# On Ranking Climate Factors Affecting the Living Organisms Based on Paired Comparison Model, a Bayesian Approach

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## Abstract

The method of paired comparisons (PC) endeavors to rank treatments presented in pairs to panelists (or respondents, judges, jurists, etc.) and they have to select the better one based on sensory evaluations. Sometimes the situations may occur when the panelists cannot discriminate between the treatments and declare a tie. In this study, an effort is made to extend the Weibull PC model to accommodate ties. The extended Weibull PC model is analyzed using Bayesian paradigm. Four different loss functions are used under noninformative (Uniform and Jeffreys) priors. The posterior and marginal posterior distributions are derived. The posterior estimates, posterior risks, preference probabilities, posterior probabilities and predictive probabilities are evaluated to know the ranking of ecological factor. The goodness of the proposed model is assessed. The entire analysis is carried out using a real data set based on the preference for the ecological factors.


## 1. Introduction

The paired comparison (PC) is well developed and reliable technique to order/ rank two or more treatments. Under this technique, pairs of treatments are presented to one or more judges for ranking according to some specific criterion. The PC technique is also considered as an efficient technique to access the preferences when the quantitative measurement of the treatments are either unavailable or accurate assessment is impossible. This technique is gaining more attention in different fields of life. The most frequent application of the PC technique is in sensory evaluations, personal rating, marketing, medicine etc. Trained panelists are using this technique in the industry to access customer's preferences and designing products. Annis & Craig (2005), Cattelan (2012) used PC technique to analyze sports related problems. (Kifyat et al 2020) developed Maxwell PC model and analyzed the preference data of brands of drinking water under Bayesian paradigm. (Sung & Wu, 2018) established a PC technique that can be used as an alternative to the likert scale. (Ullah & Aslam, 2020) analyzed the Weibull PC model using noninformative priors under Bayesian technique. A real data set of cellphone brands is considered for analyzing the costumer's preferences for the cellphone brands.

## 2. Background

The literature different reflects a variety of PC models. (Thurstone, 1927) provided fundamental framework for the PC technique assuming the responses of the panelists to follow the normal distribution and derived his PC model based on the differences in the choice behaviors. (Bradley & Terry, 1952) assumed that the judges responses follow the logistic distribution and suggested a model. (Abbas & Aslam, 2009) considered Cauchy distribution to develop PC model. (Ullah et al., 2020) used the Weibull distribution to developed PC model. (Rao & Kupper, 1967) extended (Bradley & Terry, 1952) model for accommodating data with no preferences i.e. tie in the data. They introduced a threshold parameter in such a way that if the difference in the magnitude of the responses is less than the threshold parameter, it will indicate that the judges are unable to differentiate between the two treatments and hence will declare a tie. (Davidson, 1970) also extended the (Bradley & Terry, 1952) model by introducing a tie parameter in the model. (Aslam, 2002) performed the Bayesian analysis of the two PC models with ties. (van Barren, 1978) suggested six extensions of the Bradley and Terry model and introduced a tie parameter and order effect of the presentation of treatments. For more details one may see (Schauberger & Tutz, 2017); (Dittrich at al 2012); (Van et al 2017).

# 3. Motivation

Modeling the reliability in the life time phenomena in engineering has great importance. The Weibull random variables can be used in such cases, so in case of paired comparison perception the preference will be given to the object having more lifetime reliability i.e. take more time to its failure, as compare to the object having less failure life time. Sometimes a situation occurs when it becomes very difficult for the panelists to show their preferences for the treatments or they are incompetent to recognize the difference and declare a tie between the competing treatments. In our study, we introduce the PC model to accommodate ties declared by the panelists. Our study is a new addition in the field of Bayesian paired comparison, it would be very useful for the researchers and practitioners to analyze their problem with newly developed model. To the best of our knowledge this type of work has not been carried out in the literature so far. We perform the Bayesian analysis of the model using noninformative priors under different loss functions. For illustration, we used real data set collected from the students of the earth science department Quaid-i-Azam University, Islamabad, Pakistan, showing their preferences for the ecological factors under study.

In this study, Section 4 defines material and methods including the development of the PC model, Bayesian analyses of the model using prior distributions such as Uniform and Jeffreys priors and graphical presentation of the marginal densities of the model's worth parameters. Results and discussions are presented in Sections 5. Section 6 concludes the entire study.

# 4. Material And Methods

## 4.1 The proposed model

The details for the development of the Weibull PC model is elaborated in (Ullah et al., 2020). According to the Weibull PC model, the probability of preferring the treatment $T_j$ over $T_k$, denoted by $\theta_{j.jk}$, may be defined as

$$\theta_{j.jk} = \frac{\gamma_j^3}{\gamma_j^3 + \gamma_k^3}$$

1,

and similarly the probability of preferring the treatment $T_k$ over $T_j$, denoted by $\theta_{k.jk}$, is defined as

$$\theta_{j.jk} = \frac{\gamma_j^3}{\gamma_j^3 + \gamma_k^3}$$

2.

(Davidson, 1970) used (Luce, 1959) choice axiom in Bradley and Terry model for accommodating ties. The Weibull PC model is extended by introducing a new tie parameter ($\varepsilon$) based on the (Davidson, 1970) criteria. We considered the ratio scale of the preference probabilities $\theta_{j.jk}$ and $\theta_{j.jk}$ as following:

$\frac{\theta_{j.jk}}{\theta_{k.jk}} = \frac{\gamma_j^3}{\gamma_k^3}, \ \theta_{j.jk} \neq 0, 1$ for all (j, k). (3)

Thus the probability of no preference is denoted by $\theta_{o.jk}$, and is defined as proportional to the geometric mean of the two preference probabilities $\theta_{j.jk}$ and $\theta_{k.jk}$, i.e.,

$$\theta_{o.jk} = \varepsilon\sqrt{\theta_{j.jk}.\theta_{k.jk}}\ \varepsilon > 0$$

4,

where $\varepsilon$ is the constant of proportionality and also known as tie parameter. As the sum of probability is equals one, so by using (3) and (4) we obtain the following probabilities;

$$\theta_{j.jk} = \frac{\gamma_j^3}{\gamma_j^3 + \gamma_k^3 + \varepsilon\sqrt{\gamma_j^3 .\ \gamma_k^3}}$$

5

$$\theta_{k.jk} = \frac{\gamma_k^3}{\gamma_j^3 + \gamma_k^3 + \varepsilon\sqrt{\gamma_j^3 .\ \gamma_k^3}}$$

6

$$\theta_{o.jk} = \frac{\varepsilon\sqrt{\gamma_j^3 .\ \gamma_k^3}}{\gamma_j^3 + \gamma_k^3 + \varepsilon\sqrt{\gamma_j^3 .\ \gamma_k^3}}$$

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Here $\gamma_j$, *j = 1,2,...,m* denotes the treatment parameters and $\varepsilon$ is the tie parameter. The expressions (5), (6) and (7) represent the extended Weibull PC model with ties. Here

$\theta_{j.jkr}$ will take the value 1 if the treatment $T_j$ is preferred to $T_k$ otherwise 0 in $r^{th}$ repetition of a comparison.

$\theta_{k.jkr}$ will take the value 1 when treatment $T_k$ is preferred to $T_j$ otherwise 0 in $r^{th}$ repetition of a comparison.

$\theta_{o.jkr}$ stands for the treatment $T_j$ is tied to the treatment $T_k$ in $r^{th}$ repetition of a comparison.

$\mathrm{n_{j.jk}} = \sum_r \theta_{j.jkr}$ is the number of times the treatment $T_j$ is preferred to $T_k$.

$\mathrm{n_{k.jk}} = \sum_r \theta_{k.jkr}$ is the number of times the treatment $T_k$ is preferred to $T_j$.

$\mathrm{n_{o.jk}} = \sum_r \theta_{o.jkr}$ is the number of times the treatment $T_j$ is tied to the treatment $T_k$.

$\mathrm{n}_{jk}$ is the total number of times the treatment $T_j$ is compared with the treatment $T_k$ and the $T_j$ and $T_k$ are tied. Such that: $\mathrm{n}_{jk} = n_{j.jk} + n_{k.jk} + n_{o.jk}$

# 4.2 Prior distributions for the prosed model

For the Bayesian analysis of the proposed model, two noninformative Uniform and Jeffreys priors are assumed (Laplace, 1812; Bayes, 1763). The uniform prior allocates identical probability to each and every unit. Symbolically we can write it as:

$$p_U(\phi) \propto 1 \qquad \text{where } \phi = (\gamma_1, \gamma_2, \ldots, \gamma_m, \varepsilon),$$

while the Jeffreys prior can be obtained as: $p_J(\phi) \propto \sqrt{\det\{I(\phi)\}}$

where$I(\phi)$ denotes the Fisher information matrix. For m = 2 we have Fisher's Information matrix as:

{\text{I(}}\gamma {\text{)=(-1}}{{\text{)}}^2}\left| \begin{gathered} E\left\{ {\frac{{{\partial ^2}\log l(.)}}{{\partial \gamma _{1}^{2}}}} \right\}{\text{ }}E\left\{ {\frac{{{\partial ^2}\log l(.)}}{{\partial {\gamma _1}\partial \varepsilon }}} \right\}{\text{ }} \hfill \\ E\left\{ {\frac{{{\partial ^2}\log l(.)}}{{\partial \varepsilon {\text{ }}\partial {\gamma _1}}}} \right\}{\text{ }}E\left\{ {\frac{{{\partial ^2}\log l(.)}}{{\partial {\varepsilon ^2}}}} \right\}{\text{ }} \hfill \\ \end{gathered} \right|

As the Jeffreys prior have long and complicated algebraic expression which is not easy for applying for m = 5, so we drive the Jeffreys prior numerically for m = 5, designed in SAS package and can be obtain from the first author on the request.

# 4.3 Bayesian Analysis of the model

The likelihood function of the observed outcomes of the trial "a" and the parameters \phi is:

l({\mathbf{a}},{\text{ }}\phi )=\prod\limits_{{j<k}}^{m} {\frac{{n{}_{{jk}}}}{{{n_{j.jk}}!.{\text{ }}{n_{k.jk}}!.{\text{ }}{n_{o.jk}}!}}} {\text{ }}\frac{{\gamma _{j}^{{{S_j}}}{\text{ }}{\varepsilon ^{{n_{o.jk}}}}}}{{{{(\gamma _{j}^{3}+\gamma _{k}^{3}+\varepsilon \sqrt {\gamma _{j}^{3}.\gamma _{k}^{3}} )}^{{n_{jk}}}}}} {\gamma _j}>0{\text{ j}}=1,2,...,m{\text{ }}\varepsilon {\text{>0}}{\text{.}}

Where {S_j}=3.{n_{j.jk}}+\frac{3}{2}{n_{o.jk}}

We impose a constraint on the parameters of the model as \sum\limits_{{j=1}}^{m} {{\gamma _j}=1}, which shows that the parameters are well defined. The joint posterior distribution for the parameters \phi given data under the noninformative priors are defined as

P(\phi \left| {\mathbf{a}} \right.)=\frac{1}{R}\prod\limits_{{j<k}}^{m} {{p_W}(\gamma )} {\text{ }}\frac{{\gamma _{j}^{{{S_j}}}{\text{ }}{\varepsilon ^{{n_{o.jk}}}}}}{{{{(\gamma _{j}^{3}+\gamma _{k}^{3}+\varepsilon \sqrt {\gamma _{j}^{3}.\gamma _{k}^{3}} )}^{{n_{jk}}}}}}{\text{, W= 1,2 }} , {\gamma _j}>0{\text{ j}}=1,2,...,m{\text{ }}\varepsilon {\text{ > 0}}{\text{.}}

Here W = 1 denotes the uniform prior, W = 2 stands for the Jeffreys prior and R is the normalizing constant and defined as

R=\int_{0}^{\infty } {\int_{0}^{1} {...\int_{0}^{{1 - {\gamma _1} - {\gamma _2}...{\gamma _{m - 2}}}} {\prod\limits_{{j<k}}^{m} {{p_W}(\gamma )} {\text{ }}\frac{{\gamma _{j}^{{{S_j}}}{\text{ }}{\varepsilon ^{{n_{o.jk}}}}}}{{{{(\gamma _{j}^{3}+\gamma _{k}^{3}+\varepsilon \sqrt {\gamma _{j}^{3}.\gamma _{k}^{3}} )}^{{n_{jk}}}}}}d{\gamma _{m - 1}}...d{\gamma _1}{d_\varepsilon }} } }.

The marginal posterior distribution for the parameter of ${\gamma _1}$using uniform prior is

$$P({\gamma _1}\left| {\mathbf{a}} \right.)=\frac{1}{R}\int_{0}^{\infty } {\int_{0}^{{1 - {\gamma _1}}} {...\int_{0}^{{1 - {\gamma _1} - ...{\gamma _{m - 2}}}} {\prod\limits_{{j<k}}^{{m - 1}} {{p_W}(\gamma )} {\text{ }}\frac{{\gamma _{j}^{{{S_j}}}{\text{ }}{\varepsilon ^{{n_{o.jk}}}}}}{{{{(\gamma _{i}^{3}+\gamma _{k}^{3}+\varepsilon \sqrt {\gamma _{j}^{3}.\gamma _{k}^{3}} )}^{{n_{jk}}}}}}d{\gamma _{m - 1}}...d{\gamma _2}{d_\varepsilon },{\text{ W=1,2}}} } } ,
0<{\gamma _1}<1,{\text{ }}\varepsilon {\text{ >0}}.$$

Similarly we can obtain the other marginal posterior distributions.

The data set given in Table 1used for the analysis is about the preference of ecological factors {Topographic Factors (TO), Edaphic Factors (ED), Pyric Factors (PY), Limiting Factors (LI) and Biotic factors (BI)} which is collected from the students of earth science department of Quaid-i-Azam University, Islamabad, Pakistan.

Table 1
Data Set Showing the Preference for the Ecological Factors

| Pairs (j, k) | Pairs of Ecological Factors | $n_{ij}$ | $n_{ji}$ | $n_{oij}$ | Pairs(j,k) | Pairs of Ecological Factors | $n_{ij}$ | $n_{ji}$ | $n_{oij}$ |
|---|---|---|---|---|---|---|---|---|---|
| (1, 2) | TO, ED | 15 | 13 | 2 | (2, 4) | ED, LI | 18 | 9 | 3 |
| (1, 3) | TO, PY | 17 | 10 | 3 | (2, 5) | ED, BI | 17 | 11 | 2 |
| (1, 4) | TO, LI | 11 | 18 | 1 | (3, 4) | PY, LI | 14 | 13 | 3 |
| (1, 5) | TO, BI | 15 | 12 | 3 | (3, 5) | PY, BI | 13 | 15 | 2 |
| (2, 3) | ED, PY | 16 | 12 | 2 | (4, 5) | LI, BI | 11 | 16 | 3 |

## 4.4 Graphical Representation of the Marginal Distributions

The graphs of the marginal posterior distributions for the worth parameters $\phi$ are given in Figs. 1 and 2 respectively.

Figures 3 and 4 represent the marginal posterior distributions for the parameters $\phi$ using the Jeffreys priors.

Glancing through the above figures, we observe that all graphs are showing approximately symmetrical behavior under the both of the priors.

### 5. Results And Discussions

## 5.1 Loss functions, posterior estimates and posterior risks

For the Bayesian analysis of the model under study, we used different loss functions. Loss function shows the difference between the parameter and the estimate. It is used for the estimation of the parameters. For the accuracy of the method used to analyze the data set, loss function plays an important role. Under the loss function the Bayesian estimator minimizes the expected loss based on the posterior distribution. The expected value calculated is considered as Bayes estimates (BEs) while the expected loss is represented by the posterior risk (PR). Smaller value of the posterior risk is the indication of higher reliability. In this section we present the derivation of BEs and PRs under various loss functions using the Uniform and Jeffreys priors. We use four loss functions, namely the Squared Error Loss Function (SELF), Quadratic Loss Function (QLF), DeGroot Loss Function (DLF) and Precautionary Loss Function (PLF), (Sindhu et al 2019). The Bayes estimators along with their posterior risks are used (Ullah et al 2020).

The BEs and PRs using various loss functions under Uniform and Jeffreys priors are evaluated and are given in Tables 2 and 3. Results show that PRs for the parameters of $\phi$ under SELF are smaller as compared to those for the remaining loss functions, i.e., DLF, PLF and QLF. The results clearly show that SELF is the most suitable loss function for the estimation of model's parameters $\phi$.

Table 2
BEs and PRs (in parentheses) under different loss functions using uniform prior

| Parameters | Loss Functions | | | |
|---|---|---|---|---|
| | SELF | PLF | QLF | DLF |
| ${\gamma _1}$ | 0.2028<br>(0.00082) | 0.2031<br>(0.00902) | 0.2027<br>(0.00181) | 0.2032<br>(0.00200) |
| ${\gamma _2}$ | 0.2176<br>(0.00015) | 0.2177<br>(0.00687) | 0.2169<br>(0.00335) | 0.2183<br>(0.00314) |
| ${\gamma _3}$ | 0.1872<br>(0.00016) | 0.1875<br>(0.00843) | 0.1872<br>(0.00158) | 0.1880<br>(0.00450) |
| ${\gamma _4}$ | 0.1936<br>(0.00013) | 0.1933<br>(0.00656) | 0.1935<br>(0.00172) | 0.1942<br>(0.00339) |
| ${\gamma _5}$ | 0.1988<br>(0.00029) | 0.1984<br>(0.00343) | 0.1997<br>(0.00195) | 0.1963<br>(0.00175) |
| $\varepsilon$ | 0.1810<br>(0.0015) | 0.1810<br>(0.00804) | 0.1809<br>(0.04522) | 0.1813<br>(0.04205) |

Table 3
BEs & PRs (in parentheses) under different loss functions using the Jeffreys prior

| Parameters | Loss Functions | | | |
|---|---|---|---|---|
| | SELF | PLF | QLF | DLF |
| ${\gamma _1}$ | 0.2026 | 0.2028 | 0.2024 | 0.2030 |
| | (0.00076) | (0.00388) | (0.00172) | (0.00191) |
| ${\gamma _2}$ | 0.2171 | 0.2174 | 0.2170 | 0.2177 |
| | (0.00047) | (0.00697) | (0.00340) | (0.00320) |
| ${\gamma _3}$ | 0.1875 | 0.1879 | 0.1878 | 0.1883 |
| | (0.00035) | (0.00845) | (0.00448) | (0.00449) |
| ${\gamma _4}$ | 0.1942 | 0.1942 | 0.1940 | 0.1945 |
| | (0.00037) | (0.00640) | (0.00371) | (0.00329) |
| ${\gamma _5}$ | 0.1986 | 0.1972 | 0.1988 | 0.1965 |
| | (0.00066) | (0.00334) | (0.00187) | (0.00167) |
| $\varepsilon$ | 0.1839 | 0.1877 | 0.1837 | 0.1917 |
| | (0.00338) | (0.00769) | (0.04314) | (0.0405) |

The posterior estimates for the parameters $\phi$ using uniform prior under SELF are 0.2026, 0.22171, 0.1875, 0.11942, 0.1986 and 0.1839, respectively. Form the estimates, it is clear that the ranking of the ecological factors is as: ${\text{ED}} \to TO \to BI \to LI \to PY{\text{,}}$ indicating that Edaphic factor (ED) is preferred the most and Limiting factor (LI) is the least preferred factor.

## 5.2 The preference probabilities

The probability that defines the chance of preferring the ecological factor $T_j$ over $T_k$ in a single comparison. We denote preference probability by $p_{i.ij}$. The posterior estimates are used to calculate preference probabilities. Since the posterior estimates obtained under SELF have minimum posterior risks, so these estimates are used to find the preference probabilities for the worth parameters $\phi$ based on the noninformative priors and are given in Table 4. The value $p_{1.12}$ = 0.4104 for the factors pair (TO, ED) that indicates that TO has 41.04% probability of being preferred against ED and it is $p_{2.12}$= 0.5070 indicating 50.70% preference in the favor of ED against TO and there are 8.26% probability that none of the ecological factor will be preferred. In the similar ways, we can interpret the remaining preference probabilities.

Table 4
The preference probabilities using the uniform prior

| | (1,2) | (1,3) | (1,4) | (1,5) | (2,3) | (2,4) | (2,5) | (3,4) | (3,5) | (4,5) |
|---|---|---|---|---|---|---|---|---|---|---|
| $p_{j.jk}$ | 0.4104 | 0.5135 | 0.4904 | 0.4722 | 0.5614 | 0.5387 | 0.5206 | 0.4354 | 0.4174 | 0.4403 |
| $p_{k.jk}$ | 0.5070 | 0.4039 | 0.4267 | 0.4448 | 0.3574 | 0.3794 | 0.397 | 0.4816 | 0.4999 | 0.4767 |
| $p_{o.jk}$ | 0.0826 | 0.0826 | 0.0829 | 0.0830 | 0.0812 | 0.0819 | 0.0824 | 0.0830 | 0.0827 | 0.0830 |
| The preference probabilities using the Jeffreys prior | | | | | | | | | | |
| $p_{j.jk}$ | 0.4107 | 0.5111 | 0.487 | 0.4716 | 0.558 | 0.5343 | 0.519 | 0.4338 | 0.4182 | 0.4425 |
| $p_{k.jk}$ | 0.5054 | 0.4051 | 0.4289 | 0.4442 | 0.3595 | 0.3824 | 0.3973 | 0.4820 | 0.4974 | 0.4733 |
| $p_{o.jk}$ | 0.0839 | 0.0838 | 0.0841 | 0.0842 | 0.0825 | 0.0833 | 0.0837 | 0.0842 | 0.0844 | 0.0842 |

When we observe the ranking order, we found that there exists complete coordination between the posterior estimates and the preference probabilities.

# 5.3 The Predictive Probabilities

The predictive probabilities describes the preference of ecological factor $T_j$ over $T_k$ in a single future comparison of the two factors ($T_j$, $T_k$). The predictive probability of these two factors ($T_1$, $T_2$) is denoted by $p_{12}$ and can be calculated as;

{P_{{\text{12}}}}=\int_{0}^{1} {\int_{0}^{{1 - {\gamma _1}}} {\int_{0}^{{1 - {\gamma _1} - {\gamma _2}}} {\int_{0}^{{1 - {\gamma _1} - {\gamma _2} - {\gamma _3}}} {\int_{0}^{\infty } {{\theta _{1.12}}p() } } } } \phi \left| {a)d\varepsilon d{\gamma _4}d{\gamma _3}d{\gamma _2}d{\gamma _1}} \right. , {\gamma _j} \geqslant 0,i=1,2,...5.{\text{ }}\sum\limits_{{j=1}}^{5} {{\gamma _j} \leqslant 1,} {\text{ }}\gamma >0,{\text{ }}\varepsilon >0.

Where {\theta _{1.12}} is the model preference probability given in (5), similarly the predictive probabilities of $P_{21}$ and $P_{012}$ can be calculated as following;

{P_{{\text{21}}}}=\int_{0}^{1} {\int_{0}^{{1 - {\gamma _1}}} {\int_{0}^{{1 - {\gamma _1} - {\gamma _2}}} {\int_{0}^{{1 - {\gamma _1} - {\gamma _2} - {\gamma _3}}} {\int_{0}^{\infty } {{\theta _{2.12}}p() } } } } \phi \left| {a)d\varepsilon d{\gamma _4}d{\gamma _3}d{\gamma _2}d{\gamma _1}} \right. , {\gamma _j} \geqslant 0,j=1,2,...5.{\text{ }}\sum\limits_{{j=1}}^{5} {{\gamma _j} \leqslant 1,} {\text{ }}\gamma >0,{\text{ }}\varepsilon >0,

{P_{012}}=\int_{0}^{1} {\int_{0}^{{1 - {\gamma _1}}} {\int_{0}^{{1 - {\gamma _1} - {\gamma _2}}} {\int_{0}^{{1 - {\gamma _1} - {\gamma _2} - {\gamma _3}}} {\int_{0}^{\infty } {{\theta _{0.12}}p() } } } } \phi \left| {a)d\varepsilon d{\gamma _4}d{\gamma _3}d{\gamma _2}d{\gamma _1}} \right. , {\gamma _j} \geqslant 0,j=1,2,...5.{\text{ }}\sum\limits_{{j=1}}^{5} {{\gamma _j} \leqslant 1,} {\text{ }}\gamma >0,{\text{ }}\varepsilon >0.

Where {\theta _{2.12}}and {\theta _{0.12}}are defined in (6) and (7) respectively. The predictive probabilities under both noninformative priors are obtained and given in Table 5.

Table 5
The predictive probabilities using the uniform prior.

| | (1,2) | (1,3) | (1,4) | (1,5) | (2,3) | (2,4) | (2,5) | (3,4) | (3,5) | (4,5) |
|---|---|---|---|---|---|---|---|---|---|---|
| $p_{jk}$ | 0.4109 | 0.5133 | 0.4900 | 0.4713 | 0.5596 | 0.5370 | 0.5188 | 0.4350 | 0.4164 | 0.4394 |
| $p_{jk}$ | 0.5055 | 0.4033 | 0.4262 | 0.4445 | 0.3586 | 0.3803 | 0.3978 | 0.4814 | 0.5000 | 0.4766 |
| $p_{ojk}$ | 0.0836 | 0.0834 | 0.0838 | 0.0842 | 0.0818 | 0.0827 | 0.0834 | 0.0836 | 0.0836 | 0.084 |
| The predictive probabilities using the Jeffreys prior. | | | | | | | | | | |
| $p_{jk}$ | 0.4122 | 0.5117 | 0.4887 | 0.4709 | 0.5570 | 0.5347 | 0.5173 | 0.4356 | 0.4178 | 0.4401 |
| $p_{jk}$ | 0.5047 | 0.4054 | 0.4280 | 0.4455 | 0.3617 | 0.3831 | 0.3998 | 0.4814 | 0.4991 | 0.4761 |
| $p_{ojk}$ | 0.0831 | 0.0829 | 0.0833 | 0.0836 | 0.0813 | 0.0822 | 0.0829 | 0.083 | 0.0831 | 0.0838 |

The predictive probability $p_{12}$ predicts the future preference in the contest of ecological factors in their single future comparisons. The predictive probability for preferring the ecological factor TO on ED is 0.4109, which indicates that there are 41.09% chances that the ecological factor TO will be preferred to ED in the single future comparison. The remaining predictive probabilities can be also interpreted on the same lines. From the results obtained, it is evident that the predictive probabilities do agree with the ranking order given by the posterior means under both of the priors. Similar results are observed on the basis of the preference probabilities.

## 5.4 Bayesian Hypothesis Testing

Bayesian hypothesis testing is a simple and straightforward procedure. The posterior probabilities are calculated, and decision between the hypotheses is directly made. For the comparison of the worth of any two ecological factors $T_j$ and Tk, we consider the following hypotheses:

$${H_{jk}}:{\text{ }}{\gamma _j} \geqslant {\gamma _k}{\text{ Vs }}{H_{kj}}:{\text{ }}{\gamma _k}>{\gamma _j} , j \ne k{\text{ }}1,2,...,5.$$

We represent posterior probability for ${H_{jk}}$ by ${p_{jk}}=P({\gamma _j}>{\gamma _k})$ and for ${H_{kj}}$, we use ${q_{kj}}=1 - {p_{jk}}$, so the posterior probability for $p_{12}$ is $H_{12}$ which is defined as:

$${p_{12}}=p({H_{12}})={\text{p}}({\gamma _1} \geqslant {\gamma _2})={\text{p}}({\gamma _1} - {\gamma _2} \geqslant 0)$$

$${\text{p}}(\delta >0\left| Y \right.)=\int\limits_{{\delta =0}}^{1} {\int\limits_{{\varphi =\delta }}^{{\frac{{1+\delta }}{2}}} {\int\limits_{{{\gamma _3}=0}}^{{1+\delta - 2\varphi }} {\int\limits_{{{\gamma _4}=0}}^{{1 - \delta - (\delta - \varphi ) - {\gamma _3}}} {\int\limits_{{\varepsilon =0}}^{\infty } {p\left( {\delta ,\varphi ,{\gamma _3},{\gamma _{4,}}\varepsilon \left| {\text{y}} \right.} \right)} } } } } {\text{ }}d\varepsilon {\text{ d}}{\gamma _4}{\text{d}}{\gamma _{\text{3}}}d\varphi d\delta \quad {\text{ where }}\delta ={\gamma _1} - {\gamma _2}{\text{, }}\varphi {\text{ = }}{\gamma _1} \text{ and } {q_{12}}=p({H_{21}})=1 - p({H_{12}})$$

The hypothesis with higher probability will be accepted. The posterior probabilities of the hypotheses {H_{jk}}{\text{ and }}{H_{kj}}{\text{ (j<k=1,2}}...{\text{,5)}} are computed for the noninformative priors and given in Table 6. The posterior probability $p_{12}$ = 0.0735 for the ecological factors pair (TO, ED) indicates that the probability for the ecological factor TO is very small, so we shall reject the null hypothesis and accept the alternative hypothesis indicating a higher preference for the factor ED.

Table 6
The posterior probabilities using the uniform prior.

| Pairs | $p_{12}$ | $p_{13}$ | $p_{14}$ | $p_{15}$ | $p_{23}$ | $p_{24}$ | $p_{25}$ | $p_{34}$ | $p_{35}$ | $p_{45}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $H_{jk}$ | 0.0735 | 0.5450 | 0.3257 | 0.1844 | 0.7532 | 0.7165 | 0.6842 | 0.2026 | 0.0655 | 0.0783 |
| $H_{kj}$ | 0.9265 | 0.4550 | 0.6743 | 0.8156 | 0.2468 | 0.2835 | 0.3158 | 0.7974 | 0.9345 | 0.9217 |
| The posterior probabilities using the Jeffreys prior. | | | | | | | | | | |
| $H_{jk}$ | 0.0719 | 0.5292 | 0.3125 | 0.1754 | 0.7342 | 0.6980 | 0.6674 | 0.1991 | 0.0638 | 0.0755 |
| $H_{kj}$ | 0.9281 | 0.4708 | 0.6875 | 0.8246 | 0.2658 | 0.3020 | 0.3326 | 0.8009 | 0.9362 | 0.9245 |

From the results, we see that the hypotheses $H_{21}$, $H_{13}$, $H_{41}$, $H_{51}$ $H_{23}$, $H_{24,}$ $H_{25,}$ $H_{34}$, $H_{35}$ and $H_{54}$ are accepted while all the remaining are rejected. On the same lines we can interpret the remaining probabilities. Our results also indicate that ecological factor ED is preferred the most and ecological factor LI is preferred least, same ranking is observed through the posterior estimates which shows complete coordination among the results.

## 5.5 Plausibility of the model

In order to check the plausibility of the new developed model, we use the Chi-square test of goodness of fit. Let {\overset{\lower0.5em\hbox{$\smash{\scriptscriptstyle\frown}$}}{n} _{jk}}, {\overset{\lower0.5em\hbox{$\smash{\scriptscriptstyle\frown}$}}{n} _{kj}}and {\overset{\lower0.5em\hbox{$\smash{\scriptscriptstyle\frown}$}}{n} _{0.jk}}denote the expected frequencies that can be obtained as {\overset{\lower0.5em\hbox{$\smash{\scriptscriptstyle\frown}$}}{n} _{jk}}={n_{jk}}{\theta _{j.jk}}, {\overset{\lower0.5em\hbox{$\smash{\scriptscriptstyle\frown}$}}{n} _{kj}}={n_{kj}}{\theta _{k.kj}} and {\overset{\lower0.5em\hbox{$\smash{\scriptscriptstyle\frown}$}}{n} _{o.jk}}={n_{o.jk}}{\theta _{o.jk}}, where {\theta _{j.jk}}, {\theta _{k.jk}}and {\theta _{o.jk}} are defined in (5), (6) and (7), respectively. We define the following hypotheses.

$H_o$: The model is plausible for any value of $\gamma = \gamma_o$.

$H_1$: The model is not plausible for any value of $\gamma$.

The $\chi^2$ test is applied to test the goodness of fit of the model.

{\chi ^2}=\sum\limits_{{j<k}}^{5} {\left[ {\frac{{{{\left\{ {{n_{jk}} - {{\overset{\lower0.5em\hbox{$\smash{\scriptscriptstyle\frown}$}}{n} }_{jk}}} \right\}}^2}}}{{{{\overset{\lower0.5em\hbox{$\smash{\scriptscriptstyle\frown}$}}{n} }_{jk}}}}}+\frac{{{{\left\{ {{n_{kj}} -

{{\overset{\lower0.5em\hbox{$\smash{\scriptscriptstyle\frown}$}}{n} }_{kj}}} \right\}}^2}}}
{{{{\overset{\lower0.5em\hbox{$\smash{\scriptscriptstyle\frown}$}}{n} }_{kj}}}}+\frac{{{{\left\{ {{n_{o.jk}} -
{{\overset{\lower0.5em\hbox{$\smash{\scriptscriptstyle\frown}$}}{n} }_{ojk}}} \right\}}^2}}}
{{{{\overset{\lower0.5em\hbox{$\smash{\scriptscriptstyle\frown}$}}{n} }_{o.jk}}}}} \right]} with degrees of freedom, m(m - 2).

The obtained value of the Chi Square statistic for the Extended Weibull PC model is 7.227, with p-value is 0.94946. So according to decision rule, we conclude that the model under study is plausible and fit for the PC data.

# 6. Conclusion

In this study, we extended the Weibull PC model to accommodate ties. The model is analyzed under the Bayesian paradigm using noninformative Uniform and Jeffreys priors. Four different loss functions i.e., SELF, QLF, DLF and PLF are used for the analysis. A real data set of the preference for the ecological factors is collected from the students of earth science department Quaid-i-Azam University, Islamabad, Pakistan. The Bayesian analysis of the model constitutes the estimation of BEs along with their PRs, the preference probabilities, the predictive probability, hypothesis testing and test for the plausibility of the model. Graphs of the model parameters showed approximately a symmetrical behavior around their BEs for both of the priors. Our results indicate that BEs obtained under SELF have minimum PRs, so on the basis of BEs the ecological factors may be ranked as: {\text{ED}} \to TO \to BI \to LI \to PY{\text{.}} Our model gives encouraging results and presents the ranking which is generally prevails in our society. It is also worth mentioning that results obtained from the both priors agree a lot. The estimate of tie parameter is small and approximately identical under both of the noninformative priors. The p-value found for the model also justifies the right direction for extending the Weibull model for paired comparisons.

# Declarations

**Author Contribution:** Khalil Ullah performed the data analysis and preparation of original draft. Muhammad Aslam did supervision and methodology validation. Nasir Abbas worked on investigation, review and validation. Syed Irfan Shah did the conceptual and graphical analysis. All the authors contributed in the interpretation discussion and refinement of the paper.

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**Data availability:** Data have been originally collected from the students of earth science department, Quaid-i-Azam University, Islamabad, Pakistan. Which is also mentioned in the manuscript.

**Conflict of interest:** The authors declare no conflicts of interest among the reviewers, organizations or institutions

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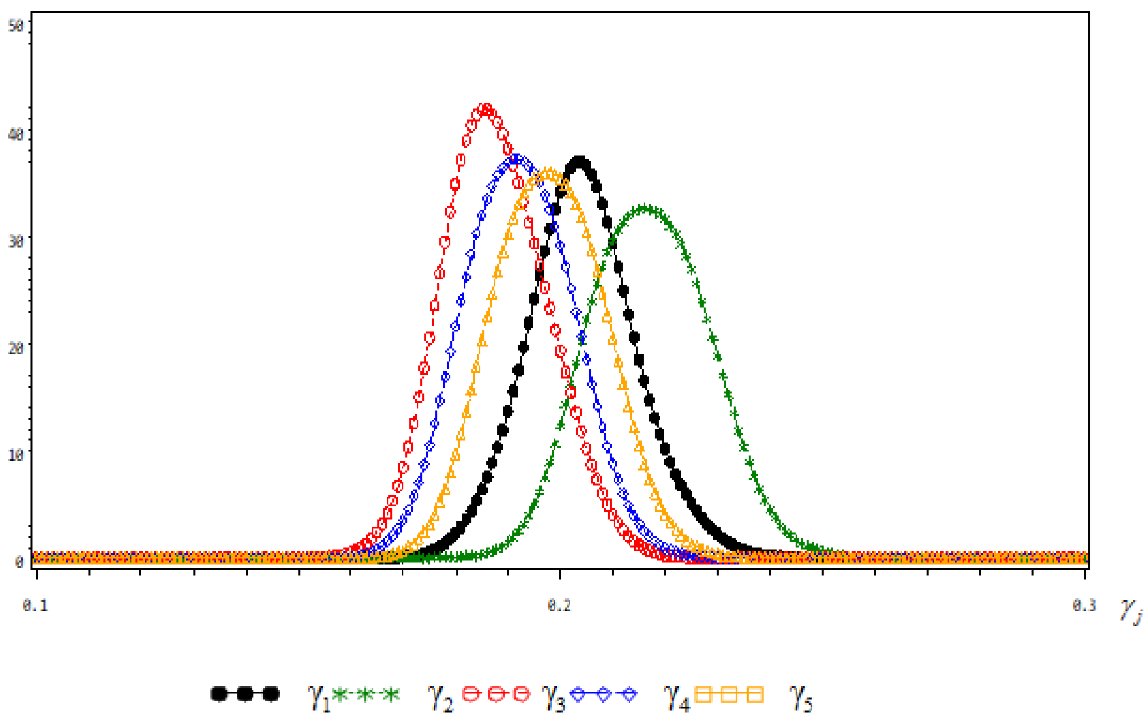
# Figures

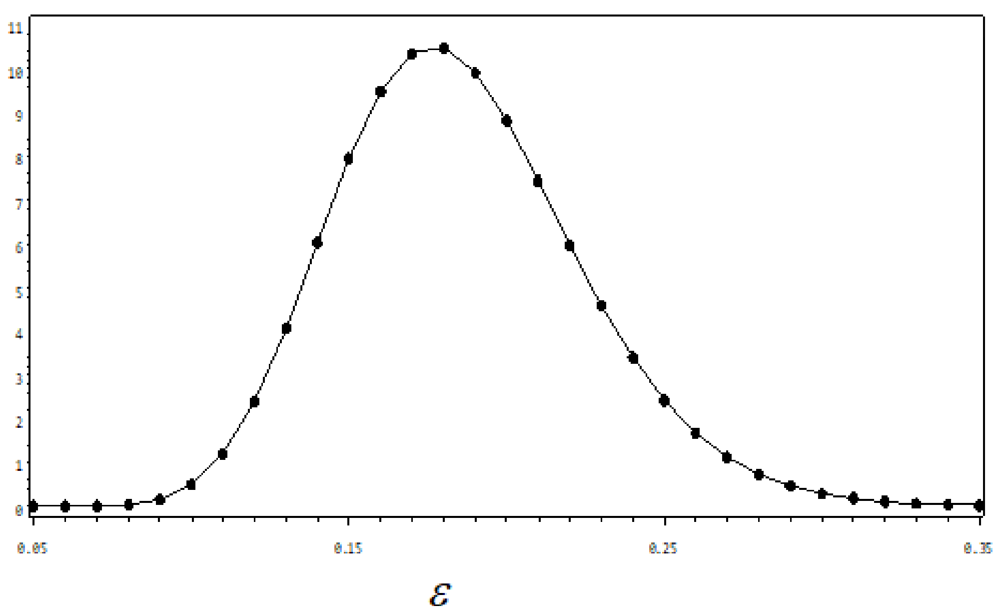
$p(\gamma_j|\mathbf{a})$ Figure 1. The marginal densities for the parameters using the uniform prior




### Figure 1

See image above for figure legend

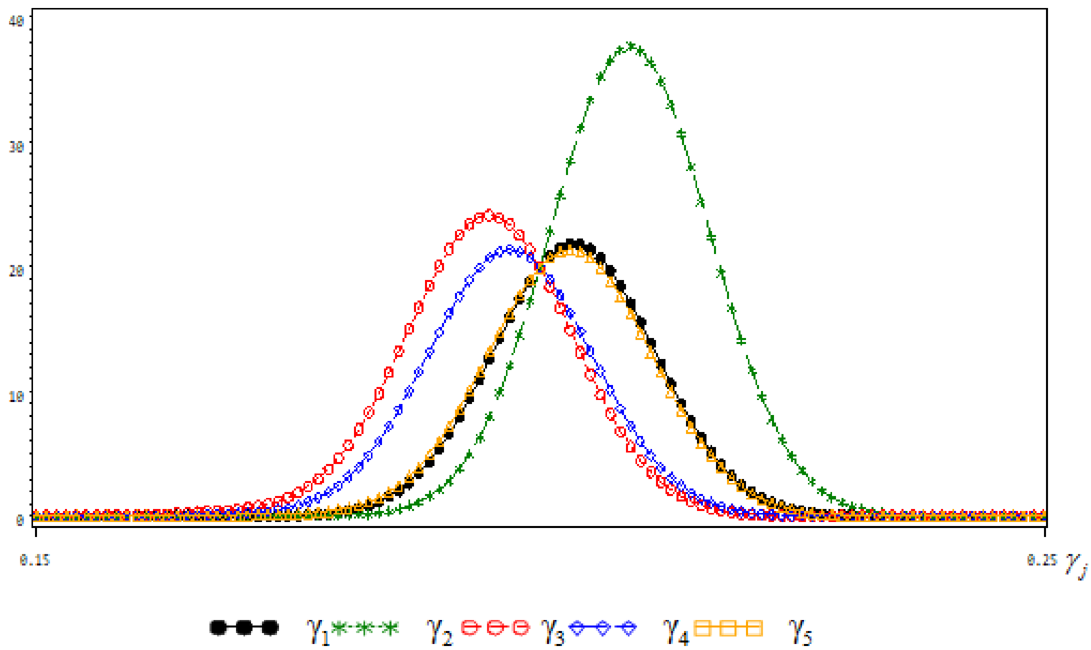
$p(\varepsilon|\mathbf{a})$ Figure 2. The marginal densities for the tie parameter using the uniform prior



## Figure 2

See image above for figure legend

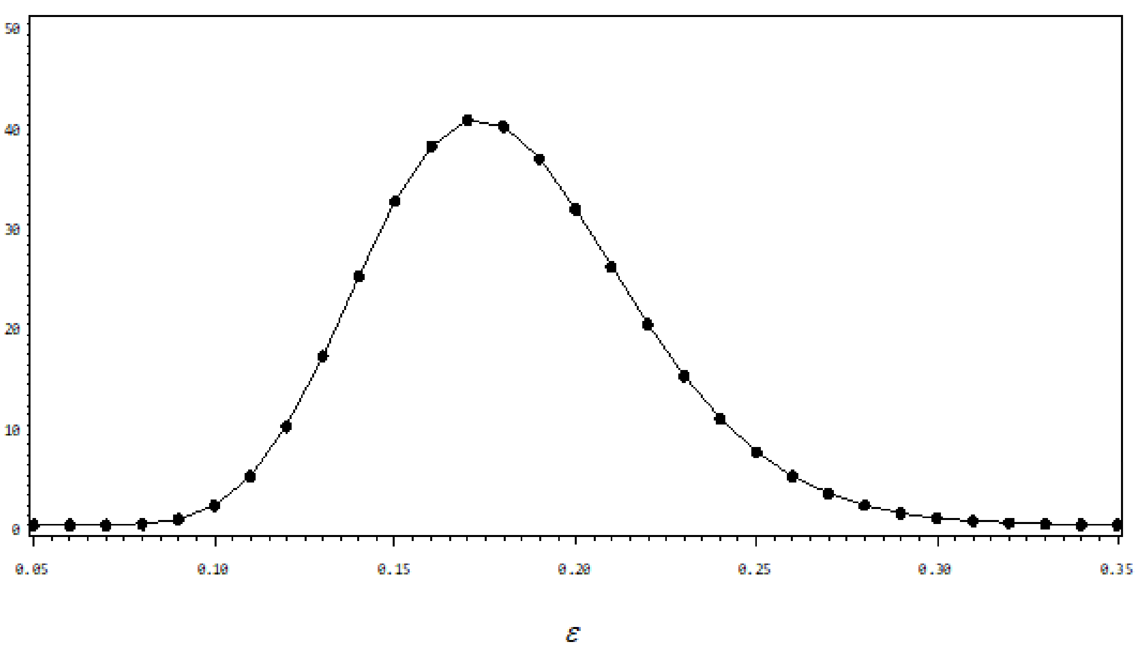
$p(\gamma_j|\mathbf{a})$ Figure 3. The marginal densities for the parameters using the Jeffreys prior




## Figure 3

See image above for figure legend

$p(\varepsilon|\mathbf{a})$ Figure 4. The marginal densities for the tie parameter using the Jeffreys prior



## Figure 4

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