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A symmetric-difference-closed orthomodular lattice that is stateless

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Abstract

This paper carries on the investigation of the orthomodular lattices that are endowed with a symmetric difference. Let us call them ODLs. Note that the ODLs may have a certain bearing on "quantum logics" - the ODLs are close to Boolean algebras though they capture the phenomenon of non-compatibility. The initial question in studying the state space of the ODLs is whether the state space can be poor. This question is of a purely combinatorial nature. In this note, we exhibit a finite ODL whose state space is empty (respectively, whose state space is a singleton).

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1 Notions

The investigation of symmetric-difference-closed orthomodular posets and lattices has recently brought a substantial contribution to the theory of orthocomplemented structures (see [1–9]). In this note, we ask on the presence of the "Greechie phenomenon" [10] in ODLs: Is there a stateless ODL? We answer this question in the positive. It should be noted that this result may support the conjecture that each compact convex set can be viewed as a state space of an ODL (compare with [7, 11]). This paper is based on the ideas of the author's thesis [3], and it extends them in places.

Let us review basic notions as we shall use them in the sequel (see [1]).

Definition 1 Let $\mathcal{L} = (X, \leq, ', \triangle, \mathbf{0}, \mathbf{1})$, where $(X, \leq, ', \mathbf{0}, \mathbf{1})$ is an orthomodular lattice and $\triangle : X^2 \to X$ is a binary operation. Then $\mathcal{L}$ is said to be an orhomodular difference lattice (an ODL) if the following conditions are satisfied $(x, y, z \in X)$:

i. $x \triangle (y \triangle z) = (x \triangle y) \triangle z$,
ii. $x \triangle \mathbf{1} = x' = \mathbf{1} \triangle x$,
iii. $x \triangle y \leq x \vee y$.

The class of ODLs forms a variety of algebras (a proper subvariety of orthomodular lattices). A basic example of ODLs is a Cartesian product of a Boolean algebra and the "Chinese lantern" MO_3 (see [1] for a thorough algebraic analysis of ODLs). In Proposition 1 that follows, we encounter several other examples. We would need the following construction (a generalization of Construction 8.5 of [1]). Let us first recall the standard construction of the horizontal sum of orthomodular lattices (see e.g., [12], p.59). If $\mathcal{L}_\alpha, \alpha \in I$ are orthomodular lattices, then the horizontal sum $\mathrm{Hor}(\mathcal{L}_\alpha, \alpha \in I)$ is defined by identifying the $\mathbf{0}s$ and $\mathbf{1}s$ in the disjoint union $\bigcup_{\alpha \in I} \mathcal{L}_\alpha$. The $\mathrm{Hor}(\mathcal{L}_\alpha, \alpha \in I)$ is obviously an orthomodular lattice. Let us formulate the following definition.

Definition 2 Let $\mathcal{B}$ be a Boolean algebra. Let $\mathcal{B}_\alpha$, $\alpha \in I$ be a collection of Boolean algebras such that each $\mathcal{B}_\alpha$, $\alpha \in I$, is a subset of $\mathcal{B}$ (not necessarily a Boolean subalgebra of $\mathcal{B}$) and, moreover, their minimal and maximal elements in each $\mathcal{B}_\alpha$ are identical to 0 and 1 of $\mathcal{B}$. Let us further require that the symmetric difference $\triangle_\alpha$ of each $\mathcal{B}_\alpha$ equals the symmetric difference $\triangle$ of $\mathcal{B}$ (i.e., if $a, b \in \mathcal{B}_\alpha$, $\alpha \in I$ then $a \triangle_\alpha b = a \triangle b$). Let us say that **the collection $\mathcal{B}_\alpha$, $\alpha \in I$ covers $\mathcal{B}$** if the following conditions are fulfilled $(\alpha, \beta \in I)$:

1. if $\mathcal{B}_\alpha \neq \mathcal{B}_\beta$, then $\mathcal{B}_\alpha \cap \mathcal{B}_\beta = \{\mathbf{0}, \mathbf{1}\}$.
2. $\cup_{\alpha \in I} \mathcal{B}_\alpha = \mathcal{B}$.

2 Results

Proposition 1 *Let $\mathcal{B}$ be a Boolean algebra with symmetric difference $\triangle$ and let $\{\mathcal{B}_\alpha, \alpha \in I\}$ cover $\mathcal{B}$. Then the horizontal sum* $\mathrm{Hor}(\mathcal{B}_\alpha, \alpha \in I)$ *allows for an operation, $\hat{\triangle}$, so that* $\mathrm{Hor}(\mathcal{B}_\alpha, \alpha \in I)$ *endowed with $\hat{\triangle}$ becomes an ODL.*

Proof Consider the elements $b_1, b_2 \in \mathrm{Hor}(\mathcal{B}_\alpha, \alpha \in I)$. If there is an $\alpha \in I$ with $b_1, b_2 \in \mathcal{B}_\alpha$, then we set $b_1 \hat{\triangle} b_2 = b_1 \triangle_\alpha b_2$ $(= b_1 \triangle b_2)$. If it is not the case, then there is exactly one $\alpha, \alpha \in I$, with $b_1 \triangle b_2 \in \mathcal{B}_\alpha$ and we set $b_1 \hat{\triangle} b_2 = b_1 \triangle b_2 \in \mathcal{B}_\alpha$. Since

$b_1 \vee b_2 = \mathbf{1}$ whenever b_1 and b_2 are nonzero and belong to distinct algebras $\mathcal{B}_{\alpha_1}, \mathcal{B}_{\alpha_2}$, we see that the above defined operation $\hat{\triangle}$ makes $\mathrm{Hor}(\mathcal{B}_\alpha, \alpha \in I)$ an ODL. $\square$

Let us call the above construction the *ODL horizontal sum*. It should be noted that the essential property of a collection $\mathcal{B}_\alpha$, $\alpha \in I$ is the property 1. of Definition 2. Indeed, if $\mathcal{B}_\alpha$, $\alpha \in I$ fulfils the property 1. of Definition 2 and if there is a $b \in \mathcal{B}$ such that $b \in \mathcal{B} \setminus \cup_{\alpha \in I} \mathcal{B}_\alpha$, we can always add the Boolean algebra $\mathcal{B}_b = \{\mathbf{0}, b, b', \mathbf{1}\}$ to the collection $\mathcal{B}_\alpha$, $\alpha \in I$ as far as we reach the collection that covers $\mathcal{B}$. The following lemma will be also used in the desired construction of a stateless ODL.

Lemma 1 Let $\mathcal{B}$ be a Boolean algebra and let $\triangle$ be the (unique) symmetric difference of $\mathcal{B}$. Let $c_1, c_2, c_3 \in \mathcal{B}$. Assume that $\mathbf{0} < c_i < \mathbf{1}$ for all $i \in \{1,2,3\}$ and assume further that $c_1 \triangle c_2 \triangle c_3 = \mathbf{1}$. Consider the set $\mathcal{D} = \{\mathbf{0}, c_1, c_2, c_3, c_1', c_2', c_3', \mathbf{1}\}$ and endow $\mathbf{D}$ with the structure of a Boolean algebra by using the Boolean isomorphism onto the Boolean algebra $\exp\{1,2,3\}$ (thus, the elements c_1, c_2 and c_3 will become the atoms of $\mathcal{D}$). Then the symmetric difference $\triangle_{\mathcal{D}}$ on $\mathcal{D}$ agrees with $\triangle$ (i.e., if $d_1, d_2 \in \mathcal{D}$, then $d_1 \triangle_{\mathcal{D}} d_2 = d_1 \triangle d_2$)

Proof Observe first that all elements of $\mathcal{D}$ are distinct. Indeed, suppose for instance that $c_1 = c_2'$. But $c_2' \triangle c_2 = \mathbf{1}$ (see [1]) and therefore $c_1 \triangle c_2 \triangle c_2 = c_2' \triangle c_2 \triangle c_3 = \mathbf{1} \triangle c_3 = c_3' \neq \mathbf{1}$. This is absurd. In showing that $\triangle_{\mathcal{D}}$ agrees with the original $\triangle \in \mathcal{B}$, let us argue by cases.

1. Suppose that $d_1, d_2 \in \mathcal{D}$ are atoms of $\mathcal{D}$. For instance, let $d_1 = c_1$ and $d_2 = c_2$. Then $c_1 \triangle_{\mathcal{D}} c_2 = (c_1 \wedge_{\mathcal{D}} c_2') \vee_{\mathcal{D}} (c_1' \wedge_{\mathcal{D}} c_2) = c_1 \vee_{\mathcal{D}} c_2 = c_3' = \mathbf{1} \triangle c_3 = (c_1 \triangle c_2 \triangle c_3) \triangle c_3 = c_1 \triangle c_2$.
2. Suppose that $d_1, d_2 \in \mathcal{D}$, d_1 is an atom and d_2 is a coatom (in $\mathcal{D}$). If $d_1 = d_2'$, the result is trivial. Otherwise, we can choose $d_1 = c_1$ and $d_2 = c_2'$. Then $d_1 \triangle_{\mathcal{D}} d_2 = c_1 \triangle_{\mathcal{D}} c_2' = (c_1 \wedge_{\mathcal{D}} c_2) \vee_{\mathcal{D}} (c_1' \wedge_{\mathcal{D}} c_2') = (c_1 \vee_{\mathcal{D}} c_2)' = c_3$.
3. Suppose that $d_1, d_2 \in \mathcal{D}$ and both d_1, d_2 are coatoms. Then $(d_1 \triangle_{\mathcal{D}} d_2)' = d_1 \triangle_{\mathcal{D}} d_2' = d_1 \triangle d_2' = (d_1 \triangle d_2)'$ and we are done.

$\square$

The definition of a state on an ODL reads as follows.

Definition 3 Let $\mathcal{L}$ be an ODL. A mapping $s : \mathcal{L} \to [0,1]$ is said to be a state on $\mathcal{L}$ if the following conditions are satisfied ($x, y \in \mathcal{L}$):

i. $s(\mathbf{1}) = 1$,
ii. $s(x) + s(y) = s(x \vee y)$ whenever $x \leq y'$,
iii. $s(x) + s(y) \geq s(x \triangle y)$.

The state space $S(\mathcal{L})$ of an ODL $\mathcal{L}$ is a convex compact set, and hence $S(\mathcal{L})$ is big if $\mathcal{L}$ has an abundance of pure states. This does not have to be the case (observe that $\mathcal{L}$ does not have to be set-representable; see [1]). In fact, it does not have to have any state at all, as we are going to show. The following construction would serve the purpose.

Construction 1 Let $\mathcal{B}$ be the Boolean algebra of all subsets of a 13-element set, $\mathcal{B} = \exp\{1, 2, \ldots, 12, 13\}$. Let us first define 5 Boolean algebras $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4$ and $\mathcal{B}_5$ as follows:

1. $\mathcal{B}_1$ is the Boolean subalgebra of $\mathcal{B}$ with atoms $a_1 = \{1\}$, $a_2 = \{2\}$, $a_3 = \{3\}$, $a_4 = \{4\}$, $a_5 = \{5\}$, $a_6 = \{6\}$ and $a_7 = \{7, 8, 9, 10, 11, 12, 13\}$.
2. $\mathcal{B}_2$ is the Boolean subalgebra of $\mathcal{B}$ with atoms $b_1 = \{8\}$, $b_2 = \{9\}$, $b_3 = \{10\}$, $b_4 = \{11\}$, $b_5 = \{12\}$, $b_6 = \{13\}$ and $b_7 = \{1, 2, 3, 4, 5, 6, 7\}$.
3. $\mathcal{B}_3$ is the Boolean algebra constructed by Lemma 1.4 with the atoms $c_1 = \{2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13\}$, $c_2 = \{3, 4, 5, 6, 7, 10, 11, 12, 13\}$ and $c_3 = \{1, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13\}$.
4. $\mathcal{B}_4$ is the Boolean algebra constructed by Lemma 1.4 with the atoms $d_1 = \{1, 2, 4, 5, 6, 7, 8, 9, 11, 12, 13\}$, $d_2 = \{1, 2, 5, 6, 7, 8, 9, 12, 13\}$ and $d_3 = \{1, 2, 3, 5, 6, 7, 8, 9, 10, 12, 13\}$.
5. $\mathcal{B}_5$ is the Boolean algebra constructed by Lemma 1.4 with the atoms $e_1 = \{1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 13\}$, $e_2 = \{1, 2, 3, 4, 7, 8, 9, 10, 11\}$ and $e_3 = \{1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12\}$.

For every pair $d, d' \in \mathcal{B} \setminus \cup_{i \leq 5} \mathcal{B}_i$, let us consider the subalgebra of $\mathcal{B}$ containing the elements $\mathbf{0}, \mathbf{1}, d, d'$. Note that there is exactly $\frac{1}{2}(2^{13} - 2 - 3(2^3 - 2) - 2(2^7 - 2)) = 3960$ such pairs of elements d, d' that do not belong to $\cup_{i \leq 5} \mathcal{B}_i$. Let us denote the remaining subalgebras by $\mathcal{B}_6 \ldots \mathcal{B}_{3965}$.

Proposition 2 *The collection of algebras $\mathcal{B}_i$, $i \leq 3965$ of the Construction 1 covers the algebra $\mathcal{B} = exp\{1, 2, \ldots, 12, 13\}$.*

Proof The required properties can be checked by a slightly tedious but relatively straightforward verification (see also [3]). □

Theorem 3 *Let $\mathcal{L}$ be the ODL horizontal sum of the collection of Boolean subalgebras of the Construction 1. Then $\mathcal{L}$ does not have any state.*

Proof For the intuition, let us use the following graphical representation of $\mathcal{L}$ (a variant of the Greechie paste job, see [10])

Figure 1 represents maximal Boolean subalgebras of $\mathcal{L}$ of the cardinality higher than 4. The dotted lines indicate the Boolean subalgebras with $a_1 \triangle b_1 \triangle c_1 = \mathbf{1}$, and similarly for a_3, b_3, d_1, etc. Hence the construction of $\mathcal{L}$ as the ODL horizontal sum gives us the following set of equations:

$$a_1 \triangle b_1 = c_1' = c_2 \vee c_3,$$
$$a_2 \triangle b_2 = c_3' = c_1 \vee c_2,$$
$$a_3 \triangle b_3 = d_1' = d_2 \vee d_3,$$

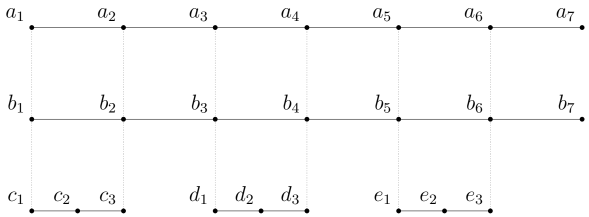

Fig. 1 Greechie diagram of Construction 1

$$a_4 \triangle b_4 = d_3' = d_1 \vee d_2,$$
$$a_5 \triangle b_5 = e_1' = e_2 \vee e_3,$$
$$a_6 \triangle b_6 = e_3' = e_1 \vee e_2$$

Looking for a contradiction, suppose that s is a state on $\mathcal{L}$. By the property of s we see that

$$\begin{aligned} s(a_1) + s(b_1) &\geq s(a_1 \triangle b_1) = s(c_2 \vee c_3) = s(c_2) + s(c_3), \\ s(a_2) + s(b_2) &\geq s(a_2 \triangle b_2) = s(c_1 \vee c_2) = s(c_1) + s(c_2), \\ s(c_1) + s(c_2) &+ s(c_3) = 1. \end{aligned}$$

This implies that

$$s(a_1) + s(a_2) + s(b_1) + s(b_2) \geq s(c_1) + s(c_2) + s(c_3) = 1.$$

Analogously, we obtain

$$s(a_3) + s(a_4) + s(b_3) + s(b_4) \geq 1,$$
$$s(a_5) + s(a_6) + s(b_5) + s(b_6) \geq 1.$$

Summing up the inequalities, we infer that $\sum_{i=1}^{6}(s(a_i)+s(b_i)) \geq 3$. But it also holds that $\sum_{i=1}^{7} s(a_i) = \sum_{i=1}^{7} s(b_i) = 1$. Together with the non-negativity of s, we have derived a contradiction. This completes the proof. $\square$

In concluding the paper, let us observe that Theorem 3 allows one to spell out another slightly bizarre result.

Theorem 4 *There is a (non-Boolean) ODL with exactly one state.*

Proof Let $\mathcal{L}$ be an ODL with no state. Let $\mathcal{K} = \mathcal{L} \times \{0,1\}$, where $\{0,1\}$ is meant as a Boolean algebra. Then $\mathcal{K}$ has exactly one state. Indeed, let us set $s(e,0) = 0$ and $s(e,1) = 1$ for any $e \in \mathcal{L}$. Then s is a two-valued state on $\mathcal{K}$. It is easily seen that this s is the only state on $\mathcal{K}$ $\square$

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Data availability statement

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

References

[1] Matoušek, M.: Orthocomplemented lattices with a symmetric difference. Algebra universalis **60**(2), 185–215 (2009)

[2] Matoušek, M., Pták, P.: Symmetric difference on orthomodular lattices and $\mathbb{Z}_2$-valued states. Commentationes Mathematicae Universitatis Carolinae **50**(4), 535–547 (2009)

[3] Voráček, V.: Combinatorial Methods in the Study of Quantum Structures. Czech Technical University in Prague, Prague (2021)

[4] Ovchinnikov, P.G.: Measures on finite concrete logics. Proceedings of the American Mathematical Society, 1957–1966 (1999)

[5] De Simone, A., Navara, M., Pták, P.: States on systems of sets that are closed under symmetric difference. Mathematische Nachrichten **288**(17-18), 1995–2000 (2015)

[6] Burešová, D., Pták, P.: Quantum logics that are symmetric-difference-closed. International Journal of Theoretical Physics (2021)

[7] Shultz, F.W.: A characterization of state spaces of orthomodular lattices. Journal of Combinatorial Theory, Series A **17**(3), 317–328 (1974)

[8] Su, J.: On set-representable orthocomplemented difference lattices. Order **37**(3), 621–636 (2020)

[9] Hroch, M., Pták, P.: States on orthocomplemented difference posets (extensions). Letters in Mathematical Physics **106**(8), 1131–1137 (2016)

[10] Greechie, R.J.: Orthomodular lattices admitting no states. Journal of Combinatorial Theory, Series A **10**(2), 119–132 (1971)

[11] Navara, M.: Descriptions of state spaces of orthomodular lattices (the hypergraph approach). Mathematica Bohemica **117**(3), 305–313 (1992)

[12] Kalmbach, G.: Orthomodular Lattices. Academic Press, London (1983)