

Qualitative analysis of a prey-predator model with prey refuge and intraspecific competition among predators

Maruthai Selvaraj Surendar

Saveetha School of Engineering, SIMATS

Muniagounder Sambath

Periyar University

Krishnan Balachandran

Bharathiar University

Yong-Ki Ma (✉ ykma@kongju.ac.kr)

Kongju National University

Research Article

Keywords: Stability analysis, Prey-predator model, Refuge, Hopf-bifurcation, Crowley- Martin functional response, Turing instability

Posted Date: June 13th, 2022

DOI: <https://doi.org/10.21203/rs.3.rs-1745611/v1>

License:  This work is licensed under a Creative Commons Attribution 4.0 International License.

[Read Full License](#)

Qualitative analysis of a prey-predator model with prey refuge and intraspecific competition among predators

Maruthai Selvaraj Surendar^{1*}, Muniagounder Sambath^{2†}, Krishnan Balachandran^{3‡},
Yong-Ki Ma^{4§}

¹ Department of Mathematics, Saveetha School of Engineering,
SIMATS, Chennai-602 105, India

² Department of Mathematics, Periyar University, Salem 636 011, India

³Department of Mathematics, Bharathiar University,
Coimbatore-641 046, India

⁴Department of Applied Mathematics, Kongju National University,
Chungcheongnam-do 32588, Republic of Korea

Abstract

In this paper we consider a prey-predator model with prey refuge and intraspecific competition between predators using the Crowley-Martin functional response and investigate the dynamic characteristics of spatial and non-spatial prey-predator systems with each analytical and numerical approach. The local stability of non-trivial interior equilibrium, the existence of a Hopf bifurcation, and stability of bifurcating periodic solutions have been obtained in the absence of diffusion. For the spatial system, the Turing and non-Turing patterns are evaluated for some set parametric beliefs, and for prey and predator inhabitants some exciting results are obtained. Numerical simulation demonstrates that both prey and predator populations will not converge to the stationary state at any foreseeable future time when the parametric values are ingested in the Turing domain.

Keywords: Stability analysis, Prey-predator model, Refuge, Hopf-bifurcation, Crowley-Martin functional response, Turing instability

2020 Mathematics Subject Classification: 35K57, 92D25, 70K50, 34D20

1 Introduction

Qualitative dynamic characteristics of interacting prey-predator models are extensively investigated by several researchers to comprehend the long-time behavior of the species. Numerous non-linear coupled ordinary differential equations are recommended and assessed

*E.Mail:mssurendar.dindigul@gmail.com (Maruthai Selvaraj Surendar)

†E.Mail:sambathbu2010@gmail.com (Muniagounder Sambath)

‡E.Mail:kb.maths.bu@gmail.com (Krishnan Balachandran)

§E.Mail:ykma@kongju.ac.kr (Yong-Ki Ma, Corresponding author)

for the interaction among prey and their potential predators. In 1925, Lotka applied a set of equations to analyze prey-predator interactions and the same set of equations were obtained in 1926 by Volterra. This model has to be known as the Lotka-Volterra model. Later on, the model was developed to include density-dependent prey's growth and a functional response form by Holling [16] which is known as the Rosenzweig-MacArthur model [32]. These are generally the basic models to describe the characteristics of normal populations of prey and predators. Through the years, prey-predator types with various kinds of efficient response features, defined by ordinary differential equations, have been examined extensively. Subsequently mathematicians and biologists tried to extend the prey-predator models with several kinds of functional responses to measure the actual phenomena of our natural ecosystem [3, 42, 43].

In addition, the most critical idea in the prey-predator model is 'functional response' that describes the rate at which a predator attacks the number of prey. The most commonly used functional responses to represent the average feeding rate of a predator are Lotka-Volterra type and Holling type:

$$\begin{aligned}
 \text{Lotka-Volterra type} & : f_1(u) = \delta u \\
 \text{Holling Type-II} & : f_2(u) = \delta u / (1 + \alpha u) \\
 \text{Holling Type-III} & : f_3(u) = \delta u^2 / (1 + \alpha u^2) \\
 \text{Holling Type-IV} & : f_4(u) = \delta u / (1 + \alpha u^2)
 \end{aligned}$$

in which u denotes the density of the prey population, δ and α are positive constants that illustrate the effects of capture rate and handling time correspondingly. It is to be noted that these four functional response features are prey-dependent and are generally not influenced by predator inhabitants. Thus to illustrate mutual disturbance among potential predators, Beddington [2] and DeAngelis [10] offered the following functional response

$$f(u, v) = \delta u / (1 + \alpha u + \beta v)$$

in which v denotes the density of the predator population and β is a positive constant describing the magnitude of interference among predators. Specifically, the Beddington-DeAngelis functional response decreases to Holling type II functional response at the moment $\beta = 0$. In the Beddington-DeAngelis functional response, disturbance among predators and handling time are believed to be distinctive. Eventually, Crowley and Martin [7] assumed that interference among predators arises no matter whether a unique predator can be searching for preys or handling preys, and in addition, they proposed a functional response,

$$f(u, v) = \delta u / ((1 + \alpha u)(1 + \beta v))$$

which is called as Crowley-Martin functional response. In fact, Crowley-Martin functional response and Beddington-DeAngelis functional response are less similar, except for the term $\alpha\beta uv$. But in some situations explaining the exact dynamics of the natural density of possible preys and predators, Crowley-Martin type functional response is superior to Beddington-DeAngelis functional response [37].

Furthermore, for the last two decades, many mathematicians and ecologists modelled the various type of prey-predator models with different kind of functional responses to reflect the exact behaviour of the natural ecosystem. For example, Allee effect, prey refuge, interference among predators, harvesting are most growing topics [5, 13, 25–28, 33, 36, 39, 40].

After the modelling process, the corresponding dynamic characteristics play a peculiar role in understanding the natural effects in future. The dynamic behaviours of prey-predator models, like local and global stability, Hopf bifurcation and Turing instability, have been investigated extensively by many authors [4, 8, 9, 17–19, 21–23, 30, 31, 35, 38, 41].

The aim of the present paper is to formulate a prey-predator model with prey refuge and intraspecific competition between predators for prey by using the Crowley-Martin functional response idea and study its corresponding qualitative dynamic characteristics.

The organization of this paper is as follows. In Section 2, we introduce a mathematical prey-predator model and in Section 3 we analyze the existence of equilibria and the local and global dynamics of equilibria. Further, the presence of Hopf bifurcation and its periodic solutions behaviours are discussed in section 4. Moreover, we examined the occurrence of Turing instability in the interior equilibrium point under diffusion effect in section 5. In Section 6, we present some numerical simulations to verify our theoretical results and find other meaningful phenomena. Finally, some concluding comments are made in Section 7.

2 Mathematical model and analysis

Bazykin [1] proposed a Crowley-Martin(C-M) type prey-predator model in which individuals from a population of two or more potential predators not only spread their time to look for and control prey but also commit some strength taking part in activities with different possible predators, causing a functional response that gives a tremendous instantaneous, per capita feeding rate:

$$f(X, Y) := \frac{\tilde{a}XY}{1 + \tilde{b}X + \tilde{c}(Y - 1)},$$

where X denotes the prey inhabitants, Y denotes predator inhabitants, and \tilde{c} (units: 1/predator) indicates the magnitude of interference among predators. The term $Y - 1$ was utilized by Beddington [2], and Crowley-Martin [7] in setting up mechanistic types in which predator inhabitants is typically expressed while counts (integers) and the system of predatory animal dependence is interference through direct activities with other potential predators. As a result, the $Y - 1$ term is used as a predator would not meddle with itself in those models. Notably, when $Y = 1$, it reduces to the traditional Holling Type II response.

Further, one of the relevant principles on the prey-predator model is the effect of prey refuge, which in turn defends a consistent proportion of prey by predation [14,15]. Therefore, there exists a significant influence of prey refuge on the coexistence amongst predator and prey. Consequently, in theoretical ecology, the result of prey refuge for the dynamics of prey-predator interaction may be analyzed as an appealing topic. Also, many authors [6, 11, 20, 34] have been figured that the prey refuge provides a stabilizing impact on prey-predator dynamic characteristics, and prey biomass could be controlled and saved via extinction because of predation. Therefore, the predator's foraging effectiveness is impacted by the prey refuge strength.

We combine the strength of prey refuge in the C-M functional response term. From the above discussion, the non-autonomous C-M type prey-predator model with prey refuge and

intraspecific competition between predators for prey takes the form:

$$\begin{cases} \frac{dX(t)}{dt} = X(r_1 - \rho X) - \frac{\delta_1(1-m)XY}{\chi_1 + \chi_2(1-m)X + \chi_3((1-m)X)(Y-1)}, \\ \frac{dY(t)}{dt} = r_2Y - \frac{\delta_2Y^2}{\chi_4 + (1-m)X} - \delta_3Y, \\ X(0) = X_0 > 0, \quad Y(0) = Y_0 > 0. \end{cases} \quad (2.1)$$

Here $X(t)$ is the biomass of prey population and $Y(t)$ is the biomass of predator population at time t respectively. It is likewise supposed the refuge protecting mX of prey, in which $m \in [0, 1)$, is constant and thus $(1-m)X$ is only prey accessible to predator. The parameters $r_1, r_2, \rho, \delta_1, \delta_2, \delta_3, \chi_1, \chi_2, \chi_3$, and χ_4 are all positive constants, with ecological meaning as follows:

Parameter	Biological/ecological meaning
r_1	Growth rate of prey population in the absence of predator,
r_2	Growth rate of predator (Based on the convention coefficients from individuals of prey into individuals of predator),
ρ	Strength of competition among individual of prey species,
δ_1	Maximum value which per capita reduction rate of prey can attain,
δ_2	Maximum value which per capita predator consumption rate can attain,
δ_3	Death rate of predator,
χ_1	Half saturation constant of prey in absence of refuge,
χ_2	Measures the handling time of prey on the feeding rate,
χ_3	Coefficient of interference(disturbance) among predator,
χ_4	Measures the extent to which environment provides protection to predator.

Assume that, in biological point of view, $\chi_1 + \chi_2(1-m)X + \chi_3((1-m)X)(Y-1) \neq 0$. Then, our system (2.1) is well-posed.

For easiness, we nondimensionalize (2.1) with the following scaling

$$u \mapsto \rho X, \quad v \mapsto Y, \quad t \mapsto t,$$

and then obtain the form

$$\begin{cases} \dot{u}(t) = u(r - u) - \frac{\delta(1-m)uv}{1 + \alpha(1-m)u + \beta((1-m)u)(v-1)}, \\ \dot{v}(t) = \xi \left(\zeta v - \frac{v^2}{\eta + (1-m)u} - \sigma v \right), \\ u(0) = u_0 > 0, \quad v(0) = v_0 > 0, \end{cases} \quad (2.2)$$

where, $r = r_1, \delta = \frac{\delta_1}{\chi_1}, \alpha = \frac{\chi_2}{\rho\chi_1}, \beta = \frac{\chi_3}{\rho\chi_1}, \xi = \frac{\delta_2}{\rho}, \zeta = \frac{r_2\rho}{\delta_2}, \eta = \rho\chi_4$, and $\sigma = \frac{\delta_3\rho}{\delta_2}$.

All of us live in a spatial universe, and the space component of environmental interactions has become a crucial factor in just how ecological communities are shaped. Numerous physical aspects of our planet, for example, ambiance, substance creation or biological

framework, can differ around the earth. So, we have to think the dynamic changes of the population depends on both space and time (spatial movement). To analyze the spatial dynamics of the prey-predator model, we consider the following system of partial differential equations with homogeneous Neumann boundary conditions.

$$\begin{cases} \frac{\partial u(x, t)}{\partial t} = u(r - u) - \frac{\delta(1 - m)uv}{1 + \alpha(1 - m)u + \beta((1 - m)u)(v - 1)} + D_1 \nabla^2 u, & x \in \Omega, \quad t > 0, \\ \frac{\partial v(x, t)}{\partial t} = \xi \left(\zeta v - \frac{v^2}{\eta + (1 - m)u} - \sigma v \right) + D_2 \nabla^2 v, & x \in \Omega, \quad t > 0, \\ \partial_\nu u = \partial_\nu v = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x, 0) = u_0(x) \geq 0, \quad v(x, 0) = v_0(x) \geq 0, & x \in \Omega. \end{cases} \quad (2.3)$$

In the above, $u(x, t)$ and $v(x, t)$ indicates the densities of prey and predator in fixed open bounded domain $\Omega \subset \mathbb{R}^N$ at time t respectively and D_1, D_2 denotes the diffusion coefficients of prey and predator. Here $\nabla^2 = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$ is the usual Laplacian operator in two-dimensional space, Ω is a bounded spatial domain with a smooth boundary $\partial\Omega$, ν is the outward unit normal vector of the boundary $\partial\Omega$. The homogeneous Neumann boundary conditions indicate that the prey-predator system is self-contained with zero population flux across the boundary.

3 Equilibrium points and stability analysis

3.1 Equilibria

System (2.2) has trivial and axial equilibrium points E_0, E_1, E_2 are always exist, where

- (i) Trivial equilibrium point: $E_0 = (0, 0)$, extinct of both prey and predator.
- (ii) Axial equilibrium points:
 $E_1 = (r, 0)$, prey only lives; $E_2 = (0, \zeta\eta - \eta\sigma)$, if $\zeta > \sigma$, predator only survive.

The non-trivial interior equilibrium point $E^*(u^*, v^*) \in \mathbb{R}_+^2$ can be obtained by solving the following system of equations:

$$u(r - u) - \frac{\delta(1 - m)uv}{1 + \alpha(1 - m)u + \beta((1 - m)u)(v - 1)} = 0, \quad \xi \left(\zeta v - \frac{v^2}{\eta + (1 - m)u} - \sigma v \right) = 0.$$

Solving these nullclines, we get a cubic equation in u ,

$$\Delta_1 u^3 + \Delta_2 u^2 + \Delta_3 u + \Delta_4 = 0. \quad (3.1)$$

where $(H) : \Delta_1 = -(m-1)^2\beta(\zeta-\sigma) < 0, \Delta_2 = (m-1)(\alpha+\beta(-1+((m-1)r+\eta)(\zeta-\sigma))) < 0, \Delta_3 = -1-(m-1)^2\delta(\zeta-\sigma)-(m-1)r(\alpha+\beta(-1+\eta(\zeta-\sigma))) > 0, \Delta_4 = r+(m-1)\delta\eta(\zeta-\sigma) > 0$, provided the cubic equation (3.1) has atmost one positive solution u^* (say). Using the value of u^* , we get the value of v^* as:

$$v^* = (\zeta - \sigma)(\eta + (1 - m)u^*),$$

$\zeta > \sigma$ provided v^* also positive.

In future, we assume that, all $E^*(u^*, v^*)$ ensures the above conditions (H) and $\zeta > \sigma$.

3.2 Local stability analysis

The dynamic behavior of the equilibrium points can be studied by computing the eigenvalues of the Jacobian matrix J of the system (2.2), namely,

$$J = \begin{pmatrix} r - 2u - \frac{\delta(1-m)v}{((m-1)u(\alpha + (v-1)\beta) - 1)^2} & \frac{\delta(1-m)(-1 + (m-1)u(\alpha - \beta))u}{((m-1)u(\alpha + (v-1)\beta) - 1)^2} \\ \frac{-\xi(m-1)v^2}{(\eta + (1-m)u)^2} & \xi \left(\zeta - \frac{2v}{\eta + (1-m)u} - \sigma \right) \end{pmatrix}. \quad (3.2)$$

The existence and local stability of the equilibrium solutions can be stated as follows:

Theorem: 3.1.

- (i) *The trivial equilibrium point $E_0 = (0, 0)$ is always unstable.*
- (ii) *If $\zeta < \sigma$ then the axial equilibrium point $E_1 = (r, 0)$ is locally asymptotically stable.*
- (iii) *The axial equilibrium point $E_2 = (0, \eta(\zeta - \sigma))$ is always unstable.*

Proof.

- (i) The Jacobian matrix of the system (2.2) evaluated at the equilibrium point $E_0 = (0, 0)$ is given by,

$$J|_{E_0} = \begin{pmatrix} r & 0 \\ 0 & \xi(\zeta - \sigma) \end{pmatrix},$$

$trJ|_{E_0} = r + \xi(\zeta - \sigma)$ and $detJ|_{E_0} = r\xi(\zeta - \sigma)$. The eigen values corresponding to $J|_{E_0}$ are $\lambda_1 = r$ (> 0) and $\lambda_2 = \xi(\zeta - \sigma)$. Thus E_0 is unstable when $\zeta > \sigma$ ($\lambda_2 > 0$), and E_0 is saddle point when $\zeta < \sigma$ ($\lambda_2 < 0$). Therefore $E_0 = (0, 0)$ is always unstable.

- (ii) The Jacobian matrix of the system (2.2) evaluated at the axial equilibrium point $E_1 = (r, 0)$ is given by,

$$J|_{E_1} = \begin{pmatrix} -r & \frac{-(m-1)r\delta}{(m-1)(\alpha-\beta)r-1} \\ 0 & \xi(\zeta - \sigma) \end{pmatrix},$$

$trJ|_{E_1} = -r + \xi(\zeta - \sigma)$ and $detJ|_{E_1} = -r\xi(\zeta - \sigma)$. When $\zeta < \sigma$, $detJ|_{E_1} > 0$ and $trJ|_{E_1} < 0$. Therefore, E_1 is locally asymptotically stable when $\zeta < \sigma$.

- (iii) The Jacobian matrix of the system (2.2) evaluated at the axial equilibrium point $E_2 = (0, \eta(\zeta - \sigma))$ is given by,

$$J|_{E_2} = \begin{pmatrix} r + \delta\eta(m-1)(\zeta - \sigma) & 0 \\ -\xi(m-1)(\zeta - \sigma)^2 & -\xi(\zeta - \sigma) \end{pmatrix},$$

then the eigen values of the above Jacobi matrix $J|_{E_2}$ are $\mu_1 = r + \delta\eta(1-m)(\zeta - \sigma)$ and $\mu_2 = -\xi(\zeta - \sigma)$. When $\zeta < \sigma$ ($\mu_2 < 0$) and $r < \delta\eta(1-m)(\zeta - \sigma)$ ($\mu_1 < 0$), both eigen values are negative. So the axial equilibrium point E_2 is locally asymptotically stable when $\zeta < \sigma$ and $r < \delta\eta(1-m)(\zeta - \sigma)$. This is not possible since $\zeta < \sigma \implies \delta\eta(1-m)(\zeta - \sigma) < 0$. Also our assumption that $r > 0$. This shows that locally asymptotically stable situation never happens. Therefore E_2 is always unstable. Also note that, when $\zeta < \sigma \implies E_2 < 0$ (i.e., $E_2 \notin \{\{0\} \cup \mathbb{R}_+\}^2$, predator also dies out). \square

3.3 Interior equilibrium qualitative behaviours

The Jacobian matrix evaluated at the coexistence equilibrium point $E^*(u^*, v^*)$ is

$$J|_{E^*} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \quad (3.3)$$

where $a_{11} = r - 2u^* - \frac{\delta(1-m)v^*}{((m-1)u^*(\alpha + (v^*-1)\beta) - 1)^2}$, $a_{12} = \frac{\delta(1-m)(-1 + (m-1)u^*(\alpha - \beta))u^*}{((m-1)u^*(\alpha + (v^*-1)\beta) - 1)^2}$,
 $a_{21} = \frac{-\xi(m-1)v^{*2}}{(\eta + (1-m)u^*)^2}$, $a_{22} = \xi \left(\zeta - \frac{2v^*}{\eta + (1-m)u^*} - \sigma \right)$.

Then trace and determinant of the Jacobian matrix (3.3) is,

$$T = \text{tr}J|_{E^*} = a_{11} + a_{22} \text{ and } D = \text{det}J|_{E^*} = a_{11}a_{22} - a_{21}a_{12}.$$

Therefore the characteristic equation of the linearized system of (2.2) at $E^* = (u^*, v^*)$ is

$$\lambda^2 - T\lambda + D = 0. \quad (3.4)$$

The qualitative behaviours of the interior equilibrium point $E^*(u^*, v^*)$ stated as follows:

Case(i): If $T < 0$ and $D > 0$, then the characteristic roots of (3.4) are either both are negative reals or complex conjugate with negative real part. Therefore, E^* is either stable node ($T^2 > 4D$) or stable spiral ($T^2 < 4D$).

Case(ii): If $T > 0$ and $D > 0$, then the characteristic roots of (3.4) are either both are positive reals or complex conjugate with positive real part. So, E^* is either unstable node ($T^2 > 4D$) or unstable spiral ($T^2 < 4D$).

Case(iii): If $D < 0$ and $T \neq 0$, then the characteristic roots of (3.4) be both are real with with opposite signs. Therefore E^* is saddle point.

Case(iv): If $D < 0$ and $T = 0$, then the characteristic roots of (3.4) be both are real numbers with same magnitude and opposite signs. So, E^* is saddle-node.

Case(v): If $D > 0$ and $T = 0$, then the characteristic roots of (3.4) be purely complex conjugate. Therefore, E^* is centre.

3.4 Global stability analysis

Now, we give the result to attain global stability in the non-trivial interior equilibrium $E^* = (u^*, v^*)$ for both local(2.2) and spatial(2.3) system.

Theorem: 3.2. *If $(\delta(1-m)v^* - u^*)\alpha < \frac{1}{1-m} + \beta u^*(v^*-1)$ and $(1-m)u^*(\beta - \alpha) < 1$ holds, then interior equilibrium $E^* = (u^*, v^*)$ of the local system (2.2) is globally asymptotically stable.*

Proof. The proof is usually developed by applying Lyapunov function. We will consider the subsequent positive definite Lyapunov function in \mathbb{R} about the equilibrium E:

$$\begin{aligned} V_1 &= u - u^* - u^* \ln\left(\frac{u}{u^*}\right) \quad \text{and} \\ V_2 &= v - v^* - v^* \ln\left(\frac{v}{v^*}\right). \end{aligned}$$

This kind of Lyapunov function was first utilized in [12], and later it was broadly exploited by several researchers. We can simply verify that $\frac{\partial V_1}{\partial u} > 0$ for $u > u^*$ and $\frac{\partial V_1}{\partial u} < 0$ for $0 < u < u^*$, and $\frac{\partial V_2}{\partial v} > 0$ for $v > v^*$ and $\frac{\partial V_2}{\partial v} < 0$ for $0 < v < v^*$. If we estimate $\frac{dV_1}{dt}$ via (2.2) we get,

$$\begin{aligned} \frac{dV_1(u)}{dt} &= (u - u^*) \frac{\dot{u}}{u} = \left(1 - \frac{u^*}{u}\right) \left\{ u(r - u) - \frac{\delta(1 - m)uv}{\Theta} \right\}, \\ &= (u - u^*) \left\{ -(u - u^*) - \left(\frac{\delta(1 - m)v}{\Theta} - \frac{\delta(1 - m)v^*}{\widehat{\Theta}} \right) \right\}, \\ &= (u - u^*)^2 \left\{ \frac{\delta(1 - m)(\alpha(1 - m)v^* - \beta(1 - m)v^*(1 - v))}{\Theta \widehat{\Theta}} - 1 \right\} \\ &\quad - \frac{\delta(1 - m)(1 + \alpha(1 - m)u^* - \beta(1 - m)u^*)(u - u^*)(v - v^*)}{\Theta \widehat{\Theta}}, \end{aligned} \tag{3.5}$$

where $\Theta = 1 + \alpha(1 - m)u + \beta((1 - m)u)(v - 1)$, $\widehat{\Theta} = 1 + \alpha(1 - m)u^* + \beta((1 - m)u^*)(v^* - 1)$. Similarly,

$$\begin{aligned} \frac{dV_2(v)}{dt} &= (v - v^*) \frac{\dot{v}}{v} = \left(1 - \frac{v^*}{v}\right) \left\{ \xi \left(\zeta v - \frac{v^2}{\Phi} - \sigma v \right) \right\}, \\ &= (v - v^*) \left\{ \frac{-\xi v}{\Phi} + \frac{\xi v^*}{\widehat{\Phi}} \right\}, \\ &= \frac{-\xi(v - v^*)^2}{\Phi} + \frac{\xi(1 - m)v^*(u - u^*)(v - v^*)}{\Phi \widehat{\Phi}}. \end{aligned} \tag{3.6}$$

where $\Phi = \eta + (1 - m)u$, $\widehat{\Phi} = \eta + (1 - m)u^*$.

Define the Lyapunov function $V = V_1(u) + \frac{1}{C}V_2(v)$

where $C = \frac{\xi v^*}{\delta(1 + \alpha(1 - m)u^* - \beta(1 - m)u^*)}$. Generating $\frac{dV}{dt}$ via (3.5) and (3.6) gives,

$$\begin{aligned} \frac{dV}{dt} &= (u - u^*)^2 \left\{ \frac{\delta(1 - m)(1 + \alpha(1 - m)u^* - \beta(1 - m)v^*(1 - v))}{\Theta \widehat{\Theta}} - 1 \right\} \\ &\quad - (v - v^*)^2 \left\{ \frac{\delta(1 + \alpha(1 - m)u^* - \beta(1 - m)u^*)}{v^* \widehat{\Phi}} \right\}. \end{aligned} \tag{3.7}$$

The coefficient of $(u - u^*)^2$ is,

$$\frac{\delta(1 - m)(1 + \alpha(1 - m)u^* - \beta(1 - m)v^*(1 - v))}{\Theta \widehat{\Theta}} - 1 \leq \frac{\delta(1 - m)(1 + \alpha(1 - m)u^* - \beta(1 - m)v^*(1 - v))}{\widehat{\Theta}} - 1$$

The coefficient of $(v - v^*)^2$ is,

$$-\frac{\delta(1 + \alpha(1 - m)u^* - \beta(1 - m)u^*)}{v^* \widehat{\Phi}} \leq -\frac{\delta(1 + \alpha(1 - m)u^* - \beta(1 - m)u^*)}{v^*}.$$

Therefore if $(\delta(1-m)v^* - u^*)\alpha < \frac{1}{1-m} + \beta u^*(v^* - 1)$ and $(1-m)u^*(\beta - \alpha) < 1$ holds consequently from (3.7), we get $\frac{dV}{dt} < 0$. Hence, by Lyapunov's asymptotic stability theorem the interior equilibrium E^* of the system(2.2) is globally asymptotically stable. \square

Now, we select the Lyapunov function for the diffusion system (2.3)

$$E(t) = \iint_{\Omega} V(u, v) dA. \quad (3.8)$$

So, differentiating $E(t)$ with respect to t along the solutions of the system (2.3), we get

$$\frac{dE(t)}{dt} = \iint_{\Omega} \left(\frac{\partial V}{\partial u} d_1 \nabla^2 u + \frac{\partial V}{\partial v} d_2 \nabla^2 v \right) dA + \iint_{\Omega} \frac{dV}{dt} dA.$$

Considering the zero-flux boundary conditions $\partial_{\nu} u = \partial_{\nu} v = 0$, $x \in \partial\Omega$, and apply Green's first identity in the plane, we get

$$\begin{aligned} \frac{dE(t)}{dt} &= \iint_{\Omega} \frac{dV}{dt} dA - \left\{ \frac{d_1 u^*}{u^2} \iint_{\Omega} \left(\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right) dA + \frac{d_2 v^*}{v^2} \iint_{\Omega} \left(\left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 \right) dA \right\} \\ &\leq \iint_{\Omega} \frac{dV}{dt} dA \leq 0. \end{aligned} \quad (3.9)$$

Therefore the equilibrium E^* of the spatial system (2.3) is globally asymptotically stable.

4 Existence of Hopf bifurcation around E^*

We analyze the Hopf bifurcation occurring at $E^*(u^*, v^*)$ by choosing ξ as the bifurcation parameter. Also, note that $E^*(u^*, v^*)$ is independent of ξ . The characteristic equation of the system(2.2) at $E^*(u^*, v^*)$ is,

$$\lambda^2 - T(\xi)\lambda + D(\xi) = 0. \quad (4.1)$$

The two roots of (4.1) are given as $\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2} = p_1(\xi) \pm ip_2(\xi)$.

Let $\xi_0 = \left\{ \frac{-r+2u^* + \frac{\delta(1-m)v^*}{((m-1)u^*(\alpha+(v^*-1)\beta)-1)^2}}{\left(\zeta - \frac{2v^*}{\eta+(1-m)u^*} - \sigma \right)} \right\}$. In order to see the instability of the system (2.2), let us state the following theorem:

Theorem: 4.1. (Hopf Bifurcation Theorem [24]) Let $T(\xi)$ and $D(\xi)$ are smooth functions of ξ in an open interval about $\xi_0 \in \mathbb{R}$ such that the characteristic equation has a pair of imaginary eigen values $\lambda = p_1(\xi) \pm ip_2(\xi)$ with $p_1(\xi)$ and $p_2(\xi) \in \mathbb{R}$ so that λ become purely imaginary at $\xi = \xi_0$ and $\frac{dp_1}{d\xi}|_{\xi=\xi_0} \neq 0$, then a Hopf bifurcation occurs around E^* at $\xi = \xi_0$. (i.e, stability changes of $E^*(x^*, y^*)$ accompanied by the formation of a limit cycle when $\xi = \xi_0$).

Theorem: 4.2. When ξ passes through ξ_0 , the local system (2.2) attains a Hopf bifurcation around $E^*(x^*, y^*)$ provided $T(\xi_0) = 0$ and $D(\xi_0) > 0$.

Proof. At $\xi = \xi_0$, the characteristic equation (4.1) becomes,

$$\lambda^2 + D(\xi_0) = 0. \quad (4.2)$$

The roots of the equation (4.2) are, $\lambda_1 = i\sqrt{D(\xi_0)}$ and $\lambda_2 = -i\sqrt{D(\xi_0)}$. Thus at E^* a couple of purely imaginary eigenvalues has existed. Likewise, $T(\xi)$ and $D(\xi)$ are smooth functions of ξ , therefore in the neighborhood of ξ_0 , the roots of the equation (4.2) are of the form $\lambda_1 = p_1(\xi) + ip_2(\xi)$ and $\lambda_2 = p_1(\xi) - ip_2(\xi)$, where $p_i(\xi)$ are real functions for $i = 1, 2$. Subsequent, let us validate the transversality condition: $\frac{d}{d\xi}(Re\lambda_i(\xi))|_{\xi=\xi_0} \neq 0$, $i = 1, 2$.

By putting $\lambda(\xi) = p_1(\xi) + ip_2(\xi)$ into the characteristic equation of the Jacobi matrix $J|_{E^*}$ (3.4), we get

$$(p_1(\xi) + ip_2(\xi))^2 + T(p_1(\xi) + ip_2(\xi)) + D = 0. \quad (4.3)$$

Differentiating both sides with respect to ξ , we get,

$$2(p_1(\xi) + ip_2(\xi))(p_1(\xi) + ip_2(\xi)) + T(p_1(\xi) + ip_2(\xi)) + \dot{T}(p_1(\xi) + ip_2(\xi)) + \dot{D} = 0. \quad (4.4)$$

Collate the real and imaginary parts from both sides, we obtain:

$$2p_1(\xi)p_1(\xi) - 2p_2(\xi)p_2(\xi) + T(p_1(\xi)) + \dot{T}(p_1(\xi)) + D = 0, \quad (4.5)$$

$$2p_1(\xi)p_2(\xi) + 2p_2(\xi)p_1(\xi) + T(p_2(\xi)) + \dot{T}(p_2(\xi)) = 0. \quad (4.6)$$

The equation (4.5) becomes,

$$\begin{aligned} \dot{p}_1(\xi)(2p_1(\xi) + T) + \dot{p}_2(\xi)(-2p_2(\xi)) + p_1(\xi)\dot{T} + \dot{D} &= 0 \\ \implies \dot{p}_1 B_1 - \dot{p}_2 B_2 + B_3 &= 0. \end{aligned} \quad (4.7)$$

and equation (4.6) becomes,

$$\begin{aligned} \dot{p}_1(\xi)(2p_2(\xi)) + \dot{p}_2(\xi)(2p_1(\xi) + T) + p_2(\xi)\dot{T} &= 0 \\ \implies \dot{p}_1 B_2 + \dot{p}_2 B_1 + B_4 &= 0. \end{aligned} \quad (4.8)$$

where, $B_1 = (2p_1(\xi) + T)$, $B_2 = (-2p_2(\xi))$, $B_3 = p_1(\xi)\dot{T} + \dot{D}$, $B_4 = p_2(\xi)\dot{T}$.

Multiply (4.7) by B_1 and (4.8) by B_2 and, then adding, we get,

$$\begin{aligned} \dot{p}_1(B_1^2 + B_2^2) + B_1 B_3 + B_2 B_4 &= 0 \\ \implies \dot{p}_1 &= \frac{-(B_1 B_3 + B_2 B_4)}{B_1^2 + B_2^2} \end{aligned}$$

At $\xi = \xi_0$,

Case (i): $p_1 = 0$, $p_2 = \sqrt{D}$.

$$\begin{aligned} \text{Then } B_1 = T \neq 0, B_2 = 2\sqrt{D}, B_3 = \dot{D}, B_4 = \dot{T}\sqrt{D}. \\ \therefore B_1 B_3 + B_2 B_4 \neq 0 \implies \dot{p}_1(\xi)|_{\xi=\xi_0} \neq 0 \end{aligned}$$

Case (ii): $p_1 = 0$, $p_2 = -\sqrt{D}$.

$$\begin{aligned} \text{Then } B_1 = T \neq 0, B_2 = -2\sqrt{D}, B_3 = \dot{D}, B_4 = -\dot{T}\sqrt{D}. \\ \therefore B_1 B_3 + B_2 B_4 \neq 0 \implies \dot{p}_1(\xi)|_{\xi=\xi_0} \neq 0 \end{aligned}$$

□

Theorem: 4.3. *Let $D(\xi) > 0$, Then*

- (i) *The interior equilibrium point $E^* = (u^*, v^*)$ of the system (2.2) is locally asymptotically stable, when $\xi > \xi_0$ and unstable, when $\xi < \xi_0$.*
- (ii) *The local system (2.2) pass through Hopf bifurcation at interior equilibrium $E^* = (u^*, v^*)$ when $\xi = \xi_0$.*

4.1 Stability behaviour of Hopf bifurcation

Right now, we might discuss the behaviour of Hopf bifurcation. For this specific purpose, we propose the perturbation $u = u_1 + u_1^{\xi_0}$ and $v = v_1 + v_1^{\xi_0}$ in our local system (2.2). Then broadening in Taylor series, we obtain:

$$\begin{aligned} \dot{u}_1 &= a_{10}u_1 + a_{01}v_1 + a_{20}u_1^2 + a_{11}u_1v_1 + a_{30}u_1^3 + a_{21}u_1^2v_1 + \dots, \\ \dot{v}_1 &= b_{10}u_1 + b_{01}v_1 + b_{20}u_1^2 + b_{11}u_1v_1 + b_{30}u_1^3 + b_{21}u_1^2v_1 + \dots, \end{aligned}$$

where,

$$a_{10} = r - 2u - \frac{\delta(1-m)v}{((m-1)u(\alpha+(v-1)\beta)-1)^2}, \quad a_{01} = \frac{\delta(1-m)(-1+(m-1)u(\alpha-\beta))u}{((m-1)u(\alpha+(v-1)\beta)-1)^2}, \quad b_{10} = \frac{-\xi_0(m-1)v^2}{(\eta+(1-m)u)^2},$$

$$b_{01} = \xi_0 \left(\zeta - \frac{2v}{\eta+(1-m)u} - \sigma \right). \text{ Therefore } a_{10} + b_{01} = 0 \text{ and } a_{10}b_{01} - a_{01}b_{10} > 0.$$

Various other coefficients will be determined by:

$$\begin{aligned} a_{20} &= 1/2 \left(\frac{\partial^2 F}{\partial u^2} \right) \Big|_{\xi=\xi_0} = -1 - \frac{(m-1)^2 v^* (\alpha + (v^* - 1)\beta) \delta}{((m-1)u^* (\alpha + (v^* - 1)\beta) - 1)^3}, \\ a_{11} &= \left(\frac{\partial^2 F}{\partial u \partial v} \right) \Big|_{\xi=\xi_0} = \frac{(m-1)((m-1)u^* (\alpha - (v^* + 1)\beta) - 1) \delta}{((m-1)u^* (\alpha + (v^* - 1)\beta) - 1)^3}, \\ a_{30} &= 1/6 \left(\frac{\partial^3 F}{\partial u^3} \right) \Big|_{\xi=\xi_0} = \frac{(m-1)^3 v^* (\alpha + (v^* - 1)\beta)^2 \delta}{((m-1)u^* (\alpha + (v^* - 1)\beta) - 1)^4}, \\ a_{21} &= 1/2 \left(\frac{\partial^3 F}{\partial u^2 \partial v} \right) \Big|_{\xi=\xi_0} = \frac{\left\{ -(m-1)^2 (\alpha(-1 + (m-1)u^* \alpha) + \beta - 2v^* \beta) \right.}{(-1 + (m-1)u^* (\alpha + (v^* - 1)\beta))^4}, \\ &\quad \left. - 2(m-1)u^* \alpha \beta - (m-1)u^* (v^{*2} - 1)\beta^2 \right\} \delta}{(-1 + (m-1)u^* (\alpha + (v^* - 1)\beta))^4}, \\ b_{20} &= 1/2 \left(\frac{\partial^2 G}{\partial u^2} \right) \Big|_{\xi=\xi_0} = \frac{-2(m-1)^2 v^* \xi}{((1-m)u^* + \eta)^3}, \\ b_{11} &= \left(\frac{\partial^2 G}{\partial u \partial v} \right) \Big|_{\xi=\xi_0} = \frac{-2(m-1)v^* \xi}{((1-m)u^* + \eta)^2}, \\ b_{30} &= 1/6 \left(\frac{\partial^3 G}{\partial u^3} \right) \Big|_{\xi=\xi_0} = \frac{-(m-1)^3 v^{*2} \xi}{((1-m)u^* + \eta)^4}, \\ b_{21} &= 1/2 \left(\frac{\partial^3 G}{\partial u^2 \partial v} \right) \Big|_{\xi=\xi_0} = \frac{-2(1-m)^2 v^* \xi}{((1-m)u^* + \eta)^3}. \end{aligned}$$

where,

$$\begin{aligned} F &= u(r - u) - \frac{\delta(1-m)uv}{1 + \alpha(1-m)u + \beta((1-m)u)(v-1)}, \\ G &= \xi \left(\zeta v - \frac{v^2}{\eta + (1-m)u} - \sigma v \right). \end{aligned}$$

The first Lyapunov number to conclude the dynamics (stable or unstable) of limit cycle arising through Hopf bifurcation has by [29]:

$$\begin{aligned} \Lambda &= \frac{-3\pi}{2a_{01}(a_{10}b_{01} - a_{01}b_{10})^{3/2}} ([a_{10}b_{10}a_{11}^2 + a_{10}a_{01}(b_{11}^2 + a_{20}b_{11}) - 2a_{10}a_{01}a_{20}^2 \\ &\quad - a_{01}^2(2a_{20}b_{20} + b_{11}b_{20}) - (a_{01}b_{10} - 2a_{10}^2)a_{11}a_{20}] \\ &\quad - (a_{10}^2 + a_{01}b_{10})[-3a_{01}a_{30} + 2a_{10}(a_{21} + b_{12}) + (b_{10}a_{12} - a_{01}b_{21})]). \end{aligned}$$

Theorem: 4.4. *When $\Lambda < 0$, the direction of Hopf bifurcation is supercritical and the bifurcated periodic solutions are stable; when $\Lambda > 0$, the direction of Hopf bifurcation is subcritical and the bifurcated periodic solutions are unstable.*

5 Diffusion-driven instability

Through this section, we concentrate the prey-predator system with self diffusion and examine the occurrence of Turing instability in the equilibrium point under diffusion effect (diffusion driven instability).

Theorem(4.3) states that whenever $\xi > \xi_0$, the non-trivial interior equilibrium E^* is locally asymptotically stable for the non-diffusion system (2.2).

Currently, we consider the influences of diffusion on the stable non-trivial interior equilibrium E^* of (2.3) under the supposition $\xi > \xi_0$. Subsequently for the diffusion system(2.3), we should consider the one-dimensional space $\Omega = (0, \pi)$ with smooth boundary $\partial\Omega$.

$$\begin{aligned} u_t &= u(r - u) - \frac{\delta(1 - m)uv}{1 + \alpha(1 - m)u + \beta((1 - m)u)(v - 1)} + d_1 u_{xx}, \quad x \in (0, \pi), \quad t > 0, \\ v_t &= \xi \left(\zeta v - \frac{v^2}{\eta + (1 - m)u} - \sigma v \right) + d_2 v_{xx}, \quad x \in (0, \pi), \quad t > 0, \\ u_x(0, t) &= u_x(\pi, t) = 0, \quad v_x(0, t) = v_x(\pi, t) = 0, \quad t > 0. \end{aligned} \quad (5.1)$$

This is actually notable operator $u \rightarrow -u_{xx}$ with Neumann boundary conditions. The analogous eigenvalues and normalized eigenfunctions are

$$\varrho_0 = 0, \quad \varphi_0(x) = \sqrt{\frac{1}{\pi}}, \quad \varrho_k = k^2, \quad \varphi_k(x) = \sqrt{\frac{2}{\pi}} \cos(kx), \quad \text{where } k = 1, 2, 3, \dots$$

Linearizing the above diffusion system (5.1) at E^* , we get

$$\begin{bmatrix} u_t \\ v_t \end{bmatrix} = L \begin{bmatrix} u \\ v \end{bmatrix} + J \begin{bmatrix} u \\ v \end{bmatrix} + D \begin{bmatrix} u_{xx} \\ v_{xx} \end{bmatrix},$$

where, $D = \text{diag}(d_1, d_2)$ and J is the Jacobian matrix pointed out in Sec 3.3. L indicates a linear operator whose domain is $D_L = X_{\mathbb{C}} := X \oplus iX = \{u_1 + iu_2 : u_1, u_2 \in X\}$, where $X := \{(u, v) \in H^2[(0, \pi)] \times H^2[(0, \pi)] : u_x(0, t) = u_x(\pi, t) = v_x(0, t) = v_x(\pi, t) = 0\}$ which is a real valued Sobolev space.

Consider the characteristic equation $L(\varphi, \psi)^T = \varrho(\varphi, \psi)^T$ and let

$$(\varphi, \psi)^T = \sum_{k=0}^{\infty} (a_k, b_k)^T \cos(kx). \quad (5.2)$$

Let $J_k = J - k^2 D$. Then we get $\sum_{k=0}^{\infty} (J_k - \varrho I)(a_k, b_k)^T \cos(kx) = 0$,

Thus every eigenvalues of L are got it by the eigenvalues of J_k for $k = 1, 2, 3, \dots$. The characteristic equation of J_k is

$$\varrho^2 - T_k \varrho + D_k = 0, \quad k = 1, 2, 3, \dots \quad (5.3)$$

where,

$$\begin{aligned} T_k &= \text{tr} J_k = T - (d_1 + d_2)k^2, \\ \text{and } D_k &= \det J_k = D + d_1 d_2 k^4 + (\delta d_1 - \delta_0 d_2)k^2. \end{aligned}$$

Simply by examining the distribution of characteristic roots of J_k , we get the imminent conclusion.

Theorem: 5.1. Assume that $\xi > \xi_0$ and $D(\xi) > 0$. Then

(i) The equilibrium $E^* = (u^*, v^*)$ of the non-diffused system (2.2) is locally asymptotically stable.

(ii) The equilibrium $E^* = (u^*, v^*)$ of the diffused system (5.1) is locally asymptotically stable if and only if the following conditions are hold.

$$(C1) \quad d_1 \geq \frac{d_2 \xi_0}{\xi},$$

$$(C2) \quad d_1 < \frac{d_2 \xi_0}{\xi} \quad \text{and} \quad \frac{d_1}{d_2} > \frac{\xi_0 \xi + 2D - 2\sqrt{D(\xi \xi_0 + D)}}{\xi^2}.$$

and for the diffused system (5.1), the solution E^* is unstable (that is, Turing instability occurs), if

$$(C3) \quad 0 < \frac{d_1}{d_2} < \frac{\xi_0 \xi + 2D - 2\sqrt{D(\xi \xi_0 + D)}}{\xi^2}. \quad (\text{Turing Instability domain})$$

Proof. Since $\xi > \xi_0 (T < 0)$, signifies that $T_k < 0$ for all $k \geq 0$. As well by the meaning of T_k , we have the relationship, for every $k \geq 0$, $T_{k+1} < T_k$. We calculte D_k like a quadratic function in k^2 , that is $D_k = D(k^2) = d_1 d_2 (k^2)^2 + (\xi d_1 - \xi_0 d_2) k^2 + D$. Consequently the real part of the characteristic values signs of (5.3) are discovered by the sign of $D(k^2)$, separately. The symmetric axis of the graph $(k^2, D(k^2))$ is $k_{min}^2 = -\frac{(\xi d_1 - \xi_0 d_2)}{2d_1 d_2}$. We know that, when $D_k < 0$ ($D(k^2) < 0$), the characteristic roots of J_k (5.3) be, two real roots with opposite signs. Notice that in $D(k^2)$, $d_1 d_2 > 0$ and $k^2 > 0$. Therefore, whenever $\xi d_1 - \xi_0 d_2 < 0$, $D(k^2)$ attains bare minimum at $k^2 = k_{min}^2$.

Therefore, when (C3) holds, $D(k_{min}^2)$ be negative; But it means that any one of the characteristic roots of J_k have positive real part, that is E^* will be the unstable solution of (5.1). As a result we consider that, whenever (C3) retains, Turing instability occurs. (C1) implies that $D_k > 0$ for all $k \geq 0$ (since $D_0 = D > 0$) and (C2) implies that $D(k_{min}^2)$ be positive, therefore all the characteristic roots of J_k has negative real parts. In this manner, any of the conditions (C1) and (C2) make sure that the characteristic roots of J_k have negative real parts. Consequently, any one of the conditions of (C1) and (C2) holds, then E^* be the stable equilibrium solution of (5.1). \square

6 Numerical simulation

In this section, we performed a few numerical simulations to verify our analytical findings by using MATLAB. We consider the system (2.2) with fixed parameters $\{r = 0.7, \delta = 1.5, \alpha = 0.8, \beta = 0.4, \xi = 1.8, \zeta = 1.7, \eta = 0.4, \sigma = 1.4, 0 \leq m = 0.35 < 1\}$. For these parametric values, the corresponding nullclines shown in figure 1. The pink and orange curves represent the prey and predator nullclines, respectively. Intersecting points of nullclines are equilibrium points E_0, E_1, E_2 and $E^*(u^*, v^*) = (0.51525, 0.22047)$. Note that, for the above set of parameters, violate one condition $\Delta_3 > 0$ in (H). Even though there exists a unique E^* in \mathbb{R}_+^2 . Therefore, the set of conditions we mentioned in (H) are necessary but not sufficient for the existence and uniqueness of $E^* \in \mathbb{R}_+^2$.

Through vector fields (blue arrows) in Figure 1, we may understand the stability nature of the equilibria. For this set of parameters, $E_0 = (0, 0)$ is unstable. In particular, for this case, E_0 is a nodal source. Also, where $\zeta = 1.7 > \sigma = 1.4$ therefore $E_1 = (r, 0) = (1.7, 0)$ and $E_2 = (0, \zeta\eta - \eta\sigma) = (0, 0.12)$ also unstable (saddle point).

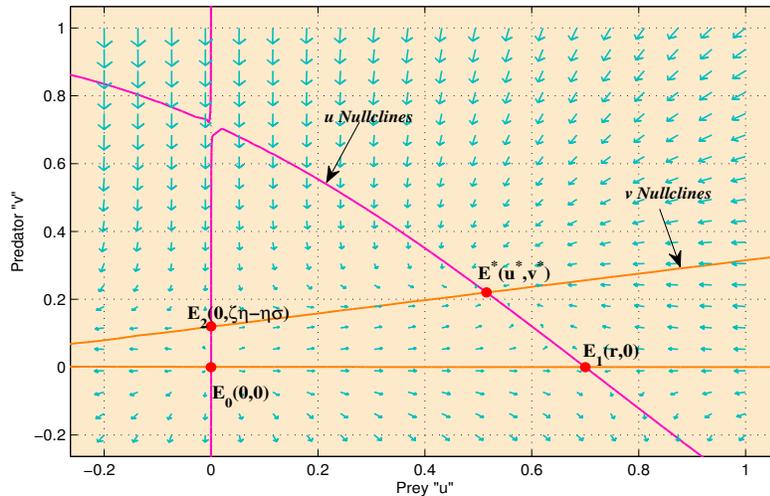


Figure 1. Nullclines of the prey-predator system (2.2) with parameters $\{r = 0.7, \delta = 1.5, \alpha = 0.8, \beta = 0.4, \xi = 1.8, \zeta = 1.7, \eta = 0.4, \sigma = 1.4, m = 0.35\}$.

By fixing the same set of above parameters other than ζ and σ , with condition $\zeta = 1.4 < \sigma = 1.7$, our resultant phase plane is shown in Figure 2. Since $\zeta < \sigma$, the axial equilibrium $E_2 = (0, \zeta\eta - \eta\sigma) = (0, -0.12) \notin \{\{0\} \cup \mathbb{R}\}_+^2$. We drawn different phase trajectories start from various initial points. From the phase plane, we get $E_1 = (r, 0) = (0.7, 0)$ is locally asymptotically stable (nodal sink). Also Figure 2 shows that E_0 is unstable (saddle point) and E_2 is unstable (nodal source). Also note that, for this set of parametric values oppose the conditions $\Delta_1 < 0, \Delta_3 > 0$ in (H) . Therefore no interior equilibrium $E^*(u^*, v^*)$ exist in \mathbb{R}_+^2 . But, there are three non-trivial equilibria (yellow dots) exist which are flow on II and III quadrant of the phase plane. From the biological perspective these are meaningless. As per the following set: $\{r = 1, \delta = 0.8, \alpha = 10, \beta = 0.1, \xi = 0.15, \zeta = 7.1, \eta = 0.01, \sigma = 6, 0 \leq m = 0.01 < 1\}$, possess all conditions of (H) and $\zeta > \sigma$. Then, there exist a unique non-trivial interior equilibrium $E^*(u^*, v^*) = (0.920642, 1.01358) \in \mathbb{R}_+^2$ which is shown in Figure 3. For these parameter values, the trivial equilibrium point $E_0 = (0, 0)$ is unstable and axial equilibrium $E_1 = (1, 0)$ and $E_2 = (0, 0.011)$ are saddle points. Further, the above set of parameters follows the conditions $(\delta(1-m)v^* - u^*)\alpha < \frac{1}{1-m} + \beta u^*(v^* - 1)$ and $(1-m)u^*(\beta - \alpha) < 1$. As per Theorem 3.2, figure 3 depicts that different phase trajectories start from different initial points, but ultimately all the trajectories goes to the same interior equilibrium point $E^* = (0.920642, 1.01358)$. It shows E^* is globally asymptotically stable.

Now, consider the prey-predator model (2.2) together with the following set of parameters $\{r = 1, \delta = 8, \alpha = 10, \beta = 0.1, \zeta = 7.1, \eta = 0.01, \sigma = 6, m = 0.01\}$. Then our ODE

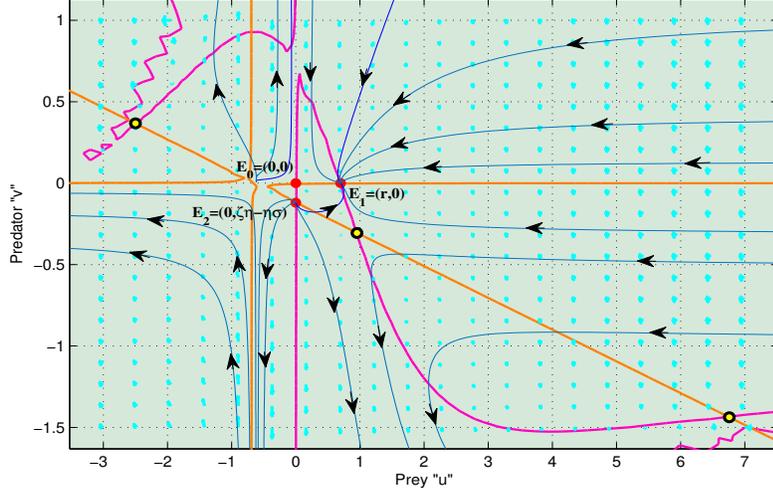


Figure 2. Phase plane of the prey-predator system (2.2) with parameters $\{r = 0.7, \delta = 1.5, \alpha = 0.8, \beta = 0.4, \xi = 1.8, \zeta = 1.4, \eta = 0.4, \sigma = 1.7, m = 0.35\}$.

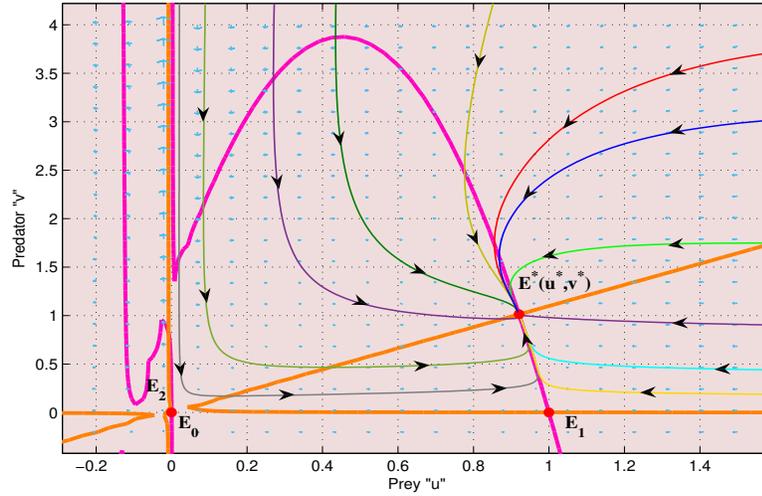


Figure 3. Existence and uniqueness of $E^*(u^*, v^*)$. The parametric values are taken as $\{r = 1, \delta = 0.8, \alpha = 10, \beta = 0.1, \xi = 0.15, \zeta = 7.1, \eta = 0.01, \sigma = 6, m = 0.01\}$. And the steady state $E^* = (0.920642, 1.01358)$ is globally asymptotically stable. Since $(\delta(1-m)v^* - u^*)\alpha < \frac{1}{1-m} + \beta u^*(v^* - 1)$ and $(1-m)u^*(\beta - \alpha) < 1$.

model becomes,

$$\begin{cases} \frac{du}{dt} &= u(1-u) - \frac{8(1-0.01)uv}{1+10(1-0.01)u+0.1((1-0.01)u)(v-1)}, \\ \frac{dv}{dt} &= \xi \left(7.1v - \frac{v^2}{0.01+(1-0.01)u} - 6v \right), \\ u(0) &= u_0 > 0, \quad v(0) = v_0 > 0. \end{cases} \quad (6.1)$$

Underneath the set of parameters, the above system (6.1) has a unique non-trivial positive interior equilibrium $E^*(u^*, v^*) = (0.315572, 0.3554658)$. Pertaining to considering the preceding parametric values, the crucial point value be $\xi_0 = 0.183713$, which is independent of ξ . Further, we change the bifurcation parameter ξ only.

By fixing $\xi = 0.2$, we have $\xi > \xi_0 = 0.183713$, and it follows from Theorem(4.3), $E^*(u^*, v^*)$ is locally asymptotically stable (see Figure 4).

When fixing $\xi = \xi_0 = 0.183713$, implies that $T(\xi_0) = 0$. Subsequently by Theorem(4.3), E^* will lose its stability and a Hopf bifurcation arises, that is, a family group of intermittent periodic solutions bifurcating from the interior equilibrium E^* (see Figure 5).

Through $\xi = 0.15$, we certainly have $\xi < \xi_0$, and that follows by Theorem(4.3), $E^*(u^*, v^*)$ is unstable (see Figure 6).

Moreover, when ξ goes through ξ_0 from the right side of ξ_0 , the equilibrium point E^* loses its steadiness and a Hopf bifurcation occurs, as appeared in Figure 5 & Figure 6. For the above set of parameters, we get $\Lambda = -9.01885 < 0$, therefore from Theorem 4.4, the direction of the Hopf bifurcation at $\xi = \xi_0$ is supercritical and that bifurcating periodic solutions be asymptotically stable which is shown in phase diagram of Figure 5 and Figure 6.

Formation of limit cycle around the interior equilibrium point with initial data $(0.4, 0.3)$ and $(0.5, 0.5)$ which is inside and outside of the limit cycle respectively shown in figure 7. The diagram shows that the limit cycle is stable.

To verify the occurrence of Turing instability of the diffusive prey-predator system (5.1), we consider the following set: $\{r = 1, \delta = 8, \alpha = 10, \beta = 0.1, \zeta = 7.1, \eta = 0.01, \sigma = 6, m = 0.01\}$ with Neumann boundary condition on one dimensional spatial domain $\Omega = (0, 60)$. We modify the diffusion co-efficients d_1 and d_2 only.

Within the parameters $d_1 = 1, d_2 = 2$ and $\xi = 0.2$ ($0.2 = \xi > \xi_0 = 0.183713$), also $1 = d_1 < \frac{d_2 \xi_0}{\xi} = 1.83713$ and $\frac{d_1}{d_2} = 0.5 > \frac{\xi_0 \xi + 2D - 2\sqrt{D(\xi \xi_0 + D)}}{\xi^2} = 0.071$, that is, (C2) holds. By Theorem 5.1, the homogeneous equilibrium solution E^* of the system (5.1) is stable. The stability behaviour and the corresponding contour diagram is shown in the Figure 8.

Under the parameters $d_1 = 0.01, d_2 = 1$ and $\xi = 0.2$ ($\xi > \xi_0$), Then $0.01 = \frac{d_1}{d_2} < \frac{\xi_0 \xi + 2D - 2\sqrt{D(\xi \xi_0 + D)}}{\xi^2} = 0.071$, that is, (C3) holds. By Theorem 5.1, the homogeneous equilibrium solution E^* of the system (5.1) is unstable. That is Turing instability occurs. The unstable Turing nature and the corresponding contour diagram is shown in Figure 9.

7 Conclusion

This paper discussed the dynamical behavior of a prey-predator model incorporating prey refuge and interference among predators. Here the interaction between the prey and the

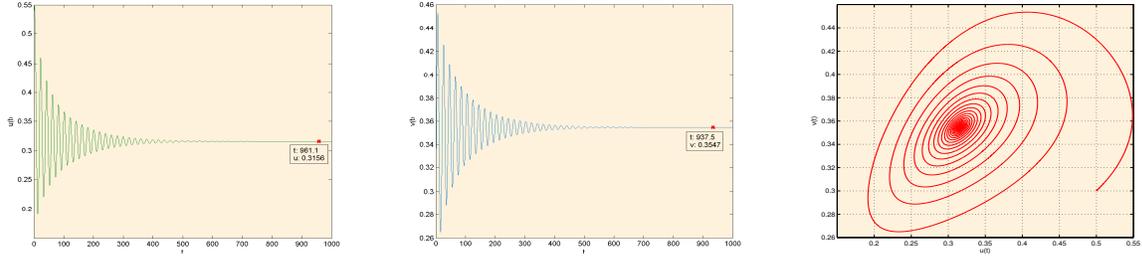


Figure 4. Stability behavior with respect to time t and phase portraits of the system (2.2) with $\xi > \xi_0$ and initial data $(u_0, v_0) = (0.5, 0.3)$.

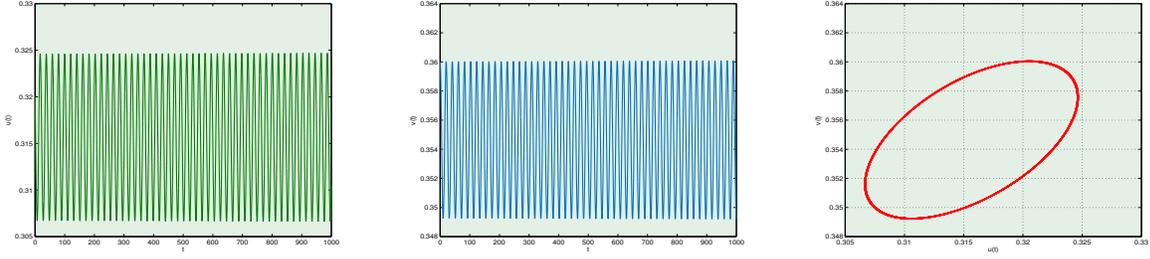


Figure 5. Periodic behavior with respect to time t and phase portraits of the system (2.2) with $\xi = \xi_0$ and initial data $(u_0, v_0) = (0.32, 0.36)$.

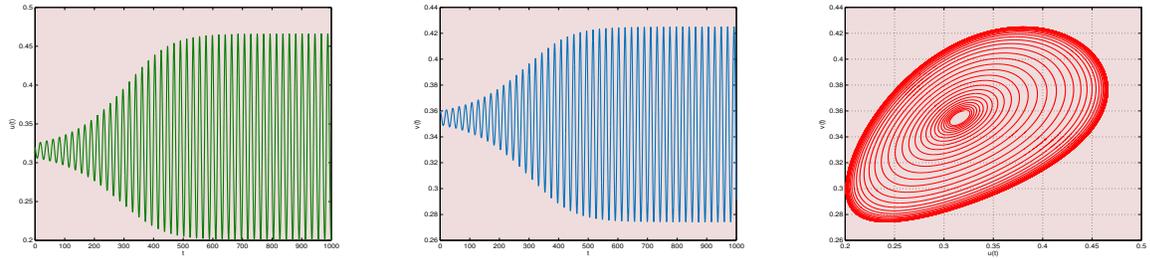


Figure 6. Unstable behavior with respect to time t and phase portraits of the system (2.2) with $\xi < \xi_0$ and initial data $(u_0, v_0) = (0.32, 0.36)$.

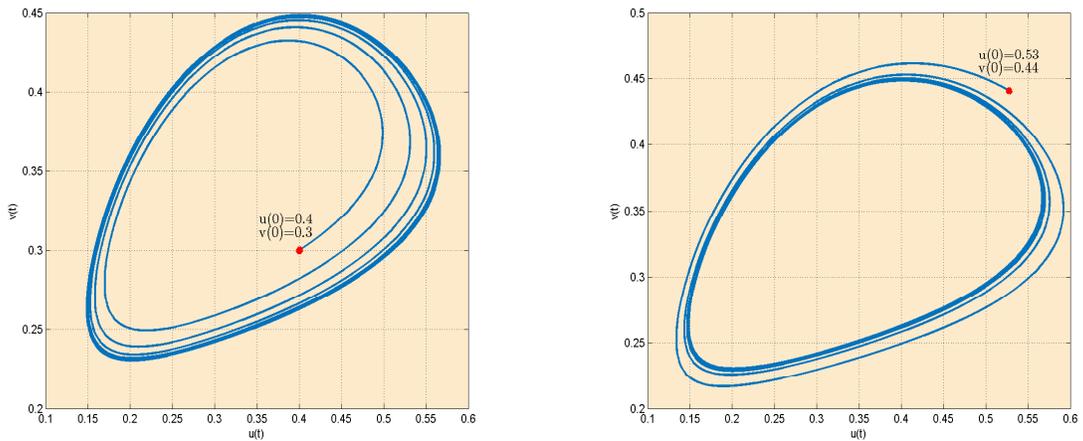


Figure 7. Existence of stable limit cycle around the interior equilibrium $E^* = (0.315572, 0.354658)$.

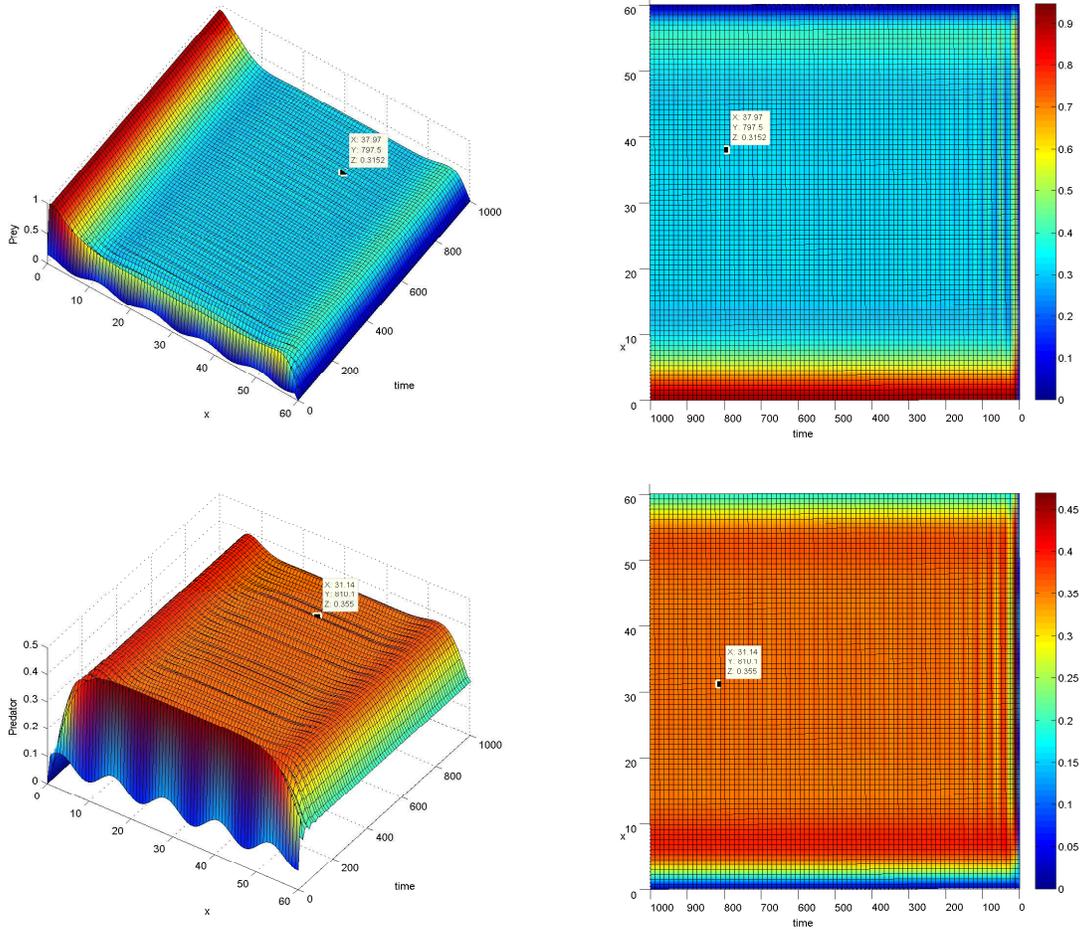


Figure 8. Numerical simulations of the diffusion system (5.1) with parameter restriction $\xi = 0.2$ ($0.2 = \xi > \xi_0 = 0.183713$), $d_1 = 1$ and $d_2 = 2$, with initial condition $(t = 0, 0.1 + 0.05(\cos(\frac{1}{2}u)), 0.1 + 0.03 \sin(\frac{1}{2}v))$.

predator is governed by the Crowley-Martin response function. The innovation of this work should be to analyze the refuge function during intraspecific competition between predators for prey. Existence criteria of biological meaningful axial and interior equilibrium points have been acquired, and their stability evaluation has also been performed. Also, the existence criteria states the notion that whenever the growth rate of predator (based on the convention coefficients from individuals of prey into individuals of a predator) is greater than the death rate of predator ($\zeta = r_2\rho/\delta_2 > \sigma = \delta_3\rho/\delta_2$), then only predator will survive ($v^* > 0$).

Also, the parameter ξ , which is independent of E^* , has an influential role in exhibits a Hopf-bifurcation around E^* . Biologically Theorem 4.3 states that, when $D > 0$ and the proportion between the maximal per capita predator consumption rate and intensity of competition among individual of prey species ($\xi = \delta_2/\rho$) is greater than the critical value ξ_0 , the prey-predator inhabitants will be secure for any initial interior population. Let prey refuge and interference among predators also could be high measures, even though whenever the ratio between maximal per capita predator consumption rate and intensity of

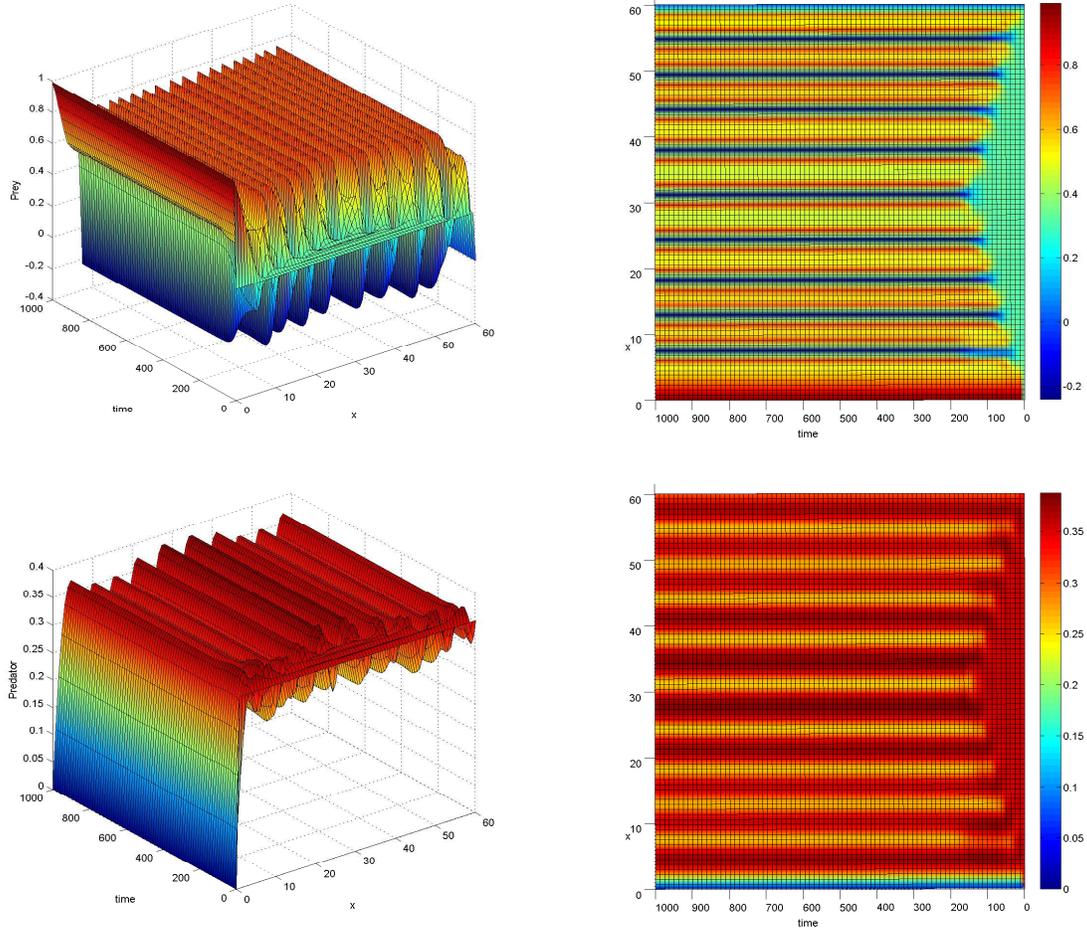


Figure 9. Numerical simulations of the diffusion system (5.1) with parameter restriction $\xi = 0.2$ ($0.2 = \xi > \xi_0 = 0.183713$), $d_1 = 0.01$ and $d_2 = 1$, with initial condition ($t = 0, 0.31, 0.35$).

competition among individual of prey species more petite than the critical value ξ_0 , then we cannot conclude the population size exactly; it will be unstable. Whenever the ratio between δ_2 and ρ exactly equaled to the critical value ξ_0 , the prey-predator inhabitant's dynamics change periodically. The occurrence of Hopf bifurcation usually addresses this kind of notion. The corresponding Hopf bifurcation type and main results are presented in Theorem 4.4.

Further, we studied the dynamical changes within the prey-predator populace according to both space and time motions. With these spatial movements, we considered the diffusion system (5.1) and analyzed the diffusion-driven instability of the spatial system (5.1) in detail. Due to the occurrence of diffusion, the stability of the interior equilibria E^* could be changed from stable to unstable even though $\xi > \xi_0$ is satisfied. The corresponding results are presented in Theorem 5.1. Further, we verified all our analytical results with numerical examples in Section 6. One can include the stochastic noise effect in the above model through the habitat-dependent parameters in future work.

Acknowledgment

The work of Yong-Ki Ma was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. 2021R1F1A1048937).

Data Availability

Our paper contains numerical experimental results, and values for these experiments are included in the paper. The data is freely available.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

- [1] A.D.Bazykin, Nonlinear Dynamics of Interacting Populations, *World Scientific*, Singapore, 1998.
- [2] J.R.Beddington, Mutual interference between parasites or predators and its effect on searching efficiency, *The Journal of Animal Ecology*, 44(1975) 331-340.
- [3] R.S.Cantrell and C.Cosner, On the dynamics of predator-prey models with the Beddington-DeAngelis functional response, *Journal of Mathematical Analysis and Applications*, 257 (2001) 206-222.
- [4] P.A.Braza, The bifurcation structure of the Holling-Tanner model for predator-prey interactions using two-timing, *SIAM Journal on Applied Mathematics*, 63 (2003) 889-904.
- [5] X.Cao and W.Jiang, Turing-Hopf bifurcation and spatiotemporal patterns in a diffusive predator-prey system with Crowley-Martin functional response, *Nonlinear Analysis: Real World Applications*, 43 (2018) 428-450.
- [6] J.B.Collings, Bifurcation and stability analysis of a temperature-dependent mite predator-prey interaction model incorporating a prey refuge, *Bulletin of mathematical biology*, 57 (1995) 63-76.
- [7] P.H.Crowley and E.K.Martin, Functional responses and interference within and between year classes of a dragonfly population, *Journal of the North American Benthological Society*, 8 (1989) 211-221.
- [8] C. Celik, Stability and Hopf bifurcation in a delayed ratio dependent Holling-Tanner type model, *Applied Mathematics and Computation*, 255 (2015) 228-237.
- [9] S.Chen and J.Shi, Global stability in a diffusive Holling-Tanner predator-prey model, *Applied Mathematics Letters*, 25 (2012) 614-618.
- [10] D.L.DeAngelis, R.A.Goldstein and R.V.O'Neill, A model for trophic interaction, *Ecology*, 56 (1975) 881-892.
- [11] J.Ghosh, B.Sahoo and S.Poria, Prey-predator dynamics with prey refuge providing additional food to predator, *Chaos, Solitons & Fractals*, 96, (2017) 110-119.

- [12] B.S.Goh, Global stability in two-species interactions, *Journal of Mathematical Biology*, 3 (1976) 313-318.
- [13] L.N.Guin, S.Pal, S.Chakravarty and S.Djilali, Pattern dynamics of a reaction-diffusion predator-prey system with both refuge and harvesting, *International Journal of Biomathematics*, (2020) p.2050084.
- [14] M.P.Hassell, The Dynamics of Arthropod Predator-Prey Systems, Princeton University Press, Princeton, 1978.
- [15] M.P.Hassell and R.M.May, Stability in insect host-parasite models, *The Journal of Animal Ecology*, (1973) 693-726.
- [16] C.S.Holling, The functional response of predators to prey density and its role in mimicry and population regulation, *The Memoirs of the Entomological Society of Canada*, 97 (1965) 5-60.
- [17] S.B.Hsu and T.W.Hwang, Global stability for a class of predator-prey systems, *SIAM Journal on Applied Mathematics*, 55 (1995) 763-783.
- [18] S.B.Hsu and T.W.Huang, Hopf bifurcation analysis for a predator-prey system of Holling and Leslie type, *Taiwanese Journal of Mathematics*, 3 (1999) 35-53.
- [19] C.Gokila, M.Sambath, K.Balachandran and Y.-K.Ma, Analysis of stochastic predator-prey model with disease in the prey and Holling type II functional response, *Advances in Mathematical Physics*, 2020 (2020) 3632091.
- [20] V.Krivan, Effects of optimal antipredator behavior of prey on predator-prey dynamics: the role of refuges, *Theoretical Population Biology*, 53 (1998) 131-142.
- [21] X.Li , W.Jiang and J.Shi, Hopf bifurcation and Turing instability in the reaction-diffusion Holling-Tanner predator-prey model, *IMA Journal of Applied Mathematics*, 76 (2011) 1-20.
- [22] P.P.Liu and Y.Xue, Spatiotemporal dynamics of a predator-prey model , *Nonlinear Dynamics*, 69 (2012) 71-77.
- [23] Z.P.Ma, W.T.Li, Bifurcation analysis on a diffusive Holling-Tanner predator-prey model, *Applied Mathematical Modelling*, 37 (2013) 4371-4384.
- [24] J.D.Murray, *Mathematical Biology-I: An Introduction*, Springer, Berlin, 2007.
- [25] S.Mondal and G.P.Samanta, Dynamics of an additional food provided predator-prey system with prey refuge dependent on both species and constant harvest in predator, *Physica A: Statistical Mechanics and its Applications*, 534 (2019) 122301.
- [26] S. Mondal and G.P. Samanta, Provision of additional food as a tool of biological control in a delayed predator-prey interaction with prey refuge, *International Journal of Modelling and Simulation*, 42 (2022), 570-594.
- [27] A. Mougi, Predator interference and complexity-stability in food webs, *Scientific reports*, 12 (2022) 1-8.

- [28] N.E.Papanikolaou, S.Dervisoglou, A.Fantinou, T.Kypraios, V.Giakoumaki and D.Perdikis, Predator size affects the intensity of mutual interference in a predatory mirid, *Ecology and evolution*, 11 (2021) 1342-1351.
- [29] L.Perko, *Differential Equations and Dynamical Systems*, Springer, Berlin, 2013.
- [30] R.Peng and M.Wang, Global stability of the equilibrium of a diffusive Holling- Tanner prey-predator model, *Applied Mathematics Letters* , 20 (2007) 664-670.
- [31] R.Peng and M.Wang, Stationary patterns of the Holling-Tanner prey-predator model with diffusion and cross-diffusion, *Applied Mathematics and Computation*, 196 (2008) 570-577.
- [32] M.L.Rosenzweig and R.H. MacArthur, Graphical representation and stability conditions of predator-prey interactions, *The American Naturalist*, 97 (1963) 209-223.
- [33] M.Sen, M.Banerjee and A.Morozov, Bifurcation analysis of a ratio-dependent prey-predator model with the Allee effect, *Ecological Complexity*, 11 (2012) 12-27.
- [34] S. Saha and G.P. Samanta, Analysis of a predator-prey model with herd behavior and disease in prey incorporating prey refuge, *International Journal of Biomathematics*, 12 (2019) 1950007.
- [35] M.Sambath and K.Balachandran, Bifurcations in a diffusive predator-prey model with predator saturation and competition response, *Mathematical Methods in the Applied Sciences*, 38 (2015) 785-798.
- [36] S.Sharma and G.P.Samanta, A Leslie-Gower predator-prey model with disease in prey incorporating a prey refuge, *Chaos, Solitons & Fractals*, 70 (2015) 69-84.
- [37] G.T.Skalski and J.F.Gilliam, Functional responses with predator interference: viable alternatives to the Holling type II model, *Ecology*, 82 (2001) 3083-3092.
- [38] M.S.Surendar, M.Sambath and K.Balachandran, Bifurcation on diffusive Holling Tanner predator-prey model with stoichiometric density dependence, *Nonlinear Analysis: Modelling and Control*, 25(2020)225-244.
- [39] B.Tiwari and S.N.Raw, Dynamics of Leslie-Gower model with double Allee effect on prey and mutual interference among predators, *Nonlinear Dynamics*, 103 (2021) 1229-1257.
- [40] J.P.Tripathi, D.Jana, N.V.Devi, V.Tiwari and S.Abbas, Intraspecific competition of predator for prey with variable rates in protected areas, *Nonlinear Dynamics*, 102 (2020) 511-535.
- [41] X.Tang, H.Jiang, Z.Deng and T.Yu, Delay induced subcritical Hopf bifurcation in a diffusive predator-prey model with herd behavior and hyperbolic mortality, *Journal of Applied Analysis and Computation*, 7 (2017) 1385-1401.
- [42] D.Xiao and S.Ruan, Global analysis in a predator-prey system with nonmonotonic functional response, *SIAM Journal on Applied Mathematics*, 61 (2001) 1445-1472.

- [43] H.Zhu, S.A.Campbell and G.S.Wolkowicz, Bifurcation analysis of a predator-prey system with nonmonotonic functional response, *SIAM Journal on Applied Mathematics*, 63 (2003) 636-682.