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RESEARCH

Error Estimates of Hermite-Hadamard type Inequalities on Quantum Calculus

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Abstract
In this paper, we establish error estimates of Hermite-Hadamard inequalities corresponding ${}_{\zeta_1}q$-fractional calculus operators via (α, m)-convex functions. Identity is used to examine the main results of this study. The results in the literature are obtained as remarks on our general consequences.

Mathematics Subject Classification: 26A33; 81Q40

Keywords: ${}_{\gamma_1}q$-derivative; Error estimates; Hermite-Hadamard's inequality; (α, m)-convex function

1 Introduction

One of the most crucial concepts in mathematics is convexity. It has a wide range of applications (see books [1, 2]). The theory of optimization and the theory of inequalities involving convex functions have many attractive consequences. Such functions have been studied by a number of scholars, aiming to extend them in many ways using novel ideas and efficient approaches [3, 4]. Ghulam et al. [5] introduced Hadamard inequalities for strongly m-convex functions through Riemann-Liouville fractional integrals. For both s-convex and s-concave functions, Khan et al. [6] developed various new and generalized Hermite-Hadamard type inequalities. Some Jensen's and Hardy-type conditions for convex functions are enhanced by Wang et al. [7] using Riemann-Liouville delta fractional integrals. Inequalities of the Hermite-Hadamard type for integrals emerging in conformable fractional calculus were introduced by Bohner et al. [8]. Páles et al. [9] established an extension of the decomposition theorem in the context of higher-order convexity notions. Following are preliminaries related to our main results.

Definition 1.1 *The mapping $\Lambda : [\zeta_1, \zeta_2] \to \mathbb{R}$ is said to be convex if*

$$\Lambda(\vartheta\mu_1 + (1-\vartheta)\mu_2) \leq \vartheta\Lambda(\mu_1) + (1-\vartheta)\Lambda(\mu_2)$$

holds for every $\mu_1, \mu_2 \in [\zeta_1, \zeta_2]$ and $\vartheta \in [0, 1]$.

Λ is a concave function if Λ^{-1} is convex.

In the following manner, Toader et al. [10] established the class of m-convex functions defined by the following definition.

Definition 1.2 *The mapping $\Lambda : [\zeta_1, \zeta_2] \to \mathbb{R}$ is called m-convex, if the inequality*

$$\Lambda(\vartheta\mu_1 + m(1-\vartheta)\mu_2) \leq \vartheta\Lambda(\mu_1) + m(1-\vartheta)\Lambda(\mu_2)$$

satisfies for every $\mu_1, \mu_2 \in [\zeta_1, \zeta_2]$ and $m, \vartheta \in [0, 1]$.

For $m = 1$, the m-convexity of a function is reduces to the classical convexity.

Many authors have established various integral inequalities relevant to m-convex mappings owing to the extensive implications of the said convex functions. Tariq et al. [11] introduced a novel family of convex functions known as exponential type m-convex functions. Wang et al. [12] presents certain Hermite-Hadamard type inequalities for m-convex mappings involving fractional integrals with exponential kernels.

The following concept of (α, m)-convexity is given by Mihesan's et al. [13].

Definition 1.3 *The mapping $\Lambda : [\zeta_1, \zeta_2] \to \mathbb{R}$ is said to be (α, m)-convex mapping, if the following inequality*

$$\Lambda(\vartheta\mu_1 + m(1-\vartheta)\mu_2) \leq \vartheta^{\alpha}\Lambda(\mu_1) + m(1-\vartheta^{\alpha})\Lambda(\mu_2)$$

holds for all $\mu_1, \mu_2 \in [\zeta_1, \zeta_2], \vartheta \in [0, 1]$ and $(\alpha, m) \in [0, 1]^2$.

It is notable that for $\alpha = 1, (\alpha, m)$-convexity is evidently reduces to m-convexity and to classical convexity for $\alpha = m = 1$.

Safdar et al. [14] explored some conclusions for generalized (α, m)-convex functions that are differentiable in absolute sense. For (α, m)-convex functions, Erhan et al. [15] proposes a number of Hadamard-type integral inequalities.

In literature, convex functions are used to generate a variety of inequalities [16, 17]. One of the most well-known classical inequality is a Hermite-Hadamard inequality, which has a wide range of geometrical implications and applications [18] and is stated as follows:

Theorem 1.1 *If $\Lambda : [\zeta_1, \zeta_2] \to \mathbb{R}$ is a convex mapping and $\mu_1, \mu_2 \in [\zeta_1, \zeta_2]$ with $\mu_1 < \mu_2$, then the following inequality*

$$\Lambda\left(\frac{\mu_1 + \mu_2}{2}\right) \leq \frac{1}{\mu_2 - \mu_1}\int_{\mu_1}^{\mu_2} \Lambda(\nu)d\nu \leq \frac{\Lambda(\mu_1) + \Lambda(\mu_2)}{2}$$

holds. The inequalities are reversed if Λ is concave.

Using fractional integrals and derivatives, a majority of scholars investigated this inequality and produced a lot of refinements and extensions [19, 20, 21, 22].

Furthermore, certain actions in the area of **q**-analysis are being conducted, as initiated by Euler, in order to gain ability in mathematics that leads to quantum computing **q**-calculus, which is acknowledged as a connection between physics and mathematics [23, 24]. Quantum calculus has a wide spectrum of uses in quantum information theory, which is a branch of strategy that includes, along with other aspects, information theory, computer science, cryptography and philosophy [25, 26].

The **q**-parameter was utilized by Newton in his studies on infinite series. Additionally, Jackson [27, 28] was the first to propose the quantum calculus or limitless calculus in a systematic way. Since then, the scope of related research has expanded rapidly. In [29], Bermudo et al. present some variants of the Hermite-Hadamard inequality for general convex functions in the context of **q**-calculus. Chu et al. [30] illustrated the notion of ${}^{\zeta_2}D_{\mathbf{p},\mathbf{q}}$ derivative and ${}^{\zeta_2}(\mathbf{p},\mathbf{q})$-integral.

For several sorts of tasks, quantum and post-quantum integrals are often used to investigate numerous integral inequalities. For differentiable convex functions having a critical point, Jhanthanam et al. [31] establish several new conclusions on the left-hand side of the q-Hermite-Hadamard inequality. For a newly defined quantum integral, Budak et al. [32] initially obtained two new Hermite-Hadamard-type quantum inequalities and then established many quantum Hermite-Hadamard inequalities. For convex stochastic processes and coordinated stochastic processes, Sitthiwirattham et al. [33] offered some new **q** Hermite-Hadamard type inequalities, using newly defined integrals. Prabseang et al. [34] looked into several novel quantum analogues of Hermite-Hadamard inequalities for double integrals, as well as improvements of Hermite-Hadamard inequalities for **q**-differentiable convex functions. For convex functions, Ali et al. [35] also obtains some new Hermite-Hadamard inequalities using the newly specified $R(\mathbf{p},\mathbf{q})$-integral.

Now, we will go through some common **q**-calculus definitions, inequalities and the notation as follows [23]:

$$[k]_{\mathbf{q}} = \frac{1-\mathbf{q}^k}{1-\mathbf{q}} = 1+\mathbf{q}+\mathbf{q}^2+\cdots+\mathbf{q}^{k-1}$$

for $\mathbf{q} \in (0,1)$.

The **q**-Jackson integral of a mapping Λ from 0 to ζ_2 was defined by Jackson [27] as follows:

$$\int\limits_0^{\zeta_2} \Lambda(\nu) d_{\mathbf{q}}\nu = (1-\mathbf{q})\zeta_2 \sum_{k=0}^{\infty} \mathbf{q}^k \Lambda(\zeta_2 \mathbf{q}^k), \tag{1.1}$$

where $0 < \mathbf{q} < 1$, given that the sum converges absolutely.

Jackson [27] was defined the **q**-Jackson integral of a mapping on $[\zeta_1, \zeta_2]$ as follows:

$$\int\limits_{\zeta_1}^{\zeta_2} \Lambda(\nu) d_{\mathbf{q}}\nu = \int\limits_0^{\zeta_2} \Lambda(\nu) d_{\mathbf{q}}\nu - \int\limits_0^{\zeta_1} \Lambda(\nu) d_{\mathbf{q}}\nu.$$

Definition 1.4 *[36] Consider the function $\Lambda : [\zeta_1, \zeta_2] \to \mathbb{R}$ to be continuous. Then, for Λ at $\vartheta \in [\zeta_1, \zeta_2]$, the ${}_{\zeta_1}\mathbf{q}$-derivative is determined as*

$${}_{\zeta_1}D_{\mathbf{q}}\Lambda(\nu) = \frac{\Lambda(\nu) - \Lambda(\mathbf{q}\nu + (1-\mathbf{q})\zeta_1)}{(1-\mathbf{q})(\nu - \zeta_1)} \tag{1.2}$$

for $\nu \neq \zeta_1$.

Since $\Lambda : [\zeta_1, \zeta_2] \to \mathbb{R}$ is a continuous mapping, we can have

$$_{\zeta_1}D_{\mathbf{q}}\Lambda(\zeta_1) = \lim_{\nu \to \zeta_1} {_{\zeta_1}D_{\mathbf{q}}\Lambda(\nu)}.$$

If $_{\zeta_1}D_{\mathbf{q}}\Lambda(\nu)$ exists for all $\nu \in [\zeta_1, \zeta_2]$, the function Λ is well-known $_{\zeta_1}\mathbf{q}$-differentiable on $[\zeta_1, \zeta_2]$. If we choose $\zeta_1 = 0$ in (1.2), then we have $_0D_{\mathbf{q}}\Lambda(\nu) = D_{\mathbf{q}}\Lambda(\nu)$, where $D_{\mathbf{q}}\Lambda(\nu)$ is the $\mathbf{q}$-derivative of Λ at $\nu \in [0, \zeta_2]$ given by

$$D_{\mathbf{q}}\Lambda(\nu) = \frac{\Lambda(\nu) - \Lambda(\mathbf{q}\nu)}{(1-\mathbf{q})\nu}$$

for $\nu \neq 0$, see for [23] more details.

Definition 1.5 *Consider the mapping $\Lambda : [\zeta_1, \zeta_2] \to \mathbb{R}$ to be continuous. Then, for Λ at $\vartheta \in [\zeta_1, \zeta_2]$, the second $_{\zeta_1}\mathbf{q}$-derivative is given as*

$$\begin{aligned} _{\zeta_1}D^2_{\mathbf{q}}\Lambda(\nu) &= {_{\zeta_1}D_{\mathbf{q}}}\big({_{\zeta_1}D_{\mathbf{q}}\Lambda(\nu)}\big) \\ &= \frac{\Lambda(\mathbf{q}^2\vartheta\zeta_2 + (1-\vartheta\mathbf{q}^2)\zeta_1) - (1+\mathbf{q})\Lambda(\mathbf{q}\vartheta\zeta_2 + (1-\vartheta\mathbf{q})\zeta_1) + \mathbf{q}\Lambda(\vartheta\zeta_2 + (1-\vartheta)\zeta_1)}{(1-\mathbf{q})^2\mathbf{q}(\zeta_2-\zeta_1)^2\vartheta^2}. \end{aligned}$$

Definition 1.6 *[29] Consider the function $\Lambda : [\zeta_1, \zeta_2] \to \mathbb{R}$ to be continuous. Then, for Λ at $\vartheta \in [\zeta_1, \zeta_2]$, the $^{\zeta_2}\mathbf{q}$-derivative is defined as*

$$^{\zeta_2}D_{\mathbf{q}}\Lambda(\nu) = \frac{\Lambda(\mathbf{q}\nu + (1-\mathbf{q})\zeta_2) - \Lambda(\nu)}{(1-\mathbf{q})(\zeta_2 - \nu)}$$

for $\nu \neq \zeta_2$.

Definition 1.7 *[36] Consider the function $\Lambda : [\zeta_1, \zeta_2] \to \mathbb{R}$ to be continuous. Then the $_{\zeta_1}\mathbf{q}$-definite integral $[\zeta_1, \zeta_2]$ is given as*

$$\begin{aligned} \int_{\zeta_1}^{\zeta_2} \Lambda(\nu)\,{_{\zeta_1}d_{\mathbf{q}}\nu} &= (1-\mathbf{q})(\zeta_2-\zeta_1)\sum_{k=0}^{\infty}\mathbf{q}^k\Lambda(\mathbf{q}^k\zeta_2 + (1-\mathbf{q}^k)\zeta_1) \\ &= (\zeta_2-\zeta_1)\int_0^1 \Lambda\left((1-\vartheta)\zeta_1 + \vartheta\zeta_2\right) d_{\mathbf{q}}\vartheta. \end{aligned}$$

In the area of $\mathbf{q}$-calculus, the following $_{\zeta_1}\mathbf{q}$-Hermite-Hadamard inequality for convex functions was established by Alp et al. [37].

Theorem 1.2 *If $\Lambda : [\zeta_1, \zeta_2] \to \mathbb{R}$ is a convex differentiable mapping on $[\zeta_1, \zeta_2]$, then*

$$\Lambda\left(\frac{\mathbf{q}\zeta_1 + \zeta_2}{1+\mathbf{q}}\right) \leq \frac{1}{\zeta_2 - \zeta_1}\int_{\zeta_1}^{\zeta_2}\Lambda(\nu)\,{_{\zeta_1}d_{\mathbf{q}}\nu} \leq \frac{\mathbf{q}\Lambda(\zeta_1) + \Lambda(\zeta_2)}{1+\mathbf{q}}, \tag{1.3}$$

where $0 < \mathbf{q} < 1$.

In [37, 38], authors developed some estimates for the above inequality.

Furthermore, Bermudo et al. [29] provided the next definition and derived the Hermite-Hadamard inequalities.

Definition 1.8 *Consider the function $\Lambda : [\zeta_1, \zeta_2] \to \mathbb{R}$ to be continuous. Then the* ${}^{\zeta_2}\mathbf{q}$*-definite integral* $[\zeta_1, \zeta_2]$ *is given as*

$$\int_{\zeta_1}^{\zeta_2} \Lambda(\nu)^{\zeta_2} d_{\mathbf{q}}\nu = (1-\mathbf{q})(\zeta_2-\zeta_1)\sum_{k=0}^{\infty}\mathbf{q}^k \Lambda(\mathbf{q}^k\zeta_1 + (1-\mathbf{q}^k)\zeta_2)$$
$$= (\zeta_2-\zeta_1)\int_0^1 \Lambda\left(\vartheta\zeta_1 + (1-\vartheta)\zeta_2\right) d_{\mathbf{q}}\vartheta.$$

Theorem 1.3 *[29] Let $\Lambda : [\zeta_1, \zeta_2] \to \mathbb{R}$ be a convex and differentiable function on $[\zeta_1, \zeta_2]$, then we have the following* $\mathbf{q}$*-Hermite-Hadamard inequalities*

$$\Lambda\left(\frac{\zeta_1 + \mathbf{q}\zeta_2}{1+\mathbf{q}}\right) \le \frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} \Lambda(\nu)^{\zeta_2} d_{\mathbf{q}}\nu \le \frac{\Lambda(\zeta_1) + \mathbf{q}\Lambda(\zeta_2)}{1+\mathbf{q}}, \tag{1.4}$$

where $0 < \mathbf{q} < 1$.

From Theorem 1.2 and Theorem 1.3, we obtain the next inequalities.

Corollary 1.1 *[29] For any convex mapping $\Lambda : [\zeta_1, \zeta_2] \to \mathbb{R}$ and $0 < \mathbf{q} < 1$, we have*

$$\Lambda\left(\frac{\mathbf{q}\zeta_1 + \zeta_2}{1+\mathbf{q}}\right) + \Lambda\left(\frac{\zeta_1 + \mathbf{q}\zeta_2}{1+\mathbf{q}}\right)$$
$$\le \frac{1}{\zeta_2-\zeta_1}\left[\int_{\zeta_1}^{\zeta_2} \Lambda(\nu)_{\zeta_1} d_{\mathbf{q}}\nu + \int_{\zeta_1}^{\zeta_2} \Lambda(\nu)^{\zeta_2} d_{\mathbf{q}}\nu\right] \le \Lambda(\zeta_1) + \Lambda(\zeta_2) \tag{1.5}$$

and

$$\Lambda\left(\frac{\zeta_1+\zeta_2}{2}\right) \le \frac{1}{2(\zeta_2-\zeta_1)}\left[\int_{\zeta_1}^{\zeta_2} \Lambda(\nu)_{\zeta_1} d_{\mathbf{q}}\nu + \int_{\zeta_1}^{\zeta_2} \Lambda(\nu)^{\zeta_2} d_{\mathbf{q}}\nu\right] \le \frac{\Lambda(\zeta_1)+\Lambda(\zeta_2)}{2}. \tag{1.6}$$

Theorem 1.4 *(Hölder's inequality)[39, p. 604] Let $\nu > 0, 0 < \mathbf{q} < 1$ and $\rho_1 > 1$. If $\frac{1}{\rho_1} + \frac{1}{\varrho_1} + 1$, then*

$$\int_0^{\nu} \left|\Lambda(\nu)\Phi(\nu)\right| d_{\mathbf{q}}\nu \le \left(\int_0^{\nu} \left|\Lambda(\nu)\right|^{\rho_1} d_{\mathbf{q}}\nu\right)^{\frac{1}{\rho_1}} \left(\int_0^{\nu} \left|\Lambda(\nu)\Phi(\nu)\right|^{\varrho_1} d_{\mathbf{q}}\nu\right)^{\frac{1}{\varrho_1}}.$$

2 Main Results

The present section is dedicated to explore some new Hermite-Hadamard inequalities for such functions that have second ${}_{\zeta_1}\mathbf{q}$-derivatives in absolute values are convex. For the existence of the main results, we need the following lemma.

Lemma 2.1 *[40, Lemma 4.1] If $\Lambda : [\zeta_1, \zeta_2] \to \mathbb{R}$ is a twice ${}_{\zeta_1}\mathbf{q}$-differentiable mapping on (ζ_1, ζ_2), where $[\zeta_1, \zeta_2] \subset \mathbb{R}$ and ${}_{\zeta_1}D^2_{\mathbf{q}}\Lambda$ is integrable and continuous on $[\zeta_1, \zeta_2]$, then we have*

$$\frac{\mathbf{q}\Lambda(\zeta_1) + \Lambda(\zeta_2)}{1+\mathbf{q}} - \frac{1}{(\zeta_2 - \zeta_1)} \int_{\zeta_1}^{\zeta_2} \Lambda(\nu)\, {}_{\zeta_1}d_{\mathbf{q}}\nu$$
$$= \frac{\mathbf{q}^2(\zeta_2 - \zeta_1)^2}{1+\mathbf{q}} \int_0^1 \vartheta(1 - \mathbf{q}\vartheta)\, {}_{\zeta_1}D^2_{\mathbf{q}}\Lambda\left((1-\vartheta)\zeta_1 + \vartheta\zeta_2\right) d_{\mathbf{q}}\vartheta,$$

where $0 < \mathbf{q} < 1$.

The aim of this paper is to find some estimates for right-hand side of inequality (1.4).

Theorem 2.1 *If $\Lambda : [\zeta_1, \zeta_2] \to \mathbb{R}$ is a twice ${}_{\zeta_1}\mathbf{q}$-differentiable mapping on (ζ_1, ζ_2), where $[\zeta_1, \zeta_2] \subset \mathbb{R}$ and ${}_{\zeta_1}D^2_{\mathbf{q}}\Lambda$ is integrable and continuous on $[\zeta_1, \zeta_2]$, provided that $|{}_{\zeta_1}D^2_{\mathbf{q}}\Lambda|$ is (α, m)-convex on $[\zeta_1, \zeta_2]$, then the following inequality holds.*

$$\left|\frac{\mathbf{q}\Lambda(\zeta_1) + \Lambda(\zeta_2)}{1+\mathbf{q}} - \frac{1}{(\zeta_2 - \zeta_1)} \int_{\zeta_1}^{\zeta_2} \Lambda(\nu)\, {}_{\zeta_1}d_{\mathbf{q}}\nu\right|$$
$$\leq \frac{\mathbf{q}^2(\zeta_2 - \zeta_1)^2(1-\mathbf{q})^2}{(1+\mathbf{q})(1-\mathbf{q}^{\alpha+2})(1-\mathbf{q}^{\alpha+3})} \left[v_1 \left|{}_{\zeta_1}D^2_{\mathbf{q}}\Lambda(\zeta_1)\right| + m\left|{}_{\zeta_1}D^2_{\mathbf{q}}\Lambda\left(\frac{\zeta_2}{m}\right)\right|\right], \quad (2.7)$$

where

$$v_1 = \frac{(1-\mathbf{q}^{\alpha+2})(1-\mathbf{q}^{\alpha+3})}{(1-\mathbf{q}^2)(1-\mathbf{q}^3)} - 1. \quad (2.8)$$

Proof By taking first the modulus in Lemma 2.1 and then using the (α, m)-convexity of $|{}_aD^2_{\mathbf{q}}\Lambda|$, we get

$$\left|\frac{\mathbf{q}\Lambda(\zeta_1) + \Lambda(\zeta_2)}{1+\mathbf{q}} - \frac{1}{(\zeta_2 - \zeta_1)} \int_{\zeta_1}^{\zeta_2} \Lambda(\nu)\, {}_{\zeta_1}d_{\mathbf{q}}\nu\right|$$
$$\leq \frac{\mathbf{q}^2(\zeta_2 - \zeta_1)^2}{1+\mathbf{q}} \int_0^1 \vartheta(1 - \mathbf{q}\vartheta) \left|{}_{\zeta_1}D^2_{\mathbf{q}}\Lambda\left((1-\vartheta)\zeta_1 + \vartheta\zeta_2\right)\right| d_{\mathbf{q}}\vartheta$$
$$= \frac{\mathbf{q}^2(\zeta_2 - \zeta_1)^2}{1+\mathbf{q}} \int_0^1 \vartheta(1 - \mathbf{q}\vartheta) \left|{}_{\zeta_1}D^2_{\mathbf{q}}\Lambda\left((1-\vartheta)\zeta_1 + m\vartheta\frac{\zeta_2}{m}\right)\right| d_{\mathbf{q}}\vartheta$$

$$\leq \frac{\mathbf{q}^2(\zeta_2-\zeta_1)^2}{1+\mathbf{q}} \int_0^1 \vartheta\,(1-\mathbf{q}\vartheta)\left[(1-\vartheta^\alpha)\left|{}_{\zeta_1}D^2_{\mathbf{q}}\Lambda\,(\zeta_1)\right| + m\vartheta^\alpha\left|{}_{\zeta_1}D^2_{\mathbf{q}}\Lambda\left(\frac{\zeta_2}{m}\right)\right|\right] d_{\mathbf{q}}\vartheta$$

$$= \frac{\mathbf{q}^2(\zeta_2-\zeta_1)^2}{1+\mathbf{q}}\Bigg[\left|{}_{\zeta_1}D^2_{\mathbf{q}}\Lambda\,(\zeta_1)\right| \int_0^1 (\vartheta-\vartheta^{\alpha+1})(1-\mathbf{q}\vartheta)d_{\mathbf{q}}\vartheta$$

$$+ m\left|{}_{\zeta_1}D^2_{\mathbf{q}}\Lambda\left(\frac{\zeta_2}{m}\right)\right| \int_0^1 \vartheta^{\alpha+1}\,(1-\mathbf{q}\vartheta)\,d_{\mathbf{q}}\vartheta\Bigg].$$

After simplification inequality (2.7) is obtained. □

Corollary 2.1 *By choosing $\alpha = 1$ inequality (2.7) becomes*

$$\frac{\mathbf{q}\Lambda(\zeta_1)+\Lambda(\zeta_2)}{1+\mathbf{q}} - \frac{1}{(\zeta_2-\zeta_1)}\int_{\tau_1}^{\zeta_2}\Lambda(\nu)_{\zeta_1}d_{\mathbf{q}}\nu$$
$$\leq \frac{\mathbf{q}^2(\zeta_2-\zeta_1)^2}{(1+\mathbf{q})(\mathbf{q}^2+\mathbf{q}+1)(\mathbf{q}^3+\mathbf{q}^2+\mathbf{q}+1)}\left[\mathbf{q}^2\left|{}_{\zeta_1}D^2_{\mathbf{q}}\Lambda\,(\zeta_1)\right| + m\left|{}_{\zeta_1}D^2_{\mathbf{q}}\Lambda\left(\frac{\zeta_2}{m}\right)\right|\right].$$

Corollary 2.2 *By choosing $\alpha = m = 1$ inequality (2.7) becomes*

$$\frac{\mathbf{q}\Lambda(\zeta_1)+\Lambda(\zeta_2)}{1+\mathbf{q}} - \frac{1}{(\zeta_2-\zeta_1)}\int_{\zeta_1}^{\zeta_2}\Lambda(\nu)_{\zeta_1}d_{\mathbf{q}}\nu$$
$$\leq \frac{\mathbf{q}^2(\zeta_2-\zeta_1)^2}{(1+\mathbf{q})(\mathbf{q}^2+\mathbf{q}+1)(\mathbf{q}^3+\mathbf{q}^2+\mathbf{q}+1)}\left[\mathbf{q}^2\left|{}_{\zeta_1}D^2_{\mathbf{q}}\Lambda\,(\zeta_1)\right| + \left|{}_{\zeta_1}D^2_{\mathbf{q}}\Lambda\,(\zeta_2)\right|\right].$$

Remark 2.1 *Let the assumptions of Theorem 2.1 assumed to be true with the limit as $\mathbf{q} \to 1^-$ and $\alpha = m = 1$, then we get the following trapezoidal inequality*

$$\left|\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\Lambda(\nu)d\nu - \frac{\Lambda(\zeta_1)+\Lambda(\zeta_2)}{2}\right|$$
$$\leq \frac{(\zeta_2-\zeta_1)^2}{12}\left[\frac{|\Lambda''(\zeta_1)|+|\Lambda''(\zeta_2)|}{2}\right],$$

which appeared in [41, Proposition 2].

Theorem 2.2 *Suppose that $\Lambda : [\zeta_1, \zeta_2] \to \mathbb{R}$ is a twice ${}_{\zeta_1}\mathbf{q}$-differentiable mapping on (ζ_1, ζ_2) where $[\zeta_1, \zeta_2] \subset \mathbb{R}$ and ${}_{\zeta_1}D^2_{\mathbf{q}}\Lambda$ is integrable and continuous on $[\zeta_1, \zeta_2]$. If*

$|_{\zeta_1}D^2_{\mathbf{q}}\Lambda|^{\varrho_1}$, $\varrho_1 > 1$, *is* (α, m)*-convex on* $[\zeta_1, \zeta_2]$*, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{\mathbf{q}\Lambda(\zeta_1)+\Lambda(\zeta_2)}{1+\mathbf{q}} - \frac{1}{(\zeta_2-\zeta_1)}\int\limits_{\zeta_1}^{\zeta_2}\Lambda(\nu)_{\zeta_1}d_{\mathbf{q}}\nu\right| \\
&\leq \frac{\mathbf{q}^2(\zeta_2-\zeta_1)^2}{(1+\mathbf{q})^{2-\frac{1}{\varrho_1}}(1+\mathbf{q}+\mathbf{q}^2)}\left[\frac{(1-\mathbf{q}^3)(1-\mathbf{q})}{(1-\mathbf{q}^{\alpha+2})(1-\mathbf{q}^{\alpha+3})}\right]^{\frac{1}{\varrho_1}} \\
&\quad \times \left[v_1\left|_{\zeta_1}D^2_{\mathbf{q}}\Lambda(\zeta_1)\right|^{\varrho_1} + m\left|_{\zeta_1}D^2_{\mathbf{q}}\Lambda\left(\frac{\zeta_2}{m}\right)\right|^{\varrho_1}\right]^{\frac{1}{\varrho_1}}, \qquad (2.9)
\end{aligned}$$

where v_1 *is defined in* (2.8) *and* $0 < \mathbf{q} < 1$.

Proof By taking first the modulus and then using the power mean inequality in Lemma 2.1, we get

$$\begin{aligned}
&\left|\frac{\mathbf{q}\Lambda(\zeta_1)+\Lambda(\zeta_2)}{1+\mathbf{q}} - \frac{1}{(\zeta_2-\zeta_1)}\int\limits_{\zeta_1}^{\zeta_2}\Lambda(\nu)_{\zeta_1}d_{\mathbf{q}}\nu\right| \\
&\leq \frac{\mathbf{q}^2(\zeta_2-\zeta_1)^2}{1+\mathbf{q}}\int\limits_0^1 \vartheta(1-\mathbf{q}\vartheta)\left|_{\zeta_1}D^2_{\mathbf{q}}\Lambda((1-\vartheta)\zeta_1+\vartheta\zeta_2)\right| d_{\mathbf{q}}\vartheta \\
&\leq \frac{\mathbf{q}^2(\zeta_2-\zeta_1)^2}{1+\mathbf{q}}\left(\int\limits_0^1 \vartheta(1-\mathbf{q}\vartheta)d_{\mathbf{q}}\vartheta\right)^{1-\frac{1}{\varrho_1}}\left(\int\limits_0^1 \vartheta(1-\mathbf{q}\vartheta)\left|_{\zeta_1}D^2_{\mathbf{q}}\Lambda((1-\vartheta)\zeta_1+\vartheta\zeta_2)\right|^{\varrho_1} d_{\mathbf{q}}\vartheta\right)^{\frac{1}{\varrho_1}} \\
&= \frac{\mathbf{q}^2(\zeta_2-\zeta_1)^2}{1+\mathbf{q}}\left(\int\limits_0^1 \vartheta(1-\mathbf{q}\vartheta)d_{\mathbf{q}}\vartheta\right)^{1-\frac{1}{\varrho_1}}\left(\int\limits_0^1 \vartheta(1-\mathbf{q}\vartheta)\left|_{\zeta_1}D^2_{\mathbf{q}}\Lambda\left((1-\vartheta)\zeta_1+m\vartheta\frac{\zeta_2}{m}\right)\right|^{\varrho_1} d_{\mathbf{q}}\vartheta\right)^{\frac{1}{\varrho_1}}
\end{aligned}$$

By the (α, m)-convexity of $|_{\zeta_1}D^2_{\mathbf{q}}\Lambda|^{\varrho_1}$, we have

$$\begin{aligned}
&\left|\frac{\mathbf{q}\Lambda(\zeta_1)+\Lambda(\zeta_2)}{1+\mathbf{q}} - \frac{1}{(\zeta_2-\zeta_1)}\int\limits_{\zeta_1}^{\zeta_2}\Lambda(\nu)_{\zeta_1}d_{\mathbf{q}}\nu\right| \\
&\leq \frac{\mathbf{q}^2(\zeta_2-\zeta_1)^2}{1+\mathbf{q}}\left(\int\limits_0^1 \vartheta(1-\mathbf{q}\vartheta)d_{\mathbf{q}}\vartheta\right)^{1-\frac{1}{\varrho_1}} \\
&\quad \times \left(\int\limits_0^1 \vartheta(1-\mathbf{q}\vartheta)\left[(1-\vartheta^\alpha)\left|_{\zeta_1}D^2_{\mathbf{q}}\Lambda(\zeta_1)\right|^{\varrho_1} + m\vartheta^\alpha\left|_{\zeta_1}D^2_{\mathbf{q}}\Lambda\left(\frac{\zeta_2}{m}\right)\right|^{\varrho_1}\right] d_{\mathbf{q}}\vartheta\right)^{\frac{1}{\varrho_1}} \\
&= \frac{\mathbf{q}^2(\zeta_2-\zeta_1)^2}{1+\mathbf{q}}\left(\int\limits_0^1 \vartheta(1-\mathbf{q}\vartheta)d_{\mathbf{q}}\vartheta\right)^{1-\frac{1}{\varrho_1}} \\
&\quad \times \left(\left|_{\zeta_1}D^2_{\mathbf{q}}\Lambda(\zeta_1)\right|^{\varrho_1}\int\limits_0^1 (\vartheta-\vartheta^{\alpha+1})(1-\mathbf{q}\vartheta)d_{\mathbf{q}}\vartheta + m\left|_{\zeta_1}D^2_{\mathbf{q}}\Lambda\left(\frac{\zeta_2}{m}\right)\right|^{\varrho_1}\int\limits_0^1 \vartheta^{\alpha+1}(1-\mathbf{q}\vartheta)d_{\mathbf{q}}\vartheta\right)^{\frac{1}{\varrho_1}}
\end{aligned}$$

$$= \frac{\mathbf{q}^2(\zeta_2-\zeta_1)^2}{(1+\mathbf{q})}\left(\frac{1}{(1+\mathbf{q})(1+\mathbf{q}+\mathbf{q}^2)}\right)^{1-\frac{1}{\varrho_1}}$$
$$\times\left(\frac{(1-\mathbf{q})^2}{(1-\mathbf{q}^{\alpha+2})(1-\mathbf{q}^{\alpha+3})}\right)^{\frac{1}{\varrho_1}}\left[v_1\left|{}_{\zeta_1}D^2_{\mathbf{q}}\Lambda(\zeta_1)\right|^{\varrho_1}+m\left|{}_{\zeta_1}D^2_{\mathbf{q}}\Lambda\left(\frac{\zeta_2}{m}\right)\right|^{\varrho_1}\right]^{\frac{1}{\varrho_1}}.$$

After simplification the required inequality is obtained. □

Corollary 2.3 *By choosing* $\alpha=1$ *inequality* (2.9) *becomes*

$$\left|\frac{\mathbf{q}\Lambda(\zeta_1)+\Lambda(\zeta_2)}{1+\mathbf{q}}-\frac{1}{(\zeta_2-\tau_1)}\int_{\zeta_1}^{\zeta_2}\Lambda(\nu)_{\zeta_1}d_{\mathbf{q}}\nu\right|$$
$$\leq\frac{\mathbf{q}^2(\zeta_2-\zeta_1)^2}{(1+\mathbf{q})^{2-\frac{1}{\varrho_1}}(1+\mathbf{q}+\mathbf{q}^2)}\left(\frac{1}{(1+\mathbf{q}+\mathbf{q}^2+\mathbf{q}^3)}\right)^{\frac{1}{\varrho_1}}$$
$$\times\left[\mathbf{q}^2\left|{}_{\zeta_1}D^2_{\mathbf{q}}\Lambda(\zeta_1)\right|^{\varrho_1}+m\left|{}_{\zeta_1}D^2_{\mathbf{q}}\Lambda\left(\frac{\zeta_2}{m}\right)\right|^{\varrho_1}\right]^{\frac{1}{\varrho_1}}.$$

Corollary 2.4 *By choosing* $\alpha=m=1$ *inequality* (2.9) *becomes*

$$\left|\frac{\mathbf{q}\Lambda(\zeta_1)+\Lambda(\zeta_2)}{1+\mathbf{q}}-\frac{1}{(\zeta_2-\zeta_1)}\int_{\zeta_1}^{\zeta_2}\Lambda(\nu)_{\zeta_1}d_{\mathbf{q}}\nu\right|$$
$$\leq\frac{\mathbf{q}^2(\zeta_2-\zeta_1)^2}{(1+\mathbf{q})^{2-\frac{1}{\varrho_1}}(1+\mathbf{q}+\mathbf{q}^2)}\left(\frac{1}{(1+\mathbf{q}+\mathbf{q}^2+\mathbf{q}^3)}\right)^{\frac{1}{\varrho_1}}$$
$$\times\left[\mathbf{q}^2\left|{}_{\zeta_1}D^2_{\mathbf{q}}\Lambda(\zeta_1)\right|^{\varrho_1}+\left|{}_{\zeta_1}D^2_{\mathbf{q}}\Lambda(\zeta_2)\right|^{\varrho_1}\right]^{\frac{1}{\varrho_1}}.$$

Remark 2.2 *By choosing the limit as* $\mathbf{q}\to 1^-$ *and* $\alpha=m=1$ *in Theorem 2.2, we have an integral inequality, which appeared in [42, Remark 3].*

Theorem 2.3 *Suppose that* $\Lambda:[\zeta_1,\zeta_2]\to\mathbb{R}$ *is a twice* ${}_{\zeta_1}\mathbf{q}$*-differentiable function on* (ζ_1,ζ_2)*, where* $[\zeta_1,\zeta_2]\subset\mathbb{R}$ *and* ${}_{\zeta_1}D^2_{\mathbf{q}}\Lambda$ *is continuous and integrable on* $[\zeta_1,\zeta_2]$*. If* $|{}_{\zeta_1}D^2_{\mathbf{q}}\Lambda|^{\varrho_1}$ *is* (α,m)*-convex on* $[\zeta_1,\zeta_2]$ *for some* $\varrho_1>1$ *and* $\frac{1}{\rho_1}+\frac{1}{\varrho_1}=1$*, then*

$$\left|\frac{\mathbf{q}\Lambda(\zeta_1)+\Lambda(\zeta_2)}{1+\mathbf{q}}-\frac{1}{(\zeta_2-\zeta_1)}\int_{\zeta_1}^{\zeta_2}\Lambda(\nu)_{\zeta_1}d_{\mathbf{q}}\nu\right|$$
$$\leq\frac{\mathbf{q}^2(\zeta_2-\zeta_1)^2}{(1+\mathbf{q})}(v_2)^{\frac{1}{\rho_1}}$$
$$\times\left(\frac{\mathbf{q}(1-\mathbf{q}^\alpha)\left|{}_{\zeta_1}D^2_{\mathbf{q}}\Lambda(\zeta_1)\right|^{\varrho_1}+m(1-\mathbf{q})\left|{}_{\zeta_1}D^2_{\mathbf{q}}\Lambda\left(\frac{\zeta_2}{m}\right)\right|^{\varrho_1}}{1-\mathbf{q}^{\alpha+1}}\right)^{\frac{1}{\varrho_1}}, \tag{2.10}$$

where

$$v_2=(1-\mathbf{q})\sum_{k=0}^{\infty}(\mathbf{q}^k)^{\rho_1+1}(1-\mathbf{q}^{k+1})^{\rho_1} \tag{2.11}$$

for $0 < \mathbf{q} < 1$.

Proof By taking first the modulus and then using the well known Hölder's inequality in Lemma 2.1, we get

$$\begin{aligned}
&\left|\frac{\mathbf{q}\Lambda(\zeta_1)+\Lambda(\zeta_2)}{1+\mathbf{q}}-\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\Lambda(\nu)_{\zeta_1}d_{\mathbf{q}}\nu\right| \\
&\le \frac{\mathbf{q}^2(\zeta_2-\zeta_1)^2}{1+\mathbf{q}}\int_0^1 \vartheta(1-\mathbf{q}\vartheta)\left|{}_{\zeta_1}D^2_{\mathbf{q}}\Lambda\left((1-\vartheta)\zeta_1+\vartheta\zeta_2\right)\right|d_{\mathbf{q}}\vartheta \\
&\le \frac{\mathbf{q}^2(\zeta_2-\zeta_1)^2}{1+\mathbf{q}}\left(\int_0^1(\vartheta(1-\mathbf{q}\vartheta))^{\rho_1}d_{\mathbf{q}}\vartheta\right)^{\frac{1}{\rho_1}}\left(\left|{}_{\zeta_1}D^2_{\mathbf{q}}\Lambda\left((1-\vartheta)\zeta_1+\vartheta\zeta_2\right)\right|^{\varrho_1}d_{\mathbf{q}}\vartheta\right)^{\frac{1}{\varrho_1}} \\
&= \frac{\mathbf{q}^2(\zeta_2-\zeta_1)^2}{1+\mathbf{q}}\left(\int_0^1(\vartheta(1-\mathbf{q}\vartheta))^{\rho_1}d_{\mathbf{q}}\vartheta\right)^{\frac{1}{\rho_1}}\left(\left|{}_{\zeta_1}D^2_{\mathbf{q}}\Lambda\left((1-\vartheta)\zeta_1+m\vartheta\frac{\zeta_2}{m}\right)\right|^{\varrho_1}d_{\mathbf{q}}\vartheta\right)^{\frac{1}{\varrho_1}}.
\end{aligned}$$

Since $|{}_{\zeta_1}D^2_{\mathbf{q}}\Lambda|^{\varrho_1}$ is (α, m)-convex, therefore we can write

$$\begin{aligned}
&\left|\frac{\mathbf{q}\Lambda(\zeta_1)+\Lambda(\zeta_2)}{1+\mathbf{q}}-\frac{1}{(\zeta_2-\zeta_1)}\int_{\zeta_1}^{\zeta_2}\Lambda(\nu)_{\zeta_1}d_{\mathbf{q}}\nu\right| \\
&\le \frac{\mathbf{q}^2(\zeta_2-\zeta_1)^2}{1+\mathbf{q}}\left(\int_0^1(\vartheta(1-\mathbf{q}\vartheta))^{\rho_1}d_{\mathbf{q}}\vartheta\right)^{\frac{1}{\rho_1}} \\
&\times\left(\left|{}_{\zeta_1}D^2_{\mathbf{q}}\Lambda(\zeta_1)\right|^{\varrho_1}\int_0^1(1-\vartheta^{\alpha})d_{\mathbf{q}}\vartheta+m\left|{}_{\zeta_1}D^2_{\mathbf{q}}\Lambda\left(\frac{\zeta_2}{m}\right)\right|^{\varrho_1}\int_0^1\vartheta^{\alpha}d_{\mathbf{q}}\vartheta\right)^{\frac{1}{\varrho_1}}.
\end{aligned}$$

After simple calculation, we get the required inequality. □

Corollary 2.5 *By choosing* $\alpha = 1$ *inequality* (2.10) *becomes*

$$\begin{aligned}
&\left|\frac{\mathbf{q}\Lambda(\zeta_1)+\Lambda(\zeta_2)}{1+\mathbf{q}}-\frac{1}{(\zeta_2-\zeta_1)}\int_{\zeta_1}^{\zeta_2}\Lambda(\vartheta)_{\zeta_1}d_{\mathbf{q}}\nu\right| \\
&\le \frac{\mathbf{q}^2(\zeta_2-\zeta_1)^2}{(1+\mathbf{q})}(v_2)^{\frac{1}{\rho_1}}\left(\frac{\mathbf{q}\left|{}_{\zeta_1}D^2_{\mathbf{q}}\Lambda(\zeta_1)\right|^{\varrho_1}+m\left|{}_{\zeta_1}D^2_{\mathbf{q}}\Lambda\left(\frac{\zeta_2}{m}\right)\right|^{\varrho_1}}{1+\mathbf{q}}\right)^{\frac{1}{\varrho_1}},
\end{aligned}$$

where v_2 *is define in* (2.11).

Corollary 2.6 *By choosing $\alpha = m = 1$ inequality (2.10) becomes*

$$\left|\frac{\mathbf{q}\Lambda(\zeta_1)+\Lambda(\zeta_2)}{1+\mathbf{q}} - \frac{1}{(\zeta_2-\zeta_1)}\int\limits_{\zeta_1}^{\zeta_2}\Lambda(\nu)_{\zeta_1}d_{\mathbf{q}}\nu\right|$$
$$\leq \frac{\mathbf{q}^2(\zeta_2-\zeta_1)^2}{(1+\mathbf{q})}(v_2)^{\frac{1}{\rho_1}}\left(\frac{\mathbf{q}\left|{}_{\zeta_1}D_{\mathbf{q}}^2\Lambda(\zeta_1)\right|^{\varrho_1}+\left|{}_{\zeta_1}D_{\mathbf{q}}^2\Lambda(\zeta_2)\right|^{\varrho_1}}{1+\mathbf{q}}\right)^{\frac{1}{\varrho_1}},$$

where v_2 is define in (2.11).

Remark 2.3 *By choosing the limit as $\mathbf{q} \to 1^-$ and $\alpha = m = 1$ in Theorem 2.3, we have an integral inequality, which appeared in [42, Remark 4].*

In terms of the second $\mathbf{q}$-derivatives we obtain another Hermite-Hadamard-type inequality for powers.

Theorem 2.4 *Let the assumptions of Theorem 2.3 be true, then the inequality*

$$\left|\frac{\mathbf{q}\Lambda(\zeta_1)+\Lambda(\zeta_2)}{1+\mathbf{q}} - \frac{1}{\zeta_2-\zeta_1}\int\limits_{\zeta_1}^{\zeta_2}\Lambda(\nu)_{\zeta_1}d_{\mathbf{q}}\nu\right|$$
$$\leq \mathbf{q}\frac{\mathbf{q}^2(\zeta_2-\zeta_1)^2}{(1+\mathbf{q})}\left(\frac{1}{[\rho_1+1]_{\mathbf{q}}}\right)^{\frac{1}{\rho_1}}\left(v_3\left|{}_{\zeta_1}D_{\mathbf{q}}^2\Lambda(\zeta_1)\right|^{\varrho_1}+mv_4\left|{}_{\zeta_1}D_{\mathbf{q}}^2\Lambda\left(\frac{\zeta_2}{m}\right)\right|^{\varrho_1}\right)^{\frac{1}{\varrho_1}} \tag{2.12}$$

holds, where

$$v_3 = (1-\mathbf{q})\sum_{k=0}^{\infty}\mathbf{q}^k(1-\mathbf{q}^{k\alpha}(1-\mathbf{q}^{k+1})^{\rho_1}$$

and

$$v_4 = (1-\mathbf{q})\sum_{k=0}^{\infty}\mathbf{q}^{k(\alpha+1)}(1-\mathbf{q}^{k+1})^{\rho_1}.$$

Proof By taking the modulus and then using the well known Hölder's inequality in Lemma 2.1, we get

$$\left|\frac{\mathbf{q}\Lambda(\zeta_1)+\Lambda(\zeta_2)}{1+\mathbf{q}} - \frac{1}{(\zeta_2-\zeta_1)}\int\limits_{\zeta_1}^{\zeta_2}\Lambda(\nu)_{\zeta_1}d_{\mathbf{q}}\nu\right|$$
$$\leq \frac{\mathbf{q}^2(\zeta_2-\zeta_1)^2}{1+\mathbf{q}}\int\limits_0^1\vartheta(1-\mathbf{q}\vartheta)\left|{}_{\zeta_1}D_{\mathbf{q}}^2\Lambda((1-\vartheta)\zeta_1+\vartheta\zeta_2)\right|d_{\mathbf{q}}\vartheta$$

$$\leq \frac{\mathbf{q}^2(\zeta_2-\zeta_1)^2}{1+\mathbf{q}}\left(\int_0^1 \vartheta^{\rho_1} d_{\mathbf{q}}\vartheta\right)^{\frac{1}{\rho_1}}\left((1-\mathbf{q}\vartheta)^{\varrho_1}\left|{}_{\zeta_1}D_{\mathbf{q}}^2\Lambda\left((1-\vartheta)\zeta_1+\vartheta\zeta_2\right)\right|^{\varrho_1} d_{\mathbf{q}}\vartheta\right)^{\frac{1}{\varrho_1}}$$

$$= \frac{\mathbf{q}^2(\zeta_2-\zeta_1)^2}{1+\mathbf{q}}\left(\int_0^1 \vartheta^{\rho_1} d_{\mathbf{q}}\vartheta\right)^{\frac{1}{\rho_1}}\left((1-\mathbf{q}\vartheta)^{\varrho_1}\left|{}_{\zeta_1}D_{\mathbf{q}}^2\Lambda\left((1-\vartheta)\zeta_1+m\vartheta\frac{\zeta_2}{m}\right)\right|^{\varrho_1} d_{\mathbf{q}}\vartheta\right)^{\frac{1}{\varrho_1}}.$$

Since $|{}_{\zeta_1}D_{\mathbf{q}}^2\Lambda|^{\varrho_1}$ is (α,m)-convex, we have

$$\left|\frac{\mathbf{q}\Lambda(\zeta_1)+\Lambda(\zeta_2)}{1+\mathbf{q}}-\frac{1}{(\zeta_2-\zeta_1)}\int_{\zeta_1}^{\zeta_2}\Lambda(\nu)_{\zeta_1}d_{\mathbf{q}}\nu\right|$$

$$\leq \frac{\mathbf{q}^2(\zeta_2-\zeta_1)^2}{1+\mathbf{q}}\left(\int_0^1 \vartheta^{\rho_1} d_{\mathbf{q}}\vartheta\right)^{\frac{1}{\rho_1}}$$

$$\times\left(\left|{}_{\zeta_1}D_{\mathbf{q}}^2\Lambda(\zeta_1)\right|^{\varrho_1}\int_0^1(1-\mathbf{q}\vartheta)^{\varrho_1}(1-\vartheta^{\alpha})\,d_{\mathbf{q}}\vartheta\right.$$

$$\left.+\,m\left|{}_{\zeta_1}D_{\mathbf{q}}^2\Lambda\left(\frac{\zeta_2}{m}\right)\right|^{\varrho_1}\int_0^1(1-\mathbf{q}\vartheta)^{\varrho_1}\vartheta^{\alpha}d_{\mathbf{q}}\vartheta\right)^{\frac{1}{\varrho_1}}$$

$$= \frac{\mathbf{q}^2(\zeta_2-\zeta_1)^2}{(1+\mathbf{q})}\left(\frac{1}{[\rho_1+1]_{\mathbf{q}}}\right)^{\frac{1}{\rho_1}}\left(v_3\left|{}_{\zeta_1}D_{\mathbf{q}}^2\Lambda(\zeta_1)\right|^{\varrho_1}+mv_4\left|{}_{\zeta_1}D_{\mathbf{q}}^2\Lambda\left(\frac{\zeta_2}{m}\right)\right|^{\varrho_1}\right)^{\frac{1}{\varrho_1}},$$

where

$$v_3=\int_0^1(1-\mathbf{q}\vartheta)^{\varrho_1}(1-\vartheta^{\alpha})d_{\mathbf{q}}\vartheta$$

and

$$v_4=\int_0^1(1-\mathbf{q}\vartheta)^{\varrho_1}\vartheta^{\alpha}d_{\mathbf{q}}\vartheta.$$

Hence the proof is done. □

Corollary 2.7 *By choosing $\alpha = 1$ in inequality (2.12) becomes*

$$\left|\frac{\mathbf{q}\Lambda(\zeta_1)+\Lambda(\zeta_2)}{1+\mathbf{q}}-\frac{1}{(\zeta_2-\zeta_1)}\int_{\zeta_1}^{\zeta_2}\Lambda(\nu)_{\zeta_1}d_{\mathbf{q}}\nu\right|$$

$$\leq \frac{\mathbf{q}^2(\zeta_2-\zeta_1)^2}{(1+\mathbf{q})}\left(\frac{1}{[\rho_1+1]_{\mathbf{q}}}\right)^{\frac{1}{\rho_1}}\left(v_3\left|{}_{\zeta_1}D_{\mathbf{q}}^2\Lambda(\zeta_1)\right|^{\varrho_1}+mv_4\left|{}_{\zeta_1}D_{\mathbf{q}}^2\Lambda\left(\frac{\zeta_2}{m}\right)\right|^{\varrho_1}\right)^{\frac{1}{\varrho_1}},$$

where

$$v_3=(1-\mathbf{q})\sum_{k=0}^{\infty}\mathbf{q}^k(1-\mathbf{q}^k)(1-\mathbf{q}^{k+1})^{\rho_1}$$

and

$$v_4 = (1-\mathbf{q})\sum_{k=0}^{\infty}\mathbf{q}^{2k}(1-\mathbf{q}^{k+1})^{\rho_1}.$$

Corollary 2.8 *By choosing $\alpha = m = 1$ inequality (2.12) becomes*

$$\left|\frac{\mathbf{q}\Lambda(\zeta_1)+\Lambda(\zeta_2)}{1+\mathbf{q}} - \frac{1}{(\zeta_2-\zeta_1)}\int_{\zeta_1}^{\zeta_2}\Lambda(\nu)_{\zeta_1}d_{\mathbf{q}}\nu\right|$$
$$\leq \frac{\mathbf{q}^2(\zeta_2-\zeta_1)^2}{(1+\mathbf{q})}\left(\frac{1}{[\rho_1+1]_{\mathbf{q}}}\right)^{\frac{1}{\rho_1}}\left(v_3\left|{}_{\zeta_1}D_{\mathbf{q}}^2\Lambda(\zeta_1)\right|^{\varrho_1} + v_4\left|{}_{\zeta_1}D_{\mathbf{q}}^2\Lambda(\zeta_2)\right|^{\varrho_1}\right)^{\frac{1}{\varrho_1}},$$

where

$$v_3 = (1-\mathbf{q})\sum_{n=0}^{\infty}\mathbf{q}^{n}(1-\mathbf{q}^{n})(1-\mathbf{q}^{n+1})^{\rho_1}$$

and

$$v_4 = (1-\mathbf{q})\sum_{n=0}^{\infty}\mathbf{q}^{2n}(1-\mathbf{q}^{n+1})^{\rho_1}$$

.

Remark 2.4 *By choosing the limit as $\mathbf{q}\to 1^-$ and $\alpha = m = 1$ in Theorem 2.4, we have*

$$\left|\frac{\Lambda(\zeta_1)+\Lambda(\zeta_2)}{2} - \frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\Lambda(\nu)d\nu\right|$$
$$\leq \frac{(\zeta_2-\zeta_1)^2}{2}\left(\frac{1}{\rho_1+1}\right)^{\frac{1}{\rho_1}}\left(\frac{1}{(\varrho_1+1)(\varrho_1+2)}\right)^{\frac{1}{\varrho_1}}\left((\varrho_1+1)\left|\Lambda''(\zeta_1)\right|^{\varrho_1} + \left|\Lambda''(\zeta_2)\right|^{\varrho_1}\right)^{\frac{1}{\varrho_1}}.$$

3 Conclusion

In this paper, we tried to represent the relation of integral calculus without the limit known as quantum calculus. This relationship was presented in terms of error estimates of Hermite-Hadamard inequalities. The established results are correlated with the existing literature presented as remarks. Such estimates provide a future direction for the readers to establish such estimates for other convexities.

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Author's contributions
MS dealt with writing the manuscript, conceptualization, formal analysis, investigation and validation. KS performed the formal analysis, investigation and validation. SN dealt with the conceptualization, formal analysis, investigation and validation. KN performed the formal analysis, funding acquisition, validation, edition original draft preparation and writing revised version. All authors read and approved the final manuscript.

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