A generalization of intrinsic geometry and an affine connection representation of gauge fields

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Abstract

There are two ways to unify gravitational field and gauge field. One is to represent gravitational field as principal bundle connection, and the other is to represent gauge field as affine connection. Poincare gauge theory and metric-affine gauge theory adopt the first approach. This paper adopts the second.

We show a generalization of Riemannian geometry and a new affine connection, and apply them to establishing a unified coordinate description of gauge field and gravitational field. It has the following advantages.

(i) Gauge field and gravitational field have the same affine connection representation, and can be described by a unified spatial frame.

(ii) Time can be regarded as the total metric with respect to all dimensions of internal coordinate space and external coordinate space. On-shell can be regarded as gradient direction. Quantum theory can be regarded as a geometric theory of distribution of gradient directions. Hence, gravitational theory and quantum theory obtain the same view of time and space and a unified description of evolution in affine connection representation of gauge fields.

(iii) Chiral asymmetry, coupling constants, MNS mixing and CKM mixing can appear spontaneously in affine connection representation, while in $U(1) \times SU(2) \times SU(3)$ principal bundle connection representation they can just only be artificially set up. Some principles and postulates of the conventional theories that are based on principal bundle connection representation can be turned into theorems in affine connection representation, so they are not necessary to be regarded as principles or postulates anymore.

Keywords: affine connection representation of gauge fields, generalized intrinsic geometry, reference-system, simple connection, time metric, distribution of gradient directions

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1. Introduction

1.1. Background and purpose

We know that gravitational field can be described by the geometric theory on Riemannian manifold, and that gauge field can be described by the geometric theory on fibre bundle. There have been many theories trying to solve the problem that how to establish a unified description of gravitational field and gauge field.

The main difficulties of this problem in the above two geometric theories are as follows.

(i) The affine connection that describes gravitational field is different from the abstract connection that describes gauge field. Gravitational field and gauge field are not described by the same connection.

(ii) The affine connection of gravitational field has an explicit sense of view of time and space, but the abstract connection of gauge field has not. Affine connection reflects the degrees of freedom of spacetime, but abstract connection just only reflects some abstract degrees of freedom.

(iii) Time is a component of gravitational field, but not a component of gauge field. Minkowski coordinates basically regard 1-dimensional time and 3-dimensional space on an equal footing. The description of evolution of gravitational field is essentially different from that of gauge field.

There are the following two ways to solve the problem (i).

One way is to represent gravitational field as principal bundle connection. We can take the transformation group $Gravi(3, 1)$ of gravitational field as the structure group of principal bundle to establish a gauge theory of gravitational field, the local transformation group of which is in the form of $Gravi(3, 1) \otimes_s Gauge(n)$, e.g. Poincare gauge theory [1–11] and metric-affine gauge theory [12–23]. Gravitational field and gauge field can be uniformly described by connections on principal bundle. We point out that, this approach solves the problem (i) in form, but it cannot solve the problems (ii) and (iii).

The other way is to represent gauge field as affine connection. This is the approach adopted by this paper. In this way, gravitational field and gauge field can be uniformly described by affine connection, and both the problems (i) and (ii) can be solved. In addition, we find that in affine connection representation, time can be regarded as the total metric with respect to all dimensions of internal coordinate space and external coordinate space, so the problem (iii) can be solved.

The difficulty with the second approach is that, Riemannian manifold is not suitable for establishing an affine connection representation of gauge fields. In other words, an affine connection (e.g. Levi-Civita connection, spin connection, Riemann-Cartan connection) that is dependent on Christoffel symbol $\{\rho_{\mu\nu}\}$ can describe gravitational field, but cannot describe general gauge field.

We propose some new mathematical tools to overcome the above difficulty, and establish a torsion-free affine connection representation of gauge fields in a geometry that is more general than Riemannian geometry. This is the main work of this paper.

In the theory of this paper, gravitational field and gauge field are unified in affine connection representation. Thus, the problems (i), (ii) and (iii) can all be solved. In addition, there is another advantage that some principles and postulates of the conventional theories that are based on principal bundle connection representation can be turned into theorems in affine connection representation, so they are not necessary to be regarded as principles or postulates anymore.

1.2. Ideas and methods

We divide the problem of establishing affine connection representation of gauge fields into three parts as follows.

(I) Which affine connection is suitable for describing not only gravitational field, but also gauge field?

(II) How to describe the evolution of gauge field and gravitational field in affine connection representation?

(III) What are the concrete forms of electromagnetic, weak and strong interaction fields in affine connection representation?

For the problem (I). We need a new geometry in a new way to integrate Riemannian geometry and gauge field geometry. The new geometry is required to be more general than Riemannian geometry, and more concrete than the existing gauge field geometry.

It has long been known that the coefficients $G_{MN} = \delta_{AB} B_M^A B_N^B$ of metric tensor remain unchanged when orthogonal transformations act on the semi-metric B_M^A (or to say frame field) [24, 25]. So we can say a Riemannian geometric property is an invariant under identical transformations of metric, and we can also speak of it as an invariant under orthogonal transformations of semi-metric.

If we take different orthogonal transformations of semi-metric at different points of a Riemannian manifold, the Riemannian manifold remains unchanged. It is evident that local orthogonal transformations mean more general geometric properties. However, Riemannian manifold cannot reflect these geometric properties.

We see that the semi-metric B_M^A is more fundamental than the metric G_{MN} . Therefore, we need to generalize the Riemannian manifold (M, G_{MN}) to a more general manifold (M, B_M^A) , and then to regard orthogonal transformation of semi-metric as gauge transformation.

Next, we need an affine connection to describe gauge field. Riemann-Cartan geometry is a generalization of Riemannian geometry, its affine connection is independent of metric, but Riemann-Cartan connection and spin connection are both dependent on Christoffel symbol, so they are not suitable for describing gauge field.

We make a further generalization, and adopt a new torsion-free affine connection that is totally independent of Christoffel symbol, and use semi-metric to construct an affine connection representation of gauge fields. In this way, gravitational field and gauge field can be unified on a manifold (M, B_M^A) that is defined by semi-metric.

Intrinsic geometry should not be confined to the point of view of metric and Christoffel symbol. Fig.1 is an example of 2-dimensional manifold (M, B_M^A) . It shows that semi-metric is indeed possessed of an intrinsic geometric significance independent of metric, and that it reflects the degree of slackness and tightness of the distribution of a coordinate frame in another coordinate frame.

Figure 1: The intuition of intrinsic geometry of a surface

On a manifold (M, B_M^A) , we are able to use the semi-metric B_M^A to endow particle fields and gauge fields with intrinsic geometric constructions.

In the theories based on principal bundle connection representation:

(1) For particle fields. Several complex-valued functions which satisfy the Dirac equation, are sometimes used to refer to a charged lepton field l , and sometimes a neutrino field ν . It is not clear that how to distinguish these field functions l and ν by inherent mathematical constructions.

(2) For gauge fields. (i) In the electrodynamics, the electromagnetic potentials A and φ are abstract, they have no inherent mathematical constructions. In other words, the Levi-Civita connection $\Gamma_{\nu\rho}^{\mu}$ is constructed by the metric $g_{\mu\nu}$, but it is not clear that what geometric quantity A and φ are constructed by. (ii) In the QFT based on the Yang-Mills theory[26], such as Glashow-Weinberg-Salam's unified theory of weak and electromagnetic interactions[27–35], quantum chromodynamics[36–46] and various GUTs [47–52], as well as gravitational gauge GUTs [53–59], those connections which describe gauge fields are similar to the above A and φ , and they are devoid of inherent mathematical constructions.

By contrast, in the affine connection representation of this paper, various gauge fields including A and φ and various particle fields including l and ν are all constructed by semi-metric of internal coordinate space. Thus, they are not only abstract algebraic representations of group, but also possessed of concrete geometric entities.

For the problem (II). There is a fundamental difficulty that time is a component of gravitational field, but not a component of gauge field. This leads to an essential difference between the description of evolution of gravitational field and that of gauge field. In this case, it is difficult to obtain a unified theory of evolution in affine connection representation. We find that, we can define time as the total metric with respect to all dimensions of internal coordinate space and external coordinate space, and define evolution as one-parameter group of diffeomorphism, to overcome the above difficulty.

Now that gauge field and gravitational field are both represented as affine connection, then the properties that are related to gauge field, such as charge, current, mass, energy, momentum and action, must have corresponding affine connection representations. Thus, Yang-Mills equation, energy-momentum equation and Dirac equation are turned into geometric properties in gradient direction, in other words, on-shell evolution is characterized by gradient direction. Correspondingly, quantum theory can be interpreted as a geometric theory of distribution of gradient directions.

For the problem (III). Suppose Γ_{NP}^M is the required affine connection, which is constructed by semi-metric B_M^A . The basic idea is that the components of Γ_{NP}^M with $M, N \in \{4, 5, \dots, \mathfrak{D}\}$ describe electromagnetic, weak and strong interaction fields. The other components of Γ_{NP}^M describe gravitational field.

1.3. Content and organization

In this paper, we are going to show how to construct the affine connection representation of gauge fields. Sections are organized as follows.

Corresponding to the problem (I), in section 2 we make some necessary mathematical preparations, and construct a new torsion-free affine connection that is different from Levi-Civita connection, spin connection and Riemann-Cartan connection. Meanwhile, we generalize the notion of reference system, and give it a strict mathematical definition. The reference system in conventional sense is just only defined on a local coordinate neighborhood, and it has only $(1 + 3)$ dimensions. But in this paper we define the concept of reference-system over the entire manifold. It is possessed of more dimensions but different from Kaluza-Klein theory [60–62] and string theories [63–75]. Thus, both of gravitational field and gauge field are regarded as special cases of such a concept of reference-system.

Corresponding to the problem (II), in section 3 we establish the general theory of evolution in affine connection representation of gauge fields, and in section 4 we discuss the application of this general theory of evolution to $(1 + 3)$ -dimensional classical spacetime.

Corresponding to the problem (III), in sections 5 to 7 we show concrete forms of affine connection representations of electromagnetic, weak and strong interaction fields.

Some topics are organized as follows.

(1) Time is regarded as the total metric with respect to all spatial dimensions including external coordinate space and internal coordinate space, see Definition 3.1.1 and Discussion 4.3.2 for detail. The CPT inversion is interpreted as the composition of the total inversion of coordinates and the total inversion of metrics, see section 3.8 for detail. The conventional $(1 + 3)$ -dimensional Minkowski coordinate x^μ originates from the general $\mathfrak{D}$ -dimensional coordinate x^M . The construction method of extra dimensions is different from those of Kaluza-Klein theory and string theory, see section 4.2 for detail.

(2) On-shell evolution is characterized by gradient direction field. See sections 3.4, 3.5, 3.6 and 4.4 for detail.

(3) Quantum theory is regarded as a geometric theory of distribution of gradient directions. We show two dual descriptions of gradient direction. They just exactly correspond to the Schrödinger picture and the Heisenberg picture. In these points of view, the gravitational theory and quantum theory become coordinated. They have a unified description of evolution, and the definition of Feynman propagator is simplified to a stricter form. See sections 3.9 and 3.10 for detail.

(4) Yang-Mills equation originates from a geometric property of gradient direction. We show the affine connection representation of Yang-Mills equation. See section 3.5 and section 4.6 for detail.

(5) Energy-momentum equation originates from a geometric property of gradient direction. We show the affine connection representation of mass, energy, momentum and action, see section 3.6, Definition 4.4.1 and Discussion 4.4.1 for detail. Furthermore, we also show the affine connection representation of Dirac equation, see section 4.5 for detail.

(6) Gauge field and particle field are two geometric fields both constructed by semi-metric. This paper is not going to adopt the way of taking several abstract functions ν , l , A_μ^K and speaking of them directly as a neutrino field, an electric charged lepton field and a gauge field, but to show the inherent and geometric constructions of ν , l , A_μ^K concretely. See Discussion 3.5.??, sections 5, 6 and Definition 7.2 for detail.

(7) The coupling constants of interactions between gauge field and particle field originate from the metric of the internal coordinate space of manifold. See sections 5, 6 and 7 for detail.

(8) Why do not neutrinos participate in the electromagnetic interactions? And why do not right-handed neutrinos participate in the weak interactions with W bosons? In the Standard Model, they are both postulates. However, in the new framework, they are natural and geometric results of affine connection representation of gauge fields, therefore not necessary to be regarded as postulates anymore. See Proposition 5.2 and Proposition 7.1 for detail.

(9) Why do leptons and quarks both have three generations? We give an answer in section 7. It is a natural geometric result of affine connection representation of gauge fields.

2. Mathematical preparations

2.1. Geometric manifold

In order to establish the affine connection representation of gauge fields, we have to generalize Riemannian manifold to geometric manifold.

Definition 2.1.1. Let M be a $\mathfrak{D}$ -dimensional connected smooth real manifold. $\forall p \in M$, take a coordinate chart (U_p, φ_{U_p}) on a neighborhood U_p of p . They constitute a coordinate covering

$$\varphi \triangleq \{(U_p, \varphi_{U_p})\}_{p \in M},$$

which is said to be a **point-by-point covering**. For the sake of simplicity, U_p can be denoted by U , and φ_{U_p} by φ_U .

Let φ and ψ be two point-by-point coverings. For the two coordinate frames φ_U and ψ_U on the neighborhood U of point p , if

$$f_p \triangleq \varphi_U \circ \psi_U^{-1} : \psi_U(U) \rightarrow \varphi_U(U), \quad \xi^A \mapsto x^M$$

is a smooth homeomorphism, f_p is called a **local reference-system**, where ψ_U is said to be the **basis coordinate frame** of f_p , and φ_U is said to be the **performance coordinate frame** of f_p .

If every $p \in M$ is endowed with a local reference-system $f(p)$, and we require the semi-metrics B_M^A and C_A^M in section 2.2 to be smooth real functions on M , then

$$f : M \rightarrow REF, \quad p \mapsto f(p) \quad (1)$$

is said to be a **reference-system** on M , and (M, f) is said to be a **geometric manifold**.

2.2. Metric and semi-metric

In the absence of a special declaration, the indices take values as $A, B, C, D, E = 1, 2, \dots, \mathfrak{D}$ and $M, N, P, Q, R = 1, 2, \dots, \mathfrak{D}$. The derivative functions

$$b_M^A \triangleq \frac{\partial \xi^A}{\partial x^M}, \quad c_A^M \triangleq \frac{\partial x^M}{\partial \xi^A} \quad (2)$$

of $f(p)$ on U_p define the semi-metrics (or to say frame field) B_M^A and C_A^M of f on the manifold M , that are

$$B_M^A : M \rightarrow \mathbb{R}, \quad p \mapsto B_M^A(p) \triangleq (b_{f(p)})_M^A(p), \quad C_A^M : M \rightarrow \mathbb{R}, \quad p \mapsto C_A^M(p) \triangleq (c_{f(p)})_A^M(p). \quad (3)$$

Let $\delta_{AB} = \delta^{AB} = \delta_B^A = \text{Kronecker}(A, B)$ and $\varepsilon_{MN} = \varepsilon^{MN} = \varepsilon_N^M = \text{Kronecker}(M, N)$. The metric tensors of f are

$$G_{MN} = \delta_{AB} B_M^A B_N^B, \quad H_{AB} = \varepsilon_{MN} C_A^M C_B^N. \quad (4)$$

Similarly, it can also be defined that $\bar{b}_A^M \triangleq \frac{\partial \xi_A}{\partial x^M}$, $\bar{c}_M^A \triangleq \frac{\partial x_M}{\partial \xi_A}$ and corresponding $\bar{B}_A^M$, $\bar{C}_M^A$.

2.3. Gauge transformation in affine connection representation

$\forall p \in M, f(p) \triangleq \rho_U \circ \psi_U^{-1}$ induces **local reference-system transformations**

$$\begin{aligned} L_{f(p)} : k(p) &\triangleq \psi_U \circ \varphi_U^{-1} \mapsto \rho_U \circ \varphi_U^{-1} = f(p) \circ k(p), \\ R_{f(p)} : h(p) &\triangleq \varphi_U \circ \rho_U^{-1} \mapsto \varphi_U \circ \psi_U^{-1} = h(p) \circ f(p), \end{aligned}$$

and **reference-system transformations** on manifold M

$$L_f : p \mapsto L_{f(p)}, \quad R_f : p \mapsto R_{f(p)}. \quad (5)$$

We also speak of L_f and R_f as **(affine) gauge transformations**, see Discussion 4.6.3 for reasons.

- (i) If $[B_M^A]$ of f is the identity matrix, then L_f and R_f are called **identical transformations**.
- (ii) If $\forall p_1, p_2 \in M, B_M^A(p_1) = B_M^A(p_2)$, then L_f and R_f are called **flat transformations**.
- (iii) If $\delta_{AB} B_M^A B_N^B = \varepsilon_{MN}$, then L_f and R_f are called **orthogonal transformations**.

The totality of all reference-system transformations on M is denoted by $GL(M)$, which is a subgroup of $\bigotimes_{p \in M} GL(\mathfrak{D}, \mathbb{R})_p$, where $\bigotimes$ represents external direct product.

2.4. Generalized intrinsic geometry

Let the notation F_k represent L_k or R_k . The above transformations lead to the following relations of equivalence.

(i) Define a relation of equivalence $\cong$ such that $f \cong g$ and $(M, f) \cong (M, g)$ if and only if $\forall p \in M, (B_f)_M^A(p) = (B_g)_M^A(p)$. The corresponding equivalence classes are denoted by $[f]$ and $(M, [f])$.

(ii) Define a relation of equivalence $\simeq$ such that $f \simeq g$ and $(M, f) \simeq (M, g)$ if and only if there exists a flat transformation F_k such that $F_k(f) = g$. The corresponding equivalence classes are denoted by $|f|$ and $(M, |f|)$.

(iii) Define a relation of equivalence $\simeq_O$ such that $f \simeq_O g$ and $(M, f) \simeq_O (M, g)$ if and only if there exists an orthogonal transformation F_k such that $F_k(f) = g$. The corresponding equivalence classes are denoted by $[f]_O$ and $(M, [f]_O)$.

(iv) Define a relation of equivalence $\sim$ such that $f \sim g$ and $(M, f) \sim (M, g)$ if and only if there exists an arbitrary transformation F_k such that $F_k(f) = g$. The corresponding equivalence classes are denoted by $[f]_U$ and $(M, [f]_U)$.

The totality of all the geometric manifolds on M is denoted by $\mathcal{M}(M)$. The above four relations of equivalence determine the following four classifications.

$$\mathcal{M}(M)/\cong, \quad \mathcal{M}(M)/\simeq, \quad \mathcal{M}(M)/\simeq_O, \quad \mathcal{M}(M)/\sim.$$

The largest one is $\mathcal{M}(M)/\cong$, which is said to be **(generalized) intrinsic geometry**. The smallest one is $\mathcal{M}(M)/\sim$, which is said to be **universal geometry**.

Let the notation hX represent a geometric property of X . We have the following conclusions.

- (i) $\forall (M, [f]) \in \mathcal{M}(M)/\cong, h(M, [f])$ is an invariant under identical transformation.
- (ii) $\forall (M, |f|) \in \mathcal{M}(M)/\simeq, h(M, |f|)$ is an invariant under flat transformation.
- (iii) $\forall (M, [f]_O) \in \mathcal{M}(M)/\simeq_O, h(M, [f]_O)$ is an invariant under orthogonal transformation.
- (iv) $\forall (M, [f]_U) \in \mathcal{M}(M)/\sim, h(M, [f]_U)$ is an invariant under arbitrary transformation.

If detailed physical properties are described by geometric properties on $\mathcal{M}(M)/\cong$, then universal physical properties can necessarily be described by geometric properties on $\mathcal{M}(M)/\sim$. Therefore, general covariance and gauge invariance are both geometric properties on $\mathcal{M}(M)/\sim$, they remain unchanged under arbitrary transformations of reference-system.

It is evident that $f \simeq_O g$ if and only if $(G_f)_{MN} = (G_g)_{MN}$. So $\mathcal{M}(M)/\simeq_O$ is Riemannian geometry, and $(M, [f]_O)$ is just exactly Riemannian manifold.

Due to $G_{MN} = \delta_{AB} B_M^A B_N^B$ we know that $(B_f)_M^A = (B_g)_M^A$ ensures $(G_f)_{MN} = (G_g)_{MN}$, but conversely when $(G_f)_{MN} = (G_g)_{MN}$ we have $(B_f)_M^A \neq (B_g)_M^A$ in general. So (M, f) and (M, g) are the same Riemannian manifold, but different geometric manifolds.

Under an orthogonal affine gauge transformation L_k , the metric G_{MN} and the Christoffel symbol $\{\frac{M}{NP}\}$ both remain unchanged, therefore affine connection representation of gauge fields cannot be constructed with G_{MN} and $\{\frac{M}{NP}\}$ on Riemannian geometry $\mathcal{M}(M)/\simeq_O$, but can only be constructed with B_M^A and C_A^M on generalized intrinsic geometry $\mathcal{M}(M)/\cong$.

2.5. Simple connection

We do not take into account spin connection and Riemann-Cartan connection, because they are dependent on Christoffel symbol $\{\frac{M}{NP}\}$. A nice choice is to adopt the following definition, which is independent of Christoffel symbol.

Definition 2.5.1. Let there be an affine connection D . It presents as

$$D \frac{\partial}{\partial x^N} \triangleq \Gamma_{NP}^M dx^P \otimes \frac{\partial}{\partial x^M}, \quad D dx^N \triangleq -\Gamma_{MP}^N dx^P \otimes dx^M.$$

If the affine connection coefficients are defined as

$$\Gamma_{NP}^M \triangleq \frac{1}{2} C_A^M \left(\frac{\partial B_N^A}{\partial x^P} + \frac{\partial B_P^A}{\partial x^N} \right) = -\frac{1}{2} \left(B_N^A \frac{\partial C_A^M}{\partial x^P} + B_P^A \frac{\partial C_A^M}{\partial x^N} \right), \quad (6)$$

D is said to be a **simple connection**. In addition, denote $\Gamma_{MNP} \triangleq G_{MM'} \Gamma_{NP}^{M'}$. It is easy to verify that

$$\Gamma_{MNP} = \frac{1}{2} \delta_{AB} B_M^B \left(\frac{\partial B_N^A}{\partial x^P} + \frac{\partial B_P^A}{\partial x^N} \right),$$

and that D is torsion-free. If we do not require it to be torsion-free, it is fine to adopt $\Gamma_{NP}^M \triangleq C_A^M \frac{\partial B_N^A}{\partial x^P}$.

Proposition 2.5.1. Suppose there are reference-systems g and k on a manifold M . Let the semi-metrics of g be

$$(B_g)_M^A, \quad (C_g)_A^M.$$

Let the semi-metrics of k be

$$(B_k)_{M'}^M, \quad (C_k)_M^{M'}.$$

And let the semi-metrics of $g' \triangleq L_k(g)$ be

$$(B_{g'})_{M'}^A = (B_g)_M^A (B_k)_{M'}^M, \quad (C_{g'})_A^{M'} = (C_g)_A^M (C_k)_M^{M'}.$$

Then let the simple connections of (M, g) , (M, k) , (M, g') be $(\Gamma_g)_{NP}^M$, $(\Gamma_k)_{N'P'}^{M'}$, $(\Gamma_{g'})_{N'P'}^{M'}$, respectively. Thus,

$$(\Gamma_{g'})_{N'P'}^{M'} = (\Gamma_g)_{NP}^M (C_k)_M^{M'} (B_k)_{N'}^N (B_k)_{P'}^P + (\Gamma_k)_{N'P'}^{M'}. \quad (7)$$

Proof. We just need to calculate it.

$$\begin{aligned}
(\Gamma_{g'})_{N'P'}^{M'} &\triangleq \frac{1}{2}(C_{g'})_A^{M'} \left(\frac{\partial(B_{g'})_{N'}^A}{\partial x^{P'}} + \frac{\partial(B_{g'})_{P'}^A}{\partial x^{N'}} \right) \\
&= \frac{1}{2}(C_k)_M^{M'}(C_g)_A^M \left(\frac{\partial((B_k)_{N'}^N(B_g)_N^A)}{\partial x^{P'}} + \frac{\partial((B_k)_{P'}^P(B_g)_P^A)}{\partial x^{N'}} \right) \\
&= \frac{1}{2}(C_k)_M^{M'}(C_g)_A^M \left(\frac{\partial(B_k)_{N'}^N}{\partial x^{P'}}(B_g)_N^A + (B_k)_{N'}^N \frac{\partial(B_g)_N^A}{\partial x^{P'}} + \frac{\partial(B_k)_{P'}^P}{\partial x^{N'}}(B_g)_P^A + (B_k)_{P'}^P \frac{\partial(B_g)_P^A}{\partial x^{N'}} \right) \\
&= \frac{1}{2}(C_k)_M^{M'}(C_g)_A^M \left((B_k)_{N'}^N \frac{\partial(B_g)_N^A}{\partial x^{P'}} + (B_k)_{P'}^P \frac{\partial(B_g)_P^A}{\partial x^{N'}} \right) \\
&\quad + \frac{1}{2}(C_k)_M^{M'}(C_g)_A^M \left(\frac{\partial(B_k)_{N'}^N}{\partial x^{P'}}(B_g)_N^A + \frac{\partial(B_k)_{P'}^P}{\partial x^{N'}}(B_g)_P^A \right) \\
&= \frac{1}{2}(C_k)_M^{M'}(C_g)_A^M \left(\frac{\partial(B_g)_N^A}{\partial x^P} + \frac{\partial(B_g)_P^A}{\partial x^N} \right) (B_k)_{N'}^N (B_k)_{P'}^P + \frac{1}{2}(C_k)_M^{M'} \left(\frac{\partial(B_k)_{N'}^M}{\partial x^{P'}} + \frac{\partial(B_k)_{P'}^M}{\partial x^{N'}} \right) \\
&= (\Gamma_g)_{NP}^M (C_k)_M^{M'} (B_k)_{N'}^N (B_k)_{P'}^P + (\Gamma_k)_{N'P'}^{M'}. \quad \square
\end{aligned}$$

Remark 2.5.1. The above proposition is applied to Discussion 4.6.3. It induces the affine connection representation of gauge transformation.

Remark 2.5.2. When $G_{MN} = \delta_{AB} B_M^A B_N^B = \text{const}$, we have $B_M^A \neq \text{const}$ in general. So Levi-Civita connection $\Lambda_{NP}^M = 0$, and simple connection $\Gamma_{NP}^M \triangleq \frac{1}{2} C_A^M \left(\frac{\partial B_N^A}{\partial x^P} + \frac{\partial B_P^A}{\partial x^N} \right) \neq 0$. Then, the corresponding Riemannian curvatures

$$\begin{aligned}
K_{NPQ}^M &\triangleq \frac{\partial \Lambda_{NQ}^M}{\partial x^P} - \frac{\partial \Lambda_{NP}^M}{\partial x^Q} + \Lambda_{NQ}^H \Lambda_{HP}^M - \Lambda_{NP}^H \Lambda_{HQ}^M = 0, \\
R_{NPQ}^M &\triangleq \frac{\partial \Gamma_{NQ}^M}{\partial x^P} - \frac{\partial \Gamma_{NP}^M}{\partial x^Q} + \Gamma_{NQ}^H \Gamma_{HP}^M - \Gamma_{NP}^H \Gamma_{HQ}^M \neq 0.
\end{aligned}$$

This shows that simple connection reflects more curved-properties of a manifold than Levi-Civita connection.

3. The evolution in affine connection representation of gauge fields

Now that we have the required affine connection, next we have to solve the problem that how to describe the evolution in affine connection representation.

In the existing theories, time is a component of gravitational field, but not a component of gauge field. This leads to an essential difference between the description of evolution of gravitational field and that of gauge field. In this case, it is difficult to obtain a unified theory of evolution in affine connection representation. We adopt the following way to overcome this difficulty.

3.1. The relation between time and space

Definition 3.1.1. Suppose $M = P \times N$ and $r \triangleq \dim P = 3$. Let

$$A, B, M, N = 1, \dots, \mathfrak{D}; \quad s, i = 1, \dots, r; \quad a, m = r + 1, \dots, \mathfrak{D}.$$

On the geometric manifold (M, f) , the $d\xi^0$ and dx^0 which are defined by

$$\begin{aligned} (d\xi^0)^2 &\triangleq \sum_{A=1}^{\mathfrak{D}} (d\xi^A)^2 = \delta_{AB} d\xi^A d\xi^B = G_{MN} dx^M dx^N, \\ (dx^0)^2 &\triangleq \sum_{M=1}^{\mathfrak{D}} (dx^M)^2 = \varepsilon_{MN} dx^M dx^N = H_{AB} d\xi^A d\xi^B \end{aligned} \quad (8)$$

are said to be **total space metrics** or **time metrics**. We also suppose

$$\begin{aligned} (d\xi^{(P)})^2 &\triangleq \sum_{s=1}^r (d\xi^s)^2, & (d\xi^{(N)})^2 &\triangleq \sum_{a=r+1}^{\mathfrak{D}} (d\xi^a)^2, \\ (dx^{(P)})^2 &\triangleq \sum_{i=1}^r (dx^i)^2, & (dx^{(N)})^2 &\triangleq \sum_{m=r+1}^{\mathfrak{D}} (dx^m)^2. \end{aligned}$$

$d\xi^{(N)}$ and $dx^{(N)}$ are regarded as proper-time metrics. For convenience, P is said to be **external space** and N is said to be **internal space**.

Remark 3.1.1. The above definition implies a new view of time and space. The relation between time and space in this way is different from the Minkowski coordinates x^μ ($\mu = 0, 1, 2, 3$). Time and space are not the components on an equal footing anymore, but have a relation of total to component. It can be seen later that time reflects the total evolution in the full-dimensional space, while a specific spatial dimension reflects just a partial evolution in a specific direction.

The relation of the above concept to Minkowski coordinates is discussed detailedly in section 4.

3.2. Evolution path as a submanifold

Definition 3.2.1. Let there be two reference-systems f and g on a manifold M . If $\forall p \in M$, $f(p) \triangleq \phi_U \circ \psi_U^{-1}$ and $g(p) \triangleq \phi_U \circ \rho_U^{-1}$ have the same performance coordinate frame ϕ_U , namely it can be intuitively expressed as a diagram

$$\psi_U(U) \xrightarrow{f(p)} \phi_U(U) \xleftarrow{g(p)} \rho_U(U), \quad (9)$$

we say f and g **motion relatively and interact mutually**, and also say that f evolves in g , or f evolves on the geometric manifold (M, g) . Meanwhile, g evolves in f , or to say g evolves on (M, f) .

Let there be a one-parameter group of diffeomorphisms

$$\varphi_X : M \times \mathbb{R} \rightarrow M$$

acting on M , such that $\varphi_X(p, 0) = p$. Thus, φ_X determines a smooth tangent vector field X on M . If X is nonzero everywhere, we say φ_X is a **set of evolution paths**, and X is an **evolution direction field**. Let $T \subseteq \mathbb{R}$ be an interval, then the regular imbedding

$$L_p \triangleq \varphi_{X,p} : T \rightarrow M, \quad t \mapsto \varphi_X(p, t) \quad (10)$$

is said to be an **evolution path** through p . The tangent vector $\frac{d}{dt} \triangleq [L_p] = X(p)$ is called an **evolution direction** at p . For the sake of simplicity, we also denote $L_p \triangleq L_p(T) \subset M$, then

$$\pi : L_p \rightarrow M, \quad q \mapsto q \quad (11)$$

is also a regular imbedding. If it is not necessary to emphasize the point p , L_p is denoted by L concisely.

In order to describe physical evolution, next we are going to strictly describe the mathematical properties of the reference-systems f and g sending onto the evolution path L .

Definition 3.2.2. Let the coordinates of (U, ψ_U) , (U, ϕ_U) , (U, ρ_U) be ξ^A , x^M , ζ^A , respectively, and the corresponding time metrics $d\xi^0$, dx^0 , $d\zeta^0$. On $U_L \triangleq U \cap L_p$ we have parameter equations

$$\begin{aligned} \xi^A &= \xi^A(x^0), & x^M &= x^M(\xi^0), & \zeta^A &= \zeta^A(x^0), \\ \xi^0 &= \xi^0(x^0), & x^0 &= x^0(\xi^0), & \zeta^0 &= \zeta^0(x^0). \end{aligned} \quad (12)$$

Take f for example, according to Eq.(12), on U_L we define

$$\begin{aligned} b_0^A &\triangleq \frac{d\xi^A}{dx^0}, & b_0^0 &\triangleq \frac{d\xi^0}{dx^0}, & \varepsilon_0^M &\triangleq \frac{dx^M}{dx^0} = b_0^0 c_0^M = b_0^A c_A^M, \\ c_0^M &\triangleq \frac{dx^M}{d\xi^0}, & c_0^0 &\triangleq \frac{dx^0}{d\xi^0}, & \delta_0^A &\triangleq \frac{d\xi^A}{d\xi^0} = c_0^0 b_0^A = c_0^M b_M^A. \end{aligned}$$

Define $d\xi_0 \triangleq \frac{dx^0}{d\xi^0} dx^0$ and $dx_0 \triangleq \frac{d\xi^0}{dx^0} d\xi^0$, which induce $\frac{d}{d\xi_0}$ and $\frac{d}{dx_0}$, such that $\left\langle \frac{d}{d\xi_0}, d\xi_0 \right\rangle = 1$, $\left\langle \frac{d}{dx_0}, dx_0 \right\rangle = 1$. So we can also define

$$\begin{aligned} \bar{b}_A^0 &\triangleq \frac{d\xi_A}{dx_0}, & \bar{b}_0^0 &\triangleq \frac{d\xi_0}{dx_0}, & \bar{\varepsilon}_M^0 &\triangleq \frac{dx_M}{dx_0} = \bar{b}_0^0 \bar{c}_M^0 = \bar{b}_A^0 \bar{c}_M^A, \\ \bar{c}_M^0 &\triangleq \frac{dx_M}{d\xi_0}, & \bar{c}_0^0 &\triangleq \frac{dx_0}{d\xi_0}, & \bar{\delta}_A^0 &\triangleq \frac{d\xi_A}{d\xi_0} = \bar{c}_0^0 \bar{b}_A^0 = \bar{c}_M^0 \bar{b}_M^A. \end{aligned}$$

They determine the following smooth functions on the entire L , similar to section 2.2:

$$\begin{aligned} B_0^A &: L \rightarrow \mathbb{R}, \quad p \mapsto B_0^A(p) \triangleq (b_{f(p)})_0^A(p), & C_0^M &: L \rightarrow \mathbb{R}, \quad p \mapsto C_0^M(p) \triangleq (c_{f(p)})_0^M(p), \\ \bar{B}_A^0 &: L \rightarrow \mathbb{R}, \quad p \mapsto \bar{B}_A^0(p) \triangleq (\bar{b}_{f(p)})_A^0(p), & \bar{C}_M^0 &: L \rightarrow \mathbb{R}, \quad p \mapsto \bar{C}_M^0(p) \triangleq (\bar{c}_{f(p)})_M^0(p), \\ B_0^0 &: L \rightarrow \mathbb{R}, \quad p \mapsto B_0^0(p) \triangleq (b_{f(p)})_0^0(p), & C_0^0 &: L \rightarrow \mathbb{R}, \quad p \mapsto C_0^0(p) \triangleq (c_{f(p)})_0^0(p), \\ \bar{B}_0^0 &: L \rightarrow \mathbb{R}, \quad p \mapsto \bar{B}_0^0(p) \triangleq (\bar{b}_{f(p)})_0^0(p), & \bar{C}_0^0 &: L \rightarrow \mathbb{R}, \quad p \mapsto \bar{C}_0^0(p) \triangleq (\bar{c}_{f(p)})_0^0(p). \end{aligned}$$

For convenience, we still use the notations ε and δ , and have the following smooth functions.

$$\begin{aligned} \varepsilon_0^M &\triangleq B_0^0 C_0^M = B_0^A C_A^M, & \delta_0^A &\triangleq C_0^0 B_0^A = C_0^M B_M^A, & G_{00} &\triangleq B_0^0 B_0^0 = G_{MN} \varepsilon_0^M \varepsilon_0^N, \\ \bar{\varepsilon}_M^0 &\triangleq \bar{B}_0^0 \bar{C}_M^0 = \bar{B}_A^0 \bar{C}_M^A, & \bar{\delta}_A^0 &\triangleq \bar{C}_0^0 \bar{B}_A^0 = \bar{C}_M^0 \bar{B}_M^A, & G^{00} &\triangleq C_0^0 C_0^0 = G^{MN} \bar{\varepsilon}_M^0 \bar{\varepsilon}_N^0. \end{aligned}$$

It is easy to verify that $dx_0 = G_{00} dx^0$ and $\frac{d}{dx_0} = G^{00} \frac{d}{dx^0}$ are both true on L by a simple calculation.

3.3. Evolution lemma

We have the following two evolution lemmas. The affine connection representations of Yang-Mills equation, energy-momentum equation and Dirac equation are dependent on them.

Definition 3.3.1. $\forall p \in L$, suppose $T_p(M)$ and $T_p(L)$ are tangent spaces, $T_p^*(M)$ and $T_p^*(L)$ are cotangent spaces. The regular imbedding $\pi : L \rightarrow M$, $q \mapsto q$ induces the tangent map and the cotangent map

$$\begin{aligned}\pi_* : T_p(L) &\rightarrow T_p(M), \quad [\gamma_L] \mapsto [\pi \circ \gamma_L], \\ \pi^* : T_p^*(M) &\rightarrow T_p^*(L), \quad df \mapsto d(f \circ \pi).\end{aligned}\tag{13}$$

Evidently, π_* is an injection, and π^* is a surjection. $\forall \frac{d}{dt} \in T_p(L)$, $\frac{d}{dt} \in T_p(M)$, $df \in T_p^*(M)$, $df_L \in T_p^*(L)$, if and only if

$$\frac{d}{dt} = \pi_* \left(\frac{d}{dt_L} \right), \quad df_L = \pi^*(df)\tag{14}$$

are true, we denote

$$\frac{d}{dt} \cong \frac{d}{dt_L}, \quad df \simeq df_L.\tag{15}$$

Proposition 3.3.1. If $\frac{d}{dt} \cong \frac{d}{dt_L}$ and $df \simeq df_L$, then $\langle \frac{d}{dt}, df \rangle = \langle \frac{d}{dt_L}, df_L \rangle$.

Proof. We know the tangent vectors $\frac{d}{dt}$ and $\frac{d}{dt_L}$ are defined as equivalence classes $[\gamma]$ and $[\gamma_L]$ of parameter curves, the cotangent vectors df and df_L are defined as equivalence classes $[f]$ and $[f_L]$ of smooth functions.

$\frac{d}{dt} = \pi_* \left(\frac{d}{dt_L} \right)$ and $df_L = \pi^*(df)$ indicate that $\exists \gamma \in [\gamma]$, $\gamma_L \in [\gamma_L]$, $f \in [f]$, $f_L \in [f_L]$ such that $\gamma = \pi \circ \gamma_L$ and $f_L = f \circ \pi$. Thus,

$$\left\langle \frac{d}{dt}, df \right\rangle = \left\langle \frac{d}{dt_L}, df_L \right\rangle \Leftrightarrow \langle [\gamma], [f] \rangle = \langle [\gamma_L], [f_L] \rangle \Leftrightarrow \frac{d(f \circ \gamma)}{dt} = \frac{d(f_L \circ \gamma_L)}{dt},$$

where

$$f \circ \gamma = f \circ (\pi \circ \gamma_L), \quad f_L \circ \gamma_L = (f \circ \pi) \circ \gamma_L.$$

Evidently we have $f \circ \gamma = f_L \circ \gamma_L$, and it makes $\langle \frac{d}{dt}, df \rangle = \langle \frac{d}{dt_L}, df_L \rangle$ true. $\square$

Proposition 3.3.2. The following conclusions are true.

$$\begin{cases} w^M \frac{\partial}{\partial x^M} \cong w^0 \frac{d}{dx^0} & \Leftrightarrow w^M = w^0 \varepsilon_0^M, \\ w_M dx^M \simeq w_0 dx^0 & \Leftrightarrow w_M \varepsilon_0^M = w_0, \end{cases} \quad \begin{cases} \bar{w}_M \frac{\partial}{\partial x_M} \cong \bar{w}_0 \frac{d}{dx_0} & \Leftrightarrow \bar{w}_M = \bar{w}_0 \varepsilon_M^0, \\ \bar{w}^M dx_M \simeq \bar{w}^0 dx_0 & \Leftrightarrow \bar{w}^M \varepsilon_M^0 = \bar{w}^0. \end{cases}$$

Proof. The following local discussion can also be applied to the entire manifold. Take the above left-hand side equations for example.

$$\pi_* \left(\frac{d}{dx^0} \right) = \frac{dx^M}{dx^0} \frac{\partial}{\partial x^M} \Rightarrow \varepsilon_0^M \frac{\partial}{\partial x^M} \cong \frac{d}{dx^0} \Rightarrow w^0 \varepsilon_0^M \frac{\partial}{\partial x^M} \cong w^0 \frac{d}{dx^0}.$$

We know π_* is an injection, so $w^M \frac{\partial}{\partial x^M} \cong w^0 \frac{d}{dx^0} \Leftrightarrow w^M = w^0 \varepsilon_0^M$.

$$\pi^*(dx^M) = \frac{dx^M}{dx^0} dx^0 \Rightarrow dx^M \simeq \varepsilon_0^M dx^0 \Rightarrow w_M dx^M \simeq w_M \varepsilon_0^M dx^0.$$

We know π^* is a surjection, so $w_M dx^M \simeq w_0 dx^0 \Leftrightarrow w_M \varepsilon_0^M = w_0$. $\square$

3.4. On-shell evolution as a gradient

Definition 3.4.1. Let T be an smooth n -order tensor field. The restriction on (U, x^M) is $T \triangleq t \left\{ \frac{\partial}{\partial x} \otimes dx \right\}$, where $\left\{ \frac{\partial}{\partial x} \otimes dx \right\}$ represents the tensor basis generated by several $\frac{\partial}{\partial x^M}$ and dx^M , and the tensor coefficients of T are concisely denoted by $t : U \rightarrow \mathbb{R}$.

Let D be a simple connection. Consider $D\mathbf{T} \triangleq t_{;Q} dx^Q \otimes \left\{ \frac{\partial}{\partial x} \otimes dx \right\}$. Denote

$$Dt \triangleq t_{;Q} dx^Q, \quad \nabla t \triangleq t_{;Q} \frac{\partial}{\partial x_Q}.$$

$\forall p \in M$, the integral curve of ∇t , that is $L_p \triangleq \varphi_{\nabla t, p}$, is a **gradient line** of $\mathbf{T}$. It can be seen later that the above gradient operator ∇ characterizes the **on-shell evolution**.

For any evolution path L , let $U_L \triangleq U \cap L$. Denote $t_L \triangleq t|_{U_L}$ and $t_{L;0} \triangleq t_{;Q} \varepsilon_0^Q$, as well as

$$D_L t_L \triangleq t_{L;0} dx^0, \quad \nabla_L t_L \triangleq t_{L;0} \frac{d}{dx_0}.$$

Proposition 3.4.1. The following conclusions are evidently true.

- (i) $Dt \simeq D_L t_L$ if and only if L is an arbitrary evolution path.
- (ii) $\nabla t \cong \nabla_L t_L$ if and only if L is a gradient line of $\mathbf{T}$.

Definition 3.4.2. More generally, suppose there is a tensor $\mathbf{U} \triangleq u_Q dx^Q \otimes \left\{ \frac{\partial}{\partial x} \otimes dx \right\}$. In such a notation, all the indices are concisely ignored except Q . $u_Q dx^Q$ uniquely determines a **characteristic direction** $u_Q \frac{\partial}{\partial x_Q}$.

If the system of 1-order linear partial differential equations $t_{;Q} = u_Q$ has a solution t , then it is true that $Dt = u_Q dx^Q$ and $\nabla t = u_Q \frac{d}{dx_Q}$.

If it does not have a solution, we also denote the notations $Dt \triangleq u_Q dx^Q$ and $\nabla t \triangleq u_Q \frac{d}{dx_Q}$ in form for convenience.

Thus, no matter whether $t_{;Q} = u_Q$ has a solution, as long as we take the evolution direction $[L]$ as $[L] = u_Q \frac{\partial}{\partial x_Q}$, the following conclusions are always true.

$$Dt \simeq D_L t_L, \quad \nabla t \cong \nabla_L t_L, \tag{16}$$

where $D_L t_L \triangleq u_0 dx^0$, $\nabla_L t_L \triangleq u_0 \frac{d}{dx_0}$ and $u_0 \triangleq u_Q \varepsilon_0^Q$.

Remark 3.4.1. Now for any geometric property in the form of tensor $\mathbf{U}$, we are able to express its on-shell evolution in the form of ∇t .

Next, two important on-shell evolutions are discussed in the following two sections. One is the on-shell evolution of the potential field of a reference-system. The other is the one that a general charge of a reference-system evolves in the potential field of another reference-system.

3.5. On-shell evolution of potential field and affine connection representation of Yang-Mills equation

The table I of the article[76] proposes a famous correspondence between gauge field terminologies and fibre bundle terminologies. However, it does not find out the corresponding mathematical object to the source J_μ^K . In this section, we give an answer to this problem from the point of view of section 3.4, and show the affine connection representation of Yang-Mills equation.

Suppose f evolves in g according to Definition 3.2.1, that is, $\forall p \in M$,

$$(U, \xi^A) \xrightarrow{f(p)} (U, x^M) \xleftarrow{g(p)} (U, \zeta^A).$$

We always take the following notations in the coordinate frame (U, x^M) .

(i) Let the simple connections of geometric manifolds (M, f) and (M, g) be Λ_{NP}^M and Γ_{NP}^M , respectively. The colon ":" and semicolon ";" are used to express the covariant derivatives on (M, f) and (M, g) , respectively, e.g.:

$$u^Q_{;P} = \frac{\partial u^Q}{\partial x^P} + u^H \Lambda_{HP}^Q, \quad u^Q_{;P} = \frac{\partial u^Q}{\partial x^P} + u^H \Gamma_{HP}^Q.$$

Λ_{NP}^M is said to be the potential field of f , and Γ_{NP}^M is said to be the potential field of g .

(ii) Let the coefficients of Riemannian curvatures of (M, f) and (M, g) be K_{NPQ}^M and R_{NPQ}^M , respectively, i.e.

$$\begin{aligned} K_{NPQ}^M &\triangleq \frac{\partial \Lambda_{NQ}^M}{\partial x^P} - \frac{\partial \Lambda_{NP}^M}{\partial x^Q} + \Lambda_{NQ}^H \Lambda_{HP}^M - \Lambda_{NP}^H \Lambda_{HQ}^M, \\ R_{NPQ}^M &\triangleq \frac{\partial \Gamma_{NQ}^M}{\partial x^P} - \frac{\partial \Gamma_{NP}^M}{\partial x^Q} + \Gamma_{NQ}^H \Gamma_{HP}^M - \Gamma_{NP}^H \Gamma_{HQ}^M. \end{aligned}$$

(iii) The values of indices of internal space and external space are taken according to Definition 4.1.

Consider the evolution of f . Denote $K_{NPQ}^M{}^{;P} \triangleq G^{PP'} K_{NPQ}^M{}_{;P'}$. On an arbitrary evolution path L , we define

$$\rho_{N0}^M dx^0 \triangleq \pi^* \left(K_{NPQ}^M{}^{;P} dx^Q \right) \in T^*(L).$$

Then, according to Definition 3.3.1 and the evolution lemma of Proposition 3.3.2, we obtain $\rho_{N0}^M = K_{NPQ}^M{}^{;P} \varepsilon_0^Q$ and

$$K_{NPQ}^M{}^{;P} dx^Q \simeq \rho_{N0}^M dx^0.$$

According to Definition 3.4.2, let the characteristic direction of $K_{NPQ}^M{}^{;P} dx^Q$ be ∇t . Then, according to Proposition 3.4.1, $\forall p \in L$, if and only if $[L_p] = \nabla t|_p$, we have

$$\nabla t|_p = K_{NPQ}^M{}^{;P} \frac{\partial}{\partial x_Q} \cong \rho_{N0}^M \frac{d}{dx_0} = \nabla_L t|_p.$$

Applying the evolution lemma of Proposition 3.3.2 again, we obtain

$$K_{NPQ}^M{}^{;P} = \rho_{N0}^M \varepsilon_Q^0.$$

Denote $j_{NQ}^M \triangleq \rho_{N0}^M \varepsilon_Q^0$, then if and only if $[L_p] = \nabla t|_p$ we have

$$K_{NPQ}^M{}^{;P} = j_{NQ}^M, \tag{17}$$

which is said to be **(affine) Yang-Mills equation** of f . Thus, we have the following two results.

(i) We obtain the mathematical origination of charge and current. We know that the evolution path L is an imbedding submanifold of M . Thus, the charge ρ_{N0}^M originates from the pull-back π^* from M to L for the curvature divergence, and the current j_{NQ}^M originates from ∇t that is related to ρ_{N0}^M . Besides, there is a further explanation in section 4.4.

(ii) We obtain the following proposition. This shows that Yang-Mills equation originates from a geometric property in the direction of ∇t .

Proposition 3.5.1. Let ∇t be the characteristic direction of $K_{NPQ}^M \cdot^P dx^Q$. At any point p on M , the Yang-Mills equation $K_{NPQ}^M \cdot^P = j_{NQ}^M$ holds if and only if the evolution direction $[L_p] = \nabla t|_p$.

Definition 3.5.1. We speak of

$$\rho_{MN}^0 \triangleq G^{00} G_{MM'} \rho_{N0}^{M'}$$

as the **field function of a general charge**, or a **charge** for short.

3.6. On-shell evolution of general charge and affine connection representation of mass, energy, momentum and action

In order to be compatible with the affine connection representation of gauge fields, we also have to define mass, energy, momentum and action in the form associated to affine connection. We are going to show them in this section and section 4.4.

Let $F^0 \triangleq \rho_{MN}^0 dx^M \otimes dx^N$. For the sake of simplicity, denote ρ_{MN}^0 by ρ_{MN} concisely. Let D be the simple connection of (M, g) , then

$$DF^0 \triangleq D\rho_{MN} \otimes dx^M \otimes dx^N, \quad \nabla F^0 \triangleq \nabla \rho_{MN} \otimes dx^M \otimes dx^N,$$

where $D\rho_{MN} \triangleq \rho_{MN;Q} dx^Q$ and $\nabla \rho_{MN} \triangleq \rho_{MN;Q} \frac{\partial}{\partial x^Q}$. According to Proposition 3.4.1, if and only if the evolution direction is taken as $[L] = \nabla \rho_{MN}$, we have

$$D\rho_{MN} \simeq D_L \rho_{MN}, \quad \nabla \rho_{MN} \cong \nabla_L \rho_{MN},$$

that is

$$\rho_{MN;Q} dx^Q \simeq \rho_{MN;0} dx^0, \quad \rho_{MN;Q} \frac{\partial}{\partial x^Q} \cong \rho_{MN;0} \frac{d}{dx^0}. \quad (18)$$

Definition 3.6.1. For more convenience, the notation ρ_{MN} is further abbreviated as ρ . In affine connection representation, energy and momentum of ρ are defined as

$$\begin{aligned} E_0 &\triangleq \rho_{;0} \triangleq \rho_{;Q} \varepsilon_0^Q, & p_Q &\triangleq \rho_{;Q}, & H_0 &\triangleq \frac{d\rho}{dx^0}, & P_Q &\triangleq \frac{\partial \rho}{\partial x^Q}, \\ E^0 &\triangleq \rho^{;0} \triangleq \rho^{;Q} \varepsilon_Q^0, & p^Q &\triangleq \rho^{;Q}, & H^0 &\triangleq \frac{d\rho}{dx_0}, & P^Q &\triangleq \frac{\partial \rho}{\partial x_Q}. \end{aligned} \quad (19)$$

Proposition 3.6.1. Let the charge ρ of f evolves in g . At any point p on M , the equation

$$E_0 E^0 = p_Q p^Q \quad (20)$$

holds if and only if the evolution direction $[L_p] = \nabla \rho|_p$. Eq.(20) is the **(affine) energy-momentum equation** of ρ .

Proof. According to the above discussion, $\forall p \in M$, $[L_p] = \nabla \rho|_p$ is equivalent to

$$p_Q dx^Q \simeq E_0 dx^0, \quad p_Q \frac{\partial}{\partial x_Q} \cong E_0 \frac{d}{dx_0}. \quad (21)$$

Then due to Proposition 3.3.1 we obtain the directional derivative in the gradient direction $\nabla \rho$:

$$\left\langle p_Q \frac{\partial}{\partial x_Q}, p_M dx^M \right\rangle = \left\langle E_0 \frac{d}{dx_0}, E_0 dx^0 \right\rangle,$$

i.e. $G^{QM} p_Q p_M = G^{00} E_0 E_0$, or $p_Q p^Q = E_0 E^0$. $\square$

Proposition 3.6.2. Let the charge ρ of f evolves in g . At any point p on M , the equations

$$p^Q = E^0 \frac{dx^Q}{dx^0}, \quad p_Q = E_0 \frac{dx_Q}{dx_0} \quad (22)$$

hold if and only if the evolution direction $[L_p] = \nabla \rho|_p$.

Proof. Due to the evolution lemma of Proposition 3.3.2, we immediately obtain Eq.(22) from Eq.(21). $\square$

Remark. According to Proposition 4.2.2, the light velocity $c = 1$, therefore in the gradient direction $\nabla \rho$, the conclusion of Proposition 3.6.2 is consistent with the conventional

$$p = mv.$$

Thus, in affine connection representation, the energy-momentum equation and the conventional definition of momentum both originate from a geometric property in gradient direction.

Definition 3.6.2. Let $\mathcal{P}(b, a)$ be the totality of paths from a to b . And suppose $L \in \mathcal{P}(b, a)$, and the evolution parameter x^0 satisfies $t_a \triangleq x^0(a) < x^0(b) \triangleq t_b$. The **elementary affine action** of ρ is defined as

$$\mathfrak{s}(L) \triangleq \int_L D\rho = \int_L p_Q dx^Q = \int_{t_a}^{t_b} E_0 dx^0. \quad (23)$$

Thus, $\delta \mathfrak{s}(L) = 0$ if and only if L is a gradient line of ρ .

Specially, in the case where g is orthogonal, we can also define action in the following way.

On (M, g) let there be Dirac algebras γ^M and γ_N such that

$$\gamma^M \gamma^N + \gamma^N \gamma^M = 2G^{MN}, \quad \gamma_M \gamma_N + \gamma_N \gamma_M = 2G_{MN}, \quad \gamma_M \gamma^M = 1.$$

In a gradient direction of ρ , from Eq.(20) we obtain that

$$\begin{aligned} p_Q p^Q = E_0 E^0 &\Leftrightarrow \rho_{;Q} \rho^{;Q} = \rho_{;0} \rho^{;0} \\ &\Leftrightarrow G^{PQ} \rho_{;P} \rho_{;Q} = G^{00} \rho_{;0} \rho_{;0} \\ &\Leftrightarrow (\gamma^P \gamma^Q + \gamma^Q \gamma^P) \rho_{;P} \rho_{;Q} = 2 \rho_{;0} \rho_{;0} \\ &\Leftrightarrow (\gamma^P \rho_{;P})(\gamma^Q \rho_{;Q}) + (\gamma^Q \rho_{;Q})(\gamma^P \rho_{;P}) = 2 \rho_{;0} \rho_{;0} \\ &\Leftrightarrow (\gamma^P \rho_{;P})^2 = (\rho_{;0})^2. \end{aligned}$$

Take $\gamma^P \rho_{;P} = \rho_{;0}$ without loss of generality, then, in the gradient direction of ρ we have

$$\gamma^P \rho_{;P} dx^0 = \rho_{;0} dx^0 = \varepsilon_0^P \rho_{;P} dx^0 = D\rho. \quad (24)$$

So we can define the following (**orthogonal affine**) **action** of ρ .

$$s(L) \triangleq \int_L (\gamma^P \rho_{;P} dx^0 + D\rho) = \int_{t_a}^{t_b} (\gamma^P \rho_{;P} + \varepsilon_0^P \rho_{;P}) dx^0 = \int_{t_a}^{t_b} (\gamma^P \rho_{;P} + E_0) dx^0. \quad (25)$$

Remark 3.6.1 and Remark 4.5.1 explain the rationality of this definition. We have $s(L) = 2\mathfrak{s}(L)$ in the gradient direction of ρ , so $\mathfrak{s}(L)$ and $s(L)$ are consistent.

Remark 3.6.1. In the Minkowski coordinate frame of Definition 4.2.1, the evolution parameter x^0 is replaced by $\tilde{x}^\tau$, then there still exists a concept of gradient direction $\tilde{\nabla} \tilde{\rho}$. Correspondingly,

$$\mathfrak{s}(L) \triangleq \int_L D\rho = \int_L p_Q dx^Q = \int_{t_a}^{t_b} E_0 dx^0, \quad s(L) = \int_{t_a}^{t_b} (\gamma^P \rho_{;P} + E_0) dx^0$$

present as

$$\tilde{\mathfrak{s}}(L) \triangleq \int_L \tilde{D}\tilde{\rho} = \int_L \tilde{p}_\mu d\tilde{x}^\mu = \int_{\tau_a}^{\tau_b} \tilde{m}_\tau d\tilde{x}^\tau, \quad \tilde{s}(L) = \int_{\tau_a}^{\tau_b} (\gamma^\mu \tilde{\rho}_{;\mu} + \tilde{m}_\tau) d\tilde{x}^\tau,$$

where $\tilde{m}_\tau$ is the rest-mass and $\tilde{x}^\tau$ is the proper-time.

3.7. Affine connection representation of interaction

First, we define the following notations.

$$\begin{aligned} [\rho \Gamma_G] &\triangleq \frac{\partial \rho}{\partial x^G} - \rho_{;G} \triangleq \frac{\partial \rho_{MN}}{\partial x^G} - \rho_{MN;G} = \rho_{MH} \Gamma_{NG}^H + \rho_{HN} \Gamma_{MG}^H, & [\rho \Gamma^Q] &\triangleq G^{GQ} [\rho \Gamma_G], \\ [\rho \Gamma_0] &\triangleq \frac{d\rho}{dx^0} - \rho_{;0} \triangleq \frac{d\rho_{MN}}{dx^0} - \rho_{MN;0} = \rho_{MH} \Gamma_{N0}^H + \rho_{HN} \Gamma_{M0}^H, & [\rho \Gamma^0] &\triangleq G^{00} [\rho \Gamma_0], \\ [\rho R_{PQ}] &\triangleq \rho_{MH} R_{NPQ}^H + \rho_{HN} R_{MPQ}^H, & [\rho E_{PQ}] &\triangleq [\rho \Gamma_Q]_{;P} - [\rho \Gamma_P]_{;Q}, \\ [\rho B_{PQ}] &\triangleq \rho_{MH} \left(\frac{\partial \Gamma_{NQ}^H}{\partial x^P} - \frac{\partial \Gamma_{NP}^H}{\partial x^Q} \right) + \rho_{HN} \left(\frac{\partial \Gamma_{MQ}^H}{\partial x^P} - \frac{\partial \Gamma_{MP}^H}{\partial x^Q} \right), & [\rho F_{PQ}] &\triangleq \frac{\partial [\rho \Gamma_Q]}{\partial x^P} - \frac{\partial [\rho \Gamma_P]}{\partial x^Q}. \end{aligned}$$

Then, through some calculations, we can obtain that

$$\begin{aligned} F_P &\triangleq \frac{dp_P}{dx^0} = \frac{\partial E_0}{\partial x^P} - p_Q \frac{\partial \varepsilon_0^Q}{\partial x^P} + [\rho F_{PQ}] \varepsilon_0^Q, & [\rho F_{PQ}] &= [\rho E_{PQ}], \\ f_P &\triangleq p_{P;0} = E_{0;P} - p_Q \varepsilon_{0;P}^Q + [\rho R_{PQ}] \varepsilon_0^Q, & [\rho B_{PQ}] &= [\rho R_{PQ}], \end{aligned}$$

which are the affine connection representations of general Lorentz force equations. See Discussion 4.4.1 for further illustrations.

Furthermore, the affine connection representation of the conservation of energy-momentum can base on the above result to be constructed. We need to take Y_{MN} such that $Y_{MN}^{;M} = -G^{00}(E_{0;N} - p_Q \varepsilon_{0;N}^Q + [\rho R_{NQ}] \varepsilon_0^Q) - \varepsilon_M^{0;M} p_N$. And let $W_{MN} \triangleq E_0 \varepsilon_M^0 \varepsilon_N^0$, $T_{MN} \triangleq W_{MN} + Y_{MN}$. According to Proposition 3.6.2, $E_0 \varepsilon_N^0 = p_N$ in the gradient direction of ρ . Then

$$W_{MN}^{;M} = (E_0 \varepsilon_M^0 \varepsilon_N^0)^{;M} = (\varepsilon_M^0 p_N)^{;M} = p_N^{;M} \varepsilon_M^0 + \varepsilon_M^{0;M} p_N = p_N^{;0} + \varepsilon_M^{0;M} p_N = -Y_{MN}^{;M}.$$

Thus, $(W_{MN} + Y_{MN})^{;M} = 0$, that is $T_{MN}^{;M} = 0$.

3.8. Inversion transformation in affine connection representation

In affine connection representation, CPT inversion is interpreted as a total inversion of coordinates and metrics. Let $i, j = 1, 2, 3$ and $m, n = 4, 5, \dots, \mathfrak{D}$.

Let the local coordinate representation of reference-system k be $x'^j = -\delta_i^j x^i$, $x'^m = \delta_m^n x^n$, then parity inversion can be represented as

$$P \triangleq L_k : x^i \rightarrow -x^i, x^m \rightarrow x^m.$$

Let the local coordinate representation of reference-system h be $x'^j = \delta_i^j x^i$, $x'^m = -\delta_m^n x^n$, then charge conjugate inversion can be represented as

$$C \triangleq L_h : x^i \rightarrow x^i, x^m \rightarrow -x^m.$$

Time coordinate inversion can be represented as

$$T_0 : x^0 \rightarrow -x^0.$$

Total inversion of coordinates can be represented as

$$CPT_0 : x^Q \rightarrow -x^Q, x^0 \rightarrow -x^0. \quad (26)$$

The positive or negative sign of metric marks two opposite directions of evolution. Let N be a closed submanifold of M , and let its metric be $dx^{(N)}$. Denote the totality of closed submanifolds of M by $\mathfrak{B}(M)$, then **total inversion of metrics** can be expressed as

$$T^{(M)} \triangleq \prod_{N \in \mathfrak{B}(M)} (dx^{(N)} \rightarrow -dx^{(N)}). \quad (27)$$

Denote time inversion by

$$T \triangleq T^{(M)} T_0,$$

then the joint transformation of the total inversion of coordinates CPT_0 and the total inversion of metrics $T^{(M)}$ is

$$(CPT_0)(T^{(M)}) = CPT. \quad (28)$$

Summarize the above discussions, then we have

$$\begin{aligned} CPT_0 : x^Q &\rightarrow -x^Q, x^0 \rightarrow -x^0, dx^Q \rightarrow dx^Q, dx^0 \rightarrow dx^0, \\ T^{(M)} : x^Q &\rightarrow x^Q, x^0 \rightarrow x^0, dx^Q \rightarrow -dx^Q, dx^0 \rightarrow -dx^0, \\ CPT : x^Q &\rightarrow -x^Q, x^0 \rightarrow -x^0, dx^Q \rightarrow -dx^Q, dx^0 \rightarrow -dx^0. \end{aligned}$$

The CPT invariance in affine connection representation is very clear. Concretely, on (M, g) we consider the CPT transformation acting on g . Denote $s \triangleq \int_L D\rho$ and $D_P e^{is} \triangleq \left(\frac{\partial}{\partial x^P} - i[\rho \Gamma_P] \right) e^{is}$, then through simple calculations we obtain that

$$CPT : D\rho \rightarrow D\rho, \quad D_P e^{is} \rightarrow -D_P e^{is}.$$

Remark 3.8.1. In quantum mechanics there is a complex conjugation in the time inversion of wave function $T : \psi(x, t) \rightarrow \psi^*(x, -t)$. In affine connection representation, we know the complex conjugation is a straightforward mathematical result of the total inversion of metrics $T^{(M)}$.

3.9. Two dual descriptions of gradient direction field

Discussion 3.9.1. Let X and Y be non-vanishing smooth tangent vector fields on the manifold M . And let L_Y be the Lie derivative operator induced by the one-parameter group of diffeomorphism φ_Y . Then, according to a well-known theorem[77], we obtain the Lie derivative equation

$$[X, Y] = L_Y X. \quad (29)$$

Suppose $\forall p \in M$, $Y(p)$ is a unit-length vector, i.e. $\|Y(p)\| = 1$. Let the parameter of φ_Y be x^0 . Then, on the evolution path $L \triangleq \varphi_{Y,p}$, we have

$$Y \cong \frac{d}{dx^0}. \quad (30)$$

Thus, Eq.(29) can also be represented as

$$[X, Y] = \frac{d}{dx^0} X. \quad (31)$$

On the other hand, $\forall df \in T(M)$ and $df_L \triangleq \pi^*(df)$, due to (30) and Proposition 3.3.1 we have $\langle Y, df \rangle = \langle \frac{d}{dx^0}, df_L \rangle$, that is

$$Yf = \frac{d}{dx^0} f_L. \quad (32)$$

Definition 3.9.1. Let $H \triangleq \|\nabla \rho\|^{-1} \nabla \rho = \varepsilon_0^M \frac{\partial}{\partial x^M} \cong \frac{d}{dx^0}$. It is evident that $\forall p \in M$, $\|H(p)\| = 1$. If and only if taking $Y = H$, we speak of (31) and (32) as real-valued **(affine) Heisenberg equation** and **(affine) Schrödinger equation**, respectively, that is

$$[X, H] = \frac{d}{dx^0} X, \quad Hf = \frac{d}{dx^0} f_L. \quad (33)$$

Discussion 3.9.2. The above two equations both describe the gradient direction field, and thereby reflect on-shell evolution. Such two dual descriptions of gradient direction show the real-valued affine connection representation of Heisenberg picture and Schrödinger picture.

It is not hard to find out several different kinds of complex-valued representations of gradient direction. For examples, one is the affine Dirac equation in section 4.5, another is as follows.

Let $\psi \triangleq f e^{is_L}$, where it is fine to take either $s_L \triangleq s(L)$ or $s_L \triangleq \mathfrak{s}(L)$ from Definition 3.6.2. According to Eq.(33), it is easy to obtain on L , that

$$[X, H] = \frac{d}{dx^0} X, \quad H\psi = \frac{d\psi}{dx^0}. \quad (34)$$

This is consistent with the conventional Heisenberg equation and Schrödinger equation (taking the natural units that $\hbar = 1$, $c = 1$)

$$[X, -iH] = \frac{\partial}{\partial t} X, \quad -iH\psi = \frac{\partial \psi}{\partial t} \quad (35)$$

and they have a coordinate correspondence

$$\frac{\partial}{\partial(ix^k)} \leftrightarrow \frac{\partial}{\partial x^k}, \quad \frac{\partial}{\partial t} \leftrightarrow \frac{d}{dx^0}.$$

Due to Discussion 4.3.1 we know that $\frac{\partial}{\partial t} \leftrightarrow \frac{d}{dx^0}$ originates from the difference that the evolution parameter is x^τ or x^0 . Due to Discussion 4.3.2 we know that the imaginary unit i originates from the difference between the regular coordinates x^1, x^2, x^3, x^τ and the Minkowski coordinates x^1, x^2, x^3, x^0 . That is to say, the regular coordinates satisfy

$$(dx^0)^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^\tau)^2,$$

and the Minkowski coordinates satisfy

$$(dx^\tau)^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 = (dx^0)^2 + (d(ix^1))^2 + (d(ix^2))^2 + (d(ix^3))^2.$$

This causes the appearance of the imaginary unit i in the correspondence

$$ix^k \leftrightarrow x^k.$$

So Eq.(34) and Eq.(35) have exactly the same essence, and their differences only come from different coordinate representations.

The differences between coordinate representations have nothing to do with the geometric essence and the physical essence. We notice that the value of a gradient direction is dependent on the intrinsic geometry, but independent of that the equations are real-valued or complex-valued. Therefore, it is unnecessary for us to confine to such algebraic forms as real-valued or complex-valued forms, but we should focus on such geometric essence as gradient direction.

The advantage of complex-valued form is that it is convenient for describing the coherent superposition of the propagator. However, this is independent of the above discussions, and we are going to discuss it in section 3.10.

3.10. Quantum evolution as a distribution of gradient directions

From Proposition 3.6.1 we see that, in affine connection representation, the classical on-shell evolution is described by gradient direction. Then, naturally, quantum evolution should be described by the distribution of gradient directions.

The distribution of gradient directions on a geometric manifold (M, g) is effected by the bending shape of (M, g) , in other words, the distribution of gradient directions can be used to reflect the shape of (M, g) . This is the way that the quantum theory in affine connection representation describes physical reality.

In order to know the full picture of physical reality, it is necessary to fully describe the shape of the geometric manifold. For a single observation:

(1) It is the reference-system, not a point, that is used to describe the physical reality, so the coordinate of an individual point is not enough to fully describe the location information about the physical reality.

(2) Through a single observation of momentum, we can only obtain information about an individual gradient direction, this cannot reflect the full picture of the shape of the geometric manifold.

Quantum evolution provides us with a guarantee that we can obtain the distribution of gradient directions through multiple observations, so that we can describe the full picture of the shape of the geometric manifold.

Next, we are going to carry out strict mathematical descriptions for the quantum evolution in affine connection representation.

Definition 3.10.1. Let ρ be a geometric property on M , such as a charge of f . Then $H \triangleq \nabla \rho$ is a gradient direction field of ρ on (M, g) .

Let $\mathfrak{T}$ be the totality of all flat transformations $L_{[k]}$ defined in section 2.3. $\forall T \in \mathfrak{T}$, the flat transformation $T : f \mapsto T f$ induces a transformation $T^* : \rho \mapsto T^* \rho$. Denote

$$|\rho| \triangleq \{\rho_T \triangleq T^* \rho | T \in \mathfrak{T}\}, \quad |H| \triangleq \{H_T \triangleq \nabla \rho_T | T \in \mathfrak{T}\}.$$

$\forall a \in M$, the restriction of $|H|$ at a are denoted by $|H(a)| \triangleq \{H_T(a) | T \in \mathfrak{T}\}$.

We say $|H|$ is the **total distribution** of the gradient direction field H .

$\forall T \in \mathfrak{T}$, we say H_T is a **position distribution of gradient directions in momentum representation**.

$\forall a \in M$, we say $|H(a)|$ is a **momentum distribution of gradient directions in position representation**.

Remark 3.10.1. When T is fixed, H_T can reflect the shape of (M, g) . When a is fixed, $|H(a)|$ can reflect the shape of (M, g) .

However, when T and a are both fixed, $H_T(a)$ is a fixed individual gradient direction, which cannot reflect the shape of (M, g) . In other words, if the momentum p_T and the position x_a of ρ are both definitely observed, the physical reality g would be unknowable, therefore this is unacceptable. This is the embodiment of quantum uncertainty in affine connection representation.

Definition 3.10.2. Let φ_H be the one-parameter group of diffeomorphisms corresponding to H . The parameter of φ_H is x^0 . $\forall a \in M$, according to Definition 3.2.1, let $\varphi_{H,a}$ be the evolution path through a , such that $\varphi_{H,a}(0) = a$. $\forall t \in \mathbb{R}$, denote

$$\varphi_{|H|,a} \triangleq \{\varphi_{X,a} | X \in |H|\}, \quad \varphi_{|H|,a}(t) \triangleq \{\varphi_{X,a}(t) | X \in |H|\}.$$

$\forall \Omega \subseteq \mathfrak{T}$, we also denote $|H_\Omega| \triangleq \{H_T | T \in \Omega\} \subseteq |H|$ and

$$\varphi_{|H_\Omega|,a} \triangleq \{\varphi_{X,a} | X \in |H_\Omega|\} \subseteq \varphi_{|H|,a}, \quad \varphi_{|H_\Omega|,a}(t) \triangleq \{\varphi_{H,a}(t) | X \in |H_\Omega|\} \subseteq \varphi_{|H|,a}(t).$$

$\forall a \in M$, the restriction of $|H_\Omega|$ at a are denoted by $|H_\Omega(a)| \triangleq \{H_T(a) | T \in \Omega\}$.

Remark 3.10.2. At the beginning $t = 0$, intuitively, the gradient directions $|H(a)|$ of $|\rho|$ start from a and point to all directions around a uniformly. If (M, g) is not flat, when evolving to a certain time $t > 0$, the distribution of gradient directions on $\varphi_{|H|,a}(t)$ is no longer as uniform as beginning. The following definition precisely characterizes this kind of ununiformity.

Definition 3.10.3. Let the transformation $L_{g^{-1}}$ act on g , then we obtain the trivial $e \triangleq L_{g^{-1}}(g)$. Now (M, g) is sent to a flat (M, e) , and the gradient direction field $|H|$ of $|\rho|$ on (M, g) is sent to a gradient direction field $|O|$ of $|\rho|$ on (M, e) . Correspondingly, $\forall t \in \mathbb{R}$, $\varphi_{|H|,a}(t)$ is sent to $\varphi_{|O|,a}(t)$. In a word, $L_{g^{-1}}$ induces the following two maps:

$$g_*^{-1} : |H| \rightarrow |O|, \quad g_{**}^{-1} : \varphi_{|H|,a} \rightarrow \varphi_{|O|,a}.$$

$\forall T \in \mathfrak{T}$, deonte $\mathfrak{N} \triangleq \{N \in \mathfrak{T} \mid \det N = \det T\}$. Due to $\mathfrak{T} \cong GL(\mathfrak{D}, \mathbb{R})$, let $\mathfrak{U}$ be a neighborhood of T , with respect to the topology of $GL(\mathfrak{D}, \mathbb{R})$.

Take $\Omega = \mathfrak{N} \cap \mathfrak{U}$, then

$$|O_\Omega| = g_*^{-1}(|H_\Omega|), \quad \varphi_{|O_\Omega|,a} = g_{**}^{-1}(\varphi_{|H_\Omega|,a}).$$

Let μ be a Borel measure on the manifold M . $\forall t \in \mathbb{R}$, we know

$$\varphi_{|H_{\mathfrak{N}}|,a}(t) \simeq \varphi_{|O_{\mathfrak{N}}|,a}(t) \simeq \mathbb{S}^{\mathfrak{D}-1}.$$

Thus, $\varphi_{|H_{\Omega}|,a}(t) \subseteq \varphi_{|H_{\mathfrak{N}}|,a}(t)$ and $\varphi_{|O_{\Omega}|,a}(t) \subseteq \varphi_{|O_{\mathfrak{N}}|,a}(t)$ are Borel sets, so they are measurable. Denote

$$\mu_a(\varphi_{|H_{\Omega}|,a}(t)) \triangleq \mu(g_{**}^{-1}(\varphi_{|H_{\Omega}|,a}(t))) = \mu(\varphi_{|O_{\Omega}|,a}(t)).$$

When $\mathfrak{U} \rightarrow T$, we have $\Omega \rightarrow T$, $|H_{\Omega}| \rightarrow H_T$, $|H_{\Omega}(a)| \rightarrow H_T(a)$, and $\varphi_{|H_{\Omega}|,a}(t) \rightarrow b \triangleq \varphi_{H_T,a}(t)$.

For the sake of simplicity, denote $L \triangleq \varphi_{H_T,a}$. Thus, we have $a = L(0)$, $b = L(t)$, and denote $p_a \triangleq [L_a] = H_T(a)$, $p_b \triangleq [L_b] = H_T(b)$.

Because μ_a is absolutely continuous with respect to μ , Radon-Nikodym theorem[78] ensures the existence of the following limit. The Radon-Nikodym derivative

$$W_L(b, a) \triangleq \frac{d\mu_a}{d\mu_b} \triangleq \lim_{\mathfrak{U} \rightarrow T} \frac{\mu_a(\varphi_{|H_{\Omega}|,a}(t))}{\mu(\varphi_{|H_{\Omega}|,a}(t))} = \lim_{\mathfrak{U} \rightarrow T} \frac{\mu(g_{**}^{-1}(\varphi_{|H_{\Omega}|,a}(t)))}{\mu(\varphi_{|H_{\Omega}|,a}(t))} = \lim_{\mathfrak{U} \rightarrow T} \frac{\mu(\varphi_{|O_{\Omega}|,a}(t))}{\mu(\varphi_{|H_{\Omega}|,a}(t))} \quad (36)$$

is called the **distribution density of $|H|$ along L in position representation**.

On a neighborhood U of a , $\forall T \in \mathfrak{T}$, denote the normal section of $H_T(a)$ by $N_{H_T,a}$, that is

$$N_{H_T,a} \triangleq \{n \in U \mid H_T(a) \cdot (n - a) = 0\}, \quad N_{H_T,a}(t) \triangleq \{\varphi_{H_T,x}(t) \mid x \in N_{H_T,a}\}.$$

Thus, $N_{H_T,a} = N_{H_T,a}(0)$ and $N_{H_T,b} \triangleq N_{H_T,a}(t)$. If $U \rightarrow a$, we have $N_{H_T,a} \rightarrow a$ and $N_{H_T,a}(t) \rightarrow b \triangleq \varphi_{H_T,a}(t)$. The Radon-Nikodym derivative

$$Z_L(b, a) \triangleq \frac{d\mu(a)}{d\mu(b)} \triangleq \lim_{U \rightarrow a} \frac{\mu(N_{H_T,a})}{\mu(N_{H_T,b})} = \lim_{U \rightarrow a} \frac{\mu(N_{H_T,a})}{\mu(N_{H_T,a}(t))} \quad (37)$$

is called the **distribution density of $|H|$ along L in momentum representation**.

In a word, $W_L(b, a)$ and $Z_L(p_b, p_a)$ describe the density of the gradient lines that are adjacent to b in two different ways. They have the following evident property.

Proposition 3.10.1. Let L be a gradient line. $\forall a, b, c \in L$ such that $L(x_a^0) = a$, $L(x_b^0) = b$, $L(x_c^0) = c$ and $x_b^0 > x_c^0 > x_a^0$, then

$$W_L(b, a) = W_L(b, c)W_L(c, a), \quad Z_L(b, a) = Z_L(b, c)Z_L(c, a).$$

Proof. Directly compute

$$\begin{aligned} W_L(b, c)W_L(c, a) &= \frac{d\mu_a}{d\mu_c} \frac{d\mu_c}{d\mu_b} = \frac{d\mu_a}{d\mu_b} = W_L(b, a), \\ Z_L(b, c)Z_L(c, a) &= \frac{d\mu(a)}{d\mu(c)} \frac{d\mu(c)}{d\mu(b)} = \frac{d\mu(a)}{d\mu(b)} = Z_L(b, a). \quad \square \end{aligned}$$

Definition 3.10.4. If L is a gradient line of some $\rho' \in |\rho|$, we also say L is a gradient line of $|\rho|$.

Remark 3.10.3. For any a and b , it anyway makes sense to discuss the gradient line of $|\rho|$ from a to b . It is because even if the gradient line of ρ starting from a does not pass through b , it just only

needs to carry out a certain flat transformation T defined in section 2.3 to obtain a $\rho' \triangleq T_*\rho$, thus the gradient line of ρ' starting from a can just exactly pass through b . Due to $\rho, \rho' \in |\rho|$, we do not distinguish them, it is just fine to uniformly use $|\rho|$. Intuitively speaking, when $|\rho|$ takes two different initial momentums, $|\rho|$ presents as ρ and ρ' , respectively.

Discussion 3.10.1. With the above preparations, we obtain a new way to describe the construction of the propagator strictly.

For any path L that starts at a and ends at b , we denote $||L|| \triangleq \int_L dx^0$ concisely. Let $\mathcal{P}(b, a)$ be the totality of all the paths from a to b . Denote

$$\mathcal{P}(b, x_b^0; a, x_a^0) \triangleq \{L \mid L \in \mathcal{P}(b, a), ||L|| = x_b^0 - x_a^0\}.$$

$\forall L \in \mathcal{P}(b, x_b^0; a, x_a^0)$, we can let $L(x_a^0) = a$ and $L(x_b^0) = b$ without loss of generality. Thus, $\mathcal{P}(b, x_b^0; a, x_a^0)$ is the totality of all the paths from $L(x_a^0) = a$ to $L(x_b^0) = b$.

Abstractly, the propagator is defined as the Green function of the evolution equation. Concretely, the propagator still needs a constructive definition. One method is the Feynman path integral[79]

$$K(b, x_b^0; a, x_a^0) \triangleq \int_{\mathcal{P}(b, x_b^0; a, x_a^0)} e^{is} dL. \quad (38)$$

However, there are so many redundant paths in $\mathcal{P}(b, x_b^0; a, x_a^0)$ that: (i) it is difficult to generally define a measure dL on $\mathcal{P}(b, x_b^0; a, x_a^0)$, (ii) it may cause unnecessary infinities when carrying out some calculations.

In order to solve this problem, we try to reduce the scope of summation from $\mathcal{P}(b, x_b^0; a, x_a^0)$ to $H(b, x_b^0; a, x_a^0)$, where $H(b, x_b^0; a, x_a^0)$ is the totality of all the gradient lines of $|\rho|$ from $L(x_a^0) = a$ to $L(x_b^0) = b$. Thus, the (38) is turned into

$$K(b, x_b^0; a, x_a^0) = \int_{H(b, x_b^0; a, x_a^0)} \Psi(L) e^{is} dL.$$

We notice that as long as we take the probability amplitude $\Psi(L)$ of the gradient line L such that $[\Psi(L)]^2 = W_L(b, a)$ in position representation, or take $[\Psi(L)]^2 = Z_L(b, a)$ in momentum representation, it can exactly be consistent with the Copenhagen interpretation. This provides the following new constructive definition for the propagator.

Definition 3.10.5. Suppose $|\rho|$ is defined as Definition 3.10.1, and denote $H \triangleq \nabla \rho$.

Let $\mathcal{L}(b, a)$ be the totality of all the gradient lines of $|\rho|$ from a to b . Denote

$$H(b, x_b^0; a, x_a^0) \triangleq \{L \mid L \in \mathcal{L}(b, a), ||L|| = x_b^0 - x_a^0\}.$$

Let $\mathcal{L}(p_b, p_a)$ be the totality of all the gradient lines of $|\rho|$, whose starting-direction is p_a and ending-direction is p_b . Denote

$$H(p_b, x_b^0; p_a, x_a^0) \triangleq \{L \mid L \in \mathcal{L}(p_b, p_a), ||L|| = x_b^0 - x_a^0\}.$$

Let dL be a Borel measure on $H(b, x_b^0; a, x_a^0)$. In consideration of Remark 4.5.1, we let s be the affine action $s(L)$ in Definition 3.6.2. We say the intrinsic geometric property

$$K(b, x_b^0; a, x_a^0) \triangleq \int_{H(b, x_b^0; a, x_a^0)} \sqrt{W_L(b, a)} e^{is} dL \quad (39)$$

is the **propagator** of $|\rho|$ from (a, x_a^0) to (b, x_b^0) in **position representation**. If we let dL be a Borel measure on $H(p_b, x_b^0; p_a, x_a^0)$, then we say

$$\mathcal{K}(p_b, x_b^0; p_a, x_a^0) \triangleq \int_{H(p_b, x_b^0; p_a, x_a^0)} \sqrt{Z_L(b, a)} e^{is} dL \quad (40)$$

is the **propagator** of $|\rho|$ from (p_a, x_a^0) to (p_b, x_b^0) in **momentum representation**.

Discussion 3.10.2. Now (39) and (40) are strictly defined, but the Feynman path integral (38) has not been possessed of a strict mathematical definition until now. This makes it impossible at present to obtain (e.g. in position representation) a strict mathematical proof of

$$\int_{H(b, x_b^0; a, x_a^0)} \sqrt{W_L(b, a)} e^{is} dL = \int_{\mathcal{P}(b, x_b^0; a, x_a^0)} e^{is} dL.$$

We notice that the distribution densities $W_L(b, a)$ and $Z_L(b, a)$ of gradient directions establish an association between probability interpretation and geometric interpretation of quantum evolution. Therefore, we can base on probability interpretation to intuitively consider both sides of "=" as the same thing.

Discussion 3.10.3. The quantization methods of QFT are successful, and they are also applicable in affine connection representation for orthogonal f and g , but in this paper we do not discuss them. We try to propose some more ideas for general f and g to understand the quantization of field in affine connection representation.

(1) If we take

$$\mathfrak{s} = \int_L D\rho = \int_L p_Q dx^Q = \int_L E_0 dx^0$$

according to Definition 3.6.2, where D is the simple connection of (M, g) , then consider the distribution of $H \triangleq \nabla\rho$, we know that

$$K(b, x_b^0; a, x_a^0) \triangleq \int_{\nabla\rho(b, x_b^0; a, x_a^0)} \sqrt{W_L(b, a)} e^{is} dL, \quad \mathcal{K}(p_b, x_b^0; p_a, x_a^0) \triangleq \int_{\nabla\rho(p_b, x_b^0; p_a, x_a^0)} \sqrt{Z_L(b, a)} e^{is} dL$$

describe the quantization of energy-momentum. Every gradient line in $\nabla\rho(b, x_b^0; a, x_a^0)$ corresponds to a set of eigenvalues of energy and momentum.

(2) If we take

$$\mathfrak{s} = \int_L Dt = \int_L K_{NPQ}^M :^P dx^Q = \int_L \rho_{N0}^M dx^0,$$

according to section 3.5, where D is the simple connection of (M, f) , then consider the distribution of $H \triangleq \nabla t$, we know that

$$K(b, x_b^0; a, x_a^0) \triangleq \int_{\nabla t(b, x_b^0; a, x_a^0)} \sqrt{W_L(b, a)} e^{is} dL, \quad \mathcal{K}(p_b, x_b^0; p_a, x_a^0) \triangleq \int_{\nabla t(p_b, x_b^0; p_a, x_a^0)} \sqrt{Z_L(b, a)} e^{is} dL$$

describe the quantizations of charge and current. It should be emphasized that this is not the quantization of the energy-momentum of the field, but the quantization of the field itself, which presents as quantized (e.g. discrete) charges and currents.

4. Affine connection representation of gauge fields in classical spacetime

The new framework established in section 3 is discussed in the $\mathfrak{D}$ -dimensional general coordinate x^M , which is more general than the $(1+3)$ -dimensional conventional Minkowski coordinate x^μ . This is different from the extra dimensions in Kaluza-Klein theory and string theory.

(1) The evolution parameter of the $\mathfrak{D}$ -dimensional general coordinate x^M ($M = 1, 2, \dots, \mathfrak{D}$) is x^0 . That is to say, $(dx^0)^2 = \sum_{M=1}^{\mathfrak{D}} (dx^M)^2 = \sum_{i=1}^3 (dx^i)^2 + (dx^\tau)^2$, and the parameter equation of an evolution path L is represented as $x^M = x^M(x^0)$.

(2) The evolution parameter of the $(1+3)$ -dimensional Minkowski coordinate x^μ ($\mu = 0, 1, 2, 3$) is x^τ . That is to say, $(dx^\tau)^2 = (dx^0)^2 - \sum_{i=1}^3 (dx^i)^2$, and the parameter equation of L is represented as $x^\mu = x^\mu(x^\tau)$.

(3) x^μ is strictly constructed from x^M in section 4.2. So x^M is more fundamental than x^μ .

Definition 4.1. Suppose $M = P \times N$, $\mathfrak{D} \triangleq \dim M$ and $r \triangleq \dim P = 3$. According to Definition 3.1.1, we have a submanifold of external space P and a submanifold of internal space N . P inherits coordinates $\{\xi^s\}\{x^i\}$ from M , and N inherits coordinates $\{\xi^a\}\{x^m\}$ from M . The values of indices are specified as follows.

- (1) Complete indices are $A, B, C, D = 1, 2, \dots, \mathfrak{D}$; $M, N, P, Q = 1, 2, \dots, \mathfrak{D}$.
- (2) Indices of ξ are $s, t, u, v = 1, 2, \dots, r$; $a, b, c, d = r+1, r+2, \dots, \mathfrak{D}$.
- (3) Indices of x are $i, j, k, l = 1, 2, \dots, r$; $m, n, p, q = r+1, r+2, \dots, \mathfrak{D}$.
- (4) Regular and Minkowski indices of ξ are $S, T, U, V = 1, 2, \dots, r, \tau$; $\alpha, \beta, \gamma, \delta = 0, 1, 2, \dots, r$.
- (5) Regular and Minkowski indices of x are $I, J, K, L = 1, 2, \dots, r, \tau$; $\mu, \nu, \rho, \sigma = 0, 1, 2, \dots, r$.

4.1. Existence and uniqueness of classical spacetime submanifold

Definition 4.1.1. Let there be a smooth tangent vector field X on (M, f) . If $\forall p \in M$, $X(p) = b^A \frac{\partial}{\partial \xi^A} \Big|_p = c^M \frac{\partial}{\partial x^M} \Big|_p$ satisfies that b^a are not all zero and c^m are not all zero, then we say X is **internal-directed**. For any evolution path $L \triangleq \varphi_{X,p}$, we also say L is **internal-directed**.

Proposition 4.1.1. Suppose $M = P \times N$ and X is a smooth tangent vector field on M . Fix a point $o \in M$. If X is internal-directed, then:

(1) There exist a unique $(r+1)$ -dimensional regular submanifold $\gamma : \tilde{M} \rightarrow M$, $p \mapsto p$ and a unique smooth tangent vector field $\tilde{X}$ on $\tilde{M}$ such that:

- (i) $P \times \{o\}$ is a closed submanifold of $\tilde{M}$.
- (ii) The tangent map $\gamma_* : T(\tilde{M}) \rightarrow T(M)$ satisfies that $\forall q \in \tilde{M}$, $\gamma_* : \tilde{X}(q) \mapsto X(q)$.

Such an $\tilde{M}$ is called a **classical spacetime submanifold** determined by X through o .

(2) Let

$$\varphi_X : M \times \mathbb{R} \rightarrow M$$

be the one-parameter group of diffeomorphisms corresponding to X , and

$$\varphi_{\tilde{X}} : \tilde{M} \times \mathbb{R} \rightarrow \tilde{M}$$

the one-parameter group of diffeomorphisms corresponding to $\tilde{X}$. Thus, we have

$$\varphi_{\tilde{X}} = \varphi_X|_{\tilde{M} \times \mathbb{R}}.$$

Proof. We take two steps to prove.

Step 1. To construct $\tilde{M}$. We define a closed submanifold $P \times \{o\}$ on M through o via the parameter equation $x^m = x_o^m$. Then let $\varphi_X : M \times \mathbb{R} \rightarrow M$ be the one-parameter group of diffeomorphisms corresponding to X . Restrict φ_X to $P \times \{o\}$ and we obtain

$$\varphi_X|_{P \times \{o\}} : P \times \{o\} \times \{t\} \mapsto P' \times \{o'\},$$

where points o and o' are on the same orbit $L_o \triangleq \varphi_{X,o} \cdot P \times \{o\}$ and $P' \times \{o'\}$ are both homeomorphic to P . In this sense we do not distinguish P and P' , we have

$$\varphi_X|_{P \times \{o\} \times \mathbb{R}} : P \times \{o\} \times \mathbb{R} \rightarrow P \times L_o.$$

Then consider all of such $\{o\}$ on the entire orbit L_o , and we obtain a map

$$\varphi_X|_{P \times L_o \times \mathbb{R}} : P \times L_o \times \mathbb{R} \rightarrow P \times L_o.$$

Denote $\tilde{M} \triangleq P \times L_o$. So we know that $\tilde{M}$ is determined by X and the fixed point o , hence it is unique. Thus,

$$\varphi_X|_{\tilde{M} \times \mathbb{R}} : \tilde{M} \times \mathbb{R} \rightarrow \tilde{M}$$

constitutes a one-parameter group of diffeomorphisms on $\tilde{M}$.

Step 2. To construct $\gamma : \tilde{M} \rightarrow M$ and $\tilde{X}$. Because X is internal-directed, the restriction of X to $L_o \triangleq \varphi_{X,o} : \mathbb{R} \rightarrow M$ is non-vanishing everywhere and L_o is an injection. The image set of L_o can also be denoted by L_o . $\forall t \in \mathbb{R}, q \triangleq \varphi_{X,o}(t) \in L_o$, we can define a closed submanifold N_q on M through q via the parameter equation $x^i = x_q^i$, and N_q is homeomorphic to N . Due to the one-to-one correspondence between q and N_q , $L_o \rightarrow N$ is a regular imbedding. Furthermore:

(1) $\gamma : P \times L_o \rightarrow P \times N$ is a regular imbedding, that is, $\gamma : \tilde{M} \rightarrow M$. Hence the tangent map $\gamma_* : T(\tilde{M}) \rightarrow T(M)$ is an injection. Therefore, the smooth tangent vector $\tilde{X}$ which satisfies $\forall q \in \tilde{M}, \gamma_* : \tilde{X}(q) \mapsto X(q)$ is uniquely defined by X via γ_*^{-1} .

(2) We notice that $o \in \tilde{M}$, hence $L_o \triangleq \varphi_{X,o}$ is an orbit of $\varphi_X|_{\tilde{M} \times \mathbb{R}}$. In consideration of that L_o uniquely determines γ , and γ uniquely determines γ_* , and γ_*^{-1} uniquely determines $\tilde{X}$, so finally $\tilde{X}$ is uniquely determined by $\varphi_X|_{\tilde{M} \times \mathbb{R}}$. Thus, we have $\varphi_{\tilde{X}} = \varphi_X|_{\tilde{M} \times \mathbb{R}}$. $\square$

Remark 4.1.1. The above proposition constructs a $(3+1)$ -dimensional submanifold of M , that is $\tilde{M}$. We know $\tilde{M}$ inherits a part of intrinsic geometric properties of M , so $\tilde{M}$ describes the physical properties in the classical $(3+1)$ -dimensional spacetime. It should be noticed that:

- (1) $\tilde{M}$ is not independent of M , but a regular submanifold of M .
- (2) Not all of the intrinsic geometric properties of M are inherited by $\tilde{M}$. The physical properties that are reflected by $\tilde{M}$ are not complete.
- (3) The correspondence between $\tilde{X}$ and the restriction of X to $\tilde{M}$ is one-to-one. For convenience, next we are not going to distinguish the notations X and $\tilde{X}$ on $\tilde{M}$, but uniformly denote them by X .
- (4) An arbitrary path $\tilde{L} : T \rightarrow \tilde{M}, t \mapsto p$ on $\tilde{M}$ uniquely corresponds to a path $L \triangleq \gamma \circ \tilde{L} : T \rightarrow M, t \mapsto p$ on M . Evidently the image sets of L and $\tilde{L}$ are the same, that is, $L(T) = \tilde{L}(T)$. For convenience, later we are not going to distinguish the notations L and $\tilde{L}$ on $\tilde{M}$, but uniformly denote them by L .

4.2. Gravitational field on classical spacetime submanifold

Proposition 4.2.1. Let there be a geometric manifold (M, f) and its classical spacetime submanifold $\tilde{M}$. And let $L \triangleq \varphi_{\tilde{X},a}$ be an evolution path on $\tilde{M}$. Suppose $p \in L$ and U is a coordinate neighborhood of p . According to Definition 3.2.2, suppose the $f(p)$ on U and the $f_L(p)$ on $U_L \triangleq U \cap L$ satisfy that

$$f(p), f_L(p) : \xi^A = \xi^A(x^M) = \xi^A(x^0), \quad \xi^0 = \xi^0(x^0). \quad (41)$$

Thus, according to the notations in Definition 4.1, it is true that:

(1) There exists a unique local reference-system $\tilde{f}(p)$ on $\tilde{U} \triangleq U \cap \tilde{M}$ such that

$$\tilde{f}(p), \tilde{f}_L(p) : \xi^U = \xi^U(x^K) = \xi^U(x^0), \quad \xi^0 = \xi^0(x^0) \quad (42)$$

$$\text{and } (dx^\tau)^2 = \sum_{m=r+1}^{\mathfrak{D}} (dx^m)^2, \quad (d\xi^\tau)^2 = \sum_{a=r+1}^{\mathfrak{D}} (d\xi^a)^2.$$

(2) If L is internal-directed, then the above coordinate frames $(\tilde{U}, \xi^U)$ and $(\tilde{U}, x^K)$ of $\tilde{f}(p)$ uniquely determine the coordinate frames $(\tilde{U}, \tilde{\xi}^\alpha)$ and $(\tilde{U}, \tilde{x}^\mu)$ such that

$$\tilde{f}(p), \tilde{f}_L(p) : \tilde{\xi}^\alpha = \tilde{\xi}^\alpha(\tilde{x}^\mu) = \tilde{\xi}^\alpha(\tilde{x}^\tau), \quad \tilde{\xi}^\tau = \tilde{\xi}^\tau(\tilde{x}^\tau) \quad (43)$$

and the coordinates satisfy

$$\tilde{\xi}^s = \xi^s, \quad \tilde{\xi}^\tau = \xi^\tau, \quad \tilde{\xi}^0 = \xi^0, \quad \tilde{x}^i = x^i, \quad \tilde{x}^\tau = x^\tau, \quad \tilde{x}^0 = x^0.$$

That is to say, $\tilde{f}(p), \tilde{f}_L(p)$ have two coordinate representations: (42) and (43).

Proof. (1) According to the proof of the previous proposition, L is a regular submanifold of N . Let the metrics on N be

$$(dx^\tau)^2 = \sum_{m=r+1}^{\mathfrak{D}} (dx^m)^2, \quad (d\xi^\tau)^2 = \sum_{a=r+1}^{\mathfrak{D}} (d\xi^a)^2.$$

Review Definition 3.2.2 and we know the regular imbedding $\pi : L \rightarrow N$, $q \mapsto q$ induces the parameter equations $x^m = x_\tau^m(x^\tau)$ and $\xi^a = \xi_\tau^a(\xi^\tau)$ of L . Then substitute them into $f(p), f_L(p)$ and we obtain

$$\begin{aligned} \xi^A = \xi^A(x^M) = \xi^A(x^0) &\Leftrightarrow \begin{cases} \xi^u = \xi^u(x^k, x_\tau^m(x^\tau)) = \xi_L^u(x^0) \\ \xi_\tau^a(\xi^\tau) = \xi^a(x^k, x_\tau^m(x^\tau)) = \xi_L^a(x^0) \end{cases} \\ &\Leftrightarrow \begin{cases} \xi^u = \xi^u(x^k, x_\tau^m(x^\tau)) = \xi_L^u(x^0) \\ \xi^\tau = (\xi_\tau^a)^{-1} \circ \xi^a(x^k, x_\tau^m(x^\tau)) = (\xi_\tau^a)^{-1} \circ \xi_L^a(x^0) \end{cases} \\ &\Leftrightarrow \begin{cases} \xi^u = \xi^u(x^k, x^\tau) = \xi_L^u(x^0) \\ \xi^\tau = \xi^\tau(x^k, x^\tau) = \xi_L^\tau(x^0) \end{cases} \\ &\Leftrightarrow \xi^U = \xi^U(x^K) = \xi_L^U(x^0), \quad \text{abbreviated to } \xi^U = \xi^U(x^K) = \xi^U(x^0). \end{aligned}$$

(2) As same as the above, we also obtain $x^K = x^K(\xi^U) = x^K(\xi^0)$. The relation between two parameters x^0 and x^τ of L can be expressed as $x^\tau = x_L^\tau(\xi^0(x^0))$, and the relation between two

parameters ξ^0 and ξ^τ can be expressed as $\xi^\tau = \xi_L^\tau(x^0(\xi^0))$. Substitute them into $\tilde{f}(p)$, $\tilde{f}_L(p)$, then

$$\begin{aligned} \xi^U = \xi^U(x^K) = \xi^U(x^0) &\Leftrightarrow \begin{cases} \xi^u = \xi^u(x^k, x_L^\tau(\xi^0(x^0))) = \xi_L^u(x^0((x_L^\tau)^{-1}(x^\tau)) \\ \xi_L^\tau(x^0(\xi^0)) = \xi^\tau(x^k, x_L^\tau(\xi^0(x^0))) = \xi_L^\tau(x^0((x_L^\tau)^{-1}(x^\tau)) \\ \xi^0(\xi_L^{\tau^{-1}}(\xi^\tau)) = (x_L^\tau)^{-1}(x^\tau) \end{cases} \\ &\Leftrightarrow \begin{cases} \xi^u = \xi^u(x^k, x_L^\tau(\xi^0(x^0))) = \xi_L^u(x^0((x_L^\tau)^{-1}(x^\tau)) \\ \xi^0 = \xi^0((\xi_L^\tau)^{-1}(\xi^\tau(x^k, x_L^\tau(\xi^0(x^0)))) = (x_L^\tau)^{-1}(x^\tau), \\ \xi^\tau = \xi_L^\tau(x^0((x_L^\tau)^{-1}(x^\tau))) \end{cases} \end{aligned} \quad (44)$$

which can be abbreviated to

$$\begin{cases} \xi^u = \tilde{\xi}^u(x^k, x^0) = \tilde{\xi}_L^u(x^\tau) \\ \xi^0 = \tilde{\xi}^0(x^k, x^0) = \tilde{\xi}_L^0(x^\tau) \\ \xi^\tau = \tilde{\xi}^\tau(x^\tau) \end{cases}$$

Denote

$$\tilde{\xi}^s \triangleq \xi^s, \quad \tilde{\xi}^\tau \triangleq \xi^\tau, \quad \tilde{\xi}^0 \triangleq \xi^0, \quad \tilde{x}^i \triangleq x^i, \quad \tilde{x}^\tau \triangleq x^\tau, \quad \tilde{x}^0 \triangleq x^0,$$

hence we have $\tilde{\xi}^\alpha = \tilde{\xi}^\alpha(\tilde{x}^\mu) = \tilde{\xi}_L^\alpha(\tilde{x}^\tau)$, $\tilde{\xi}^\tau = \tilde{\xi}^\tau(\tilde{x}^\tau)$, abbreviated to

$$\tilde{\xi}^\alpha = \tilde{\xi}^\alpha(\tilde{x}^\mu) = \tilde{\xi}^\alpha(\tilde{x}^\tau), \quad \tilde{\xi}^\tau = \tilde{\xi}^\tau(\tilde{x}^\tau).$$

□

Definition 4.2.1. Due to the above local reference-system $\tilde{f}(p)$, we thereby have a reference-system

$$\tilde{f} : \tilde{M} \rightarrow REF_{\tilde{M}}, \quad p \mapsto \tilde{f}(p) \in REF_p.$$

In fact, $\tilde{f}(p)$ is a reference system in conventional sense. We say $\tilde{f}$ is a **classical spacetime reference-system** on $\tilde{M}$, and $(\tilde{M}, \tilde{f})$ is a **gravitational manifold**.

$(\tilde{U}, \xi^U)$ and $(\tilde{U}, x^K)$ are called **regular coordinate frames**, meanwhile $(\tilde{U}, \tilde{\xi}^\alpha)$ and $(\tilde{U}, \tilde{x}^\mu)$ are called **Minkowski coordinate frames**.

Remark 4.2.1. $\tilde{f}$ is uniquely determined by f , and $\tilde{f}$ encapsulates the internal space of f . Although $(\tilde{M}, \tilde{f})$ can reflect the intrinsic geometric properties of external space of (M, f) , it cannot totally reflect all the intrinsic geometric properties of internal space of (M, f) . There is a further illustration in section 4.6.

Definition 4.2.2. Suppose we have a geometric manifold $(\tilde{M}, \tilde{g})$. $F_{\tilde{g}}$ is a transformation induced by $\tilde{g}$.

- (1) According to Discussion 4.3.2, it is easy to know that $\tilde{\delta}_{\alpha\beta}\tilde{B}_\mu^\alpha\tilde{B}_\nu^\beta = \tilde{\varepsilon}_{\mu\nu}$ if and only if $d\tilde{\zeta}^\tau = d\tilde{x}^\tau$. Thus, we denote $d\tilde{\zeta}^\tau$ and $d\tilde{x}^\tau$ uniformly by $d\tau$, and say $\tilde{g}$ is **orthogonal**.
- (2) If the semi-metrics $\tilde{B}_\mu^\alpha$ and $\tilde{C}_\alpha^\mu$ of $\tilde{g}$ are constants on $\tilde{M}$, then we say $\tilde{g}$ is **flat**.
- (3) If $\tilde{g}$ is both orthogonal and flat, then we say $\tilde{g}$ is an **inertial-system**, $F_{\tilde{g}}$ is a **Lorentz transformation**, and $(\tilde{M}, \tilde{g})$ is the **Minkowski spacetime**.

Proposition 4.2.2. Let L be a path on $\tilde{M}$, and $\tilde{U}$ a coordinate neighborhood. Denote $\tilde{U}_L \triangleq \tilde{U} \cap L$, and $c \triangleq |dx^i/dx^0|$. Suppose L satisfies that $\forall q \in \tilde{U}_L$, the tangent vector $[L_q] \in T_q(M)$ is not internal-directed. Then, it is true that:

- (i) we have $c = 1$ on path $\tilde{U}_L$.
- (ii) $c = 1$ remains unchanged under orthogonal transformations.
- (iii) $c = 1$ remains unchanged under Lorentz transformations.

Proof. (i) In $(\tilde{U}, x^K)$, $\tilde{U}_L$ can be described by equations $x^i = x^i(x^0)$ and $x^\tau = \text{const}$ with respect to the parameter x^0 . We notice that $x^\tau = \text{const}$, so there does not exist an equation of $\tilde{U}_L$ with respect to the parameter $\tilde{x}^\tau$ in Minkowski coordinate frame $(\tilde{U}, \tilde{x}^\mu)$. Therefore, we always have $d\tilde{x}^\tau = dx^\tau = 0$ on $\tilde{U}_L$. Thus,

$$c = |dx^i/dx^0| = \left| \pm dx^i / \sqrt{(dx^i)^2 + (dx^\tau)^2} \right| = |\pm dx^i/dx^i| = 1.$$

- (ii) According to Definition 4.2.2, an orthogonal transformation satisfies $d\tilde{\zeta}^\tau = d\tilde{x}^\tau = 0$, hence

$$c' = |d\zeta^s/d\zeta^0| = |\pm d\zeta^s/d\zeta^s| = 1 = c.$$

- (iii) It is naturally true for the Lorentz transformation as an orthogonal one. This conclusion reflects the principle of constancy of light velocity. $\square$

4.3. Mathematical conversion between two coordinate representations

Discussion 4.3.1. According to Definition 3.2.2, suppose the semi-metrics of $(\tilde{M}, \tilde{f})$ with respect to the regular coordinate x^K are B_I^S and C_I^I , such that

$$\begin{aligned} d\xi^S &= B_I^S dx^I \simeq B_0^S dx^0, & dx^I &= C_S^I d\xi^S \simeq C_0^I d\xi^0, \\ C_0^I \frac{\partial}{\partial x^I} &\cong C_0^0 \frac{d}{dx^0} = \frac{d}{d\xi^0}, & B_0^S \frac{\partial}{\partial \xi^S} &\cong B_0^0 \frac{d}{d\xi^0} = \frac{d}{dx^0}. \end{aligned} \quad (45)$$

The semi-metrics of $(\tilde{M}, \tilde{f})$ with respect to the Minkowski coordinate $\tilde{x}^\mu$ are $\tilde{B}_\mu^\alpha$ and $\tilde{C}_\alpha^\mu$, such that

$$\begin{aligned} d\tilde{\xi}^\alpha &= \tilde{B}_\mu^\alpha d\tilde{x}^\mu \simeq \tilde{B}_\tau^\alpha d\tilde{x}^\tau, & d\tilde{x}^\mu &= \tilde{C}_\alpha^\mu d\tilde{\xi}^\alpha \simeq \tilde{C}_\tau^\mu d\tilde{\xi}^\tau, \\ \tilde{C}_\tau^\mu \frac{\partial}{\partial \tilde{x}^\mu} &\cong \tilde{C}_\tau^\tau \frac{d}{d\tilde{x}^\tau} = \frac{d}{d\tilde{\xi}^\tau}, & \tilde{B}_\tau^\alpha \frac{\partial}{\partial \tilde{\xi}^\alpha} &\cong \tilde{B}_\tau^\tau \frac{d}{d\tilde{\xi}^\tau} = \frac{d}{d\tilde{x}^\tau}. \end{aligned} \quad (46)$$

Applying the chain rule of differentiation, it is not hard to obtain the following relations between the regular semi-metrics and the Minkowski semi-metrics via the formula (44).

$$\begin{aligned} \tilde{B}_i^s &= -B_i^s, & \tilde{B}_\tau^s &= B_\tau^s, & \tilde{B}_0^s &= B_0^s, & \tilde{C}_s^i &= -C_s^i, & \tilde{C}_\tau^i &= C_\tau^i, & \tilde{C}_0^i &= C_0^i, \\ \tilde{B}_i^0 &= -\frac{B_i^\tau}{\delta_0^\tau}, & \tilde{B}_\tau^0 &= \frac{B_\tau^\tau}{\delta_0^\tau}, & \tilde{B}_0^0 &= \frac{B_0^\tau}{\delta_0^\tau}, & \tilde{C}_s^0 &= -\frac{C_s^\tau}{\varepsilon_0^\tau}, & \tilde{C}_\tau^0 &= \frac{C_\tau^\tau}{\varepsilon_0^\tau}, & \tilde{C}_0^0 &= \frac{C_0^\tau}{\varepsilon_0^\tau}, \\ \tilde{C}_\tau^i &= \frac{C_0^i}{\delta_0^\tau}, & \tilde{C}_\tau^\tau &= \frac{C_0^\tau}{\delta_0^\tau}, & \tilde{C}_\tau^0 &= \frac{C_0^0}{\delta_0^\tau}, & \tilde{B}_\tau^s &= \frac{B_0^s}{\varepsilon_0^\tau}, & \tilde{B}_\tau^\tau &= \frac{B_0^\tau}{\varepsilon_0^\tau}, & \tilde{B}_\tau^0 &= \frac{B_0^0}{\varepsilon_0^\tau}, \\ B_i^s &= -\tilde{B}_i^s, & B_0^s &= \tilde{B}_0^s, & B_\tau^s &= \tilde{B}_\tau^s, & C_s^i &= -\tilde{C}_s^i, & C_0^i &= \tilde{C}_0^i, & C_\tau^i &= \tilde{C}_\tau^i, \\ B_i^\tau &= -\frac{\tilde{B}_i^0}{\tilde{\delta}_0^\tau}, & B_0^\tau &= \frac{\tilde{B}_0^0}{\tilde{\delta}_0^\tau}, & B_\tau^\tau &= \frac{\tilde{B}_\tau^0}{\tilde{\delta}_0^\tau}, & C_s^\tau &= -\frac{\tilde{C}_s^0}{\tilde{\varepsilon}_0^\tau}, & C_0^\tau &= \frac{\tilde{C}_0^0}{\tilde{\varepsilon}_0^\tau}, & C_\tau^\tau &= \frac{\tilde{C}_\tau^0}{\tilde{\varepsilon}_0^\tau}, \\ C_0^i &= -\frac{\tilde{C}_\tau^i}{\tilde{\delta}_0^\tau}, & C_0^0 &= \frac{\tilde{C}_\tau^0}{\tilde{\delta}_0^\tau}, & C_0^\tau &= \frac{\tilde{C}_\tau^\tau}{\tilde{\delta}_0^\tau}, & B_0^s &= \frac{\tilde{B}_\tau^s}{\tilde{\varepsilon}_0^\tau}, & B_0^0 &= \frac{\tilde{B}_\tau^0}{\tilde{\varepsilon}_0^\tau}, & B_0^\tau &= \frac{\tilde{B}_\tau^\tau}{\tilde{\varepsilon}_0^\tau}, \end{aligned}$$

where

$$\begin{aligned}\varepsilon_J^I &\triangleq C_S^I B_J^S, \quad \delta_T^S = B_I^S C_T^I, \quad \varepsilon_0^I \triangleq B_0^0 C_0^I = B_0^S C_S^I, \quad \delta_0^S \triangleq C_0^0 B_0^S = C_0^I B_I^S. \\ \tilde{\varepsilon}_\nu^\mu &\triangleq \tilde{C}_\alpha^\mu \tilde{B}_\nu^\alpha, \quad \tilde{\delta}_\beta^\alpha = \tilde{B}_\mu^\alpha \tilde{C}_\beta^\mu, \quad \tilde{\varepsilon}_\tau^\mu \triangleq \tilde{B}_\tau^\tau \tilde{C}_\tau^\mu = \tilde{B}_\tau^\alpha \tilde{C}_\alpha^\mu, \quad \tilde{\delta}_\tau^\alpha \triangleq \tilde{C}_\tau^\tau \tilde{B}_\tau^\alpha = \tilde{C}_\tau^\mu \tilde{B}_\mu^\alpha.\end{aligned}$$

The evolution lemma of Proposition 3.3.2 can be expressed in Minkowski coordinate as

$$\begin{aligned}w^\mu \frac{\partial}{\partial \tilde{x}^\mu} &\cong w^\tau \frac{d}{d\tilde{x}^\tau} \Leftrightarrow w^\mu = w^\tau \tilde{\varepsilon}_\tau^\mu, & \bar{w}_\mu \frac{\partial}{\partial \tilde{x}_\mu} &\cong \bar{w}_\tau \frac{d}{d\tilde{x}_\tau} \Leftrightarrow \bar{w}_\mu = \bar{w}_\tau \tilde{\varepsilon}_\tau^\mu, \\ w_\mu d\tilde{x}^\mu &\simeq w_\tau d\tilde{x}^\tau \Leftrightarrow \tilde{\varepsilon}_\tau^\mu w_\mu = w_\tau, & \bar{w}^\mu d\tilde{x}_\mu &\simeq \bar{w}^\tau d\tilde{x}_\tau \Leftrightarrow \tilde{\varepsilon}_\tau^\tau \bar{w}^\mu = \bar{w}^\tau.\end{aligned}$$

where

$$\tilde{\varepsilon}_\nu^\mu \triangleq \tilde{B}_\alpha^\mu \tilde{C}_\nu^\alpha = \varepsilon_\nu^\mu, \quad \tilde{\delta}_\beta^\alpha \triangleq \tilde{C}_\mu^\alpha \tilde{B}_\beta^\mu = \delta_\beta^\alpha, \quad \tilde{\varepsilon}_\mu^\tau \triangleq \tilde{B}_\tau^\tau \tilde{C}_\mu^\tau = \tilde{B}_\alpha^\tau \tilde{C}_\mu^\alpha, \quad \tilde{\delta}_\alpha^\tau \triangleq \tilde{C}_\tau^\tau \tilde{B}_\alpha^\tau = \tilde{C}_\mu^\tau \tilde{B}_\alpha^\mu.$$

Discussion 4.3.2. Define $(d\xi^\tau)^2 \triangleq \sum_{a=r+1}^{\mathfrak{D}} (d\xi^a)^2$ and $(dx^\tau)^2 \triangleq \sum_{m=r+1}^{\mathfrak{D}} (dx^m)^2$. According to Definition 3.1.1, time metrics $d\xi^0$ and dx^0 on $(\tilde{M}, f)$ satisfy

$$\begin{aligned}(d\xi^0)^2 &\triangleq \sum_{s=1}^r (d\xi^s)^2 + (d\xi^\tau)^2 = \delta_{ST} d\xi^S d\xi^T = G_{IJ} dx^I dx^J, & G_{IJ} &\triangleq \delta_{ST} B_I^S B_J^T, \\ (dx^0)^2 &\triangleq \sum_{i=1}^r (dx^i)^2 + (dx^\tau)^2 = \varepsilon_{IJ} dx^I dx^J = H_{ST} d\xi^S d\xi^T, & H_{ST} &\triangleq \varepsilon_{IJ} C_S^I C_T^J.\end{aligned}$$

Thus, we obtain proper-time metrics $d\tilde{\xi}^\tau$ and $d\tilde{x}^\tau$ in Minkowski coordinates as below:

$$\begin{aligned}(d\tilde{\xi}^\tau)^2 &= (d\xi^0)^2 - \sum_{s=1}^r (d\xi^s)^2 = \tilde{\delta}_{\alpha\beta} d\tilde{\xi}^\alpha d\tilde{\xi}^\beta = \tilde{G}_{\mu\nu} d\tilde{x}^\mu d\tilde{x}^\nu, & \tilde{G}_{\mu\nu} &\triangleq \tilde{\delta}_{\alpha\beta} \tilde{B}_\mu^\alpha \tilde{B}_\nu^\beta, \\ (d\tilde{x}^\tau)^2 &= (dx^0)^2 - \sum_{i=1}^r (dx^i)^2 = \tilde{\varepsilon}_{\mu\nu} d\tilde{x}^\mu d\tilde{x}^\nu = \tilde{H}_{\alpha\beta} d\tilde{\xi}^\alpha d\tilde{\xi}^\beta, & \tilde{H}_{\alpha\beta} &\triangleq \tilde{\varepsilon}_{\mu\nu} \tilde{C}_\alpha^\mu \tilde{C}_\beta^\nu.\end{aligned}$$

It is easy to know there are relations between the regular metric G_{IJ} and the Minkowski metric $\tilde{G}_{\mu\nu}$ as below:

$$\begin{aligned}G_{\tau\tau} &= \frac{\tilde{G}_{\tau\tau}}{\tilde{G}_{00}} G_{00} = \frac{\tilde{\delta}_\tau^0 \tilde{\delta}_\tau^0}{\tilde{\varepsilon}_\tau^0 \tilde{\varepsilon}_\tau^0} \frac{\tilde{G}_{\tau\tau}}{\tilde{G}_{00}} \tilde{G}_{\tau\tau}, & \tilde{G}_{00} &= \frac{G_{00}}{G_{\tau\tau}} \tilde{G}_{\tau\tau} = \frac{\delta_0^\tau \delta_0^\tau}{\varepsilon_0^\tau \varepsilon_0^\tau} \frac{G_{00}}{G_{\tau\tau}} G_{00}, \\ G_{i\tau} &= -\frac{\tilde{G}_{i0}}{\tilde{G}_{00}} \tilde{\varepsilon}_\tau^0 G_{00} = -\frac{\tilde{\delta}_\tau^0 \tilde{\delta}_\tau^0}{\tilde{\varepsilon}_\tau^0 \tilde{\varepsilon}_\tau^0} \frac{\tilde{G}_{\tau\tau}}{\tilde{G}_{00}} \tilde{G}_{i0}, & \tilde{G}_{i0} &= -\frac{G_{i\tau}}{G_{\tau\tau}} \varepsilon_0^\tau \tilde{G}_{\tau\tau} = -\frac{\delta_0^\tau \delta_0^\tau}{\varepsilon_0^\tau \varepsilon_0^\tau} \frac{G_{00}}{G_{\tau\tau}} G_{i\tau}, \\ G_{\tau j} &= -\frac{\tilde{G}_{0j}}{\tilde{G}_{00}} \tilde{\varepsilon}_\tau^0 G_{00} = -\frac{\tilde{\delta}_\tau^0 \tilde{\delta}_\tau^0}{\tilde{\varepsilon}_\tau^0 \tilde{\varepsilon}_\tau^0} \frac{\tilde{G}_{\tau\tau}}{\tilde{G}_{00}} \tilde{G}_{0j}, & \tilde{G}_{0j} &= -\frac{G_{\tau j}}{G_{\tau\tau}} \varepsilon_0^\tau \tilde{G}_{\tau\tau} = -\frac{\delta_0^\tau \delta_0^\tau}{\varepsilon_0^\tau \varepsilon_0^\tau} \frac{G_{00}}{G_{\tau\tau}} G_{\tau j}, \\ G_{ij} &= -\frac{\tilde{G}_{ij}}{\tilde{G}_{00}} G_{00} = -\frac{\tilde{\delta}_\tau^0 \tilde{\delta}_\tau^0}{\tilde{\varepsilon}_\tau^0 \tilde{\varepsilon}_\tau^0} \frac{\tilde{G}_{\tau\tau}}{\tilde{G}_{00}} \tilde{G}_{ij}, & \tilde{G}_{ij} &= -\frac{G_{ij}}{G_{\tau\tau}} \tilde{G}_{\tau\tau} = -\frac{\delta_0^\tau \delta_0^\tau}{\varepsilon_0^\tau \varepsilon_0^\tau} \frac{G_{00}}{G_{\tau\tau}} G_{ij}.\end{aligned}$$

Denote $\tilde{G}_{\tau\tau} \triangleq \tilde{B}_\tau^\tau \tilde{B}_\tau^\tau$, $\tilde{G}^{\tau\tau} \triangleq \tilde{C}_\tau^\tau \tilde{C}_\tau^\tau$, then it is easy to obtain $\tilde{G}_{\tau\tau} = \frac{\delta_0^\tau \delta_0^\tau}{\varepsilon_0^\tau \varepsilon_0^\tau} G_{00}$, $G_{00} = \frac{\tilde{\delta}_\tau^0 \tilde{\delta}_\tau^0}{\tilde{\varepsilon}_\tau^0 \tilde{\varepsilon}_\tau^0} \tilde{G}_{\tau\tau}$.

4.4. Affine connection representation of classical spacetime evolution

Let $\tilde{D}$ be the simple connection on $\tilde{M}$, and denote $\tilde{t}_{L;\tau} \triangleq \tilde{t}_{;\sigma}\tilde{\varepsilon}_\tau^\sigma$, then the absolute differential and gradient of section 3.4 can be expressed on $\tilde{M}$ in Minkowski coordinate as

$$\begin{aligned}\tilde{D}\tilde{t} &\triangleq \tilde{t}_{;\sigma}d\tilde{x}^\sigma, \quad \tilde{D}_L\tilde{t}_L \triangleq \tilde{t}_{L;\tau}d\tilde{x}^\tau, \\ \tilde{\nabla}\tilde{t} &\triangleq \tilde{t}_{;\sigma}\frac{\partial}{\partial\tilde{x}_\sigma}, \quad \tilde{\nabla}_L\tilde{t}_L \triangleq \tilde{t}_{L;\tau}\frac{d}{d\tilde{x}_\tau}.\end{aligned}$$

Evidently, $\tilde{D}\tilde{t} \simeq \tilde{D}_L\tilde{t}_L$ if and only if L is an arbitrary path. $\tilde{\nabla}\tilde{t} \simeq \tilde{\nabla}_L\tilde{t}_L$ if and only if L is the gradient line.

Definition 4.4.1. Similar to section 3.6, suppose a charge $\tilde{\rho}$ of $\tilde{f}$ evolves on $(\tilde{M}, \tilde{g})$. We have the following definitions.

- (1) $\tilde{m}^\tau \triangleq \tilde{\rho}^{;\tau}$ and $\tilde{m}_\tau \triangleq \tilde{\rho}_{;\tau}$ are said to be the **rest mass** of $\tilde{\rho}$.
- (2) $\tilde{p}^\mu \triangleq -\tilde{\rho}^{;\mu}$ and $\tilde{p}_\mu \triangleq -\tilde{\rho}_{;\mu}$ are said to be the **energy-momentum** of $\tilde{\rho}$, and $\tilde{E}^0 \triangleq \tilde{\rho}^{;0}$, $\tilde{E}_0 \triangleq \tilde{\rho}_{;0}$ are said to be the **energy** of $\tilde{\rho}$.
- (3) $\tilde{M}^\tau \triangleq \frac{d\tilde{\rho}}{d\tilde{x}^\tau}$ and $\tilde{M}_\tau \triangleq \frac{d\tilde{\rho}}{d\tilde{x}^\tau}$ are said to be the **canonical rest mass** of $\tilde{\rho}$.
- (4) $\tilde{P}^\mu \triangleq -\frac{\partial\tilde{\rho}}{\partial\tilde{x}_\mu}$ and $\tilde{P}_\mu \triangleq -\frac{\partial\tilde{\rho}}{\partial\tilde{x}^\mu}$ are said to be the **canonical energy-momentum** of $\tilde{\rho}$, and $\tilde{H}^0 \triangleq \frac{\partial\tilde{\rho}}{\partial\tilde{x}_0}$, $\tilde{H}_0 \triangleq \frac{\partial\tilde{\rho}}{\partial\tilde{x}^0}$ are said to be the **canonical energy** of $\tilde{\rho}$.

Discussion 4.4.1. Similar to Proposition 3.6.1, $\forall p \in \tilde{M}$, if and only if the evolution direction $[L_p] = \tilde{\nabla}\tilde{\rho}|_p$, the directional derivative is

$$\left\langle \tilde{m}_\tau \frac{d}{d\tilde{x}_\tau}, \tilde{m}_\tau d\tilde{x}^\tau \right\rangle = \left\langle \tilde{p}_\mu \frac{\partial}{\partial\tilde{x}_\mu}, \tilde{p}_\mu d\tilde{x}^\mu \right\rangle,$$

that is $\tilde{G}^{\tau\tau}\tilde{m}_\tau\tilde{m}_\tau = \tilde{G}^{\mu\nu}\tilde{p}_\mu\tilde{p}_\nu$, or

$$\tilde{m}_\tau\tilde{m}^\tau = \tilde{p}_\mu\tilde{p}^\mu,$$

which is the affine connection representation of energy-momentum equation.

In addition, similar to Proposition 3.6.2, according to the evolution lemma, $\forall p \in \tilde{M}$, if and only if the evolution direction $[L_p] = \tilde{\nabla}\tilde{\rho}|_p$, we have $\tilde{p}_\mu = -\tilde{m}_\tau \frac{d\tilde{x}_\mu}{d\tilde{x}^\tau}$, that is $\tilde{E}_0 = \tilde{m}_\tau \frac{d\tilde{x}_0}{d\tilde{x}^\tau} = \tilde{m}_\tau \frac{dx_0}{dx^\tau}$ and $\tilde{p}_i = -\tilde{m}_\tau \frac{d\tilde{x}_i}{d\tilde{x}^\tau} = \tilde{m}_\tau \frac{d\tilde{x}_i}{d\tilde{x}^\tau} = \tilde{m}_\tau \frac{dx_i}{dx^\tau} = \tilde{E}_0 \frac{dx_i}{dx_0}$. This can also be regarded as the origin of $p = mv$.

Similar to discussions of section 3.7, denote

$$\begin{aligned}[\tilde{\rho}\tilde{\Gamma}_\omega] &\triangleq [\tilde{\rho}_{\mu\nu}\tilde{\Gamma}_\omega] \triangleq \frac{\partial\tilde{\rho}}{\partial\tilde{x}^\omega} - \tilde{\rho}_{;\omega} \triangleq \frac{\partial\tilde{\rho}_{\mu\nu}}{\partial\tilde{x}^\omega} - \tilde{\rho}_{\mu\nu;\omega} = \tilde{\rho}_{\mu\chi}\tilde{\Gamma}_{\nu\omega}^\chi + \tilde{\rho}_{\chi\nu}\tilde{\Gamma}_{\mu\omega}^\chi, & [\tilde{\rho}\tilde{\Gamma}^\omega] &\triangleq \tilde{G}^{\chi\omega}[\tilde{\rho}\tilde{\Gamma}_\chi], \\ [\tilde{\rho}\tilde{\Gamma}_\tau] &\triangleq [\tilde{\rho}_{\mu\nu}\tilde{\Gamma}_\tau] \triangleq \frac{d\tilde{\rho}}{d\tilde{x}^\tau} - \tilde{\rho}_{;\tau} \triangleq \frac{d\tilde{\rho}_{\mu\nu}}{d\tilde{x}^\tau} - \tilde{\rho}_{\mu\nu;\tau} = \tilde{\rho}_{\mu\chi}\tilde{\Gamma}_{\nu\tau}^\chi + \tilde{\rho}_{\chi\nu}\tilde{\Gamma}_{\mu\tau}^\chi, & [\tilde{\rho}\tilde{\Gamma}^\tau] &\triangleq \tilde{G}^{\tau\tau}[\tilde{\rho}\tilde{\Gamma}_\tau], \\ [\tilde{\rho}\tilde{R}_{\rho\sigma}] &\triangleq \tilde{\rho}_{\mu\chi}\tilde{R}_{\nu\rho\sigma}^\chi + \tilde{\rho}_{\chi\nu}\tilde{R}_{\mu\rho\sigma}^\chi, & [\tilde{\rho}\tilde{E}_{\rho\sigma}] &\triangleq [\tilde{\rho}\tilde{\Gamma}_\sigma]_{;\rho} - [\tilde{\rho}\tilde{\Gamma}_\rho]_{;\sigma}, \\ [\tilde{\rho}\tilde{B}_{\rho\sigma}] &\triangleq \tilde{\rho}_{\mu\chi}\left(\frac{\partial\tilde{\Gamma}_{\nu\sigma}^\chi}{\partial\tilde{x}^\rho} - \frac{\partial\tilde{\Gamma}_{\nu\rho}^\chi}{\partial\tilde{x}^\sigma}\right) + \tilde{\rho}_{\chi\nu}\left(\frac{\partial\tilde{\Gamma}_{\mu\sigma}^\chi}{\partial\tilde{x}^\rho} - \frac{\partial\tilde{\Gamma}_{\mu\rho}^\chi}{\partial\tilde{x}^\sigma}\right), & [\tilde{\rho}\tilde{F}_{\rho\sigma}] &\triangleq \frac{\partial[\tilde{\rho}\tilde{\Gamma}_\sigma]}{\partial\tilde{x}^\rho} - \frac{\partial[\tilde{\rho}\tilde{\Gamma}_\rho]}{\partial\tilde{x}^\sigma}.\end{aligned}$$

Then for the same reason as section 3.7, based on Definition 4.4.1, we can strictly prove the affine connection representation of general Lorentz force equation of $\tilde{\rho}$, which is

$$\tilde{f}_\rho \triangleq \tilde{p}_{\rho;\tau} = \tilde{m}_{\tau;\rho} - \tilde{p}_\sigma\tilde{\varepsilon}_{\tau;\rho}^\sigma + [\tilde{\rho}\tilde{R}_{\rho\sigma}]\tilde{\varepsilon}_\tau^\sigma. \quad (47)$$

And similarly, we also have $\tilde{T}_{\mu\nu}{}^{;\mu} = 0$. In the mass-point model, $\tilde{m}_{\tau;\rho}$ and $\tilde{\varepsilon}_{\tau;\rho}^\sigma$ do not make sense, so Eq.(47) turns into

$$\tilde{f}_\rho = [\tilde{\rho}\tilde{R}_{\rho\sigma}]\tilde{\varepsilon}_\tau^\sigma.$$

This is the affine connection representation of the force of interaction (e.g. the Lorentz force $\mathbf{f} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ or $f_\rho = j^\sigma F_{\rho\sigma}$ of the electrodynamics).

Definition 4.4.2. It is similar to Definition 3.6.2. Let $\tilde{\mathcal{P}}(b, a)$ be the totality of paths on $\tilde{M}$ from point a to point b . And let $L \in \tilde{\mathcal{P}}(b, a)$, and parameter $\tilde{x}^\tau$ satisfy $\tau_a \triangleq \tilde{x}^\tau(a) < \tilde{x}^\tau(b) \triangleq \tau_b$. The affine connection representation of action in Minkowski coordinates can be defined as

$$\tilde{s}(L) \triangleq \int_L \tilde{D}\tilde{\rho} = \int_L \tilde{p}_\mu d\tilde{x}^\mu = \int_{\tau_a}^{\tau_b} \tilde{m}_\tau d\tilde{x}^\tau, \quad \tilde{s}(L) \triangleq \int_{\tau_a}^{\tau_b} (\gamma^\mu \tilde{\rho}_{;\mu} + \tilde{m}_\tau) d\tilde{x}^\tau. \quad (48)$$

There are more illustrations in Remark 4.5.1.

4.5. Affine connection representation of Dirac equation

Discussion 4.5.1. Define Dirac algebras γ^μ and γ^α such that

$$\gamma^\mu = \tilde{C}_\alpha^\mu \gamma^\alpha, \quad \gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha = 2\tilde{\delta}^{\alpha\beta}, \quad \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\tilde{G}^{\mu\nu}.$$

Suppose $(\tilde{M}, \tilde{g})$ is orthogonal. According to Definition 4.2.2 and Discussion 4.3.2, $\tilde{G}_{\tau\tau} = 1$. Due to Discussion 4.4.1, in a gradient direction of $\tilde{\rho} \triangleq \tilde{\rho}_{\omega\nu}$, we have

$$\begin{aligned} \tilde{\rho}_{;\mu} \tilde{\rho}^{;\mu} = \tilde{\rho}_{;\tau} \tilde{\rho}^{;\tau} &\Leftrightarrow \tilde{G}^{\mu\nu} \tilde{\rho}_{;\mu} \tilde{\rho}_{;\nu} = \tilde{m}_\tau^2 \\ &\Leftrightarrow (\gamma^\mu \tilde{\rho}_{;\mu})(\gamma^\nu \tilde{\rho}_{;\nu}) + (\gamma^\nu \tilde{\rho}_{;\nu})(\gamma^\mu \tilde{\rho}_{;\mu}) = 2\tilde{m}_\tau^2 \\ &\Leftrightarrow (\gamma^\mu \tilde{\rho}_{;\mu})(\gamma^\nu \tilde{\rho}_{;\nu}) = \tilde{m}_\tau^2 \\ &\Leftrightarrow (\gamma^\mu \tilde{\rho}_{;\mu})^2 = \tilde{m}_\tau^2. \end{aligned}$$

Without loss of generality, take $\gamma^\mu \tilde{\rho}_{;\mu} = \tilde{m}_\tau$, that is

$$\gamma^\mu \tilde{\rho}_{\omega\nu;\mu} = \tilde{m}_{\omega\nu\tau}. \quad (49)$$

Next, denote

$$[g\tilde{\Gamma}_\mu]^{\omega\nu} \triangleq \sum_\sigma \tilde{G}^{\nu\nu'} \tilde{\Gamma}_{\nu'\sigma\mu} + \sum_\kappa \tilde{G}^{\omega\omega'} \tilde{\Gamma}_{\omega'\kappa\mu}, \quad \tilde{D}_\mu^{\omega\nu} \triangleq \partial_\mu - [g\tilde{\Gamma}_\mu]^{\omega\nu}.$$

From Eq.(49), it is obtained that

$$\begin{aligned} \sum_{\omega,\nu} \gamma^\mu \tilde{\rho}_{\omega\nu;\mu} &= \sum_{\omega,\nu} \tilde{m}_{\omega\nu\tau} \Leftrightarrow \sum_{\omega,\nu} \gamma^\mu \left(\partial_\mu \rho_{\omega\nu} - \tilde{\rho}_{\omega\chi} \tilde{\Gamma}_{\nu\mu}^\chi - \tilde{\rho}_{\chi\nu} \tilde{\Gamma}_{\omega\mu}^\chi \right) = \sum_{\omega,\nu} \tilde{m}_{\omega\nu\tau} \\ &\Leftrightarrow \sum_{\omega,\nu} \gamma^\mu \left(\partial_\mu \rho_{\omega\nu} - \tilde{\rho}_{\omega\nu} \sum_\sigma \tilde{\Gamma}_{\sigma\mu}^\nu - \tilde{\rho}_{\omega\nu} \sum_\kappa \tilde{\Gamma}_{\kappa\mu}^\omega \right) = \sum_{\omega,\nu} \tilde{m}_{\omega\nu\tau} \\ &\Leftrightarrow \sum_{\omega,\nu} \gamma^\mu \left(\partial_\mu \rho_{\omega\nu} - \tilde{\rho}_{\omega\nu} [g\tilde{\Gamma}_\mu]^{\omega\nu} \right) = \sum_{\omega,\nu} \tilde{m}_{\omega\nu\tau} \\ &\Leftrightarrow \sum_{\omega,\nu} \gamma^\mu \left(\partial_\mu - [g\tilde{\Gamma}_\mu]^{\omega\nu} \right) \tilde{\rho}_{\omega\nu} = \sum_{\omega,\nu} \tilde{m}_{\omega\nu\tau}, \end{aligned} \quad (50)$$

that is

$$\sum_{\omega,\nu} \gamma^\mu \tilde{D}_\mu^{\omega\nu} \tilde{\rho}_{\omega\nu} = \sum_{\omega,\nu} \tilde{m}_{\omega\nu\tau}, \quad \tilde{D}_\mu^{\omega\nu} \triangleq \partial_\mu - [g\tilde{\Gamma}_\mu]^{\omega\nu}, \quad (51)$$

denoted concisely by

$$\gamma^\mu \tilde{D}_\mu \tilde{\rho} = \tilde{m}_\tau, \quad \tilde{D}_\mu \triangleq \partial_\mu - [g\tilde{\Gamma}_\mu].$$

We speak of Eq.(49) and (51) as **affine Dirac equations**.

Discussion 4.5.2. Next, we construct a kind of complex-valued representation of affine Dirac equation. The restriction of the charge $\tilde{\rho}_{\omega\nu}$ to $(\tilde{U}, \tilde{x}^\mu)$ is a function $\tilde{\rho}_{\omega\nu}(\tilde{x}^\mu)$ with respect to the coordinates $(\tilde{x}^\mu) \triangleq (\tilde{x}^0, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3)$. Let

$$\tilde{P}_{\omega\nu}(\tilde{x}^0) \triangleq \int_{(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3)} \tilde{\rho}_{\omega\nu}(\tilde{x}^\mu) d^3 \tilde{x}.$$

Suppose a function $f_{\omega\nu} = f_{\omega\nu}(\tilde{x}^\mu)$ on $(\tilde{U}, \tilde{x}^\mu)$ satisfies that

$$\tilde{\rho}_{\omega\nu} = (f_{\omega\nu})^2 \tilde{P}_{\omega\nu}, \quad \int_{(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3)} (f_{\omega\nu})^2 d^3 \tilde{x} = 1, \quad \varepsilon_\tau^\mu \frac{\partial f_{\omega\nu}}{\partial \tilde{x}^\mu} = 0, \quad \gamma^\mu \frac{\partial f_{\omega\nu}}{\partial \tilde{x}^\mu} = 0.$$

We define $\psi_{\omega\nu}$ and $\tilde{\mathbb{M}}_{\omega\nu\tau}$ in the following way.

$$\begin{aligned} \tilde{y}_{\omega\nu} &\triangleq \int_L d\tilde{\rho}_{\omega\nu} = \int_L \frac{d\tilde{\rho}_{\omega\nu}}{d\tilde{x}^\tau} d\tilde{x}^\tau = \int_L \left(\frac{d(f_{\omega\nu}^2)}{d\tilde{x}^\tau} \tilde{P}_{\omega\nu} + f_{\omega\nu}^2 \frac{d\tilde{P}_{\omega\nu}}{d\tilde{x}^\tau} \right) d\tilde{x}^\tau = f_{\omega\nu}^2 \int_L \frac{d\tilde{P}_{\omega\nu}}{d\tilde{x}^\tau} d\tilde{x}^\tau \triangleq f_{\omega\nu}^2 \tilde{Y}_{\omega\nu}, \\ \psi_{\omega\nu} &\triangleq f_{\omega\nu} e^{i\tilde{Y}_{\omega\nu}}, \quad \tilde{m}_{\omega\nu\tau} \triangleq \tilde{\rho}_{\omega\nu;\tau} = (f_{\omega\nu}^2)_{,\tau} \tilde{P}_{\omega\nu} + f_{\omega\nu}^2 \tilde{P}_{\omega\nu;\tau} = f_{\omega\nu}^2 \tilde{P}_{\omega\nu;\tau} \triangleq f_{\omega\nu}^2 \tilde{\mathbb{M}}_{\omega\nu\tau}. \end{aligned}$$

In the QFT propagator, we usually take S in the path integral $\int e^{iS} \mathcal{D}\psi$ of a fermion in the form of

$$- \int \left(i\bar{\psi} \gamma^\mu D_\mu \psi - \bar{\psi} \tilde{\mathbb{M}}_\tau \psi \right) d^4 \tilde{x},$$

where S and $d^4 \tilde{x}$ are both covariant. We believe that the external spatial integral $\int_{(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3)} d^3 \tilde{x}$ is not an essential part for evolution, so for the sake of simplicity, we do not take into account the external spatial part $\int_{(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3)} d^3 \tilde{x}$, but only consider the evolution part $\int_L d\tilde{x}^0$. Meanwhile, in order to remain the covariance, $\int_L d\tilde{x}^0$ has to be replaced by $\int_L d\tilde{x}^\tau$. Thus, in affine connection representation of gauge fields, we shall consider an action in the form of

$$- \int_L \left(i\bar{\psi} \gamma^\mu D_\mu \psi - \bar{\psi} \tilde{\mathbb{M}}_\tau \psi \right) d\tilde{x}^\tau.$$

Concretely speaking, denote

$$\tilde{D}_{\omega\nu\mu} \triangleq \frac{\partial}{\partial \tilde{x}^\mu} - i[\tilde{P}\tilde{\Gamma}_\mu]_{\omega\nu}, \quad [\tilde{P}\tilde{\Gamma}_\mu]_{\omega\nu} \triangleq \sum_\sigma \tilde{P}_{\omega\nu} \tilde{\Gamma}_{\sigma\mu}^\nu + \sum_\kappa \tilde{P}_{\omega\nu} \tilde{\Gamma}_{\kappa\mu}^\omega.$$

From Eq.(48), we have

$$\tilde{s}_{\omega\nu}(L) \triangleq \int_L (\gamma^\mu \tilde{\rho}_{\omega\nu;\mu} + \tilde{m}_{\omega\nu\tau}) d\tilde{x}^\tau.$$

And from Eq.(50) we know $\sum_{\omega,\nu} \gamma^\mu \tilde{\rho}_{\omega\nu;\mu} = \sum_{\omega,\nu} \gamma^\mu \tilde{D}_\mu^{\omega\nu} \tilde{\rho}_{\omega\nu}$. Then it is obtained that

$$\begin{aligned} \tilde{s}_{\tilde{\rho}}(L) &\triangleq \sum_{\omega,\nu} \tilde{s}_{\omega\nu}(L) \\ &= \int_L \sum_{\omega,\nu} (\gamma^\mu \tilde{\rho}_{\omega\nu;\mu} + \tilde{m}_{\omega\nu\tau}) d\tilde{x}^\tau \\ &= \int_L \sum_{\omega,\nu} \left(\gamma^\mu \tilde{D}_\mu^{\omega\nu} \tilde{\rho}_{\omega\nu} + \tilde{m}_{\omega\nu\tau} \right) d\tilde{x}^\tau \\ &= \int_L \sum_{\omega,\nu} \left(\gamma^\mu \left(\partial_\mu \tilde{\rho}_{\omega\nu} - [g\tilde{\Gamma}_\mu]^{\omega\nu} \tilde{\rho}_{\omega\nu} \right) + \tilde{m}_{\omega\nu\tau} \right) d\tilde{x}^\tau \\ &= \int_L \sum_{\omega,\nu} \left(\gamma^\mu \left(\partial_\mu \tilde{y}_{\omega\nu} - [g\tilde{\Gamma}_\mu]^{\omega\nu} \tilde{\rho}_{\omega\nu} \right) + \tilde{m}_{\omega\nu\tau} \right) d\tilde{x}^\tau \\ &= \int_L \sum_{\omega,\nu} \left(\gamma^\mu \left(\partial_\mu (f_{\omega\nu}^2 \tilde{Y}_{\omega\nu}) - [g\tilde{\Gamma}_\mu]^{\omega\nu} f_{\omega\nu}^2 \tilde{P}_{\omega\nu} \right) + f_{\omega\nu}^2 \tilde{\mathbb{M}}_{\omega\nu\tau} \right) d\tilde{x}^\tau \\ &= \int_L \sum_{\omega,\nu} \left(\gamma^\mu \left(\partial_\mu \tilde{Y}_{\omega\nu} - [\tilde{P}\tilde{\Gamma}_\mu]_{\omega\nu} \right) f_{\omega\nu}^2 + f_{\omega\nu}^2 \tilde{\mathbb{M}}_{\omega\nu\tau} \right) d\tilde{x}^\tau \\ &= \int_L \sum_{\omega,\nu} \left(f_{\omega\nu} e^{-i\tilde{Y}_{\omega\nu}} \gamma^\mu \left(f_{\omega\nu} e^{i\tilde{Y}_{\omega\nu}} \partial_\mu \tilde{Y}_{\omega\nu} - [\tilde{P}\tilde{\Gamma}_\mu]_{\omega\nu} f_{\omega\nu} e^{i\tilde{Y}_{\omega\nu}} \right) + f_{\omega\nu} e^{-i\tilde{Y}_{\omega\nu}} \tilde{\mathbb{M}}_{\omega\nu\tau} f_{\omega\nu} e^{i\tilde{Y}_{\omega\nu}} \right) d\tilde{x}^\tau \\ &= \int_L \sum_{\omega,\nu} \left(-\bar{\psi}_{\omega\nu} i\gamma^\mu \left(e^{i\tilde{Y}_{\omega\nu}} \partial_\mu f_{\omega\nu} + f_{\omega\nu} e^{i\tilde{Y}_{\omega\nu}} i\partial_\mu \tilde{Y}_{\omega\nu} - i[\tilde{P}\tilde{\Gamma}_\mu]_{\omega\nu} \psi_{\omega\nu} \right) + \bar{\psi}_{\omega\nu} \tilde{\mathbb{M}}_{\omega\nu\tau} \psi_{\omega\nu} \right) d\tilde{x}^\tau \\ &= \int_L \sum_{\omega,\nu} \left(-\bar{\psi}_{\omega\nu} i\gamma^\mu \left(\partial_\mu (f_{\omega\nu} e^{i\tilde{Y}_{\omega\nu}}) - i[\tilde{P}\tilde{\Gamma}_\mu]_{\omega\nu} \psi_{\omega\nu} \right) + \bar{\psi}_{\omega\nu} \tilde{\mathbb{M}}_{\omega\nu\tau} \psi_{\omega\nu} \right) d\tilde{x}^\tau \\ &= \int_L \sum_{\omega,\nu} \left(-\bar{\psi}_{\omega\nu} i\gamma^\mu \left(\partial_\mu - i[\tilde{P}\tilde{\Gamma}_\mu]_{\omega\nu} \right) \psi_{\omega\nu} + \bar{\psi}_{\omega\nu} \tilde{\mathbb{M}}_{\omega\nu\tau} \psi_{\omega\nu} \right) d\tilde{x}^\tau \\ &= \int_L \sum_{\omega,\nu} \left(-\bar{\psi}_{\omega\nu} i\gamma^\mu \tilde{D}_{\omega\nu\mu} \psi_{\omega\nu} + \bar{\psi}_{\omega\nu} \tilde{\mathbb{M}}_{\omega\nu\tau} \psi_{\omega\nu} \right) d\tilde{x}^\tau. \end{aligned} \tag{52}$$

Hence,

$$\delta \tilde{s}_{\tilde{\rho}}(L) = 0 \quad \Rightarrow \quad i\gamma^\mu \tilde{D}_{\omega\nu\mu} \psi_{\omega\nu} = \tilde{\mathbb{M}}_{\omega\nu\tau} \psi_{\omega\nu}. \tag{53}$$

Thus, we have obtained a complex-valued representation of gradient direction of $\tilde{\rho}_{\omega\nu}$. It can be denoted concisely by

$$i\gamma^\mu \tilde{D}_\mu \psi = \tilde{\mathbb{M}}_\tau \psi.$$

Remark 4.5.1. From the above discussion, we know in the gradient direction of $\rho_{\omega\nu}$, that

$$-\sum_{\omega,\nu} \bar{\psi}_{\omega\nu} i\gamma^\mu \tilde{D}_{\omega\nu\mu} \psi_{\omega\nu} d\tilde{x}^\tau = \sum_{\omega,\nu} \tilde{D} \tilde{\rho}_{\omega\nu}.$$

This shows that $s(L)$ and $\tilde{s}(L)$ in Definition 3.6.2 and Remark 3.6.1 are indeed applicable for constructing propagator by $e^{is(L)}$ and $e^{i\tilde{s}(L)}$ in affine connection representation of gauge fields. Therefore, the idea in Discussion 3.10.3 are reasonable.

4.6. From classical spacetime back to full-dimensional space

Discussion 4.6.1. Now there is a problem. As is shown in Remark 4.2.1, $(\tilde{M}, \tilde{f})$ and $(\tilde{M}, \tilde{g})$ cannot totally reflect all the intrinsic geometric properties of internal space of (M, f) and (M, g) . Concretely speaking:

In the previous section, we discuss the affine Dirac equation $\gamma^\mu \tilde{\rho}_{\omega\nu;\mu} = \tilde{m}_{\omega\nu\tau}$ on $(\tilde{M}, \tilde{g})$. Similar to section 3.5, on $(\tilde{M}, \tilde{f})$ we have the affine Yang-Mills equation $\tilde{K}_{\nu\rho\sigma}^\mu{}^{;\rho} = \tilde{\rho}_\nu^\mu \gamma_\sigma$ on $(\tilde{M}, \tilde{f})$. Suppose there is no gravitaional field, or to say, the external space is flat, then remaining non-vanish equations are just only

$$\gamma^\mu \tilde{\rho}_{00;\mu} = \tilde{m}_{00\tau}, \quad \tilde{K}_{0\rho\sigma}^0{}^{;\rho} = \tilde{\rho}_0^0 \gamma_\sigma.$$

There are multiple internal charges

$$\rho_{mn} (m, n = 4, 5, \dots, \mathfrak{D})$$

on (M, f) . We intend to use these ρ_{mn} to describe leptons and hadrons. However, via encapsulation of classical spacetime, $(\tilde{M}, \tilde{f})$ remains only one internal charge $\tilde{\rho}_{00}$, it falls short. It is impossible for the only one real-valued field function $\tilde{\rho}_{00}$ to describe so many leptons and hadrons.

On the premise of not abandoning the $(1+3)$ -dimensional spacetime, if we want to describe gauge fields, there is a method that to use some non-coordinate abstract degrees of freedom that is irrelevant to view of time and space on the phase $e^{iT_a\theta^a}$ of a complex-valued field function ψ , just like that in Standard Model. This way is effective, but not natural. It is not satisfying for a theory to adopt a coordinate representation for external space but a non-coordinate representation for internal space.

A logically more natural way is required to abandon the framework of $(1+3)$ -dimensional spacetime $(\tilde{M}, \tilde{f})$ and $(\tilde{M}, \tilde{g})$. We should put internal space and external space together to describe their unified geometry with the same spatial frame. On (M, f) and (M, g) , there are enough real-valued field functions ρ_{mn} to describe leptons and hadrons, and enough internal components Γ_{nP}^m of affine connection to describe gauge potentials.

Therefore, only on the full-dimensional (M, f) and (M, g) can total advantages of affine connection representation of gauge fields be brought into full play, and thereby show complete details of geometric properties of gauge field. So we are going to stop the discussions about the classical spacetime $\tilde{M}$, but to focus on the full-dimensional manifold M .

Discussion 4.6.2. On M , due to $\Gamma_{NP}^M = \frac{1}{2} C_A^M \left(\frac{\partial B_N^A}{\partial x^P} + \frac{\partial B_P^A}{\partial x^N} \right)$ and $G_{MN} = \delta_{AB} B_M^A B_N^B$ we know that gauge field and gravitational field can both be described by the unified spatial frames B_M^A and C_A^M of a reference-system f or g . Reference-system is the common origination of gauge field and gravitational field.

We adopt the components of Γ_{NP}^M with $M, N \in \{4, 5, \dots, \mathfrak{D}\}$ to describe typical gauge fields such as electromagnetic, weak and strong interaction fields. The other components of Γ_{NP}^M describe gravitational field. Simple connection Γ_{NP}^M is the common affine connection representation of gauge field and gravitational field.

We adopt the components of ρ_{MN} with $M, N \in \{4, 5, \dots, \mathfrak{D}\}$ to describe the charges of leptons and hadrons. The meanings of the other components of ρ_{MN} are not clear at present, maybe they could be used to describe dark matter.

On orthogonal (M, g) and (M, f) , there are full-dimensional field equations, i.e. affine Dirac equation and affine Yang-Mills equation

$$\gamma^P \rho_{MN;P} = \rho_{MN;0}, \quad K_{NPQ}^M \cdot^P = \rho_N^M \gamma_Q, \quad (54)$$

which reflect the on-shell evolution directions $\nabla \rho$ and ∇t , respectively. Their quantum evolutions are described by the propagators in Definition 3.10.5 or Discussion 3.10.3.

Discussion 4.6.3. Gauge invariance and general covariance on (M, g) are unified into invariance under the reference-system transformation $L_k : g \rightarrow g'$.

On an orthogonal (M, g) , Eq.(52) presents as a full-dimensional action

$$s_\rho(L) = \int_L \sum_{M,N} (\gamma^P \rho_{MN;P} + \varepsilon_0^P \rho_{MN;P}) dx^0 = -i \int_L \sum_{M,N} \bar{\psi}_{MN} (\gamma^P D_{MNP} + \varepsilon_0^P D_{MNP}) \psi_{MN} dx^0.$$

If and only if $L_k : g \rightarrow g'$ is an orthogonal transformation, L_k sends $s_\rho(L)$ to

$$s'_\rho(L) = \int_L \sum_{M,N} (\gamma^{P'} \rho_{MN;P'} + \varepsilon_{0'}^{P'} \rho_{MN;P'}) dx^{0'} = -i \int_L \sum_{M,N} \bar{\psi}'_{MN} (\gamma^{P'} D'_{MNP'} + \varepsilon_{0'}^{P'} D'_{MNP'}) \psi'_{MN} dx^{0'}.$$

It is evident that

(i) Because ρ_{MN} is determined by the reference-system f but not g , ρ_{MN} remains unchanged under the transformation $L_k : g \rightarrow g'$.

(ii) L_k makes $\rho_{MN;P}$, ψ_{MN} and D_{MNP} changed, i.e.

$$\begin{aligned} L_k : \rho_{MN;P} &\triangleq \frac{\partial \rho_{MN}}{\partial x^P} - \left(\rho_{MH}(\Gamma_g)_H^N + \rho_{HN}(\Gamma_g)_M^H \right) \\ &\mapsto \rho_{MN;P'} = \frac{\partial \rho_{MN}}{\partial x^{P'}} - \left(\rho_{MH}(\Gamma_{g'})_{NP'}^H + \rho_{HN}(\Gamma_{g'})_{MP'}^H \right), \\ L_k : \psi_{MN} &\triangleq f_{MN} e^{iY_{MN}} = f_{MN} \exp \left\{ i \int_L P_{MN;0} dx^0 + \left(P_{MH}(\Gamma_g)_H^N + P_{HN}(\Gamma_g)_M^H \right) dx^P \right\} \\ &\mapsto \psi'_{MN} = f_{MN} \exp \left\{ i \int_L P_{MN;0'} dx^{0'} + \left(P_{MH}(\Gamma_{g'})_{NP'}^H + P_{HN}(\Gamma_{g'})_{MP'}^H \right) dx^{P'} \right\}, \\ L_k : D_{MNP} &\triangleq \frac{\partial}{\partial x^P} - i \left(\sum_H P_{MN}(\Gamma_g)_H^N + \sum_G P_{MN}(\Gamma_g)_G^M \right) \\ &\mapsto D'_{MNP'} = \frac{\partial}{\partial x^{P'}} - i \left(\sum_H P_{MN}(\Gamma_{g'})_{HP'}^N + \sum_G P_{MN}(\Gamma_{g'})_{GP'}^M \right), \end{aligned}$$

where $(\Gamma_{g'})_{NP'}^M = (B_k)_{M'}^M (C_k)_N^{N'} (\Gamma_{g'})_{N'P'}^{M'}$, and $L_k : (\Gamma_g)_{NP}^M \mapsto (\Gamma_{g'})_{N'P'}^{M'}$ transforms according to Proposition 2.5.1, that is

$$L_k : (\Gamma_g)_{NP}^M \mapsto (\Gamma_{g'})_{N'P'}^{M'} = (\Gamma_g)_{NP}^M (C_k)_M^{M'} (B_k)_{N'}^N (B_k)_{P'}^P + (\Gamma_k)_{N'P'}^{M'}.$$

Therefore, in affine connection representation of gauge fields, the gauge transformations $\psi \mapsto \psi'$ and $D \mapsto D'$ essentially boil down to the reference-system transformation L_k .

(iii) From the real-valued representation of action $s_{MN}(L) = \int_L (\gamma^P \rho_{MN;P} + \varepsilon_0^P \rho_{MN;P}) dx^0$, it can be seen evidently that

$$L_k : s_\rho(L) \mapsto s'_\rho(L) = s_\rho(L).$$

Remark 1. The above (ii) and (iii) show the unified gauge invariance and general covariance under the reference-system transformation L_k .

Remark 2. For a general (M, g) , g is not necessarily orthogonal, so the corresponding action should be described by

$$s_{MN}(L) = \int_L (B_0^0 \gamma^P \rho_{MN;P} + \varepsilon_0^P \rho_{MN;P}) dx^0.$$

In this general case, Definition 3.6.2 and the method in Discussion 3.10.3 are also available and effective, where we take

$$s_{MN}(L) = \int_L D \rho_{MN}.$$

Remark 3. We see that the real-valued representation of action is more concise than the complex-valued representation of action. Hence, it is more convenient to adopt real-valued representations for field function, field equation and action.

The research objects in the following sections are based on the following definition. We are going to use the simple connection Γ_{NP}^M to show the affine connection representations of electromagnetic, weak and strong interaction fields, and to adopt the real-valued representation $\rho_{MN;P}$ to discuss the interactions between gauge fields and elementary particles.

Definition 4.6.1. Let there be a geometric manifold (M, f) , such that $M = P \times N$, $r \triangleq \dim P = 3$ and $\mathfrak{D} \triangleq \dim M = 5$ or 6 or 8.

(1) $\forall p \in M$, suppose the coordinate representation of $f(p)$ takes the form of

$$\xi^a = \xi^a(x^m), \quad \xi^s = \delta_i^s x^i, \quad (s, i = 1, 2, 3; a, m = 4, 5, \dots, \mathfrak{D}),$$

and f satisfies the **internal standard conditions** that

$$(i) G_{mn} = \text{const}, \quad (ii) \text{ when } m \neq n, G_{mn} = 0. \quad (55)$$

We say f is a typical gauge field, and $L_{[f]}$ is a typical gauge transformation.

(2) $\forall p \in M$, suppose the coordinate representation of $f(p)$ takes the form of

$$\xi^a = \xi^a(x^M), \quad \xi^s = \xi^s(x^i), \quad (s, i = 1, 2, 3; a = 4, 5, \dots, \mathfrak{D}; M = 1, 2, \dots, \mathfrak{D}).$$

We say f is a typical gauge field **with gravitation**, and L_f is a gravitational gauge transformation.

5. Affine connection representation of the gauge field of weak-electromagnetic interaction

Definition 5.1. Suppose a geometric manifold (M, f) satisfies $\mathfrak{D} = r + 2 = 5$. $\forall p \in M$, the coordinate representation of $f(p)$ takes the form of

$$\xi^a = \xi^a(x^m), \quad \xi^s = \delta_i^s x^i, \quad (s, i = 1, 2, 3; a, m = 4, 5),$$

and f satisfies the internal standard conditions (55) of Definition 4.6.1(1) and

$$G^{(\mathfrak{D}-1)(\mathfrak{D}-1)} = G^{\mathfrak{D}\mathfrak{D}}.$$

We say f is a **weak and electromagnetic unified field**. The reason for such naming lies in the following proposition.

Proposition 5.1. Let the simple connection of the above (M, f) be Λ_{NP}^M and Λ_{MNP} . And let the coefficients of curvature of (M, f) be K_{NPQ}^M and K_{MNPQ} . Denote

$$\left\{ \begin{array}{l} B_P \triangleq \frac{1}{\sqrt{2}} (\Lambda_{\mathfrak{D}\mathfrak{D}P} + \Lambda_{(\mathfrak{D}-1)(\mathfrak{D}-1)P}) \\ A_P^3 \triangleq \frac{1}{\sqrt{2}} (\Lambda_{\mathfrak{D}\mathfrak{D}P} - \Lambda_{(\mathfrak{D}-1)(\mathfrak{D}-1)P}) \end{array} \right\}, \quad \left\{ \begin{array}{l} A_P^1 \triangleq \frac{1}{\sqrt{2}} (\Lambda_{(\mathfrak{D}-1)\mathfrak{D}P} + \Lambda_{\mathfrak{D}(\mathfrak{D}-1)P}) \\ A_P^2 \triangleq \frac{1}{\sqrt{2}} (\Lambda_{(\mathfrak{D}-1)\mathfrak{D}P} - \Lambda_{\mathfrak{D}(\mathfrak{D}-1)P}) \end{array} \right\}.$$

$$\left\{ \begin{array}{l} B_{PQ} \triangleq \frac{1}{\sqrt{2}} (K_{\mathfrak{D}\mathfrak{D}PQ} + K_{(\mathfrak{D}-1)(\mathfrak{D}-1)PQ}) \\ F_{PQ}^3 \triangleq \frac{1}{\sqrt{2}} (K_{\mathfrak{D}\mathfrak{D}PQ} - K_{(\mathfrak{D}-1)(\mathfrak{D}-1)PQ}) \end{array} \right\}, \quad \left\{ \begin{array}{l} F_{PQ}^1 \triangleq \frac{1}{\sqrt{2}} (K_{(\mathfrak{D}-1)\mathfrak{D}PQ} + K_{\mathfrak{D}(\mathfrak{D}-1)PQ}) \\ F_{PQ}^2 \triangleq \frac{1}{\sqrt{2}} (K_{(\mathfrak{D}-1)\mathfrak{D}PQ} - K_{\mathfrak{D}(\mathfrak{D}-1)PQ}) \end{array} \right\}.$$

And denote $g \triangleq \sqrt{(G^{(\mathfrak{D}-1)(\mathfrak{D}-1)})^2 + (G^{\mathfrak{D}\mathfrak{D}})^2}$. Thus the following equations hold, and they are all intrinsic geometric properties of (M, f) .

$$\left\{ \begin{array}{l} B_{PQ} = \frac{\partial B_Q}{\partial x^P} - \frac{\partial B_P}{\partial x^Q}, \\ F_{PQ}^3 = \frac{\partial A_Q^3}{\partial x^P} - \frac{\partial A_P^3}{\partial x^Q} + g (A_P^1 A_Q^2 - A_P^2 A_Q^1), \\ F_{PQ}^1 = \frac{\partial A_Q^1}{\partial x^P} - \frac{\partial A_P^1}{\partial x^Q} + g (A_P^2 A_Q^3 - A_P^3 A_Q^2), \\ F_{PQ}^2 = \frac{\partial A_Q^2}{\partial x^P} - \frac{\partial A_P^2}{\partial x^Q} - g (A_P^3 A_Q^1 - A_P^1 A_Q^3). \end{array} \right.$$

Proof. According to the definition, the semi-metric of f satisfies that

$$B_m^s = 0, \quad C_a^i = 0. \quad B_i^s = \delta_i^s, \quad B_i^a = 0, \quad C_s^i = \delta_s^i, \quad C_s^m = 0.$$

The metric of f satisfies that

$$G_{mn} = 0 (m \neq n), \quad G_{mn} = \text{const}, \quad G_{MN} \triangleq \delta_{AB} B_M^A B_N^B, \quad G^{MN} = \delta^{AB} C_A^M C_B^N.$$

Concretely:

$$\begin{cases} G_{ij} = \delta_{st} B_i^s B_j^t + \delta_{ab} B_i^a B_j^b = \delta_{st} \delta_i^s \delta_j^t = \delta_{ij} \\ G_{in} = \delta_{st} B_i^s B_n^t + \delta_{ab} B_i^a B_n^b = 0 \\ G_{mj} = \delta_{st} B_m^s B_j^t + \delta_{ab} B_m^a B_j^b = 0 \\ G_{mn} = B_m^{\mathfrak{D}-1} B_n^{\mathfrak{D}-1} + B_m^{\mathfrak{D}} B_n^{\mathfrak{D}} \end{cases}, \begin{cases} G^{ij} = \delta^{st} C_s^i C_t^j = \delta^{st} \delta_s^i \delta_t^j = \delta^{ij} \\ G^{in} = \delta^{st} C_s^i C_t^n = 0 \\ G^{mj} = \delta^{st} C_s^m C_t^j = 0 \\ G^{mn} = C_{\mathfrak{D}-1}^m C_{\mathfrak{D}-1}^n + C_{\mathfrak{D}}^m C_{\mathfrak{D}}^n \end{cases}.$$

Calculate the simple connection of f , that is $\Lambda_{NP}^M \triangleq \frac{1}{2} C_A^M \left(\frac{\partial B_N^A}{\partial x^P} + \frac{\partial B_P^A}{\partial x^N} \right)$ and $\Lambda_{MNP} \triangleq G_{MM'} \Lambda_{NP}^{M'}$, then we obtain

$$\begin{cases} \Lambda_{NP}^i = 0 \\ \Lambda_{jk}^m = 0 \\ \Lambda_{nP}^m = \frac{1}{2} C_a^m \left(\frac{\partial B_n^a}{\partial x^P} + \frac{\partial B_P^a}{\partial x^n} \right) \\ \Lambda_{Np}^m = \frac{1}{2} C_a^m \left(\frac{\partial B_N^a}{\partial x^p} + \frac{\partial B_p^a}{\partial x^N} \right) \end{cases}, \begin{cases} \Lambda_{iNP} = G_{iM'} \Lambda_{NP}^{M'} = G_{ii'} \Lambda_{NP}^{i'} = 0 \\ \Lambda_{mjk} = G_{mM'} \Lambda_{jk}^{M'} = G_{mm'} \Lambda_{jk}^{m'} = 0 \\ \Lambda_{mnP} = \frac{1}{2} \delta_{ab} B_m^b \left(\frac{\partial B_n^a}{\partial x^P} + \frac{\partial B_P^a}{\partial x^n} \right) \\ \Lambda_{mNp} = \frac{1}{2} \delta_{ab} B_m^b \left(\frac{\partial B_N^a}{\partial x^p} + \frac{\partial B_p^a}{\partial x^N} \right) \end{cases}.$$

Calculate the coefficients of curvature of f , that is

$$\begin{aligned} K_{nPQ}^m &\triangleq \frac{\partial \Lambda_{nQ}^m}{\partial x^P} - \frac{\partial \Lambda_{nP}^m}{\partial x^Q} + \Lambda_{HP}^m \Lambda_{nQ}^H - \Lambda_{nP}^H \Lambda_{HQ}^m, \\ K_{mnPQ} &\triangleq G_{mM'} K_{nPQ}^{M'} = G_{mm'} K_{nPQ}^{m'}, \end{aligned}$$

then we obtain

$$\begin{aligned} K_{(\mathfrak{D}-1)(\mathfrak{D}-1)PQ} &= \frac{\partial \Lambda_{(\mathfrak{D}-1)(\mathfrak{D}-1)Q}}{\partial x^P} - \frac{\partial \Lambda_{(\mathfrak{D}-1)(\mathfrak{D}-1)P}}{\partial x^Q} + G^{\mathfrak{D}\mathfrak{D}} \left(\Lambda_{(\mathfrak{D}-1)\mathfrak{D}P} \Lambda_{(\mathfrak{D}-1)Q} - \Lambda_{(\mathfrak{D}-1)P} \Lambda_{(\mathfrak{D}-1)\mathfrak{D}Q} \right), \\ K_{\mathfrak{D}(\mathfrak{D}-1)PQ} &= \frac{\partial \Lambda_{\mathfrak{D}(\mathfrak{D}-1)Q}}{\partial x^P} - \frac{\partial \Lambda_{\mathfrak{D}(\mathfrak{D}-1)P}}{\partial x^Q} + G^{\mathfrak{D}\mathfrak{D}} \left(\Lambda_{\mathfrak{D}\mathfrak{D}P} \Lambda_{(\mathfrak{D}-1)Q} - \Lambda_{(\mathfrak{D}-1)P} \Lambda_{\mathfrak{D}\mathfrak{D}Q} \right) \\ &\quad + G^{(\mathfrak{D}-1)(\mathfrak{D}-1)} \left(\Lambda_{(\mathfrak{D}-1)P} \Lambda_{(\mathfrak{D}-1)(\mathfrak{D}-1)Q} - \Lambda_{(\mathfrak{D}-1)(\mathfrak{D}-1)P} \Lambda_{(\mathfrak{D}-1)\mathfrak{D}Q} \right), \\ K_{(\mathfrak{D}-1)\mathfrak{D}PQ} &= \frac{\partial \Lambda_{(\mathfrak{D}-1)\mathfrak{D}Q}}{\partial x^P} - \frac{\partial \Lambda_{(\mathfrak{D}-1)\mathfrak{D}P}}{\partial x^Q} + G^{\mathfrak{D}\mathfrak{D}} \left(\Lambda_{(\mathfrak{D}-1)\mathfrak{D}P} \Lambda_{\mathfrak{D}\mathfrak{D}Q} - \Lambda_{\mathfrak{D}\mathfrak{D}P} \Lambda_{(\mathfrak{D}-1)\mathfrak{D}Q} \right) \\ &\quad + G^{(\mathfrak{D}-1)(\mathfrak{D}-1)} \left(\Lambda_{(\mathfrak{D}-1)(\mathfrak{D}-1)P} \Lambda_{(\mathfrak{D}-1)\mathfrak{D}Q} - \Lambda_{(\mathfrak{D}-1)\mathfrak{D}P} \Lambda_{(\mathfrak{D}-1)(\mathfrak{D}-1)Q} \right), \\ K_{\mathfrak{D}\mathfrak{D}PQ} &= \frac{\partial \Lambda_{\mathfrak{D}\mathfrak{D}Q}}{\partial x^P} - \frac{\partial \Lambda_{\mathfrak{D}\mathfrak{D}P}}{\partial x^Q} + G^{(\mathfrak{D}-1)(\mathfrak{D}-1)} \left(\Lambda_{\mathfrak{D}(\mathfrak{D}-1)P} \Lambda_{(\mathfrak{D}-1)\mathfrak{D}Q} - \Lambda_{(\mathfrak{D}-1)\mathfrak{D}P} \Lambda_{\mathfrak{D}(\mathfrak{D}-1)Q} \right). \end{aligned}$$

Hence,

$$\begin{aligned} B_{PQ} &\triangleq \frac{1}{\sqrt{2}} \left(K_{\mathfrak{D}\mathfrak{D}PQ} + K_{(\mathfrak{D}-1)(\mathfrak{D}-1)PQ} \right) \\ &= \frac{1}{\sqrt{2}} \frac{\partial \left(\Lambda_{\mathfrak{D}\mathfrak{D}Q} + \Lambda_{(\mathfrak{D}-1)(\mathfrak{D}-1)Q} \right)}{\partial x^P} - \frac{1}{\sqrt{2}} \frac{\partial \left(\Lambda_{\mathfrak{D}\mathfrak{D}P} + \Lambda_{(\mathfrak{D}-1)(\mathfrak{D}-1)P} \right)}{\partial x^Q} = \frac{\partial B_Q}{\partial x^P} - \frac{\partial B_P}{\partial x^Q}. \end{aligned}$$

$$\begin{aligned}
F_{PQ}^3 &\triangleq \frac{1}{\sqrt{2}} (K_{\mathfrak{D}\mathfrak{D}PQ} - K_{(\mathfrak{D}-1)(\mathfrak{D}-1)PQ}) \\
&= \frac{1}{\sqrt{2}} \left(\frac{\partial \Lambda_{\mathfrak{D}\mathfrak{D}Q}}{\partial x^P} - \frac{\partial \Lambda_{\mathfrak{D}\mathfrak{D}P}}{\partial x^Q} + G^{(\mathfrak{D}-1)(\mathfrak{D}-1)} (\Lambda_{\mathfrak{D}(\mathfrak{D}-1)P} \Lambda_{(\mathfrak{D}-1)\mathfrak{D}Q} - \Lambda_{(\mathfrak{D}-1)\mathfrak{D}P} \Lambda_{\mathfrak{D}(\mathfrak{D}-1)Q}) \right) \\
&\quad - \frac{1}{\sqrt{2}} \left(\frac{\partial \Lambda_{(\mathfrak{D}-1)(\mathfrak{D}-1)Q}}{\partial x^P} - \frac{\partial \Lambda_{(\mathfrak{D}-1)(\mathfrak{D}-1)P}}{\partial x^Q} + G^{\mathfrak{D}\mathfrak{D}} (\Lambda_{(\mathfrak{D}-1)\mathfrak{D}P} \Lambda_{\mathfrak{D}(\mathfrak{D}-1)Q} - \Lambda_{\mathfrak{D}(\mathfrak{D}-1)P} \Lambda_{(\mathfrak{D}-1)\mathfrak{D}Q}) \right) \\
&= \frac{\partial A_Q^3}{\partial x^P} - \frac{\partial A_P^3}{\partial x^Q} + g (\Lambda_{\mathfrak{D}(\mathfrak{D}-1)P} \Lambda_{(\mathfrak{D}-1)\mathfrak{D}Q} - \Lambda_{(\mathfrak{D}-1)\mathfrak{D}P} \Lambda_{\mathfrak{D}(\mathfrak{D}-1)Q}) \\
&= \frac{\partial A_Q^3}{\partial x^P} - \frac{\partial A_P^3}{\partial x^Q} + g (A_P^1 A_Q^2 - A_P^2 A_Q^1).
\end{aligned}$$

Similarly we also obtain $F_{PQ}^1 = \frac{\partial A_Q^1}{\partial x^P} - \frac{\partial A_P^1}{\partial x^Q} + g (A_P^2 A_Q^3 - A_P^3 A_Q^2)$ and $F_{PQ}^2 = \frac{\partial A_Q^2}{\partial x^P} - \frac{\partial A_P^2}{\partial x^Q} - g (A_P^3 A_Q^1 - A_P^1 A_Q^3)$. $\square$

Remark 5.1. Comparing the above conclusion and $U(1) \times SU(2)$ principal bundle theory, we know this proposition indicates that the reference-system f indeed can describe weak and electromagnetic field.

The following proposition shows an advantage of affine connection representation, that is to say, affine connection representation implies the chiral asymmetry of weak interaction, but $U(1) \times SU(2)$ principal bundle connection representation cannot imply it automatically.

Definition 5.2. Suppose f and g both satisfy Definition 5.1. According to Definition 3.2.1 and Definition 3.5.1, let ρ_{mn} of f evolve on (M, g) . Then $l \triangleq (\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}, \rho_{\mathfrak{D}\mathfrak{D}})$ is called an **electric charged lepton**, and $\nu \triangleq (\rho_{\mathfrak{D}(\mathfrak{D}-1)}, \rho_{(\mathfrak{D}-1)\mathfrak{D}})$ is called a **neutrino**. l and ν are collectively called **leptons**, denoted by L . And $L \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is called a **left-handed lepton**, $L \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is called a **right-handed lepton**, denoted by

$$\begin{cases} l_L \triangleq \frac{1}{\sqrt{2}} (\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} + \rho_{\mathfrak{D}\mathfrak{D}}), & \begin{cases} \nu_L \triangleq \frac{1}{\sqrt{2}} (\rho_{\mathfrak{D}(\mathfrak{D}-1)} + \rho_{(\mathfrak{D}-1)\mathfrak{D}}), \\ \nu_R \triangleq \frac{1}{\sqrt{2}} (\rho_{\mathfrak{D}(\mathfrak{D}-1)} - \rho_{(\mathfrak{D}-1)\mathfrak{D}}). \end{cases} \\ l_R \triangleq \frac{1}{\sqrt{2}} (\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} - \rho_{\mathfrak{D}\mathfrak{D}}), & \end{cases}$$

On (M, g) we define

$$\begin{cases} W_P^1 \triangleq \frac{1}{\sqrt{2}} (\Gamma_{(\mathfrak{D}-1)\mathfrak{D}P} + \Gamma_{\mathfrak{D}(\mathfrak{D}-1)P}), & \begin{cases} Z_P \triangleq \frac{1}{\sqrt{2}} (\Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-1)P} + \Gamma_{\mathfrak{D}\mathfrak{D}P}), \\ A_P \triangleq \frac{1}{\sqrt{2}} (\Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-1)P} - \Gamma_{\mathfrak{D}\mathfrak{D}P}), \end{cases} \\ W_P^2 \triangleq \frac{1}{\sqrt{2}} (\Gamma_{(\mathfrak{D}-1)\mathfrak{D}P} - \Gamma_{\mathfrak{D}(\mathfrak{D}-1)P}), & \end{cases}$$

and say the intrinsic geometric property A_P is the **electromagnetic potential**, Z_P is **Z potential**, W_P^1 and W_P^2 are **W potentials**. Then denote

$$\begin{cases} W_P^+ \triangleq \frac{1}{\sqrt{2}} (W_P^1 - iW_P^2), & \begin{cases} l_L^+ = \frac{1}{\sqrt{2}} (l_L - il_R), & \begin{cases} l_R^+ \triangleq \frac{1}{\sqrt{2}} (l_R - il_L), \\ l_R^- \triangleq \frac{1}{\sqrt{2}} (l_R + il_L). \end{cases} \\ W_P^- \triangleq \frac{1}{\sqrt{2}} (W_P^1 + iW_P^2), & \begin{cases} l_L^- = \frac{1}{\sqrt{2}} (l_L + il_R), \end{cases} \end{cases}
\end{cases}$$

Proposition 5.2. If (M, g) satisfies the symmetry condition $\Gamma_{(\mathfrak{D}-1)\mathfrak{D}P} = \Gamma_{\mathfrak{D}(\mathfrak{D}-1)P}$, then the intrinsic geometric properties l and ν of f satisfy the following conclusions on (M, g) .

$$\begin{cases} l_{L;P} = \partial_P l_L - gl_L Z_P - gl_R A_P - g\nu_L W_P^1, \\ l_{R;P} = \partial_P l_R - gl_R Z_P - gl_L A_P, \\ \nu_{L;P} = \partial_P \nu_L - g\nu_L Z_P - gl_L W_P^1, \\ \nu_{R;P} = \partial_P \nu_R - g\nu_R Z_P. \end{cases}$$

Proof. Due to $\rho_{mn;P} = \partial_P \rho_{mn} - \rho_{Hn} \Gamma_{mP}^H - \rho_{mH} \Gamma_{nP}^H = \partial_P \rho_{mn} - \rho_{hn} \Gamma_{mP}^h - \rho_{mh} \Gamma_{nP}^h$, we have

$$\begin{aligned} & \begin{cases} \rho_{(\mathfrak{D}-1)(\mathfrak{D}-1);P} = \partial_P \rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} - 2\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} \Gamma_{(\mathfrak{D}-1)P}^{\mathfrak{D}-1} - (\rho_{\mathfrak{D}(\mathfrak{D}-1)} + \rho_{(\mathfrak{D}-1)\mathfrak{D}}) \Gamma_{(\mathfrak{D}-1)P}^{\mathfrak{D}}, \\ \rho_{\mathfrak{D}\mathfrak{D};P} = \partial_P \rho_{\mathfrak{D}\mathfrak{D}} - 2\rho_{\mathfrak{D}\mathfrak{D}} \Gamma_{\mathfrak{D}P}^{\mathfrak{D}} - (\rho_{\mathfrak{D}(\mathfrak{D}-1)} + \rho_{(\mathfrak{D}-1)\mathfrak{D}}) \Gamma_{\mathfrak{D}P}^{\mathfrak{D}-1}, \\ \rho_{\mathfrak{D}(\mathfrak{D}-1);P} = \partial_P \rho_{\mathfrak{D}(\mathfrak{D}-1)} - (\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} \Gamma_{\mathfrak{D}P}^{\mathfrak{D}-1} + \rho_{\mathfrak{D}\mathfrak{D}} \Gamma_{(\mathfrak{D}-1)P}^{\mathfrak{D}}) - \rho_{\mathfrak{D}(\mathfrak{D}-1)} \left(\Gamma_{(\mathfrak{D}-1)P}^{\mathfrak{D}-1} + \Gamma_{\mathfrak{D}P}^{\mathfrak{D}} \right), \\ \rho_{(\mathfrak{D}-1)\mathfrak{D};P} = \partial_P \rho_{(\mathfrak{D}-1)\mathfrak{D}} - (\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} \Gamma_{\mathfrak{D}P}^{\mathfrak{D}-1} + \rho_{\mathfrak{D}\mathfrak{D}} \Gamma_{(\mathfrak{D}-1)P}^{\mathfrak{D}}) - \rho_{(\mathfrak{D}-1)\mathfrak{D}} \left(\Gamma_{(\mathfrak{D}-1)P}^{\mathfrak{D}-1} + \Gamma_{\mathfrak{D}P}^{\mathfrak{D}} \right). \end{cases} \\ \Rightarrow & \begin{cases} l_{L;P} = \partial_P l_L - \sqrt{2}\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} \Gamma_{(\mathfrak{D}-1)P}^{\mathfrak{D}-1} - \sqrt{2}\rho_{\mathfrak{D}\mathfrak{D}} \Gamma_{\mathfrak{D}P}^{\mathfrak{D}} - g\nu_L W_P^1, \\ l_{R;P} = \partial_P l_R - \sqrt{2}\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} \Gamma_{(\mathfrak{D}-1)P}^{\mathfrak{D}-1} + \sqrt{2}\rho_{\mathfrak{D}\mathfrak{D}} \Gamma_{\mathfrak{D}P}^{\mathfrak{D}}, \\ \nu_{L;P} = \partial_P \nu_L - gl_L W_P^1 - \nu_L \left(\Gamma_{(\mathfrak{D}-1)P}^{\mathfrak{D}-1} + \Gamma_{\mathfrak{D}P}^{\mathfrak{D}} \right), \\ \nu_{R;P} = \partial_P \nu_R - \nu_R \left(\Gamma_{(\mathfrak{D}-1)P}^{\mathfrak{D}-1} + \Gamma_{\mathfrak{D}P}^{\mathfrak{D}} \right). \end{cases} \\ \Rightarrow & \begin{cases} l_{L;P} = \partial_P l_L - gl_L Z_P - gl_R A_P - g\nu_L W_P^1, \\ l_{R;P} = \partial_P l_R - gl_R Z_P - gl_L A_P, \\ \nu_{L;P} = \partial_P \nu_L - g\nu_L Z_P - gl_L W_P^1, \\ \nu_{R;P} = \partial_P \nu_R - g\nu_R Z_P. \end{cases} \end{aligned}$$

□

Remark 5.2. The above proposition shows that the chiral asymmetry of leptons is a direct mathematical result of affine connection representation, and that it is not necessary to be postulated artificially like that in $U_Y(1) \times SU_L(2)$ theory.

Discussion 5.1. According to section 2.4, the intrinsic geometry is the largest geometry on geometric manifold, and its geometric properties are the richest, so that any irregular smooth shape can be precisely characterized by B_M^A and C_A^M . Its equivalent transformation group is the trivial subgroup $\{e\}$ that has only one element. We notice that:

(1) Principal bundle connection representation starts from a very large group, and reduces symmetries in way of "symmetry breaking" to approach the target geometry.

(2) Affine connection representation starts from the smallest group $\{e\}$, and adds symmetries in way of "symmetry condition" to approach the target geometry.

Such two ways must lead to the same destination. They both go towards the same specific geometry.

Remark 5.3. Proposition 5.2 shows that:

(1) In affine connection representation of gauge fields, the coupling constant g is possessed of a geometric meaning, that is in fact the metric of internal space. But it does not have such a clear geometric meaning in $U(1) \times SU(2)$ principal bundle connection representation.

(2) At the most fundamental level, the coupling constant of Z_P and that of A_P are equal, i.e.

$$g_Z = g_A = g.$$

Suppose there is a kind of medium. Z boson and photon move in it. Suppose Z field has interaction with the medium, but electromagnetic field A has no interaction with the medium. Then we have coupling constants

$$\tilde{g}_Z \neq \tilde{g}_A$$

in the medium, and the Weinberg angle appears.

It is quite reasonable to consider a Higgs boson as a zero-spin pair of neutrinos, because in Standard Model, Higgs boson only couples with Z field and W field, but does not couple with electromagnetic field and gluon field. If so, Higgs boson would lose its fundamentality and it would not have enough importance in a theory at the most fundamental level.

(3) The mixing of leptons of three generations do not appear in Proposition 5.2, but it can spontaneously appear in Proposition 7.1 due to the affine connection representation of the gauge field that is given by Definition 7.1.

6. Affine connection representation of the gauge field of strong interaction

Definition 6.1. Suppose a geometric manifold (M, f) satisfies $\mathfrak{D} = r + 3 = 6$. $\forall p \in M$, the coordinate representation of $f(p)$ takes the form of

$$\xi^a = \xi^a(x^m), \quad \xi^s = \delta_i^s x^i, \quad (s, i = 1, 2, 3; a, m = 4, 5, 6),$$

and f satisfies the internal standard conditions (55) of Definition 4.6.1(1) and

$$G^{(\mathfrak{D}-2)(\mathfrak{D}-2)} = G^{(\mathfrak{D}-1)(\mathfrak{D}-1)} = G^{\mathfrak{D}\mathfrak{D}}.$$

We say f is a **strong interaction field**.

Definition 6.2. Suppose f and g both satisfy Definition 6.1. According to Definition 3.2.1 and Definition 3.5.1, let ρ_{mn} of f evolve on (M, g) . Define

$$\begin{cases} d_1 \triangleq (\rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)}, \rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}), \\ d_2 \triangleq (\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}, \rho_{\mathfrak{D}\mathfrak{D}}), \\ d_3 \triangleq (\rho_{\mathfrak{D}\mathfrak{D}}, \rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)}), \end{cases} \quad \begin{cases} u_1 \triangleq (\rho_{(\mathfrak{D}-2)(\mathfrak{D}-1)}, \rho_{(\mathfrak{D}-1)(\mathfrak{D}-2)}), \\ u_2 \triangleq (\rho_{(\mathfrak{D}-1)\mathfrak{D}}, \rho_{\mathfrak{D}(\mathfrak{D}-1)}), \\ u_3 \triangleq (\rho_{\mathfrak{D}(\mathfrak{D}-2)}, \rho_{(\mathfrak{D}-2)\mathfrak{D}}). \end{cases}$$

We say d_1 and u_1 are **red color charges**, d_2 and u_2 are **blue color charges**, d_3 and u_3 are **green color charges**. Then d_1, d_2, d_3 are called **down-type color charges**, uniformly denoted by d , and u_1, u_2, u_3 are called **up-type color charges**, uniformly denoted by u . d and u are collectively called

color charges, denoted by q . We say $q \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a **left-handed color charge**, and $q \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is a **right-handed color charge**. They are

$$\begin{cases} d_{1L} \triangleq \frac{1}{\sqrt{2}} (\rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}), \\ d_{2L} \triangleq \frac{1}{\sqrt{2}} (\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} + \rho_{\mathfrak{D}\mathfrak{D}}), \\ d_{3L} \triangleq \frac{1}{\sqrt{2}} (\rho_{\mathfrak{D}\mathfrak{D}} + \rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)}), \end{cases} \quad \begin{cases} d_{1R} \triangleq \frac{1}{\sqrt{2}} (\rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)} - \rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}), \\ d_{2R} \triangleq \frac{1}{\sqrt{2}} (\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} - \rho_{\mathfrak{D}\mathfrak{D}}), \\ d_{3R} \triangleq \frac{1}{\sqrt{2}} (\rho_{\mathfrak{D}\mathfrak{D}} - \rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)}), \end{cases}$$

$$\begin{cases} u_{1L} \triangleq \frac{1}{\sqrt{2}} (\rho_{(\mathfrak{D}-2)(\mathfrak{D}-1)} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-2)}), \\ u_{2L} \triangleq \frac{1}{\sqrt{2}} (\rho_{(\mathfrak{D}-1)\mathfrak{D}} + \rho_{\mathfrak{D}(\mathfrak{D}-1)}), \\ u_{3L} \triangleq \frac{1}{\sqrt{2}} (\rho_{\mathfrak{D}(\mathfrak{D}-2)} + \rho_{(\mathfrak{D}-2)\mathfrak{D}}), \end{cases} \quad \begin{cases} u_{1R} \triangleq \frac{1}{\sqrt{2}} (\rho_{(\mathfrak{D}-2)(\mathfrak{D}-1)} - \rho_{(\mathfrak{D}-1)(\mathfrak{D}-2)}), \\ u_{2R} \triangleq \frac{1}{\sqrt{2}} (\rho_{(\mathfrak{D}-1)\mathfrak{D}} - \rho_{\mathfrak{D}(\mathfrak{D}-1)}), \\ u_{3R} \triangleq \frac{1}{\sqrt{2}} (\rho_{\mathfrak{D}(\mathfrak{D}-2)} - \rho_{(\mathfrak{D}-2)\mathfrak{D}}). \end{cases}$$

On (M, g) we denote

$$\begin{aligned} g_s &\triangleq \sqrt{(G^{(\mathfrak{D}-1)(\mathfrak{D}-1)})^2 + (G^{\mathfrak{D}\mathfrak{D}})^2} = \sqrt{(G^{(\mathfrak{D}-1)(\mathfrak{D}-1)})^2 + (G^{(\mathfrak{D}-2)(\mathfrak{D}-2)})^2} \\ &= \sqrt{(G^{(\mathfrak{D}-2)(\mathfrak{D}-2)})^2 + (G^{\mathfrak{D}\mathfrak{D}})^2}. \end{aligned}$$

$$\begin{cases} U_P^1 \triangleq \frac{1}{\sqrt{2}} (\Gamma_{(\mathfrak{D}-2)(\mathfrak{D}-2)P} + \Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-1)P}), \\ V_P^1 \triangleq \frac{1}{\sqrt{2}} (\Gamma_{(\mathfrak{D}-2)(\mathfrak{D}-2)P} - \Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-1)P}), \end{cases} \quad \begin{cases} X_P^{23} \triangleq \frac{1}{\sqrt{2}} (\Gamma_{(\mathfrak{D}-2)(\mathfrak{D}-1)P} + \Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-2)P}), \\ Y_P^{23} \triangleq \frac{1}{\sqrt{2}} (\Gamma_{(\mathfrak{D}-2)(\mathfrak{D}-1)P} - \Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-2)P}), \end{cases}$$

$$\begin{cases} U_P^2 \triangleq \frac{1}{\sqrt{2}} (\Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-1)P} + \Gamma_{\mathfrak{D}\mathfrak{D}P}), \\ V_P^2 \triangleq \frac{1}{\sqrt{2}} (\Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-1)P} - \Gamma_{\mathfrak{D}\mathfrak{D}P}), \end{cases} \quad \begin{cases} X_P^{31} \triangleq \frac{1}{\sqrt{2}} (\Gamma_{(\mathfrak{D}-1)\mathfrak{D}P} + \Gamma_{\mathfrak{D}(\mathfrak{D}-1)P}), \\ Y_P^{31} \triangleq \frac{1}{\sqrt{2}} (\Gamma_{(\mathfrak{D}-1)\mathfrak{D}P} - \Gamma_{\mathfrak{D}(\mathfrak{D}-1)P}), \end{cases}$$

$$\begin{cases} U_P^3 \triangleq \frac{1}{\sqrt{2}} (\Gamma_{\mathfrak{D}\mathfrak{D}P} + \Gamma_{(\mathfrak{D}-2)(\mathfrak{D}-2)P}), \\ V_P^3 \triangleq \frac{1}{\sqrt{2}} (\Gamma_{\mathfrak{D}\mathfrak{D}P} - \Gamma_{(\mathfrak{D}-2)(\mathfrak{D}-2)P}), \end{cases} \quad \begin{cases} X_P^{12} \triangleq \frac{1}{\sqrt{2}} (\Gamma_{\mathfrak{D}(\mathfrak{D}-2)P} + \Gamma_{(\mathfrak{D}-2)\mathfrak{D}P}), \\ Y_P^{12} \triangleq \frac{1}{\sqrt{2}} (\Gamma_{\mathfrak{D}(\mathfrak{D}-2)P} - \Gamma_{(\mathfrak{D}-2)\mathfrak{D}P}). \end{cases}$$

We notice that there are just only three independent ones in $U_P^1, U_P^2, U_P^3, V_P^1, V_P^2$ and V_P^3 . Without loss of generality, let

$$\begin{cases} R_P \triangleq a_R U_P^1 + b_R U_P^2 + c_R U_P^3, \\ S_P \triangleq a_S U_P^1 + b_S U_P^2 + c_S U_P^3, \\ T_P \triangleq a_T U_P^1 + b_T U_P^2 + c_T U_P^3, \end{cases} \quad \begin{cases} U_P^1 \triangleq \alpha_R R_P + \alpha_S S_P + \alpha_T T_P, \\ U_P^2 \triangleq \beta_R R_P + \beta_S S_P + \beta_T T_P, \\ U_P^3 \triangleq \gamma_R R_P + \gamma_S S_P + \gamma_T T_P, \end{cases}$$

where the coefficients matrix is non-singular.

Proposition 6.1. Let λ_a ($a = 1, 2, \dots, 8$) be the Gell-Mann matrices, and $T_a \triangleq \frac{1}{2}\lambda_a$ the generators of $SU(3)$ group. When (M, g) satisfies the symmetry condition $\Gamma_{(\mathfrak{D}-2)(\mathfrak{D}-2)P} + \Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-1)P} + \Gamma_{\mathfrak{D}\mathfrak{D}P} = 0$, denote

$$A_P \triangleq \frac{1}{2} \begin{pmatrix} A_P^{11} & A_P^{12} & A_P^{13} \\ A_P^{21} & A_P^{22} & A_P^{23} \\ A_P^{31} & A_P^{32} & A_P^{33} \end{pmatrix}, \quad \begin{aligned} A_P^{11} &\triangleq S_P + \frac{1}{\sqrt{6}}T_P, & A_P^{12} &\triangleq X_P^{12} - iY_P^{12}, & A_P^{13} &\triangleq X_P^{31} - iY_P^{31}, \\ A_P^{21} &\triangleq X_P^{12} + iY_P^{12}, & A_P^{22} &\triangleq -S_P + \frac{1}{\sqrt{6}}T_P, & A_P^{23} &\triangleq X_P^{23} - iY_P^{23}, \\ A_P^{31} &\triangleq X_P^{31} + iY_P^{31}, & A_P^{32} &\triangleq X_P^{23} + iY_P^{23}, & A_P^{33} &\triangleq -\frac{2}{\sqrt{6}}T_P. \end{aligned}$$

Thus, $A_P = T_a A_P^a$ if and only if $A_P^1 = X_P^{12}$, $A_P^2 = Y_P^{12}$, $A_P^3 = S_P$, $A_P^4 = X_P^{31}$, $A_P^5 = Y_P^{31}$, $A_P^6 = X_P^{23}$, $A_P^7 = Y_P^{23}$ and $A_P^8 = T_P$.

Proof. We just need to substitute the Gell-Mann matrices

$$\begin{aligned} \lambda_1 &\triangleq \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 \triangleq \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 \triangleq \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 \triangleq \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\ \lambda_5 &\triangleq \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 \triangleq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 \triangleq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 \triangleq \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \end{aligned}$$

into $A_P = T_a A_P^a$ and directly verify them. $\square$

Remark 6.1. On one hand, the above proposition shows that Definition 6.1 is an affine connection representation of strong interaction field. It does not define the gauge potentials abstractly like QCD based on $SU(3)$ theory, but gives concrete intrinsic geometric constructions to the gauge potentials. On the other hand, the above proposition implies that if we take appropriate symmetry conditions, the algebraic properties of $SU(3)$ group can be described by the transformation group $GL(3, \mathbb{R})$ of internal space of g . In other words, the exponential map $\exp : GL(3, \mathbb{R}) \rightarrow U(3)$, $[B_m^a] \mapsto e^{iT_m^a B_m^a}$ defines a covering homomorphism, and $SU(3)$ is a subgroup of $U(3)$.

7. Affine connection representation of the unified gauge field

Definition 7.1. Suppose a geometric manifold (M, f) satisfies $\mathfrak{D} = r + 5 = 8$. $\forall p \in M$, the coordinate representation of $f(p)$ takes the form of

$$\xi^a = \xi^a(x^m), \quad \xi^s = \delta_i^s x^i, \quad (s, i = 1, 2, 3; a, m = 4, 5, 6, 7, 8),$$

and f satisfies the internal standard conditions (55) of Definition 4.6.1(1) and

$$G^{(\mathfrak{D}-4)(\mathfrak{D}-4)} = G^{(\mathfrak{D}-3)(\mathfrak{D}-3)}, \quad G^{(\mathfrak{D}-2)(\mathfrak{D}-2)} = G^{(\mathfrak{D}-1)(\mathfrak{D}-1)} = G^{\mathfrak{D}\mathfrak{D}}.$$

We say f is a **typical unified gauge field**.

Definition 7.2. Suppose f and g both satisfy Definition 7.1. According to Definition 3.2.1 and Definition 3.5.1, let ρ_{mn} of f evolve on (M, g) . Define

$$\begin{cases} l \triangleq (\rho_{(\mathfrak{D}-4)(\mathfrak{D}-4)}, \rho_{(\mathfrak{D}-3)(\mathfrak{D}-3)}), \\ d_1 \triangleq (\rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)}, \rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}), \\ d_2 \triangleq (\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}, \rho_{\mathfrak{D}\mathfrak{D}}), \\ d_3 \triangleq (\rho_{\mathfrak{D}\mathfrak{D}}, \rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)}), \end{cases} \quad \begin{cases} \nu \triangleq (\rho_{(\mathfrak{D}-3)(\mathfrak{D}-4)}, \rho_{(\mathfrak{D}-4)(\mathfrak{D}-3)}), \\ u_1 \triangleq (\rho_{(\mathfrak{D}-2)(\mathfrak{D}-1)}, \rho_{(\mathfrak{D}-1)(\mathfrak{D}-2)}), \\ u_2 \triangleq (\rho_{(\mathfrak{D}-1)\mathfrak{D}}, \rho_{\mathfrak{D}(\mathfrak{D}-1)}), \\ u_3 \triangleq (\rho_{\mathfrak{D}(\mathfrak{D}-2)}, \rho_{(\mathfrak{D}-2)\mathfrak{D}}). \end{cases}$$

And Denote

$$\begin{cases} l_L \triangleq \frac{1}{\sqrt{2}} (\rho_{(\mathfrak{D}-4)(\mathfrak{D}-4)} + \rho_{(\mathfrak{D}-3)(\mathfrak{D}-3)}), \\ l_R \triangleq \frac{1}{\sqrt{2}} (\rho_{(\mathfrak{D}-4)(\mathfrak{D}-4)} - \rho_{(\mathfrak{D}-3)(\mathfrak{D}-3)}), \end{cases} \quad \begin{cases} \nu_L \triangleq \frac{1}{\sqrt{2}} (\rho_{(\mathfrak{D}-3)(\mathfrak{D}-4)} + \rho_{(\mathfrak{D}-4)(\mathfrak{D}-3)}), \\ \nu_R \triangleq \frac{1}{\sqrt{2}} (\rho_{(\mathfrak{D}-3)(\mathfrak{D}-4)} - \rho_{(\mathfrak{D}-4)(\mathfrak{D}-3)}), \end{cases}$$

$$\begin{cases} d_{1L} \triangleq \frac{1}{\sqrt{2}} (\rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}), \\ d_{2L} \triangleq \frac{1}{\sqrt{2}} (\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} + \rho_{\mathfrak{D}\mathfrak{D}}), \\ d_{3L} \triangleq \frac{1}{\sqrt{2}} (\rho_{\mathfrak{D}\mathfrak{D}} + \rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)}), \end{cases} \quad \begin{cases} d_{1R} \triangleq \frac{1}{\sqrt{2}} (\rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)} - \rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}), \\ d_{2R} \triangleq \frac{1}{\sqrt{2}} (\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} - \rho_{\mathfrak{D}\mathfrak{D}}), \\ d_{3R} \triangleq \frac{1}{\sqrt{2}} (\rho_{\mathfrak{D}\mathfrak{D}} - \rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)}), \end{cases}$$

$$\begin{cases} u_{1L} \triangleq \frac{1}{\sqrt{2}} (\rho_{(\mathfrak{D}-2)(\mathfrak{D}-1)} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-2)}), \\ u_{2L} \triangleq \frac{1}{\sqrt{2}} (\rho_{(\mathfrak{D}-1)\mathfrak{D}} + \rho_{\mathfrak{D}(\mathfrak{D}-1)}), \\ u_{3L} \triangleq \frac{1}{\sqrt{2}} (\rho_{\mathfrak{D}(\mathfrak{D}-2)} + \rho_{(\mathfrak{D}-2)\mathfrak{D}}), \end{cases} \quad \begin{cases} u_{1R} \triangleq \frac{1}{\sqrt{2}} (\rho_{(\mathfrak{D}-2)(\mathfrak{D}-1)} - \rho_{(\mathfrak{D}-1)(\mathfrak{D}-2)}), \\ u_{2R} \triangleq \frac{1}{\sqrt{2}} (\rho_{(\mathfrak{D}-1)\mathfrak{D}} - \rho_{\mathfrak{D}(\mathfrak{D}-1)}), \\ u_{3R} \triangleq \frac{1}{\sqrt{2}} (\rho_{\mathfrak{D}(\mathfrak{D}-2)} - \rho_{(\mathfrak{D}-2)\mathfrak{D}}). \end{cases}$$

On (M, g) we denote

$$\begin{cases} g \triangleq \sqrt{(G^{(\mathfrak{D}-4)(\mathfrak{D}-4)})^2 + (G^{(\mathfrak{D}-3)(\mathfrak{D}-3)})^2}, \\ g_s \triangleq \sqrt{(G^{(\mathfrak{D}-1)(\mathfrak{D}-1)})^2 + (G^{\mathfrak{D}\mathfrak{D}})^2} = \sqrt{(G^{(\mathfrak{D}-1)(\mathfrak{D}-1)})^2 + (G^{(\mathfrak{D}-2)(\mathfrak{D}-2)})^2} \\ \quad = \sqrt{(G^{(\mathfrak{D}-2)(\mathfrak{D}-2)})^2 + (G^{\mathfrak{D}\mathfrak{D}})^2}, \end{cases}$$

$$\begin{cases} Z_P \triangleq \frac{1}{\sqrt{2}} (\Gamma_{(\mathfrak{D}-4)(\mathfrak{D}-4)P} + \Gamma_{(\mathfrak{D}-3)(\mathfrak{D}-3)P}), \\ A_P \triangleq \frac{1}{\sqrt{2}} (\Gamma_{(\mathfrak{D}-4)(\mathfrak{D}-4)P} - \Gamma_{(\mathfrak{D}-3)(\mathfrak{D}-3)P}), \end{cases} \quad \begin{cases} W_P^1 \triangleq \frac{1}{\sqrt{2}} (\Gamma_{(\mathfrak{D}-4)(\mathfrak{D}-3)P} + \Gamma_{(\mathfrak{D}-3)(\mathfrak{D}-4)P}), \\ W_P^2 \triangleq \frac{1}{\sqrt{2}} (\Gamma_{(\mathfrak{D}-4)(\mathfrak{D}-3)P} - \Gamma_{(\mathfrak{D}-3)(\mathfrak{D}-4)P}), \end{cases}$$

$$\begin{cases} U_P^1 \triangleq \frac{1}{\sqrt{2}} (\Gamma_{(\mathfrak{D}-2)(\mathfrak{D}-2)P} + \Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-1)P}), \\ V_P^1 \triangleq \frac{1}{\sqrt{2}} (\Gamma_{(\mathfrak{D}-2)(\mathfrak{D}-2)P} - \Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-1)P}), \end{cases} \quad \begin{cases} X_P^{23} \triangleq \frac{1}{\sqrt{2}} (\Gamma_{(\mathfrak{D}-2)(\mathfrak{D}-1)P} + \Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-2)P}), \\ Y_P^{23} \triangleq \frac{1}{\sqrt{2}} (\Gamma_{(\mathfrak{D}-2)(\mathfrak{D}-1)P} - \Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-2)P}), \end{cases}$$

$$\begin{cases} U_P^2 \triangleq \frac{1}{\sqrt{2}} (\Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-1)P} + \Gamma_{\mathfrak{D}\mathfrak{D}P}), & X_P^{31} \triangleq \frac{1}{\sqrt{2}} (\Gamma_{(\mathfrak{D}-1)\mathfrak{D}P} + \Gamma_{\mathfrak{D}(\mathfrak{D}-1)P}), \\ V_P^2 \triangleq \frac{1}{\sqrt{2}} (\Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-1)P} - \Gamma_{\mathfrak{D}\mathfrak{D}P}), & Y_P^{31} \triangleq \frac{1}{\sqrt{2}} (\Gamma_{(\mathfrak{D}-1)\mathfrak{D}P} - \Gamma_{\mathfrak{D}(\mathfrak{D}-1)P}), \\ U_P^3 \triangleq \frac{1}{\sqrt{2}} (\Gamma_{\mathfrak{D}\mathfrak{D}P} + \Gamma_{(\mathfrak{D}-2)(\mathfrak{D}-2)P}), & X_P^{12} \triangleq \frac{1}{\sqrt{2}} (\Gamma_{\mathfrak{D}(\mathfrak{D}-2)P} + \Gamma_{(\mathfrak{D}-2)\mathfrak{D}P}), \\ V_P^3 \triangleq \frac{1}{\sqrt{2}} (\Gamma_{\mathfrak{D}\mathfrak{D}P} - \Gamma_{(\mathfrak{D}-2)(\mathfrak{D}-2)P}), & Y_P^{12} \triangleq \frac{1}{\sqrt{2}} (\Gamma_{\mathfrak{D}(\mathfrak{D}-2)P} - \Gamma_{(\mathfrak{D}-2)\mathfrak{D}P}). \end{cases}$$

Definition 7.3. Define the symmetry condition of unification:

(1) Basic conditions, No.1:

$$\begin{cases} G^{(\mathfrak{D}-4)(\mathfrak{D}-4)} = G^{(\mathfrak{D}-3)(\mathfrak{D}-3)}, \\ G^{(\mathfrak{D}-2)(\mathfrak{D}-2)} = G^{(\mathfrak{D}-1)(\mathfrak{D}-1)} = G^{\mathfrak{D}\mathfrak{D}}, \end{cases}$$

(2) Basic conditions, No.2:

$$\begin{cases} \Gamma_{(\mathfrak{D}-3)(\mathfrak{D}-4)P} = \Gamma_{(\mathfrak{D}-4)(\mathfrak{D}-3)P}, \\ \Gamma_{(\mathfrak{D}-2)(\mathfrak{D}-2)P} + \Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-1)P} + \Gamma_{\mathfrak{D}\mathfrak{D}P} = 0, \end{cases}$$

(3) MNS mixing conditions of weak interaction, No.1:

$$\begin{cases} \Gamma_{(\mathfrak{D}-4)P}^{\mathfrak{D}-2} = c_{\mathfrak{D}-3}^{\mathfrak{D}-2} \Gamma_{(\mathfrak{D}-4)P}^{\mathfrak{D}-3}, & \Gamma_{(\mathfrak{D}-3)P}^{\mathfrak{D}-2} = c_{\mathfrak{D}-4}^{\mathfrak{D}-2} \Gamma_{(\mathfrak{D}-3)P}^{\mathfrak{D}-4}, & \begin{cases} c_{\mathfrak{D}-3}^{\mathfrak{D}-2} = c_{\mathfrak{D}-4}^{\mathfrak{D}-2}, \\ c_{\mathfrak{D}-3}^{\mathfrak{D}-1} = c_{\mathfrak{D}-4}^{\mathfrak{D}-1}, \\ c_{\mathfrak{D}-3}^{\mathfrak{D}} = c_{\mathfrak{D}-4}^{\mathfrak{D}}, \end{cases} \\ \Gamma_{(\mathfrak{D}-4)P}^{\mathfrak{D}-1} = c_{\mathfrak{D}-3}^{\mathfrak{D}-1} \Gamma_{(\mathfrak{D}-4)P}^{\mathfrak{D}-3}, & \Gamma_{(\mathfrak{D}-3)P}^{\mathfrak{D}-1} = c_{\mathfrak{D}-4}^{\mathfrak{D}-1} \Gamma_{(\mathfrak{D}-3)P}^{\mathfrak{D}-4}, \\ \Gamma_{(\mathfrak{D}-4)P}^{\mathfrak{D}} = c_{\mathfrak{D}-3}^{\mathfrak{D}} \Gamma_{(\mathfrak{D}-4)P}^{\mathfrak{D}-3}, & \Gamma_{(\mathfrak{D}-3)P}^{\mathfrak{D}} = c_{\mathfrak{D}-4}^{\mathfrak{D}} \Gamma_{(\mathfrak{D}-3)P}^{\mathfrak{D}-4}, \end{cases}$$

(4) MNS mixing conditions of weak interaction, No.2:

$$\begin{cases} \rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)} = \rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)}, & \rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)} = \rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)}, \\ \rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)} = \rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)}, & \rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)} = \rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)}, \\ \rho_{\mathfrak{D}(\mathfrak{D}-3)} = \rho_{\mathfrak{D}(\mathfrak{D}-4)}, & \rho_{(\mathfrak{D}-3)\mathfrak{D}} = \rho_{(\mathfrak{D}-4)\mathfrak{D}}, \end{cases}$$

(5) CKM mixing conditions of strong interaction, No.1:

$$\begin{cases} \Gamma_{(\mathfrak{D}-2)P}^{\mathfrak{D}-3} = c_{\mathfrak{D}-2}^{\mathfrak{D}-4} \Gamma_{(\mathfrak{D}-4)P}^{\mathfrak{D}-3}, & \Gamma_{(\mathfrak{D}-2)P}^{\mathfrak{D}-4} = c_{\mathfrak{D}-2}^{\mathfrak{D}-3} \Gamma_{(\mathfrak{D}-3)P}^{\mathfrak{D}-4}, & \begin{cases} c_{\mathfrak{D}-2}^{\mathfrak{D}-4} = c_{\mathfrak{D}-1}^{\mathfrak{D}-4} = c_{\mathfrak{D}}^{\mathfrak{D}-4}, \\ c_{\mathfrak{D}-2}^{\mathfrak{D}-3} = c_{\mathfrak{D}-1}^{\mathfrak{D}-3} = c_{\mathfrak{D}}^{\mathfrak{D}-3}, \end{cases} \\ \Gamma_{(\mathfrak{D}-1)P}^{\mathfrak{D}-3} = c_{\mathfrak{D}-1}^{\mathfrak{D}-4} \Gamma_{(\mathfrak{D}-4)P}^{\mathfrak{D}-3}, & \Gamma_{(\mathfrak{D}-1)P}^{\mathfrak{D}-4} = c_{\mathfrak{D}-1}^{\mathfrak{D}-3} \Gamma_{(\mathfrak{D}-3)P}^{\mathfrak{D}-4}, \\ \Gamma_{\mathfrak{D}P}^{\mathfrak{D}-3} = c_{\mathfrak{D}}^{\mathfrak{D}-4} \Gamma_{(\mathfrak{D}-4)P}^{\mathfrak{D}-3}, & \Gamma_{\mathfrak{D}P}^{\mathfrak{D}-4} = c_{\mathfrak{D}}^{\mathfrak{D}-3} \Gamma_{(\mathfrak{D}-3)P}^{\mathfrak{D}-4}, \end{cases}$$

(6) CKM mixing conditions of strong interaction, No.2:

$$\begin{cases} \rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)} = \rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)} = \rho_{\mathfrak{D}(\mathfrak{D}-3)}, & \rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)} = \rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)} = \rho_{(\mathfrak{D}-3)\mathfrak{D}}, \\ \rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)} = \rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)} = \rho_{\mathfrak{D}(\mathfrak{D}-4)}, & \rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)} = \rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)} = \rho_{(\mathfrak{D}-4)\mathfrak{D}}, \end{cases}$$

where c_n^m are constants.

Proposition 7.1. When (M, g) satisfies the symmetry conditions of Definition 7.3, denote

$$\begin{aligned}
l' &\triangleq \left(\rho_{(\mathfrak{D}-4)(\mathfrak{D}-4)} + \frac{c_{\mathfrak{D}-4}^{\mathfrak{D}-2}}{2} (\rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)} + \rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)}) \right. \\
&\quad + \frac{c_{\mathfrak{D}-4}^{\mathfrak{D}-1}}{2} (\rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)} + \rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)}) + \frac{c_{\mathfrak{D}-4}^{\mathfrak{D}}}{2} (\rho_{\mathfrak{D}(\mathfrak{D}-4)} + \rho_{(\mathfrak{D}-4)\mathfrak{D}}), \\
&\quad \rho_{(\mathfrak{D}-3)(\mathfrak{D}-3)} + \frac{c_{\mathfrak{D}-3}^{\mathfrak{D}-2}}{2} (\rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)} + \rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)}) \\
&\quad \left. + \frac{c_{\mathfrak{D}-3}^{\mathfrak{D}-1}}{2} (\rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)} + \rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)}) + \frac{c_{\mathfrak{D}-3}^{\mathfrak{D}}}{2} (\rho_{\mathfrak{D}(\mathfrak{D}-3)} + \rho_{(\mathfrak{D}-3)\mathfrak{D}}) \right), \\
\nu' &\triangleq \left(\rho_{(\mathfrak{D}-3)(\mathfrak{D}-4)} + \frac{c_{\mathfrak{D}-3}^{\mathfrak{D}-2}}{2} (\rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)} + \rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)}) \right. \\
&\quad + \frac{c_{\mathfrak{D}-3}^{\mathfrak{D}-1}}{2} (\rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)} + \rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)}) + \frac{c_{\mathfrak{D}-3}^{\mathfrak{D}}}{2} (\rho_{\mathfrak{D}(\mathfrak{D}-4)} + \rho_{(\mathfrak{D}-4)\mathfrak{D}}), \\
&\quad \rho_{(\mathfrak{D}-4)(\mathfrak{D}-3)} + \frac{c_{\mathfrak{D}-4}^{\mathfrak{D}-2}}{2} (\rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)} + \rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)}) \\
&\quad \left. + \frac{c_{\mathfrak{D}-4}^{\mathfrak{D}-1}}{2} (\rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)} + \rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)}) + \frac{c_{\mathfrak{D}-4}^{\mathfrak{D}}}{2} (\rho_{\mathfrak{D}(\mathfrak{D}-3)} + \rho_{(\mathfrak{D}-3)\mathfrak{D}}) \right).
\end{aligned}$$

Thus, the intrinsic geometric properties l and ν of f satisfy the following conclusions on (M, g) .

$$\begin{cases} l_{L;P} = \partial_P l_L - g l_L Z_P - g l_R A_P - g \nu'_L W_P^1, \\ l_{R;P} = \partial_P l_R - g l_R Z_P - g l_L A_P, \\ \nu_{L;P} = \partial_P \nu_L - g \nu_L Z_P - g l'_L W_P^1, \\ \nu_{R;P} = \partial_P \nu_R - g \nu_R Z_P. \end{cases} \quad (56)$$

Proof. First, we compute the covariant differential of ρ_{mn} of f .

$$\begin{aligned}
\rho_{mn;P} &= \partial_P \rho_{mn} - \rho_{Hn} \Gamma_{mP}^H - \rho_{mH} \Gamma_{nP}^H \\
&= \partial_P \rho_{mn} - \rho_{(\mathfrak{D}-4)n} \Gamma_{mP}^{\mathfrak{D}-4} - \rho_{(\mathfrak{D}-3)n} \Gamma_{mP}^{\mathfrak{D}-3} - \rho_{(\mathfrak{D}-2)n} \Gamma_{mP}^{\mathfrak{D}-2} - \rho_{(\mathfrak{D}-1)n} \Gamma_{mP}^{\mathfrak{D}-1} - \rho_{\mathfrak{D}n} \Gamma_{mP}^{\mathfrak{D}} \\
&\quad - \rho_{m(\mathfrak{D}-4)} \Gamma_{nP}^{\mathfrak{D}-4} - \rho_{m(\mathfrak{D}-3)} \Gamma_{nP}^{\mathfrak{D}-3} - \rho_{m(\mathfrak{D}-2)} \Gamma_{nP}^{\mathfrak{D}-2} - \rho_{m(\mathfrak{D}-1)} \Gamma_{nP}^{\mathfrak{D}-1} - \rho_{m\mathfrak{D}} \Gamma_{nP}^{\mathfrak{D}}.
\end{aligned}$$

According to Definition 7.2 and Definition 7.3, by calculation we obtain that

$$\begin{aligned}
l_{L;P} &= \partial_P l_L - g l_L Z_P - g l_R A_P - g \nu_L W_P^1 \\
&\quad - \frac{1}{2} \left[c_{\mathfrak{D}-4}^{\mathfrak{D}-2} (\rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)} + \rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)}) + c_{\mathfrak{D}-3}^{\mathfrak{D}-2} (\rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)} + \rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)}) \right] \frac{g}{\sqrt{2}} W_P^1 \\
&\quad - \frac{1}{2} \left[c_{\mathfrak{D}-4}^{\mathfrak{D}-1} (\rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)} + \rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)}) + c_{\mathfrak{D}-3}^{\mathfrak{D}-1} (\rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)} + \rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)}) \right] \frac{g}{\sqrt{2}} W_P^1 \\
&\quad - \frac{1}{2} \left[c_{\mathfrak{D}-4}^{\mathfrak{D}} (\rho_{\mathfrak{D}(\mathfrak{D}-3)} + \rho_{(\mathfrak{D}-3)\mathfrak{D}}) + c_{\mathfrak{D}-3}^{\mathfrak{D}} (\rho_{\mathfrak{D}(\mathfrak{D}-4)} + \rho_{(\mathfrak{D}-4)\mathfrak{D}}) \right] \frac{g}{\sqrt{2}} W_P^1, \\
l_{R;P} &= \partial_P l_R - g l_R Z_P - g l_L A_P,
\end{aligned}$$

$$\begin{aligned}
\nu_{L;P} &= \partial_P \nu_L - g \nu_L Z_P - g l_L W_P^1 \\
&\quad - \frac{1}{2} \left[c_{\mathfrak{D}-4}^{\mathfrak{D}-2} (\rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)} + \rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)}) + c_{\mathfrak{D}-3}^{\mathfrak{D}-2} (\rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)} + \rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)}) \right] \frac{g}{\sqrt{2}} W_P^1 \\
&\quad - \frac{1}{2} \left[c_{\mathfrak{D}-4}^{\mathfrak{D}-1} (\rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)} + \rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)}) + c_{\mathfrak{D}-3}^{\mathfrak{D}-1} (\rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)}) \right] \frac{g}{\sqrt{2}} W_P^1 \\
&\quad - \frac{1}{2} \left[c_{\mathfrak{D}-4}^{\mathfrak{D}} (\rho_{\mathfrak{D}(\mathfrak{D}-4)} + \rho_{(\mathfrak{D}-4)\mathfrak{D}}) + c_{\mathfrak{D}-3}^{\mathfrak{D}} (\rho_{(\mathfrak{D}-3)\mathfrak{D}} + \rho_{\mathfrak{D}(\mathfrak{D}-3)}) \right] \frac{g}{\sqrt{2}} W_P^1, \\
\nu_{R;P} &= \partial_P \nu_R - g \nu_R Z_P.
\end{aligned}$$

Then, according to definitions of l' and ν' , we obtain that

$$\begin{aligned}
l'_L &= l_L \\
&\quad + \frac{c_{\mathfrak{D}-4}^{\mathfrak{D}-2}}{2\sqrt{2}} (\rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)} + \rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)}) + \frac{c_{\mathfrak{D}-4}^{\mathfrak{D}-1}}{2\sqrt{2}} (\rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)} + \rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)}) + \frac{c_{\mathfrak{D}-4}^{\mathfrak{D}}}{2\sqrt{2}} (\rho_{\mathfrak{D}(\mathfrak{D}-4)} + \rho_{(\mathfrak{D}-4)\mathfrak{D}}) \\
&\quad + \frac{c_{\mathfrak{D}-3}^{\mathfrak{D}-2}}{2\sqrt{2}} (\rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)} + \rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)}) + \frac{c_{\mathfrak{D}-3}^{\mathfrak{D}-1}}{2\sqrt{2}} (\rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)} + \rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)}) + \frac{c_{\mathfrak{D}-3}^{\mathfrak{D}}}{2\sqrt{2}} (\rho_{\mathfrak{D}(\mathfrak{D}-3)} + \rho_{(\mathfrak{D}-3)\mathfrak{D}}), \\
\nu'_L &= \nu_L \\
&\quad + \frac{c_{\mathfrak{D}-3}^{\mathfrak{D}-2}}{2\sqrt{2}} (\rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)} + \rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)}) + \frac{c_{\mathfrak{D}-3}^{\mathfrak{D}-1}}{2\sqrt{2}} (\rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)} + \rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)}) + \frac{c_{\mathfrak{D}-3}^{\mathfrak{D}}}{2\sqrt{2}} (\rho_{\mathfrak{D}(\mathfrak{D}-4)} + \rho_{(\mathfrak{D}-4)\mathfrak{D}}) \\
&\quad + \frac{c_{\mathfrak{D}-4}^{\mathfrak{D}-2}}{2\sqrt{2}} (\rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)} + \rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)}) + \frac{c_{\mathfrak{D}-4}^{\mathfrak{D}-1}}{2\sqrt{2}} (\rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)} + \rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)}) + \frac{c_{\mathfrak{D}-4}^{\mathfrak{D}}}{2\sqrt{2}} (\rho_{\mathfrak{D}(\mathfrak{D}-3)} + \rho_{(\mathfrak{D}-3)\mathfrak{D}}).
\end{aligned}$$

Substitute them into the previous equations, and we obtain that

$$\begin{cases} l_{L;P} = \partial_P l_L - g l_L Z_P - g l_R A_P - g \nu'_L W_P^1, \\ l_{R;P} = \partial_P l_R - g l_R Z_P - g l_L A_P, \end{cases}, \quad \begin{cases} \nu_{L;P} = \partial_P \nu_L - g \nu_L Z_P - g l'_L W_P^1, \\ \nu_{R;P} = \partial_P \nu_R - g \nu_R Z_P. \end{cases} \quad \square$$

Remark 7.1. Reviewing Discussion 5.1, we know the above proposition shows the geometric origin of MNS mixing of weak interaction from the perspective of intrinsic geometry. In affine connection representation of gauge fields, MNS mixing spontaneously appears as an intrinsic geometric property, therefore it is not necessary to be postulated artificially like that in the Standard Model.

In conventional physics, e , μ and τ have just only ontological differences, but they have no difference in mathematical connotation. By contrast, Proposition 7.1 tells us that leptons of three generations should be constructed by different linear combinations of $\rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)}$, $\rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)}$, $\rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)}$, $\rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)}$, $\rho_{\mathfrak{D}(\mathfrak{D}-3)}$, $\rho_{(\mathfrak{D}-3)\mathfrak{D}}$, $\rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)}$, $\rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)}$, $\rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)}$, $\rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)}$, $\rho_{\mathfrak{D}(\mathfrak{D}-4)}$ and $\rho_{(\mathfrak{D}-4)\mathfrak{D}}$. Thus, e , μ and τ may have concrete and distinguishable mathematical connotations. For example, suppose a_τ , b_τ , $a_{\tau_n}^m$, $b_{\tau_n}^m$ are constants, then we can imagine that

$$\begin{cases} e \triangleq l = (\rho_{(\mathfrak{D}-4)(\mathfrak{D}-4)}, \rho_{(\mathfrak{D}-3)(\mathfrak{D}-3)}), \\ \nu_e \triangleq \nu = (\rho_{(\mathfrak{D}-3)(\mathfrak{D}-4)}, \rho_{(\mathfrak{D}-4)(\mathfrak{D}-3)}). \end{cases}$$

$$\begin{cases} \mu \triangleq a_\mu e + \frac{1}{2} \left(a_{\mu\mathfrak{D}-4}^{\mathfrak{D}-2} \rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)} + a_{\mu\mathfrak{D}-4}^{\mathfrak{D}-1} \rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)} + a_{\mu\mathfrak{D}-4}^{\mathfrak{D}} \rho_{\mathfrak{D}(\mathfrak{D}-4)}, \right. \\ \quad \left. a_{\mu\mathfrak{D}-3}^{\mathfrak{D}-2} \rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)} + a_{\mu\mathfrak{D}-3}^{\mathfrak{D}-1} \rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)} + a_{\mu\mathfrak{D}-3}^{\mathfrak{D}} \rho_{\mathfrak{D}(\mathfrak{D}-3)} \right), \\ \nu_\mu \triangleq b_\mu \nu_e + \frac{1}{2} \left(b_{\mu\mathfrak{D}-3}^{\mathfrak{D}-2} \rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)} + b_{\mu\mathfrak{D}-3}^{\mathfrak{D}-1} \rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)} + b_{\mu\mathfrak{D}-3}^{\mathfrak{D}} \rho_{\mathfrak{D}(\mathfrak{D}-4)}, \right. \\ \quad \left. b_{\mu\mathfrak{D}-4}^{\mathfrak{D}-2} \rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)} + b_{\mu\mathfrak{D}-4}^{\mathfrak{D}-1} \rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)} + b_{\mu\mathfrak{D}-4}^{\mathfrak{D}} \rho_{\mathfrak{D}(\mathfrak{D}-3)} \right), \\ \tau \triangleq a_\tau \mu + \frac{1}{2} \left(a_{\tau\mathfrak{D}-4}^{\mathfrak{D}-2} \rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)} + a_{\tau\mathfrak{D}-4}^{\mathfrak{D}-1} \rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)} + a_{\tau\mathfrak{D}-4}^{\mathfrak{D}} \rho_{(\mathfrak{D}-4)\mathfrak{D}}, \right. \\ \quad \left. a_{\tau\mathfrak{D}-3}^{\mathfrak{D}-2} \rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)} + a_{\tau\mathfrak{D}-3}^{\mathfrak{D}-1} \rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)} + a_{\tau\mathfrak{D}-3}^{\mathfrak{D}} \rho_{(\mathfrak{D}-3)\mathfrak{D}} \right), \\ \nu_\tau \triangleq b_\tau \nu_\mu + \frac{1}{2} \left(b_{\tau\mathfrak{D}-3}^{\mathfrak{D}-2} \rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)} + b_{\tau\mathfrak{D}-3}^{\mathfrak{D}-1} \rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)} + b_{\tau\mathfrak{D}-3}^{\mathfrak{D}} \rho_{(\mathfrak{D}-4)\mathfrak{D}}, \right. \\ \quad \left. b_{\tau\mathfrak{D}-4}^{\mathfrak{D}-2} \rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)} + b_{\tau\mathfrak{D}-4}^{\mathfrak{D}-1} \rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)} + b_{\tau\mathfrak{D}-4}^{\mathfrak{D}} \rho_{(\mathfrak{D}-3)\mathfrak{D}} \right). \end{cases}$$

Proposition 7.2. When (M, g) satisfies the symmetry conditions of Definition 7.3, denote

$$\begin{aligned} d'_{1L} &\triangleq \frac{1}{2\sqrt{2}} c_{\mathfrak{D}-1}^{\mathfrak{D}-3} (\rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)} + \rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)}) + \frac{1}{2\sqrt{2}} c_{\mathfrak{D}-2}^{\mathfrak{D}-3} (\rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)}) \\ &\quad + \frac{1}{2\sqrt{2}} c_{\mathfrak{D}-1}^{\mathfrak{D}-4} (\rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)} + \rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)}) + \frac{1}{2\sqrt{2}} c_{\mathfrak{D}-2}^{\mathfrak{D}-4} (\rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)}), \\ d'_{2L} &\triangleq \frac{1}{2\sqrt{2}} c_{\mathfrak{D}}^{\mathfrak{D}-3} (\rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)}) + \frac{1}{2\sqrt{2}} c_{\mathfrak{D}-1}^{\mathfrak{D}-3} (\rho_{(\mathfrak{D}-4)\mathfrak{D}} + \rho_{\mathfrak{D}(\mathfrak{D}-4)}) \\ &\quad + \frac{1}{2\sqrt{2}} c_{\mathfrak{D}}^{\mathfrak{D}-4} (\rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)}) + \frac{1}{2\sqrt{2}} c_{\mathfrak{D}-1}^{\mathfrak{D}-4} (\rho_{(\mathfrak{D}-3)\mathfrak{D}} + \rho_{\mathfrak{D}(\mathfrak{D}-3)}), \\ d'_{3L} &\triangleq \frac{1}{2\sqrt{2}} c_{\mathfrak{D}-2}^{\mathfrak{D}-3} (\rho_{(\mathfrak{D}-4)\mathfrak{D}} + \rho_{\mathfrak{D}(\mathfrak{D}-4)}) + \frac{1}{2\sqrt{2}} c_{\mathfrak{D}}^{\mathfrak{D}-3} (\rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)} + \rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)}) \\ &\quad + \frac{1}{2\sqrt{2}} c_{\mathfrak{D}-2}^{\mathfrak{D}-4} (\rho_{(\mathfrak{D}-3)\mathfrak{D}} + \rho_{\mathfrak{D}(\mathfrak{D}-3)}) + \frac{1}{2\sqrt{2}} c_{\mathfrak{D}}^{\mathfrak{D}-4} (\rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)} + \rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)}), \\ u'_{1L} &\triangleq \frac{1}{2\sqrt{2}} c_{\mathfrak{D}-2}^{\mathfrak{D}-3} (\rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)} + \rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)}) + \frac{1}{2\sqrt{2}} c_{\mathfrak{D}-2}^{\mathfrak{D}-4} (\rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)} + \rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)}) \\ &\quad + \frac{1}{2\sqrt{2}} c_{\mathfrak{D}-1}^{\mathfrak{D}-3} (\rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)}) + \frac{1}{2\sqrt{2}} c_{\mathfrak{D}-1}^{\mathfrak{D}-4} (\rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)}), \\ u'_{2L} &\triangleq \frac{1}{2\sqrt{2}} c_{\mathfrak{D}-1}^{\mathfrak{D}-3} (\rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)}) + \frac{1}{2\sqrt{2}} c_{\mathfrak{D}-1}^{\mathfrak{D}-4} (\rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)}) \\ &\quad + \frac{1}{2\sqrt{2}} c_{\mathfrak{D}}^{\mathfrak{D}-3} (\rho_{(\mathfrak{D}-4)\mathfrak{D}} + \rho_{\mathfrak{D}(\mathfrak{D}-4)}) + \frac{1}{2\sqrt{2}} c_{\mathfrak{D}}^{\mathfrak{D}-4} (\rho_{(\mathfrak{D}-3)\mathfrak{D}} + \rho_{\mathfrak{D}(\mathfrak{D}-3)}), \\ u'_{3L} &\triangleq \frac{1}{2\sqrt{2}} c_{\mathfrak{D}}^{\mathfrak{D}-3} (\rho_{(\mathfrak{D}-4)\mathfrak{D}} + \rho_{\mathfrak{D}(\mathfrak{D}-4)}) + \frac{1}{2\sqrt{2}} c_{\mathfrak{D}}^{\mathfrak{D}-4} (\rho_{(\mathfrak{D}-3)\mathfrak{D}} + \rho_{\mathfrak{D}(\mathfrak{D}-3)}) \\ &\quad + \frac{1}{2\sqrt{2}} c_{\mathfrak{D}-2}^{\mathfrak{D}-3} (\rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)} + \rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)}) + \frac{1}{2\sqrt{2}} c_{\mathfrak{D}-2}^{\mathfrak{D}-4} (\rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)} + \rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)}). \end{aligned}$$

Then the intrinsic geometric properties $d_1, d_2, d_3, u_1, u_2, u_3$ of f satisfy the following conclusions on (M, g) .

$$\begin{aligned}
d_{1L;P} &= \partial_P d_{1L} - g_s d_{1L} U_P^1 + g_s d_{2L} V_P^1 - g_s d_{3L} V_P^1 \\
&\quad - g_s u_{1L} X_P^{23} - \frac{g_s}{2} u_{2L} X_P^{31} + \frac{g_s}{2} u_{2L} Y_P^{31} - \frac{g_s}{2} u_{3L} X_P^{12} - \frac{g_s}{2} u_{3L} Y_P^{12} - g u'_{1L} W_P^1, \\
d_{2L;P} &= \partial_P d_{2L} - g_s d_{2L} U_P^2 + g_s d_{3L} V_P^2 - g_s d_{1L} V_P^2 \\
&\quad - g_s u_{2L} X_P^{31} - \frac{g_s}{2} u_{3L} X_P^{12} + \frac{g_s}{2} u_{3L} Y_P^{12} - \frac{g_s}{2} u_{1L} X_P^{23} - \frac{g_s}{2} u_{1L} Y_P^{23} - g u'_{2L} W_P^1, \\
d_{3L;P} &= \partial_P d_{3L} - g_s d_{3L} U_P^3 + g_s d_{1L} V_P^3 - g_s d_{2L} V_P^3 \\
&\quad - g_s u_{3L} X_P^{12} - \frac{g_s}{2} u_{1L} X_P^{23} + \frac{g_s}{2} u_{1L} Y_P^{23} - \frac{g_s}{2} u_{2L} X_P^{31} - \frac{g_s}{2} u_{2L} Y_P^{31} - g u'_{3L} W_P^1, \\
d_{1R;P} &= \partial_P d_{1R} - g_s d_{1L} V_P^1 + g_s d_{2L} U_P^1 - g_s d_{3L} U_P^1 \\
&\quad + g_s u_{1L} Y_P^{23} + \frac{g_s}{2} u_{2L} X_P^{31} - \frac{g_s}{2} u_{2L} Y_P^{31} - \frac{g_s}{2} u_{3L} X_P^{12} - \frac{g_s}{2} u_{3L} Y_P^{12}, \\
d_{2R;P} &= \partial_P d_{2R} - g_s d_{2L} V_P^2 + g_s d_{3L} U_P^2 - g_s d_{1L} U_P^2 \\
&\quad + g_s u_{2L} Y_P^{31} + \frac{g_s}{2} u_{3L} X_P^{12} - \frac{g_s}{2} u_{3L} Y_P^{12} - \frac{g_s}{2} u_{1L} X_P^{23} - \frac{g_s}{2} u_{1L} Y_P^{23}, \\
d_{3R;P} &= \partial_P d_{3R} - g_s d_{3L} V_P^3 + g_s d_{1L} U_P^3 - g_s d_{2L} U_P^3 \\
&\quad + g_s u_{3L} Y_P^{12} + \frac{g_s}{2} u_{1L} X_P^{23} - \frac{g_s}{2} u_{1L} Y_P^{23} - \frac{g_s}{2} u_{2L} X_P^{31} - \frac{g_s}{2} u_{2L} Y_P^{31}, \\
u_{1L;P} &= \partial_P u_{1L} - g_s u_{1L} U_P^1 - \frac{g_s}{2} u_{2L} X_P^{12} - \frac{g_s}{2} u_{2L} Y_P^{12} - \frac{g_s}{2} u_{3L} X_P^{31} + \frac{g_s}{2} u_{3L} Y_P^{31} \\
&\quad - g_s d_{1L} X_P^{23} + g_s d_{2L} Y_P^{23} - g_s d_{3L} Y_P^{23} - g d'_{1L} W_P^1, \\
u_{2L;P} &= \partial_P u_{2L} - g_s u_{2L} U_P^2 - \frac{g_s}{2} u_{3L} X_P^{23} - \frac{g_s}{2} u_{3L} Y_P^{23} - \frac{g_s}{2} u_{1L} X_P^{12} + \frac{g_s}{2} u_{1L} Y_P^{12} \\
&\quad - g_s d_{2L} X_P^{31} + g_s d_{3L} Y_P^{31} - g_s d_{1L} Y_P^{31} - g d'_{2L} W_P^1, \\
u_{3L;P} &= \partial_P u_{3L} - g_s u_{3L} U_P^3 - \frac{g_s}{2} u_{1L} X_P^{31} - \frac{g_s}{2} u_{1L} Y_P^{31} - \frac{g_s}{2} u_{2L} X_P^{23} + \frac{g_s}{2} u_{2L} Y_P^{23} \\
&\quad - g_s d_{3L} X_P^{12} + g_s d_{1L} Y_P^{12} - g_s d_{2L} Y_P^{12} - g d'_{3L} W_P^1, \\
u_{1R;P} &= \partial_P u_{1R} - g_s u_{1R} U_P^1 + \frac{g_s}{2} u_{2R} X_P^{12} + \frac{g_s}{2} u_{2R} Y_P^{12} + \frac{g_s}{2} u_{3R} X_P^{31} - \frac{g_s}{2} u_{3R} Y_P^{31}, \\
u_{2R;P} &= \partial_P u_{2R} - g_s u_{2R} U_P^2 + \frac{g_s}{2} u_{3R} X_P^{23} + \frac{g_s}{2} u_{3R} Y_P^{23} + \frac{g_s}{2} u_{1R} X_P^{12} - \frac{g_s}{2} u_{1R} Y_P^{12}, \\
u_{3R;P} &= \partial_P u_{3R} - g_s u_{3R} U_P^3 + \frac{g_s}{2} u_{1R} X_P^{31} + \frac{g_s}{2} u_{1R} Y_P^{31} + \frac{g_s}{2} u_{2R} X_P^{23} - \frac{g_s}{2} u_{2R} Y_P^{23}.
\end{aligned}$$

Proof. Substitute Definition 7.2 into ρ_{mn} and consider Definition 7.3, then compute them, and then substitute d'_{1L} , d'_{2L} , d'_{3L} , u'_{1L} , u'_{2L} , u'_{3L} into them, we finally obtain the results. $\square$

Remark 7.2. The above proposition shows the geometric origin of CKM mixing. We see that, in affine connection representation of gauge fields, d'_{1L} , d'_{2L} , d'_{3L} , u'_{1L} , u'_{2L} , u'_{3L} spontaneously appear as intrinsic geometric properties.

In addition, this proposition also shows that the new framework has an advantage that it never causes the problem that a proton decays into a lepton like some GUTs do.

Definition 7.4. If reference-system f satisfies $\rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)} = \rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} = \rho_{\mathfrak{D}\mathfrak{D}} = \rho_{(\mathfrak{D}-2)(\mathfrak{D}-1)} = \rho_{(\mathfrak{D}-1)(\mathfrak{D}-2)} = \rho_{(\mathfrak{D}-1)\mathfrak{D}} = \rho_{\mathfrak{D}(\mathfrak{D}-1)} = \rho_{\mathfrak{D}(\mathfrak{D}-2)} = \rho_{(\mathfrak{D}-2)\mathfrak{D}} = 0$, we say f is a **lepton field**, otherwise f is a **hadron field**.

Suppose f is a hadron field. For $d_1, d_2, d_3, u_1, u_2, u_3$, if f satisfies that five of them are zero and the other one is non-zero, we say f is an **individual quark**.

Proposition 7.3. There does not exist an individual quark. In other words, if any five ones of $d_1, d_2, d_3, u_1, u_2, u_3$ are zero, then $d_1 = d_2 = d_3 = u_1 = u_2 = u_3 = 0$.

Remark 7.3. For an individual down-type quark, the above proposition is evident. Without loss of generality let $u_1 = u_2 = u_3 = 0$ and $d_1 = d_2 = 0$, thus $\rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)} = \rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} = \rho_{\mathfrak{D}\mathfrak{D}} = 0$, hence we must have $d_3 = 0$.

For an individual up-type quark, this paper has not made progress on the proof yet. Anyway, it is significant that Proposition 7.3 shows the geometric connotation of the color confinement. It involves a natural geometric constraint of the curvatures among different dimensions.

Remark 7.4. On one hand, all of the above discussions in sections 5, 6 and 7 are based on the (1) of Definition 4.6.1. The weak and electromagnetic unified field in Definition 5.1, the strong interaction field in Definition 6.1 and the typical unified gauge field in Definition 7.1 are all special cases of the (1) of Definition 4.6.1. They do not contain gravitational effects.

On the other hand, For the cases containing the gravitational effects, we just need to take the (2) of Definition 4.6.1 or a general reference-system defined in Definition 2.1.1.

8. Conclusions

1. An affine connection representation of gauge fields is established in this paper. It has the following main points of view.

(i) Geometric manifold in Definition 2.1.1 is more suitable than Riemannian manifold for describing physical space.

(ii) Simple connection in Definition 2.5.1 contains more geometric information than Levi-Civita connection, and is independent of Christoffel symbol. It can uniformly describe gauge field and gravitational field.

(iii) Time is the total metric with respect to all dimensions of internal coordinate space and external coordinate space.

(iv) On-shell evolution is the evolution along gradient direction.

(v) Quantum evolution is the evolution along distribution of gradient directions. It has a geometric meaning discussed in section 3.10.

2. In the affine connection representation of gauge fields, the following unified mathematical descriptions are achieved.

(i) Gauge field and gravitational field have a unified coordinate description. They can be represented by the same affine connection. The components of Γ_{NP}^M with $M, N \in \{4, 5, \dots, \mathfrak{D}\}$ describe electromagnetic, weak and strong interaction fields. The other components of Γ_{NP}^M describe gravitational field.

(ii) Gauge field and elementary particle field have a unified geometric construction. They are both geometric entities constructed with semi-metric.

(iii) Physical evolutions of gauge field and elementary particle field have a unified geometric description. Their on-shell evolution and quantum evolution both present as geometric properties about gradient direction.

(iv) Quantum theory and gravitational theory have a unified geometric interpretation and the same view of time and space. They both reflect intrinsic geometric properties of manifold.

(v) Several postulates in Standard Model are turned into geometric results that appears spontaneously in affine connection representation, so they are not necessary to be regarded as postulates anymore.

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