I} A_i \lessdot_{\leqslant} B$, then $A_i \lessdot_{\leqslant} B$ for any $i \in I$. Conversely, let $\{A_i\}_{i\in I} \overset{dir}{\subseteq} L^X$ with $A_i \lessdot_{\leqslant} B$ for any $i \in I$. Let $x_\lambda \nleq^* B$ and $\mu \in \beta^*(L)$. Then $(A_i)_{x_\mu} \leqslant \underline{\top}_{x_\lambda}$ and $(A_i)_{x_\mu} \leq \underline{\top}_{x_\lambda}$ for any $i \in I$. We have $(\bigvee^{dir}_{i\in I} A_i)_{x_\mu} = \bigvee^{dir}_{i\in I}(A_i)_{x_\mu}$ by (7) of Proposition 1. Thus $(\bigvee^{dir}_{i\in I} A_i)_{x_\mu} \leqslant \underline{\top}_{x_\lambda}$ by (LCER5). Therefore $\bigvee^{dir}_{i\in I} A_i \lessdot_{\leqslant} B$.

Theorem 20 *Let* $(X, \leqslant_X)$ *and* $(Y, \leqslant_Y)$ *be L-convex enclosed relation spaces. If* $f : X \to Y$ *is an L-convex enclosed relation preserving mapping, then* $f : (X, \lessdot_{\leqslant_X}) \to (Y, \lessdot_{\leqslant_Y})$ *is an L-convex derived enclosed relation preserving mapping.*

Proof Let $A \lessdot_{\leqslant_Y} B$. To prove that $f^{\leftarrow}_L(A) \lessdot_{\leqslant_X} f^{\leftarrow}_L(A \vee B)$, let $x_\lambda \nleq^* f^{\leftarrow}_L(A \vee B)$ and $\mu \in \beta^*_\lambda(L)$. Then $f(x)_\mu \nleq^* A \vee B$. By $A \lessdot_{\leqslant_Y} B$, we have $A = A_{f(x)_\mu} \leqslant_Y \underline{\top}_{f(x)_\lambda}$ and $A \leq \underline{\top}_{f(x)_\lambda}$. Thus

$$f^{\leftarrow}_L(A)_{x_\mu} = f^{\leftarrow}_L(A) \leqslant_X f^{\leftarrow}_L(\underline{\top}_{f(x)_\lambda}) \leq \underline{\top}_{x_\lambda}.$$

Hence $f^{\leftarrow}_L(A) \lessdot_{\leqslant_X} f^{\leftarrow}_L(A \vee B)$. Therefore f is an L-convex derived enclosed relation preserving mapping.

Theorem 21 *We have* $\leqslant_{\lessdot_{\leqslant}} = \leqslant$ *for any L-convex derived enclosed relation space* $(X, \lessdot)$ *and* $\lessdot_{\leqslant_{\lessdot}} = \lessdot$ *for any L-convex enclosed relation space* $(X, \leqslant)$.

Proof Let $(X, \leqslant)$ be an L-convex enclosed relation space. If $A \leqslant_{\lessdot_{\leqslant}} B$, then there is a $C \in L^X$ such that $A \lessdot_{\leqslant} C$ and $A \vee C = B$. Thus $A \lessdot_{\leqslant} B$ and $A \leq B$. By $A \lessdot_{\leqslant} B$, we have $A = A_{x_\mu} \leqslant \underline{\top}_{x_\lambda}$ for any $x_\lambda \nleq^* B$ and $\mu \in \beta^*_\lambda(L)$. Hence $A \leqslant \bigwedge_{x_\lambda \nleq^* B} \underline{\top}_{x_\lambda} = B$ by (LCER3). That is, $A \leqslant B$ holds.

Conversely, if $A \leqslant B$ then $A \leq B$ by (LCER2). For any $x_\lambda \nleq^* B$ and $\mu \in \beta^*_\lambda(L)$, we have $A_{x_\mu} = A \leqslant B \leq \underline{\top}_{x_\lambda}$. Thus $A_{x_\mu} \leqslant \underline{\top}_{x_\lambda}$. Hence $A \lessdot_{\leqslant} B$. Further, by $A \lessdot_{\leqslant} B$ and $A \vee B = B$, we have $A \leqslant_{\lessdot_{\leqslant}} B$.

In conclusion, we have $A \leqslant_{\lessdot_{\leqslant}} B$ iff $A \leqslant B$ for all $A, B \in L^X$. That is, $\leqslant_{\lessdot_{\leqslant}} = \leqslant$.

Let $(X, \lessdot)$ be an L-convex derived enclosed relation space. Let $A \lessdot_{\leqslant_{\lessdot}} B$. If $x_\lambda \nleq^* B$ and $\mu \in \beta^*_\lambda(L)$, we have $A_{x_\mu} \lessdot_{\lessdot} \underline{\top}_{x_\lambda}$. Thus there is a $C \in L^X$ such that $A_{x_\mu} \lessdot C$ and $A_{x_\mu} \vee C = \underline{\top}_{x_\lambda}$. Hence $A_{x_\mu} \lessdot \underline{\top}_{x_\lambda}$ and $A_{x_\mu} \leq \underline{\top}_{x_\lambda}$. Therefore $A \lessdot B$ by (LCDER2).

Conversely, let $A \preccurlyeq B$. To prove that $A \preccurlyeq_{\preccurlyeq_{\preccurlyeq}} B$, let $x_\lambda \not\leq^* B$ and $\mu \in \beta^*_\lambda(L)$. We need to prove that $A_{x_\mu} \preccurlyeq_{\preccurlyeq} \underline{\top}_{x_\lambda}$.

Actually, by $A \preccurlyeq B \leq \underline{\top}_{x_\lambda}$, we have $A_{x_\mu} \preccurlyeq \underline{\top}_{x_\lambda}$ and $A_{x_\mu} \leq \underline{\top}_{x_\lambda}$ by (LCDER2). In addition, by $A_{x_\mu} \vee \underline{\top}_{x_\lambda} = \underline{\top}_{x_\lambda}$, we have $A_{x_\mu} \preccurlyeq_{\preccurlyeq} \underline{\top}_{x_\lambda}$. Therefore $A \preccurlyeq_{\preccurlyeq_{\preccurlyeq}} B$

In conclusion, for all $A, B \in L^X$, we have $A \preccurlyeq_{\preccurlyeq_{\preccurlyeq}} B$ iff $A \preccurlyeq B$. That is, $\preccurlyeq_{\preccurlyeq_{\preccurlyeq}} = \preccurlyeq$.

Based on Theorems 17 and 18, we obtain a factor $\mathbb{F} : L$-**CDERS**$\to L$-**CERS** defined by

$$\mathbb{F}((X, \preccurlyeq)) = (X, \preccurlyeq_{\preccurlyeq}), \quad \mathbb{F}(f) = f.$$

Based on Theorems 17–21, $\mathbb{F}$ is an isomorphic factor. Thus we have the following conclusion.

Theorem 22 *The category L-**CDERS** is isomorphic to the category L-**CERS**.*

To simply characterize L-convex derived enclosed relation, we introduce the notion of L-convex derived hull operator as follows.

Definition 8 An operator $\mathcal{D} : L^X \to L^X$ is called an L-convex derived hull operator on L^X and the pair $(X, \mathcal{D})$ is called an L-convex derived hull space if for all $A, B \in L^X$ and any $x_\lambda \in \beta^*(\underline{\top})$,

(LCDH1) $\mathcal{D}(\underline{\bot}) = \underline{\bot}$;

(LCDH2) $\mathcal{D}(A) \leq B$ iff $\bigvee_{\mu \in \beta^*_\lambda(L)}(\mathcal{D}(A_{x_\mu}) \vee A_{x_\mu}) \leq \underline{\top}_{x_\lambda}$ for any $x_\lambda \not\leq^* B$;

(LCDH3) $\mathcal{D}(\mathcal{D}(A) \vee A) \leq \mathcal{D}(A) \vee A$;

(LCDH4) $\mathcal{D}(\bigvee^{dir}_{i\in I} A_i) = \bigvee_{i\in I} \mathcal{D}(A_i)$.

Let $(X, \mathcal{D}_X)$ and $(Y, \mathcal{D}_Y)$ be L-convex derived hull spaces. A mapping $f : X \to Y$ is called an L-convex derived hull preserving mapping, if for any $A \in L^X$,

$$f^{\to}_L(\mathcal{D}_X(A)) \leq \mathcal{D}_Y(f^{\to}_L(A)) \vee f^{\to}_L(A).$$

The category of L-convex derived hull spaces and L-convex derived hull preserving mappings is denoted by L-**CDHS**.

Theorem 23 *Let $(X, \mathcal{D})$ be an L-convex derived hull space. Define a binary relation $\preccurlyeq_{\mathcal{D}}$ on L^X by:*

$$\forall A, B \in L^X, \ A \preccurlyeq_{\mathcal{D}} B \ \Leftrightarrow \ \mathcal{D}(A) \leq B.$$

Then $\preccurlyeq_{\mathcal{D}}$ is an L-convex derived enclosed relation space.

Proof (LCDER1). We have $\mathcal{D}(\underline{\bot}) = \underline{\bot}$ by (LCDH1). Thus $\underline{\bot} \preccurlyeq_{\mathcal{D}} \underline{\bot}$.

(LCDER2). It directly follows from (LCDH2).

(LCDER3). If $A \preccurlyeq_{\mathcal{D}} \bigwedge_{i\in I} B_i$ then $\mathcal{D}(A) \leq \bigwedge_{i\in I} B_i \leq B_j$ for any $j \in I$. Thus $A \preccurlyeq_{\mathcal{D}} B_j$ for any $j \in I$. Conversely, if $A \preccurlyeq_{\mathcal{D}} B_i$ for any $i \in I$, then $\mathcal{D}(A) \leq B_i$. Thus $\mathcal{D}(A) \leq \bigwedge_{i\in I} B_i$ which implies that $A \preccurlyeq_{\mathcal{D}} \bigwedge_{i\in I} B_i$.

(LCDER4). Let $A \preccurlyeq_{\mathcal{D}} B$ and let $E = \mathcal{D}(A) \vee A$. We have $E \preccurlyeq_{\mathcal{D}} E \leq B$ by (LCDH3). Thus $E \preccurlyeq_{\mathcal{D}} B$. In addition, we have $A \preccurlyeq_{\mathcal{D}} E$ by $\mathcal{D}(A) \leq E$. Therefore $A \preccurlyeq_{\mathcal{D}} E \preccurlyeq_{\mathcal{D}} A \vee B$ and $A \leq E \leq A \vee B$ as desired.

(LCDER5). It directly follows from (LCDH4).

Theorem 24 *Let $(X, \mathcal{D}_X)$ and $(Y, \mathcal{D}_Y)$ be L-convex derived hull spaces. If $f : X \to Y$ is an L-convex derived hull preserving mapping, then $f : (X, \preccurlyeq_{\mathcal{D}_X}) \to (Y, \preccurlyeq_{\mathcal{D}_Y})$ is an L-convex derived enclosed relation preserving mapping.*

Proof If $A \preccurlyeq_{\mathcal{D}_Y} B$ then $\mathcal{D}_Y(A) \leq B$. Thus

$$\begin{aligned} f^{\to}_L(\mathcal{D}_X(f^{\leftarrow}_L(A))) &\leq f^{\to}_L(f^{\leftarrow}_L(A)) \vee \mathcal{D}_Y(f^{\to}_L(f^{\leftarrow}_L(A))) \\ &\leq A \vee \mathcal{D}_Y(A) \\ &\leq A \vee B. \end{aligned}$$

Hence

$$\mathcal{D}_X(f^{\leftarrow}_L(A)) \leq f^{\leftarrow}_L(A \vee B) = f^{\leftarrow}_L(A) \vee f^{\leftarrow}_L(B)$$

followed by $f^{\leftarrow}_L(A) \preccurlyeq_{\mathcal{D}_X} f^{\leftarrow}_L(A) \vee f^{\leftarrow}_L(B)$. Therefore f is an L-convex derived enclosed relation preserving mapping.

Theorem 25 *Let $(X, \preccurlyeq)$ be an L-convex derived enclosed relation space. Define an operator $\mathcal{D}_{\preccurlyeq} : L^X \to L^X$ by*

$$\forall A \in L^X, \ \mathcal{D}_{\preccurlyeq}(A) = \bigwedge\{B \in L^X : A \preccurlyeq B\}.$$

Then $(X, \mathcal{D}_{\preccurlyeq})$ is an L-convex derived hull space.

Proof (LCDH1). We have $\mathcal{D}_{\preccurlyeq}(\underline{\bot}) \leq \underline{\bot}$ by (LCDER1). Thus $\mathcal{D}_{\preccurlyeq}(\underline{\bot}) = \underline{\bot}$.

(LCDH2). If $\mathcal{D}_{\preccurlyeq}(A) \leq B$ then $A \preccurlyeq \mathcal{D}_{\preccurlyeq}(A) \leq B$ which implies $A \preccurlyeq B$. By (LCDER2), we have $A_{x_\mu} \leq \underline{\top}_{x_\lambda}$ and $A_{x_\mu} \preccurlyeq \underline{\top}_{x_\lambda}$ for all $x_\lambda \not\leq^* B$ and $\mu \in \beta^*_\lambda(L)$. Thus $\bigvee_{\mu \in \beta^*(L)}(\mathcal{D}_{\preccurlyeq}(A_{x_\mu}) \vee A_{x_\mu}) \leq \underline{\top}_{x_\lambda}$.

Conversely, assume that $\bigvee_{\mu\in\beta^*_\lambda(L)}(\mathcal{D}_\preccurlyeq(A_{x_\mu})\vee A_{x_\mu}) \le \underline{\top}_{x_\lambda}$ for any $x_\lambda \not\le^* B$. By (LCDER3), we have $A_{x_\mu} \preccurlyeq \mathcal{D}_\preccurlyeq(A_{x_\mu})$ for all $x_\lambda \not\le^* B$ and $\mu \in \beta^*_\lambda(L)$. Thus $A_{x_\mu} \preccurlyeq \underline{\top}_{x_\lambda}$ and $A_{x_\mu} \le \underline{\top}_{x_\lambda}$. Hence $A \preccurlyeq \underline{\top}_{x_\lambda}$ by (LCDER2). Therefore $\mathcal{D}_\preccurlyeq(A) \le \bigwedge_{x_\lambda\not\le^* B}\underline{\top}_{x_\lambda} = B$ by (6) of Proposition 1.

(LCDH3). Let $x_\lambda \in \beta^*(\underline{\top})$ with $x_\lambda \not\le \mathcal{D}_\preccurlyeq(A)\vee A$. Then $x_\lambda \not\le A$ and $x_\lambda \not\le \mathcal{D}_\preccurlyeq(A)$. By $x_\lambda \not\le \mathcal{D}_\preccurlyeq(A)$, there is a $B \in L^X$ such that $x_\lambda \not\le B$ and $A \preccurlyeq B$. By (LCDER4), there is an $E \in L^X$ such that $A \preccurlyeq E \preccurlyeq B \vee A$ and $A \le E \le B \vee A$. By $A \preccurlyeq E$, we have $\mathcal{D}_\preccurlyeq(A) \vee A \le E \preccurlyeq A \vee B$. Thus

$$\mathcal{D}_\preccurlyeq(\mathcal{D}_\preccurlyeq(A) \vee A) \le (A \vee B) \not\ge x_\lambda.$$

Hence $x_\lambda \not\le \mathcal{D}_\preccurlyeq(\mathcal{D}_\preccurlyeq(A)\vee A)$. Therefore we conclude that $\mathcal{D}_\preccurlyeq(\mathcal{D}_\preccurlyeq(A) \vee A) \le \mathcal{D}_\preccurlyeq(A) \vee A$.

(LCDH4). Clearly, we have $\bigvee_{i\in I}\mathcal{D}_\preccurlyeq(A_i) \le \mathcal{D}_\preccurlyeq(\bigvee^{dir}_{i\in I}A_i)$. Conversely, let $x_\lambda \in J(L^X)$ with $x_\lambda \not\le \bigvee_{i\in I}\mathcal{D}_\preccurlyeq(A_i)$. Thus there is a $\mu \in \beta^*(\lambda)$ such that $x_\mu \not\le \bigvee_{i\in I}\mathcal{D}_\preccurlyeq(A_i)$. For any $i \in I$, by $x_\mu \not\le \mathcal{D}_\preccurlyeq(A_i)$, there is a $C_i \in L^X$ such that $x_\mu \not\le C_i$ and $A_i \preccurlyeq C_i$. Let $C = \bigvee_{i\in I}C_i$. Then $A_i \preccurlyeq C$ for any $i \in I$. So $\bigvee^{dir}_{i\in I}A_i \preccurlyeq C \not\ge x_\lambda$. Hence $x_\lambda \not\le \mathcal{D}_\preccurlyeq(\bigvee^{dir}_{i\in I}A_i)$. Therefore $\mathcal{D}_\preccurlyeq(\bigvee^{dir}_{i\in I}A_i) \le \bigvee_{i\in I}\mathcal{D}_\preccurlyeq(A_i)$.

Theorem 26 *Let $(X, \preccurlyeq_X)$ and $(Y, \preccurlyeq_Y)$ be L-convex derived enclosed relation spaces. If $f : X \to Y$ is an L-convex derived enclosed relation preserving mapping, then $f : (X, \mathcal{D}_{\preccurlyeq_X}) \to (Y, \mathcal{D}_{\preccurlyeq_Y})$ is an L-convex derived hull preserving mapping.*

Proof Let $A \in L^X$ and let $x_\lambda \in J(L^X)$ with $x_\lambda \not\le f^\leftarrow_L(\mathcal{D}_{\preccurlyeq_Y}(f^\rightarrow_L(A))) \vee f^\leftarrow_L(f^\rightarrow_L(A))$. Then

$$f^\rightarrow_L(x_\lambda) \not\le \mathcal{D}_{\preccurlyeq_Y}(f^\rightarrow_L(A)) \vee f^\rightarrow_L(A).$$

By $f^\rightarrow_L(x_\lambda) \not\le \mathcal{D}_{\preccurlyeq_Y}(f^\rightarrow_L(A))$, there is a $B \in L^X$ such that $f^\rightarrow_L(x_\lambda) \not\le B$ and $f^\rightarrow_L(A) \preccurlyeq_Y B$. Thus $x_\lambda \not\le f^\leftarrow_L(B)$ and $A \le f^\leftarrow_L(f^\rightarrow_L(A)) \preccurlyeq_X f^\leftarrow_L(B)$. Hence $A \preccurlyeq_X f^\leftarrow_L(B)$ and $x_\lambda \not\le \mathcal{D}_{\preccurlyeq_X}(A)$. So

$$\begin{aligned} &f^\rightarrow_L(\mathcal{D}_{\preccurlyeq_X}(A)) \\ \le\ &f^\rightarrow_L(f^\leftarrow_L(\mathcal{D}_{\preccurlyeq_Y}(f^\rightarrow_L(A))) \vee f^\leftarrow_L(f^\rightarrow_L(A))) \\ \le\ &\mathcal{D}_{\preccurlyeq_Y}(f^\rightarrow_L(A)) \vee f^\rightarrow_L(A). \end{aligned}$$

Therefore f is an L-convex derived hull preserving mapping.

Theorem 27 *We have $\mathcal{D}_{\preccurlyeq_\mathcal{D}} = \mathcal{D}$ for any L-convex derived hull space $(X, \mathcal{D})$ and $\preccurlyeq_{\mathcal{D}_\preccurlyeq} = \preccurlyeq$ for any L-convex derived enclosed relation space $(X, \preccurlyeq)$.*

Proof Let $(X, \mathcal{D})$ be an L-convex derived hull space and $A \in L^X$. We have

$$\mathcal{D}(A) \le \bigwedge\{B \in L^X : A \preccurlyeq_\mathcal{D} B\} = \mathcal{D}_{\preccurlyeq_\mathcal{D}}(A).$$

Conversely, we have $\mathcal{D}(A) \le \underline{\top}_{x_\lambda}$ for any $x_\lambda \not\le^* \mathcal{D}(A)$. Thus $A \preccurlyeq_\mathcal{D} \underline{\top}_{x_\lambda}$ and $\mathcal{D}_{\preccurlyeq_\mathcal{D}}(A) \le \underline{\top}_{x_\lambda}$. Hence

$$\mathcal{D}_{\preccurlyeq_\mathcal{D}}(A) \le \bigwedge_{x_\lambda\not\le\mathcal{D}(A)}\underline{\top}_{x_\lambda} = \mathcal{D}(A).$$

Therefore $\mathcal{D}_{\preccurlyeq_\mathcal{D}}(A) = \mathcal{D}(A)$ which shows that $\mathcal{D}_{\preccurlyeq_\mathcal{D}} = \mathcal{D}$.

Let $(X, \preccurlyeq)$ be an L-convex derived enclosed relation space. If $A \preccurlyeq B$ then $\mathcal{D}_\preccurlyeq(A) \le B$ and so $A \preccurlyeq_{\mathcal{D}_\preccurlyeq} B$. Conversely, if $A \preccurlyeq_{\mathcal{D}_\preccurlyeq} B$, then $\mathcal{D}_\preccurlyeq(A) \le B$ and $A \preccurlyeq \mathcal{D}_\preccurlyeq(A)$ by (LCDER3). Thus $A \preccurlyeq B$. In conclusion, we have $A \preccurlyeq B$ iff $A \preccurlyeq_{\mathcal{D}_\preccurlyeq} B$. That is, $\preccurlyeq_{\mathcal{D}_\preccurlyeq} = \preccurlyeq$.

Based on Theorems 25 and 26, we obtain a factor $\mathbb{G} : L$-**CDERS**$\to L$-**CDHS** by

$$\mathbb{G}((X, \preccurlyeq)) = (X, \mathcal{D}_\preccurlyeq), \quad \mathbb{G}(f) = f.$$

Based on Theorems 23–27, $\mathbb{G}$ is an isomorphic factor. Thus we have the following result.

Theorem 28 *The category L-***CDERS** *is isomorphic to the category L-***CDHS**.

Based on Theorem 3 and Theorems 17–22, relations between L-convex derived enclosed relation spaces and L-convex spaces are presented as follows.

Corollary 3 *(1) Let $(X, \mathcal{C})$ be an L-convex space. Define a binary operator $\preccurlyeq_\mathcal{C}$ on L^X by: for all $A, B \in L^X$,*

$$A \preccurlyeq_\mathcal{C} B \Leftrightarrow \forall x_\lambda \not\le^* B, \forall \mu \in \beta^*_\lambda(L),\ co_\mathcal{C}(A_{x_\mu}) \le \underline{\top}_{x_\lambda}.$$

Then $(X, \preccurlyeq_\mathcal{C})$ is an L-convex derived enclosed relation space.

(2) Let $(X, \preccurlyeq)$ be an L-convex derived enclosed relation space. Define an operator $co_\preccurlyeq : L^X \to L^X$ by:

$$\forall A \in L^X,\ \ co_\preccurlyeq(A) = A \vee \bigwedge\{B \in L^X : A \preccurlyeq B\}.$$

Then $co_\preccurlyeq$ is an L-convex hull operator which induces an L-convex structure denoted by $\mathcal{C}_\preccurlyeq$.

*(3) The category L-***CDERS** *is isomorphic to the category L-***CS**.

Based on Corollary 3 and Theorems 23–27, relations between *L*-convex derived enclosed relation spaces and *L*-convex spaces are presented as follows.

Corollary 4 *The mutual relations between L-convex derived hull spaces and L-convex spaces can be directly achieved as follows.*

(1) Let $(X, \mathcal{D})$ *be an L-convex derived hull space. Define an operator* $co_{\mathcal{D}} : L^X \to L^X$ *by:*

$$\forall A \in L^X,\ \ co_{\mathcal{D}}(A) = \mathcal{D}(A) \vee A.$$

Then $co_{\mathcal{D}}$ *is the L-convex hull operator of an L-convex structure denoted by* $\mathcal{C}_{\mathcal{D}}$.

(2) Let $(X, \mathcal{C})$ *be an L-convex space. Define an operator* $\mathcal{D}_{\mathcal{C}} : L^X \to L^X$ *by:*

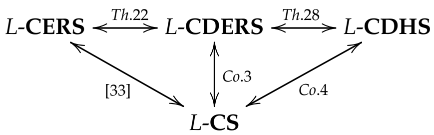
$$\forall A \in L^X,\ \ \mathcal{D}_{\mathcal{C}}(A) = \bigvee\{x_\lambda \in \beta^*(\underline{\top}) : \forall \mu \in \beta^*_\lambda(L),\ x_\mu \le co_{\mathcal{C}}(A_{x_\mu})\}.$$

Then $\mathcal{D}_{\mathcal{C}}$ *is an L-convex derived hull operator.*

*(3) The category L-***CDHS** *is isomorphic to the category L-***CS**.

We draw the following diagram to show the relations among categories mentioned in this section.

Fig. 4 Relations 2.



5 Conclusions

(1) In this paper, we introduce notions of *L*-concave derived internal relation space and *L*-concave derived internal space by which we characterize *L*-concave internal relation space and *L*-concave space. Also, we introduce *L*-convex derived enclosed relation space and *L*-convex derived hull space by which we characterize *L*-convex enclosed relation space and *L*-convex space.

(2) Solutions of Problem 1 and Problem 2 in the introduction section are presented as follows.

Fig. 5 Solution 1.

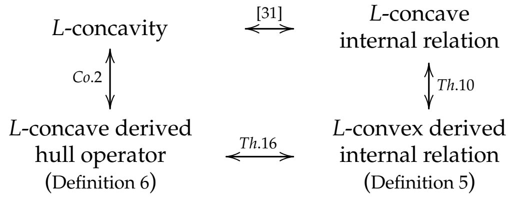
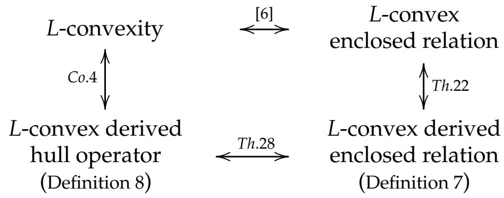


Fig. 6 Solution 2.



(3) If L has an inverse involution $'$, then a subset $\mathcal{A} \subseteq L^X$ is an *L*-concave structure iff the set $\mathcal{A}' = \{A' : A \in \mathcal{A}\}$ is an *L*-convex structure. Similarly, a binary relation $\preccurlyeq$ on L^X is an *L*-concave internal relation iff the binary relation $\preccurlyeq'$ is an *L*-convex enclosed relation, where $\preccurlyeq'$ is defined by $A \preccurlyeq' B$ iff $B' \preccurlyeq A'$ for all $A, B \in L^X$. By this consideration, we usually say that *L*-concave space and *L*-convex space are dual concepts. Similarly, *L*-concave internal relation and *L*-convex enclosed relation are dual concepts too. However, *L*-concave derived internal relation and *L*-convex derived enclosed relation can't be regarded as dual concepts in this sense. We have the following example.

Let $X = \{x\}$ and $L = \{\bot, c, a, b, d, \top\}$ be a lattice defined by Fig. 7 as follows. The inverse involution $'$ is defined by $\bot' = \top$, $a' = b$ and $c' = d$. Let $\mathcal{C} = \{x_\bot, x_c, x_a, x_\top\}$. Then the *L*-convex derived enclosed relation $\lessdot_{\mathcal{C}}$ is presented by Table. 1.

Let $\lessdot'_{\mathcal{C}}$ be defined by $A \lessdot'_{\mathcal{C}} B$ iff $B' \lessdot_{\mathcal{C}} A'$ for all $A, B \in L^X$. Suppose that $\lessdot'_{\mathcal{C}}$ is an *L*-concave derived internal relation. We have $x_c \lessdot'_{\mathcal{C}} x_c \vee x_c$, $x_a \lessdot'_{\mathcal{C}} x_c \vee x_a$ and $x_b \lessdot'_{\mathcal{C}} x_c \vee x_b$ since $x_d \lessdot_{\mathcal{C}} x_d$, $x_b \lessdot_{\mathcal{C}} x_b$ and $x_a \lessdot_{\mathcal{C}} x_a$. Thus $x_d \lessdot'_{\mathcal{C}} x_c$ by

Fig. 7 Lattice

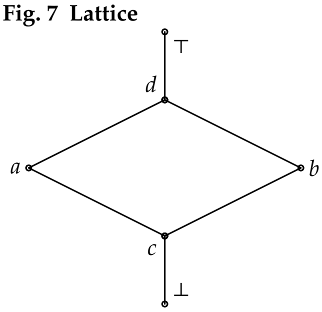


Table 1

	$x_\bot$	x_c	x_a	x_b	x_d	$x_\top$
$x_\bot$	$\preccurlyeq_C$	$\preccurlyeq_C$	$\preccurlyeq_C$	$\preccurlyeq_C$	$\preccurlyeq_C$	$\preccurlyeq_C$
x_c	$\preccurlyeq_C$	$\preccurlyeq_C$	$\preccurlyeq_C$	$\preccurlyeq_C$	$\preccurlyeq_C$	$\preccurlyeq_C$
x_a	$\preccurlyeq_C$	$\preccurlyeq_C$	$\preccurlyeq_C$	$\preccurlyeq_C$	$\preccurlyeq_C$	$\preccurlyeq_C$
x_b	$\preccurlyeq_C$	$\preccurlyeq_C$	$\preccurlyeq_C$	$\preccurlyeq_C$	$\preccurlyeq_C$	$\preccurlyeq_C$
x_d			$\preccurlyeq_C$		$\preccurlyeq_C$	$\preccurlyeq_C$
$x_\top$			$\preccurlyeq_C$		$\preccurlyeq_C$	$\preccurlyeq_C$

(LCDIR2). So $x_d \preccurlyeq_C x_c$. It is a contradiction. Hence $\preccurlyeq'_C$ isn't an L-concave derived internal relation.

Actually, $\mathcal{A} = C' = \{x_\bot, x_b, x_d, x_\top\}$ is an L-concave space. From Table. 2 and Table. 3, we find that $\preccurlyeq_{\mathcal{A}}$ is quite different from $\preccurlyeq'_C$.

Table 2

	$x_\bot$	x_c	x_a	x_b	x_d	$x_\top$
$x_\bot$	$\preccurlyeq_{\mathcal{A}}$	$\preccurlyeq_{\mathcal{A}}$	$\preccurlyeq_{\mathcal{A}}$	$\preccurlyeq_{\mathcal{A}}$	$\preccurlyeq_{\mathcal{A}}$	$\preccurlyeq_{\mathcal{A}}$
x_c				$\preccurlyeq_{\mathcal{A}}$	$\preccurlyeq_{\mathcal{A}}$	$\preccurlyeq_{\mathcal{A}}$
x_a				$\preccurlyeq_{\mathcal{A}}$	$\preccurlyeq_{\mathcal{A}}$	$\preccurlyeq_{\mathcal{A}}$
x_b				$\preccurlyeq_{\mathcal{A}}$	$\preccurlyeq_{\mathcal{A}}$	$\preccurlyeq_{\mathcal{A}}$
x_d				$\preccurlyeq_{\mathcal{A}}$	$\preccurlyeq_{\mathcal{A}}$	$\preccurlyeq_{\mathcal{A}}$
$x_\top$				$\preccurlyeq_{\mathcal{A}}$	$\preccurlyeq_{\mathcal{A}}$	$\preccurlyeq_{\mathcal{A}}$

Table 3

	$x_\bot$	x_c	x_a	x_b	x_d	$x_\top$
$x_\bot$	$\preccurlyeq'_C$	$\preccurlyeq'_C$	$\preccurlyeq'_C$	$\preccurlyeq'_C$	$\preccurlyeq'_C$	$\preccurlyeq'_C$
x_c	$\preccurlyeq'_C$	$\preccurlyeq'_C$	$\preccurlyeq'_C$	$\preccurlyeq'_C$	$\preccurlyeq'_C$	$\preccurlyeq'_C$
x_a			$\preccurlyeq'_C$	$\preccurlyeq'_C$	$\preccurlyeq'_C$	$\preccurlyeq'_C$
x_b	$\preccurlyeq'_C$	$\preccurlyeq'_C$	$\preccurlyeq'_C$	$\preccurlyeq'_C$	$\preccurlyeq'_C$	$\preccurlyeq'_C$
x_d			$\preccurlyeq'_C$	$\preccurlyeq'_C$	$\preccurlyeq'_C$	$\preccurlyeq'_C$
$x_\top$			$\preccurlyeq'_C$	$\preccurlyeq'_C$	$\preccurlyeq'_C$	$\preccurlyeq'_C$

(4) Algebraic property of convex hulls is an essential feature of convex structure which is different from other mathematic structures such as topological structures, convergence structure and matroid. With the developmen-t of fuzzy extensions of convex theory, such property has been accordingly extended by many means [1,11,17,19,32]. Thus it could be worth to discuss presentations of algebraic property in L-convex enclosed relation space, L-convex derived enclosed relation space and L-convex derived hull space.

(5) The notions L-concave derived internal relation, L-concave derived hull operator, L-convex derived enclosed relation and L-convex derived hull operator may provide some alternative ways in discussing separation axiom-s of L-convex spaces and relations among L-convex spaces, L-matroids and L-convergence spaces.

Acknowledgements This work is supported by the National Natural Science Foundation of China (11871097), University Natural Science Research Project of Anhui Province (KJ2020A0 056), Doctoral Scientific Research Foundation of Anhui Normal University (751966), Doctor-al Scientific Research Foundation of Henan U-niversity of Science and Technology (13480055) and Hunan Educational Committee Foundation (18A474).

Contributions of authors

The first author provides the main thought and purpose of the paper. In addition, he applies derived operator to L-concave internal space and L-convex enclosed relation space. For this, he introduces L-concave derived internal relation space and L-convex derived enclosed relation space. After this, he proves that categories of L-concave derived internal relation spaces and L-convex derived enclosed relation spaces, L-concave internal relation s-paces and L-convex enclosed relation spaces are isomorphic.

Based on the work of the first author, the second author applies derived operator to L-

concave hull space and *L*-convex hull space. He introduces *L*-concave derived hull space and *L*-convex derived hull space. Then he establishes relations among *L*-concave derived hull space, *L*-convex derived hull space, *L*-concave derived internal relation space and *L*-convex derived enclosed relation space.

Graphs and conclusions of this paper are contributed by both authors cooperatively.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

Human and animal rights This article does not contain any studies with human participants or animals performed by any of the authors.

Informed consent Informed consent was obtained from all individual participants included in the study.

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