On the admissibility and input-output representation for a class of Volterra integro-differential systems

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Abstract This article studies a class of controlled-observed Volterra integro-differential problems in the case that the operator of the associated Cauchy problem generate a semi-group on a Banach space and the integral part being given by a convolution of integrable functions with L^p-admissible observation operators kernel. Sufficient and/or necessary conditions for L^p-admissibility of control and observation operators are given in term of the kernels. In particular, results on the equivalence between the finite-time (or infinite-time) L^p-admissibility and the uniform L^p-admissibility are given for both control and observation operators, extending a generalization of results known to hold for the standard Cauchy problems. Particular attention is paid to the problem of obtaining the input-output representation of such systems, providing a theory which is analogue to that of Salamon-Weiss for linear systems. We mention that our approach is mainly based on the theory of infinite-dimensional L^p-regular linear systems in the Salamon-Weiss sense. These results are illustrated by an example involving heat conduction with memory given by some space fractional Laplacian kernel.

Keywords C_0-semigroups · Volterra integro-differential problems · Resolvent operators · Admissibility · Weiss conjecture · Regular linear systems
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1 introduction

Volterra integro-differential problems play an important role in the modelling of dynamical phenomena and have attracted the attention of many researchers and have been the subject of much interesting works, see e.g. [55],[59], and the references therein. In such class of evolution equations, the differential and integral operators can appear together at the same time. In several practical situations, the modelling gives rise to unbounded controlled-observed systems of form (1). This fact appears naturally when the control or the observation is exercised

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through the boundary or a point for the systems governed by partial differential equations and in many situations of Volterra integro-differential equations.

The purpose of this article is to investigate conditions for the sake of obtaining the admissibility for control and observation operators and the input-output representation for a class of Volterra integro-differential systems in Banach spaces. Input-output representation plays an important role in the controlling of observed-controlled dynamical phenomena, in particular, whose control and observation operators are restricted to be unbounded.

Let X,Y and U be Banach spaces with norms respectively denoted by $\|\cdot\|_X, \|\cdot\|_Y$ and $\|\cdot\|_U$ by and consider the following controlled-observed Volterra integro-differential system:

$$\begin{cases} \dot{x}(t) = Ax(t) + \int_0^t k(t-s)Kx(s)ds + Bu(t),\ t > 0, \\ y(t) = Cx(t),\ x(0) = x_0 \in X \end{cases} \tag{1}$$

which has a "big" intersection with a class of Volterra integral problem as demonstrated in [63]. Here the unbounded operator $(A, D(A))$ is the generator of a strongly continuous semigroup (simply, C_0-semigroup or operator semigroup; see e.g [58] for some informations) $(T(t))_{t\geq 0}$ on the state space X, and K is a linear operator in X with domains $D(K) \supset D(A)$ and $k(\cdot)$ is a real-valued scalar function. The control and observation operators B and C are (possibly unbounded) linear operators on U and $D(A)$ respectively. In (1), $u(\cdot) \in L^p_{loc}(\mathbb{R}^+; U)$ with $p \in [1, +\infty)$ is the input function which enters through the inhomogeneity Bu while $x(\cdot)$ is the X-valued controlled state and the output function $y(\cdot)$ is induced by the observation operator C and takes values in a (output) Banach space Y. Recall that all vector-valued integrals are Bochner integrals. Due to the hybrid Cauchy-Volterra context of these problems, the control and observation operators need to have the L^p-admissibility properties. Here, we are concerned with the L^p-integrable functions $k(\cdot)$. We refer to [59, section I.5] for typical kernel functions and related physical explanations.

By setting $k = 0$, (1) encompasses the subclass

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t),\ t \geq 0, \\ y(t) = Cx(t),\ x(0) = x_0 \in X, \end{cases} \tag{2}$$

of infinite dimensional linear systems. In particular, this subclass of infinite dimensional linear systems with unbounded control and observation operators is well understood and numerous results are available on L^p-admissibility and input-output representation (see e.g. [66], [65], [69], [64], [61] and the references therein).

By setting $K = A$, (1) becomes the subclass

$$\begin{cases} \dot{x}(t) = Ax(t) + \int_0^t k(t-s)Ax(s)ds + Bu(t),\ t \geq 0, \\ y(t) = Cx(t),\ x(0) = x_0 \in X, \end{cases} \tag{3}$$

From the L^p-admissibility viewpoint, this subclass of Volterra integro-differential systems has received some attention during the past two decades and few results are available (see [48], [45], [45], [25], [10], [16], [63]). However, to the best of our knowledge, the input-output representation of this subclass has not been studied in the literature.

In parallel with (1), we introduce a homogeneous (non-controlled) version of the above non-standard Volterra integro-differential problem (nsVIDP in short),

$$\dot{x}(t) = Ax(t) + \int_0^t k(t-s)Kx(s)ds,\ t \geq 0, \qquad x(0) = x_0, \tag{4}$$

and it is called a standard Volterra integro-differential problem (sVIDP in short) if the kernel $K = A$.
Recently, for a bounded analytic semigroups $(T(t))_{t\geq 0}$, this class of problems has been studied from the L^p-maximal regularity viewpoint in [2], and in particular for $K = (-A)^{\alpha}$ with $0 < \alpha < 1/2$.

To study and present behaviours of models and problems described with integral and differential equations, we first need the existence of their solutions. In this connection, many authors have studied the Volterra integro-differential problems (4) by the classical approach using the Laplace transform, and for different reasons, using the Laplace transform (see, e.g., Da Prato and Iannelli [18, 19], Lunardi [51] and Pruss [59]). Maximal L^p-regularity results are due to Clement and Da Prato [17], Prüss [59]. In addition to the classical approach, there is a semigroup approach which has been used in, e.g. Miller [55], Chen and Grimmer [12, 13], Desch and Grimmer [20], Desch and Schappacher [22], Kunisch, Di Blasio and Sinestrari, Nagel and Sinestrari [56] and Engel and Nagel [24]. It was the main objection against the semigroup approach for many years that it is not possible to obtain regularity of the solutions (see [59, pp. 338-339]. This is not true as it was proved in [3].

The L^p-admissibility problem of both control and observation operators has attracted many authors during the last 30 years and there is a wealth of literature on the Cauchy problems (CPs in short) (i.e. (4) with $k(t) = 0$) [42, 65, 68, 44, 50, 43, 61, 7, 52]. There are too many related works to list them comprehensively here. For Volterra problems, the L^p-admissibility was studied relatively recently. The first studies on L^2-admissibility of control operators for a class of the standard Volterra integral problems (sVIPs in short)

$$x(t) = x_0 + \int_0^t a(t-s)Ax(s)ds,\ t \geq 0, \tag{5}$$

began with the paper of Jung [48]. The idea of treating finite-time L^2-admissibility of control or observation operators for sVIPs (5) has been exploited past years by several authors (see e.g. [45], [46], [33], [34], [25], [10]). It is legitimate to ask if there is a relation between the L^p-admissibility with respect to CPs and sVIPs, sVIDPs respectively. Indeed, in [48], the finite-time L^2-admissibility of control operators for a class of sVIPs (5) with completely positive kernel functions $a(\cdot)$ is linked with L^2-admissibility of the well-studied CPs, i.e. $a(\cdot) = 1$. Likewise, in [45], infinite-time L^2-admissibility for observation operators for sVIPs (5) is linked with infinite-time L^2-admissibility for CPs for a large class of kernel functions and the result subsumes that of [48]. Other results are related to the case where the generator of the underlying semigroup has a Riesz basis of eigenvectors [33]. In [46], the authors gave necessary and sufficient conditions for finite-time L^2-admissibility for observation operators for a class of sVIDPs (4) when the underlying semigroup is equivalent to a contraction semigroup, which generalizes an analogous result known to hold for the CPs and it subsumes the result in [48]. Recently, the authors in [10] investigated the notion of Favard spaces with respect to resolvent operator for a class of sVIPs (5) and they established some relationships with finite-time L^p-admissibility for control operators for $p \in [1, \infty)$. This extends the results obtained for CPs in [52]. The authors [25] gave a new sufficient condition, which is also necessary if the underlying control space is UMD for the finite and infinite-time L^p-admissibility of control operators for a class of sVIDPs, with exponentially stable resolvent operators, extending a result for the CPs [7]. This, was been extended recently to more general class of the nsVIDPs [63]. Furthermore, it was proved in [10] that for sVIPs (5) with some *creep* kernels $a(\cdot)$ (see [59, Definition 4.4]), and hence for sVIDPs (4) with some appropriate kernels $k(\cdot)$, the finite-time and infinite-time L^1-admissibility of observation operators are equivalent for exponentially stable resolvent operators. In particular, for reflexive state space X it was shown in [10] that the finite-time L^1-admissibility and uniform L^1-admissibility are equivalent, extending well-known results for CPs [64, Remark 4.3.5]. To the best of the authors' knowledge, these problems are still open for Volterra systems, and in particular for (4), due to the lack of semigroup property and the presence of kernels which make them more complicated than those for the CPs. In this paper, in particular, we will give some affirmative answers to the above problems.

As for sVIPs and sVIDPs, it is tempting to hope for a transfer of L^p-admissibility for nsVIDPs (16) from property of CPs . (A reformuler)It is however the case here by proving that with some conditions if B is an L^p-admissible control operator for CPs (see. Definition 18) then in our main result (see. Theorem 3), it is shown that (16) has a X-valued solution $x(t)$ for all $u(\cdot) \in L^p_{loc}(\mathbb{R}^+, U)$ given by (17). In particular, B is uniformly L^p-admissible for nsVIDPs (16). Likewise, the observation operator C is uniformly L^p-admissible for (4) if and only if C is finite-time L^p-admissible for CPs (see Proposition 5). Moreover, under strong condition than the exponential stability of the resolvent operator associated to nsVIDPs (4), we prove that the finite-time and infinite-time L^p-admissibility of observation operators for nsVIDPs (4) are equivalent (see Proposition 5) . This constitutes a partial affirmative answer to the problem (**P2**) (see Section 4). For sVIDPs (4) with kernels $k(\cdot) \in W^{1,2}(\mathbb{R}^+)$ and $p = 2$ (resp. sVIPs (5) with some *creep* kernel functions $k(\cdot)$ and $p = 1$), and concerning the observation operators, implicitly the authors in [46] (resp. [10]) proved partially the corresponding analogue p-Weiss conjecture. Here, we will contribute again to that subject by proving that the p-Weiss conjectures for CPs and the nsVIDPs under consideration are equivalent.

It is customary in systems theory to deal with mathematical models of dynamical systems which are driven by inputs and which produce outputs. Apart from the interest per se, this is potentially useful for various control purposes. Indeed, in finite dimensional linear systems theory, it is well-known that some control or synthesis problems are easier formulated and/or solved for systems given in state space form, while other problems may be more naturally stated and/or solved for systems described in input-output form. Among others, and as for L^p-regular infinite-dimensional linear systems [69], which resemble linear finite-dimensional systems in many ways, it ??is to obtain a formula for the transfer function, for nsVIDP (1). In this paper, also we aim at presenting conditions which allow us to derive an explicit expression relating the inputs directly to the outputs for the nsVIDPs (1) by extending a part of the theory of L^p-regular linear systems to this class of nsVIDPs (1). These conditions exclude (resp. include) the sVIDPs (4) if $p \in (1, \infty)$ (resp. $p = 1$ and A generates a bounded analytic semigroup). Likewise, the basic ingredient for our result of the input-output representation for the nsVIDP (1), which is of the great use for linear (infinite dimension) systems, is the notion of L^p-regular linear systems. Theoretically, this class of systems is very appealing, and there are applications where it is natural to use it. For example, as for linear PDEs (see [11,29,29,28,31]), numerous attempts have been made to obtain similar results for delay and neutral PDEs (see [35], [6]) and bilinear PDEs (see [5]) and recently for a class of sVIDPs (4) with delay in state, input and output (see [23]).

The paper is structured as follows. Section 2 contains some preliminaries and well-known properties of resolvent operators for nsVIDPs (4) except of Proposition 3 recently proved in [63] which is promising and encourage the exploration of its use leading to the main results of this paper. In Section 3, we recall how the nsVIDPs (4) can be transformed to CPs on some appropriate product space and especially Lemma 1 which will be used constantly in this paper. In Section 4, we first introduce the definitions of the L^p-admissibility for CPs and nsVIDPs (4) and recall some required facts about L^p-regular linear systems. Then, we revisit the Cauchy transformation recalled in Section 3 in terms of the kernels

k and K. The second part is devoted to disclose the L^p-admissibility results as well as the input-output representation for the nsVIDPs (1). In the last section, we provide a heat conduction model with space fractional Laplacian memory to illustrate our previous abstract results.

2 Review on resolvent operators and some related preliminaries results

In this section, we collect some useful elementary facts about a general class of nsVIDP (4) and resolvent operators and fix the notation, which will be employed through the paper. These topics have been discussed in detail in [59, 27]. We refer to these works for reference to the literature and further results.

Throughout this section $(X, \|\cdot\|_X)$ be a Banach space, $(A, D(A))$ is a closed linear densely defined operator in X (not necessarily a generator) and $\rho(A)$ the resolvent set of the operator A if it is no empty. The function $f : \mathbb{R}^+ \to X$ is continuous and $(K(t))_{t\geq 0}$ is a family of linear operators on X with domains $D(K(t)) \supset D(A)$ satisfy the following assumption:
$(\mathbf{H1}) : D(K(t)) \supset D(A)$ for almost all t, the functions $t \longmapsto K(t)x$ with $x \in D(A)$ are strongly measurable and there is $b(\cdot) \in L^1_{loc}(\mathbb{R}^+)$ such that

$$\|K(t)x\|_X \leq b(t)(\|x\|_X + \|Ax\|_X =: \|x\|_A) \text{ for almost all } t.$$

We consider the inhomogeneous nsVIDP:

$$\begin{cases} \dot{x}(t) = Ax(t) + \int_0^t K(t-s)x(s)ds + f(t), \ t > 0, \\ x(0) = x_0 \in X. \end{cases} \tag{6}$$

In what follows, we give some results for the existence of solutions for (6)

Definition 1 ([27])

- (i) A function $x : \mathbb{R}^+ \to X$ is called a solution of (6) if x is continuously differentiable on $\mathbb{R}^+$, $x(t) \in D(A)$, $Ax(\cdot)$ is continuous and (6) holds on $\mathbb{R}^+$.
- (ii) The homogeneous part of (6) (i.e.$f = 0$) is called well-posed if, for each $v \in D(A)$, there is a unique solution $x_v(t)$ of (6) with $x_v(0) = v$ and for a sequence $(v_n) \subset D(A)$, $v_n \to 0$ in X implies $x_{v_n}(t) \to 0$ in X, uniformly on compact intervals.

Definition 2 A family of bounded linear operators $(S(t))_{t\geq 0}$, is called resolvent operator for the equation (6), if the following three conditions are satisfied:

(S1) For all $x \in X$, $S(\cdot)x$ is continuous on $\mathbb{R}^+$, $S(0) = I$ (identity operator).
(S2) $S(t)(D(A)) \subset D(A)$ for all $t \geq 0$, for $x \in D(A)$, $AS(\cdot)x$ is continuous and $S(\cdot)x$ is continuously differentiable on $\mathbb{R}^+$ for $\|\cdot\|_X$-norm.
(S3) For all $x \in D(A)$ and $t \geq 0$, the following resolvent equations

$$\dot{S}(t)x = AS(t)x + \int_0^t K(t-s)S(s)xds = S(t)Ax + \int_0^t K(s)S(t-s)xds,$$

hold.

First, we note that there can be at most one resolvent operator which is uniquely determined. The proofs of these results and further information on the resolvent can be found in [59,27]. The existence of a resolvent operator allows us to find the solution for the equation (6) where several of the properties of the resolvent has been discussed in [59,27]. In the sense specified in Definition 2 above, it can be shown under quite general hypotheses (see below) on the kernel operator $K(\cdot)$ that any solution of (6) is represented by the variation-of-constants formula (see. [27, Corollary 1])

$$x(t) = S(t)x_0 + \int_0^t S(t-s)f(s)ds, \ t \geq 0. \tag{7}$$

A function $x(t)$ defined by (7) will in general not be a solution of (6). Nevertheless we call it a mild solution of (6). Obviously (see [27]), the existence of a resolvent operator $(S(t))_{t\geq 0}$ for the equation (6) implies the well-posedness for the homogeneous part of (6). However, the converse also holds as it was proved in the following

Theorem 1 *([27, Theorem 4]) The equation (6) admits a resolvent operator* $(S(t)_{t\geq 0})$ *if and only if the homogeneous part of (6) is well-posed.*

A resolvent operator $(S(t))_{t\geq 0}$ is called exponentially bounded, if it satisfies **(S4)**: there exist $M > 0$ and $\omega \in \mathbb{R}$ such that $\|S(t)\|_{\mathscr{L}(X)} \leq Me^{\omega t}$ for all $t \geq 0$. The growth bound of $(S(t))_{t\geq 0}$ is $\omega_0(S) := \inf\left\{\omega \in \mathbb{R}, \|S(t)\|_{\mathscr{L}(X)} \leq Me^{\omega t}, t \geq 0, M > 0\right\}$ and the resolvent operator is called exponentially stable if $\omega_0(S) < 0$. Note that, contrary to the case of C_0-semigroups, a resolvent for (6) need not to be exponentially bounded: a counterexample can be found in [21], [59, pp. 45]. However, there are checkable conditions guaranteeing that (6) possesses an exponentially bounded resolvent operator (see. [22], [27]). We will use the Laplace transform at times. Suppose that $g : \mathbb{R}^+ \to X$ is measurable and there exist $M > 0$, $\gamma \in \mathbb{R}$, such that $\|g(t)\|_X \leq Me^{\gamma t}$ for almost $t \geq 0$. Then the Laplace transform $\widehat{g}(\lambda) = \int_0^\infty e^{-\lambda t} g(t)\,dt$ exists for all $\lambda \in \mathbb{C}_\gamma := \{\lambda \in \mathbb{C} : Re(\lambda) > \gamma\}$.

In order to have Laplace transform theory available for the study of (6) we have to impose a growth restriction on the function $b(t)$ from $(\mathbf{H1})$
$(\mathbf{H2})$: There is $\beta \in \mathbb{R}$ such that $b(\cdot)e^{-\beta \cdot} \in L^1(\mathbb{R}^+)$.

We notice that the hypothesis $(\mathbf{H2})$ means that $b(\cdot)$ has an absolutely convergent Laplace transform

$$\widehat{b}(\lambda) = \int_0^\infty e^{-\lambda t} b(t)\,dt, \ \lambda \in \mathbb{C}_\beta,$$

and therefore $K(\cdot)x$ admits an absolutely convergent Laplace transform

$$\widehat{K}(\lambda)x = \int_0^\infty e^{-\lambda t} K(t)x dt, \quad x \in D(A), \ \lambda \in \mathbb{C}_\beta.$$

Recall that for $K(\cdot)$ satisfying $(\mathbf{H1}) - (\mathbf{H2})$, if for each $x \in D(A)$ the function $K(\cdot)x$ lies in $W^{1,p}(\mathbb{R}^+, X)$ or it is of bounded variation on $[0,\infty)$ (resp. on $[0,\tau]$ for all τ), then it was proved in [22, Theorem 3.1] (resp. [27, Corollary 4]) that nsVIDP (4) admits a resolvent operator if (resp. iff) A generates a C_0-semigroup on X (Phrase a reformuler). Since the results in [22] have been obtained using the semigroup approach, we remark that implicitly, in these cases, the resolvent operators satisfy $(\mathbf{S4})$. This will be discussed further in more details in the next section.

Now, the following like Hille-Yosida generation theorem stated in [27, Theorem 8] is quite important for the rest of the paper and will be used in essentially in the next section. It establishes a relationship between existence of exponentially bounded resolvent operators of nsVIDPs (4) and their Laplace transforms. This has been proved previously in [19] for the sVIDPs (4).

Proposition 1 *Let* $(\mathbf{H1})-(\mathbf{H2})$ *hold. Then a necessary and sufficient condition for existence of a resolvent operator* $S(t)$ *for (6) satisfying* $(\mathbf{S4})$ *is that there are some* $\omega^V>\beta$ *and* $M\geq 1$ *such that* $\lambda I-A-\widehat{K}(\lambda)$ *with domain* $D(A)$ *is invertible for all* $Re\lambda>\omega^V$ *and the operator*

$$H(\lambda):=\left(\lambda I-A-\widehat{K}(\lambda)\right)^{-1}, \tag{8}$$

called the resolvent associated to $(S(t))_{t\geq 0}$

$$\widehat{S}(\lambda)=H(\lambda) \text{ and } \left\|H^{(n)}(\lambda)\right\|_{\mathscr{L}(X)}\leq Mn!\left(Re\lambda-\omega^V\right)^{-(n+1)} \text{ for all } n\in\mathbb{N}. \tag{9}$$

Let E and F be linear operators in a vector space Z with $D(F)\subset Z=D(E)$. We say that E commutes with F if:
(i) $z\in D(F)$ implies that $Ez\in D(F)$.
(ii) $\forall z\in D(F), EFz=FEz$.
In case $D(E)=Z$ this definition of commutativity coincides with the obvious one: $EF=FE$.

Note that for $K(t)=0$, Proposition 1 becomes the celebrated generation theorem for C_0-semigroups. Moreover, it is well-known that for CPs, sVIPs (5) and sVIDPs (4), the semigroups or the resolvent operators automatically commutes with A. For the nsVIDPs (4), we have the following result (see [27, Theorem 7]).

Proposition 2 *Let* $(\mathbf{H1})-(\mathbf{H2})$ *hold and suppose that* $(S(t))_{t\geq 0}$ *is a resolvent operator for (6) satisfying* $(\mathbf{S4})$. *Then* $S(t)$ *commutes with A on* $D(A)$ *for all* $t\geq 0$ *if and only if there is* $\mu\in\rho(A)$ *such that* $R(\mu,A)$ *commutes with* $(K(t),D(A))$ *for almost all* $t\geq 0$ *(resp. with* $S(t)$ *for* $t\geq 0$ *and* $\widehat{S}(\lambda)$ *for* $Re(\lambda)>\omega^V$*).*

Now since A is a closed operator, we may consider $(X_1,\|\cdot\|_A)$ the domain of A equipped with the graph-norm continuously embedded in X. If the resolvent set $\rho(A)$ of A is non empty, then $A_1: D(A^2)\to X_1$, with $A_1x=Ax$, is a closed operator in X_1 and $\rho(A_1)=\rho(A)$. Besides, we may consider X_{-1}^A the completion of X with respect to the norm: $\|x\|_{-1}=\left\|(\mu_0 I-A)^{-1}x\right\|_X$ for some $\mu_0\in\rho(A)$ and all $x\in X$. These spaces are independent of the choice of μ_0 and related by the following continuous and dense injections

$$X_1\overset{d}{\hookrightarrow}X\overset{d}{\hookrightarrow}X_{-1}^A.$$

Furthermore, the operator $A: X_1\to X$ is continuous and densely defined, its (unique) extension to X makes of it a closed operator in X_{-1}^A , and it is called A_{-1} and we have $\rho(A)=\rho(A_{-1})$. For more details (see. [56], [24]) for a construction.

As for the sVIPs (5) and sVIDPs (4), the following crucial result [63, Proposition 2.6], which is essentially based on the Proposition 2, presents conditions under which the resolvent operator $(S(t))_{t\geq 0}$ of (6) has an extension to X_{-1}^A which is itself a resolvent operator. These extensions will play a very vital role especially in the study of the admissibility of control operators in Section 4.

Proposition 3 *Let* $(\mathbf{H1})-(\mathbf{H2})$ *hold and suppose that* $(S(t))_{t\geq 0}$ *is a the resolvent operator for (6) satisfying* $(\mathbf{S4})$. *If* $S(t)$ *commutes with* A *for all* $t \geq 0$ *then the following assertions hold:*

(i) The operator $K(t)$ *admits an extension* $\overline{K}(t)$ *to* X_{-1}^A *satisfying:* $D(\overline{K}(t)) \supset X$ *for almost all* t *and for all* $x \in X$

$$\left\|\overline{K}(t)x\right\|_{-1} \leq b(t)\left(\|x\|_{-1} + \|A_{-1}x\|_{-1}\right), \tag{10}$$

(ii) $R(\mu, A_{-1})$ *commutes with* $(\overline{K}(t), X)$ *for some* $\mu \in \rho(A)$ *and for almost all* t.

(iii) The resolvent operator $(S(t))_{t\geq 0}$ *admits an extension to an operator* $(S_{-1}(t))_{t\geq 0}$ *on* X_{-1}^A *which inherits all properties of* $(S(t))_{t\geq 0}$.

(vi) $(S_{-1}(t))_{t\geq 0}$ *is exactly the resolvent operator of the following nsVIDP in* X_{-1}^A

$$\begin{cases} \dot{x}(t) = A_{-1}x(t) + \int_0^t \overline{K}(t-s)x(s)ds, \ t \geq 0, \\ x(0) = x \in X_{-1}^A. \end{cases} \quad . \tag{11}$$

Appealing now to Proposition 3(i), $\overline{K}(t)x$ is Laplace transformable for every $\mathrm{Re}\lambda > \beta$ and $x \in X$. So, by virtue of Propositions 1 and 3, for all $x \in X_{-1}^A$ and $Re(\lambda) > \omega^V$, the Laplace-transform $\widehat{S_{-1}}(\lambda)x$ exists and it is given by

$$\widehat{S_{-1}}(\lambda)x = \left(\lambda I - A_{-1} - \widehat{\overline{K}}(\lambda)\right)^{-1} x =: H_{-1}(\lambda)x, \tag{12}$$

3 An equivalent Cauchy problem: A Review

It is well-known that a class of nsVIDPs (6) with A generator of a C_0-semigroup on X and some appropriate kernel operators $K(\cdot)$ can be converted to some abstract CP on a product space (see. e.g. [24, VI.7]). This technique has been widely used (see. e.g. [13], [12], [22], [46], [3], [25], [16]) and will be applied in the next section. To that purpose, we are concerned by a class of nsVIDPs (4) where the operator kernel $K(\cdot) = k(\cdot)K$ and the kernel function $k(\cdot)$ lies in $W^{1,p}(\mathbb{R}^+)$. In the next section, we will see how can get rid of the condition imposed on k. Now, let us introduce the product space $\mathscr{X} := X \times L^p(\mathbb{R}^+; X)$ $1 \leq p < +\infty$, which is a Banach space with the norm

$$\left\|\begin{pmatrix} x \\ f \end{pmatrix}\right\|_{\mathscr{X}}^2 = \| x \|_X^2 + \| f \|_{L^p(\mathbb{R}^+,X)}^2, \ x \in X, f \in L^p(\mathbb{R}^+; X).$$

Next, we define on $\mathscr{X}$ the matrix operator

$$\mathscr{T}(t) := \begin{pmatrix} T(t) & F(t) \\ 0 & W(t) \end{pmatrix}, \ t \geq 0,$$

where $(W(t))_{t\geq 0}$, the left shift C_0-semigroup and $F(t)$ are defined on $L^p(\mathbb{R}^+; X)$ as follows

$$(W(t)f)(\tau) := f(t+\tau), \ \ \tau \geq 0, \ F(t)f := \int_0^t T(t-s)f(s)ds, \ f \in L^p(\mathbb{R}^+; X).$$

Then $(\mathscr{T}(t))_{t\geq 0}$ forms a C_0-semigroup on $\mathscr{X}$ with the generator given by

$$\mathscr{A} := \begin{pmatrix} A & \delta_0 \\ 0 & \dfrac{d}{ds} \end{pmatrix}, \ \ D(\mathscr{A}) := D(A) \times W^{1,p}(\mathbb{R}^+; X), \tag{13}$$

where $\frac{d}{ds}$ denotes the generator of the semigroup $(W(t))_{t\geq 0}$ with $D(\frac{d}{ds})$=$W^{1,p}(\mathbb{R}^+;X)$ the vector-valued Sobolev space and δ_0 the Dirac distribution, i.e $\delta_0(f)=f(0)$ for each $f\in W^{1,p}(\mathbb{R}^+;X)$. Finally, we define the time-varying operator $(\mathscr{K}(\cdot),D(A))$ as follows $\mathscr{K}(\cdot)x:=k(\cdot)Kx,\ x\in D(A)$, and

$$\mathscr{M}\begin{pmatrix} x \\ f \end{pmatrix} := \begin{pmatrix} 0 & 0 \\ \mathscr{K} & 0 \end{pmatrix}\begin{pmatrix} x \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ \mathscr{K}x \end{pmatrix}, \quad \begin{pmatrix} x \\ f \end{pmatrix} \in D(\mathscr{M}) := D(\mathscr{A}),$$

and

$$\mathscr{A}^V := \mathscr{A} + \mathscr{M} = \begin{pmatrix} A & \delta_0 \\ \mathscr{K} & \dfrac{d}{ds} \end{pmatrix}, \quad D(\mathscr{A}^V) := D(\mathscr{A}). \tag{14}$$

Now we rewrite formally the nsVIDP (6) as the following equivalent Cauchy problem:

$$\begin{cases} \dot{\mathscr{Z}}(t) = \mathscr{A}^V \mathscr{Z}(t),\ t\geq 0, \\ \mathscr{Z}(0) = \begin{pmatrix} x \\ f \end{pmatrix} \in \mathscr{X}. \end{cases}$$

According to [22, Theorems 2.1-3.1], the nsVIDP (4) admits a unique resolvent operator $(S(t))_{t\geq 0}$. In fact, it is shown that $\mathscr{A}^V$ generates the C_0-semigroup $(\mathscr{T}^V(t))$ on $\mathscr{X}$ and that $S(t)x$ is the first component of $\mathscr{T}^V(t)\begin{pmatrix} x \\ 0 \end{pmatrix}$. Furthermore, it is observed that, in [59, pp. 338-339], the semigroup $\mathscr{T}^V(t)$ has the following formal representation on $D(\mathscr{A})$ in terms of $S(t)$:

$$\mathscr{T}^V(t) = \begin{pmatrix} S(t) & [S*](t) \\ \int_0^t k(t-\tau+\cdot)KS(\tau)d\tau & W(t)+[R(\cdot)*](t) \end{pmatrix}\ t\geq 0, \tag{15}$$

where

$$[S*](t)(f) = (S*f)(t),\ R(r)(t) = \int_0^t k(r+t-\tau)KS(\tau)d\tau.$$

Observe that in this case, the resolvent operator $(S(t))_{t\geq 0}$ of (6) is exponentially bounded, this extends the result in [46] with $p=2$, which has been obtained by a direct method. Indeed, the semigroup $(\mathscr{T}^V(t))_{t\geq 0}$ is always exponentially bounded (see. e.g. [24, Proposition 5.5 p.39]), thus there exist constants $M\geq 1$ and $\gamma\in\mathbb{R}$ such that $\left\|\mathscr{T}^V(t)\right\|\leq Me^{\gamma t}$ implies in turn that $(S(t))_{t\geq 0}$ satisfies (**S4**) and $\omega_0(S)\leq \omega_0\left(\mathscr{A}^V\right)$ (the growth bound of $(\mathscr{T}^V(t))_{t\geq 0}$). Thus it is permissible to consider its Laplace transform. Combining this with Proposition 1, for all Re$\lambda > \max(\omega_0\left(\mathscr{A}^V\right), \omega^V)$ and $x_0\in X$, we obtain

$$\int_0^\infty e^{-\lambda t} S(t)x_0 dt = \left(\lambda I - A - \widehat{K}(\lambda)\right)^{-1} x_0.$$

Finally the following result [63], initially proved for the sVIDs [15], [25], will be very useful for the next section.

Lemma 1 *For Re$\lambda>0$ large enough, we have*

$$R(\lambda,\mathscr{A}^V) = \begin{pmatrix} H(\lambda) & H(\lambda)\delta_0 R(\lambda,\frac{d}{ds}) \\ R(\lambda,\frac{d}{ds})\mathscr{K}H(\lambda) & R(\lambda,\frac{d}{ds})\mathscr{K}H(\lambda)\delta_0 R(\lambda,\frac{d}{ds}) + R(\lambda,\frac{d}{ds}) \end{pmatrix},$$

where $R(\lambda,\frac{d}{ds})$ is the resolvent of $(W(t))_{t\geq 0}$.

4 L^p-admissibility and Input-Output Representation

First, we state in a precise way the new and weak hypotheses on the memory kernels which guarantee the existence of a unique solution to (4). Second, we will give the main results of this paper by exhibiting sufficient and necessary conditions for the control and observation operators for a class of nsVIDPs (1) to be L^p-admissible for all $p \in [1,+\infty)$. The idea is that for some special types of kernels k and K, the L^p-admissibility of control and observation operators for the nsVIDPs (1.1) may follow from that of the systems (18) and (20) respectively. Here, occasionally using the L^p-admissibility of observation operators for the semigroups, we will have the opportunity to reinvestigate the above Cauchy transformation of nsVIDPs (4) and in that case the assumption $k(\cdot) \in W^{1,p}(\mathbb{R}^+)$ no longer required. Finally, based on the L^p-regular systems theory, we aim at giving conditions which allow to derive the explicit expression relating the inputs directly to the outputs for the nsVIDPs (1).

As in Section 3, we shall assume that that A generates a C_0-semigroup $(T(t))_{t\geq 0}$ on X and $B \in \mathscr{L}(U, X_{-1}^A)$ and $K \in \mathscr{L}(X_1, X)$. Thereby, for K being an L^p-admissible observation operator for $(T(t))_{t\geq 0}$ (see. Definition 4), a weaker condition on $k(\cdot)$ is displayed and will show (see. Proposition 4) that (16) admits a resolvent operator $(S(t))_{t\geq 0}$ satisfying **(S4)** with only $k(\cdot) \in L^p(\mathbb{R}^+)$, in particular $K(\cdot) := k(\cdot)K$ satisfies $(\mathbf{H2})$ for all $\beta > 0$. For the extension of Lemma 1 to the case where $k(\cdot) \in L^p(\mathbb{R}^+)$, the reader is refereed to [63, Lemma 3.1]. In addition, if $R(\mu, A)$ commutes with K for some $\mu \in \rho(A)$ then according to Proposition 2, the resolvent operator $(S(t))_{t\geq 0}$ commutes with A, and by virtue of Proposition 3(vi) the unique extension of the resolvent operator $(S(t))_{t\geq 0}$ to X_{-1}^A denoted by $(S_{-1}(t))_{t\geq 0}$ exists and also commutes with A_{-1} and satisfies **(S4)**.

This motivates us to make the following assumptions:
$(\mathbf{H3})$: $k(\cdot) \in L^p(\mathbb{R}^+)$ and K is an L^p-admissible observation operator for $(T(t))_{t\geq 0}$,
$(\mathbf{H4})$: $R(\mu, A)$ commutes with K for all $\mu \in \rho(A)$.

Remark 1 Here, the sVIDPs (4) (i.e. $K = A$) with $p \in (1,\infty)$ are not concerned because A will never be finite-time L^p-admissible as observation for itself unless it is bounded, and this is of no interest for us. Generally, and in particular for A, the finite-time L^∞-admissible observation operators are necessarily bounded (see. [66, Proposition 6.5]). If $p = 1$ and the (unbounded) operator A generates an analytic semigroups, then sVIDPs (4) can be considered if and only if A has the finite-time L^1-maximal regularity, equivalently if A has finite-time L^1-admissibility for itself. This was proved in [49, Theorem 3.6] (without explicitly using the notion of admissibility) by rescaling the semigroup.

Now, if $(\mathbf{H3}) - (\mathbf{H4})$ hold and $f = Bu$ in (7) then from Section 2 the mild solution of

$$\begin{cases} \dot{x}(t) = Ax(t) + \int_0^t k(t-s)Kx(s)ds + Bu(t), \ t \geq 0, \\ x(0) = x_0 \in X, \end{cases} \tag{16}$$

is formally given by the following variation-of-constants formula

$$x(t) = S(t)x_0 + \int_0^t S_{-1}(t-s)Bu(s)ds, \tag{17}$$

which is actually the classical solution if $B \in \mathscr{L}(U,X), x_0 \in D(A)$ and u sufficiently smooth (see. [27, Theorem 3]).

In many practical examples the control operator B is unbounded, hence (16)-(17) are viewed on the extrapolation space X_{-1}^A. So, an additional assumption on B will be needed to ensure that the solution given by (17) takes value in the state space X for every $t \geq 0$, $x_0 \in X$ and every $u \in L^p(\mathbb{R}^+;U)$ or $L^p_{loc}(\mathbb{R}^+;U)$. Therefore, in the same spirit of CPs (i.e. $k(\cdot) = 0$) and of sVIDPs (4) (see [65,25]), we need the following definitions generalized in a most natural way from the linear control system,

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \ t > 0, \\ x(0) = x_0. \end{cases} \tag{18}$$

Definition 3 Let $(\mathbf{H3}) - (\mathbf{H4})$ hold and $p \in [1, +\infty)$. The control operator B is called

(i) (infinite-time) L^p-admissible operator for $(S(t))_{t \geq 0}$, if there exists a constant $M > 0$ such that

$$\left\| \Phi_t^{S,B} u := (S_{-1} * Bu)(t) \right\|_X \leq M \|u\|_{L^p(\mathbb{R}^+;U)} \text{ for all } u \in L^p\left(\mathbb{R}^+;U\right) \text{ and } t > 0.$$

(ii) L^p-admissible operator for $(S(t))_{t\geq 0}$ in finite-time $t_0 > 0$ if there exists a constant $M_{t_0} > 0$ such that: $\left\| \Phi_{t_0}^{S,B} u \right\|_X \leq M_{t_0} \|u\|_{L^p([0,t_0];U)}$ for all $u \in L^p([0,t_0];U)$.
An operator B is called finite-time L^p-admissible operator for $(S(t))_{t\geq 0}$ if it is L^p-admissible in some time $t_0 \geq 0$, and it is called uniformly L^p-admissible operator for $(S(t))_{t\geq 0}$ if it is L^p-admissible in time t for all $t \geq 0$.

Note that, for the sVIDPs (4), the definition of L^p-admissible control operator for $(S(t))_{t\geq 0}$ was introduced in [45] for $p = 2$ and it implies the finite-time L^2-admissibility condition considered in [48]. Our definitions of finite and infinite-time L^p-admissible control operators for $(S(t))_{t\geq 0}$ correspond to that of the semigroups and resolvent operators (see [65, 10]), also imply that of [48] when $p = 2$. It is obvious that L^p-admissibility is a substantial concept and B is L^p-admissible in any time $t > 0$ if it is infinite-time L^p-admissible. But, it is well-known that the property $(\mathbf{P1})$: the finite-time L^p-admissibility and the uniform L^p-admissibility are equivalent for the semigroups [66] while it is not clear (due to the lack of semigroup property) whether this remains true for resolvent operators. Moreover, it is well-known that the property $(\mathbf{P2})$: the finite-time and the infinite-time L^p-admissibility are equivalent for exponentially stable semigroups for all $p \in [1, \infty[$ (see [64], [26]). Occasionally, it should be observed that the real difficulty in studding the L^p-admissibility in particular, for nsVIDPs (1) is associated with the properties $(\mathbf{P1}) - (\mathbf{P2})$, since it is essential if we want to talk about the X-valued solution of (16) (see (17)). The properties **(P1)-(P2)** remain open to our knowledge, and it is natural to ask whether these properties hold for the resolvent operators. In particular, **(P2)** is more delicate due to the lack of semigroup property. However, the authors in [46] gave a partial negative to **(P2)** by exhibiting an example of finite dimensional sVIDP (4) whose underlying semigroup is exponentially stable (see [46, Example 5.1]). Surprisingly, in [46, Example 5.1], (see [14]) the resolvent operator is not exponentially stable, and thereby their example is not genuinely a counterexample, and the property **(P2)** still open to our knowledge. In [10], the properties **(P1)-(P2)** were been partially verified, when $p = 1$, for a class of sVIPs (5) with some *creep* kernels function $a(\cdot)$ which may contains a large class of the sVIDPs (4). We note that sVIPs (5) with kernel $a(t) = 1 + \dfrac{\alpha}{\beta}(1 - e^{-\beta t}), \alpha, \beta > 0$, which is a *creep* function, corresponding to the sVIDPs (4) with the exponential kernel $k(t) = \alpha e^{-\beta t}$, forms an important class of memory kernels.

In Theorem 3 below, sufficient conditions for the L^p-admissibility of B will be presented for a class of nsVIDPs (4). This, in particular shows that the property **(P1)** takes place for such class.

Of course, the nsVIDPs (1) controlled with L^p-admissible control operators B are exactly (16) coupled with the following observation equation:

$$y(t,x_0,u) = Cx(t) \;\; t \geq 0, \tag{19}$$

where the X-valued function $x(t)$ is given by (17) and $C \in \mathscr{L}(X_1,Y)$ is called the observation operator where Y is another Banach space.

For bounded (resp. unbounded) operator C, the output function $y(\cdot,x_0,u)$ is well defined for all $x_0 \in X$ and $u(\cdot) \in L^p(\mathbb{R}^+,U)$ (resp. $x_0 \in D(A)$ and u sufficiently smooth [58]). For unbounded operator C, to guarantee that $y(\cdot,x_0,u)$ and in particular $y(\cdot,x_0,0)$ of (16) makes sense and it is L^p-integrable on the right half-line $\mathbb{R}^+$ or on some finite interval $[0,\tau]$, the following definitions generalized in a most natural way from that of the semigroup i.e. $k(t) \equiv 0$ (see [66]) will be needed,

$$\begin{cases} \dot{x}(t) = Ax(t), \;\; t > 0 \\ y(t) = Cx(t), x(0) = x_0. \end{cases} \tag{20}$$

Before going further, as for the control operators, we introduce the L^p-admissibility version of observation operators for (16)-(19).

Definition 4 Let (**H3**) holds and $p \in [1,+\infty)$. The observation operator C is called

(i) (infinite-time) L^p-admissible operator for $(S(t))_{t\geq 0}$, if there exists a constant $M > 0$ such that

$$\int_0^{+\infty} \|CS(s)x\|_Y^p ds \leq M^p \|x\|_X^p \text{ for all } x \in D(A).$$

(ii) L^p-admissible operator for $(S(t))_{t\geq 0}$ in finite-time if $\tau_0 > 0$ if there exists a constant $M_{\tau_0} > 0$ such that:

$$\int_0^{\tau_0} \|CS(s)x\|_Y^p ds \leq M_{\tau_0}^p \|x\|_X^p \text{ for all } x \in D(A).$$

An operator C is finite-time L^p-admissible operator for $(S(t))_{t\geq 0}$ if it is L^p-admissible for some $\tau_0 > 0$, and it is called uniformly L^p-admissible operator for $(S(t))_{t\geq 0}$ if it is L^p-admissible in time t for all $t \geq 0$.

Note that L^p-admissibility of C in time τ_0 guarantees that the mapping $\mathbb{L} : X_1 \to L^p(0,\tau_0;Y)$, given by

$$(\mathbb{L}x_0)(t) := CS(t)x_0 \;\; x_0 \in D(A) \;\text{ and }\; t \in [0,\tau_0],$$

possesses a unique extension (again denoted by $\mathbb{L}$) to a linear bounded operator from X to $L^p(0,\tau_0;Y)$. In this case the output equation in (19) can be interpreted as

$$y(t) = (\mathbb{L}x_0)(t), \;\; t \in [0,\tau_0],$$

which makes sense for every $x_0 \in X$. In Lemma 6, we will give a representation of $\mathbb{L}$ similar to that of the systems (20) given in [62].

As for the control operators, it is well-known that the finite-time L^p-admissibility of observation operator for the semigroups does not depend on time and that the finite-time and the infinite-time L^p-admissibility are equivalent for exponentially stable semigroups, while it is not clear whether these remain true for the resolvent operators. In Proposition 5, sufficient conditions will be presented for the L^p-admissibility of observation operators, in particular the properties **(P1)** and **(P2)**, for the class of nsVIDPs under consideration are true.

Note that our definition of L^p-admissible observation operators for (4) is different from the one used in [46]. In fact, in [46], an operator C is called finite-time L^p-admissible for $(S(t))_{t\geq 0}$ if and only if there exist constant M and $\omega \in \mathbb{R}$ such that

$$\int_0^t \|CS(s)x_0\|_Y^p ds \leq M^p e^{\omega t}\|x_0\|_X^p \text{ for all } x_0 \in D(A), \;\; t \geq 0.$$

Clearly, our definition of the finite-time L^p-admissibility of C is weaker than of [46], but the converse implication is true. More, the above estimation looks stronger than the corresponding one for nsVIDP (16), i.e. Definition 4(ii). However this is not the case at least for the class under consideration, and it will be proved in Proposition 5(iii). Moreover, in order to guarantee that the controlled output function $y(\cdot,x_0,u)$ lies locally in $L^p(\mathbb{R}^+,Y)$ when the (non null) input $u(\cdot)$ lies locally in $L^p(\mathbb{R}^+,U)$, we make heavy use of the notion of L^p-regular linear systems. Of course, by setting $k(t)\equiv 0$ in (16) the resolvent $(S(t))_{t\geq 0}$ coincides with $(T(t))_{t\geq 0}$ and that the notion of L^p-admissibility of both control and observation operators for the semigroup $(T(t))_{t\geq 0}$ comes simply from Definitions 3 and 4 respectively.

We now are ready to recall in a very sketchy way, the machinery of L^p-regular linear systems in the sense of Salamon-Weiss [60], [66], which will be used as the main tool for our results.

Definition 5 The operator triple $((T(t))_{t\geq 0},B,C)$ is L^p-well-posed if the following assertions hold:

(i) B and C are finite-time L^p-admissible control and observation operators for $(T(t))_{t\geq 0}$,
(ii) there exists a bounded linear (the so-called extended input-output) operator $\mathbb{F}^{T,B,C}$ from $L^p([0,\tau];U)$ to $L^p([0,\tau];Y))$ for any $\tau>0$ which verifies some well-known composition property.
(iii) A L^p-well-posed triple $((T(t))_{t\geq 0},B,C)$ is called L^p-regular (with feedthrough zero) if the limit $\lim\limits_{\tau\to 0}\dfrac{1}{\tau}\displaystyle\int_0^\tau(\mathbb{F}^{T,B,C}u_z)(\sigma)d\sigma = 0$, where $u_z(s)=z$ for $s\geq 0$ and $z\in U$.

In order to recall the representation theorem of the regular linear systems, we recall the Λ-extension of C with respect to A introduced in [69],(see also [41]) defined as follow

$$\begin{aligned} D(C_\Lambda^A) &:= \{x\in X : \lim_{s\to+\infty} sCR(s,A)x \text{ exists in } Y\},\\ C_\Lambda^A x &:= \lim_{s\to+\infty} sCR(s,A)x, \end{aligned} \tag{21}$$

with the Λ-extension, for the L^p-well-posed triple $((T(t))_{t\geq 0},B,C)$, the L^p-regularity condition is equivalent to $\text{Range}(\lambda - A_{-1})^{-1}B \subset D(C_\Lambda^A)$ for some (and hence for all) $\lambda\in\rho(A)$ (see [69]).

Regular linear systems have a nice property that is for a L^p-regular triple $((T(t))_{t\geq 0},B,C)$, its associated extended input-output operator $\mathbb{F}^{T,B,C}$ has the following representation:

$$(\mathbb{F}^{T,B,C}u)(t) = C_\Lambda^A \Phi_t^{T,B}u,$$

for any $u\in L^p([0,\tau],U)$ and almost every $t\in[0,\tau]$. Moreover, it is well-known that if the operator C (resp. B) is bounded then the triple $((T(t))_{t\geq 0},B,C)$ is regular if and only if B (resp. C) is L^p-admissible for $(T(t))_{t\geq 0}$, and for regular linear systems there is a way to represent them in input-output form. From now on, we will frequently use the above representation for operator $\mathbb{F}^{T,B,C}$. Analogously, we say that the triple $((S(t))_{t\geq 0},B,C)$ is

L^p-regular if both B and C are uniformly L^p-admissible for $(S(t))_{t\geq 0}$, $\Phi_t^{S,B}u \in D(C_\Lambda^A)$ for t-a.e. and if there exist $\tau > 0$ and $\kappa_\tau > 0$ such that

$$\begin{cases} \forall u \in L^p(\mathbb{R}^+;U), \ \left\|(\mathbb{F}^{S,B,C}u)(\cdot) := C_\Lambda^A \Phi_\cdot^{S,B} u\right\|_{L^p([0,\tau],U)} \leq \kappa_\tau \|u\|_{L^p([0,\tau],U)}, \\ \lim\limits_{t\to 0} \dfrac{1}{t} \displaystyle\int_0^t (\mathbb{F}^{S,B,C}u_z)(\sigma)d\sigma = 0. \end{cases}$$

We call $\mathbb{F}^{S,B,C}$ the extended input-output operator associated to the triple $((S(t))_{t\geq 0}, B, C)$.

Now, we are ready to announce our main result. It is an analogue of [69, Theorem 2.3] for the nsVIDPs (1)

Theorem 2 *Let* (**H3**) $-$ (**H4**) *hold. Assume that* $((T(t))_{t\geq 0}, B, K)$ *is an* L^p*- regular linear system.*

(i) *For any* $x_0 \in X, u(\cdot) \in L^p_{loc}(\mathbb{R}^+;U)$*, there is a unique function* $x(\cdot) \in C(\mathbb{R}^+, X)$ *solution of* (16) *given by* (17).
If in addition $p \geq 2$ *and* $((T(t))_{t\geq 0}, B, C)$ *is* L^p*-regular, then* $x(t) \in D(C_\Lambda^A)$ *for almost all* t *and* $C_\Lambda^A x(\cdot) \in L^p_{loc}(\mathbb{R}^+;Y)$*, and*

$$\Sigma \begin{cases} \dot{x}(t) = A_{-1}x(t) + \displaystyle\int_0^t k(t-s)\overline{K}x(s)ds + Bu(t) \quad (in\ X_{-1}^A), \\ y(t) = C_\Lambda^A x(t) \qquad t-a.e., \\ x(0) = x_0. \end{cases}$$

In particular $((S(t))_{t\geq 0}, B, C)$ *is* L^p*-regular and for all* $T > 0$*,* $x_0 \in X$ *and* $u(\cdot) \in L^p([0,T];U)$*, there exists* $C_T > 0$ *such that*

$$\|x(T)\|_X^p + \|y\|^p_{L^p([0,T];Y)} \leq C_T\left(\|x_0\|_X^p + \|u\|^p_{L^p([0,T];U)}\right).$$

(ii) *The transfer function* G *of* Σ *is given for* $s \in \mathbb{C}$ *(with* $Re(s)$ *great enough) by*

$$G(s) = C_\Lambda^A\left(sI - A_{-1} - \widehat{k}(s)\overline{K}\right)^{-1}B,$$

where $\overline{K}$ *is the extension of* K *given by Proposition 3 (i).*

For this, some preparations are needed. First, let us consider the operator $\mathscr{B} := (B,0)^T$ and observe that the operator $\mathscr{A}^V$ given in (14) can be rewritten in the form:

$$\mathscr{A}^V = \mathscr{A}_0 + \mathscr{P}, \mathscr{A}_0 = \begin{pmatrix} A & 0 \\ 0 & \dfrac{d}{ds} \end{pmatrix}, \text{ and } \mathscr{P} = \begin{pmatrix} 0 & \delta_0 \\ \mathscr{K} & 0 \end{pmatrix},$$

with $D(\mathscr{P}) = D(\mathscr{A}_0) = D(A) \times W^{1,p}(\mathbb{R}^+;X)$. The fact that $\mathscr{A}_0$ is diagonal with diagonal domain, it is a generator of the semigroup $(\mathscr{T}_0(t))_{t\geq 0}$ given by

$$\mathscr{T}_0(t) = \begin{pmatrix} T(t) & 0 \\ 0 & W(t) \end{pmatrix},$$

and it is immediate to see that $\mathscr{X}_{-1}^{\mathscr{A}_0} = X_{-1}^A \times (L^p(\mathbb{R}^+;X))_{-1}^{d/ds}$ and the extrapolated semigroup of $(\mathscr{T}_0(t))_{t\geq 0}$ is given by

$$\mathscr{T}_{0,-1}(t) = \begin{pmatrix} T_{-1}(t) & 0 \\ 0 & W_{-1}(t) \end{pmatrix}.$$

We will prove in Proposition 4 that under (**H3**) the operator $(\mathscr{A}^V, D(\mathscr{A}_0))$ is a generator of a C_0-semigroup on $\mathscr{X}$ which will be denoted from now $(\widetilde{\mathscr{T}}^V(t))_{t\geq 0}$, in particular the nsVIDP (4) admits a resolvent operators.

The fact that Range$(\mathscr{B}) \subset \mathscr{X}_{-1}^{\mathscr{A}_0}$, so $\mathscr{B}$ can be regarded as a unbounded control operator associated to $\mathscr{A}_0$. However, it is not easy to see that $\mathscr{B}$ is a control operator associated to $\mathscr{A}^V$, that is Range$(\mathscr{B}) \subset \mathscr{X}_{-1}^{\mathscr{A}^V}$. In order to bypass this difficulty we will use the notion of L^p-regular triple recalled above.

Before proving our main result, let us record the following lemma that we will use several times.

Lemma 2 (i) *If B is finite-time L^p-admissible control for $(T(t))_{t\geq 0}$ (resp. (**H3**) holds) then $\mathscr{B}$ and $\mathscr{P}$ is finite-time L^p-admissible control (resp. observation) operator for $(\mathscr{T}_0(t))_{t\geq 0}$. In particular the operator $\mathscr{A}^V$ generates a C_0-semigroup on $\mathscr{X}$ denoted by $(\widetilde{\mathscr{T}}^V(t))_{t\geq 0}$. Moreover, the Λ-extension of $\mathscr{P}$ with respect to $\mathscr{A}_0$ satisfies $D(K_\Lambda^A) \times W^{1,p}(\mathbb{R}^+, X) \subset D(\mathscr{P}_\Lambda^{\mathscr{A}_0})$ and*

$$\mathscr{P}_\Lambda^{\mathscr{A}_0} = \begin{pmatrix} 0 & \delta_0 \\ k(\cdot)K_\Lambda^A & 0 \end{pmatrix} \quad \text{on } D(K_\Lambda^A) \times W^{1,p}(\mathbb{R}^+, X).$$

*Let (**H3**) holds.*

(ii) *If $((T(t))_{t\geq 0}, B, K)$ is L^p-regular, then $((\mathscr{T}_0(t))_{t\geq 0}, \mathscr{B}, \mathscr{P})$ is so. In particular $\mathscr{B}$ is finite-time L^p-admissible control operator for $(\widetilde{\mathscr{T}}^V(t))_{t\geq 0}$.*

(iii) *If $((T(t))_{t\geq 0}, B, K)$ is L^p-regular, then $(\widetilde{\mathscr{T}}^V(t))_{t\geq 0}, \mathscr{B}, \mathscr{P})$ is so.*

Proof (i) The L^p-admissibility of both $\mathscr{B}$ and $\mathscr{P}$ is obvious due to the simple expression of $(\mathscr{T}_0(t))_{t\geq 0}$. The fact that $\mathscr{P}$ is L^p-admissible for $(\mathscr{T}_0(t))_{t\geq 0}$, is well-known that the operator $\mathscr{A}^V$ generates a C_0-semigroup on $\mathscr{X}$ denoted by $(\widetilde{\mathscr{T}}^V(t))_{t\geq 0}$ (see [24],[37]). Finally, a simple computation gives the required characterization of $\mathscr{P}_\Lambda^{\mathscr{A}_0}$.

(ii) Assume that the triple $((T(t))_{t\geq 0}, B, K)$ is L^p-regular, in particular the statement (i) guarantees that both $\mathscr{B}$ and $\mathscr{P}$ are finite-time L^p-admissible for $(\mathscr{T}_0(t))_{t\geq 0}$. Therefore, it remains only to verify that for all $u \in L^p(\mathbb{R}^+, U)$, $\Phi_t^{\mathscr{T}_0,\mathscr{B}} u \in D(\mathscr{P}_\Lambda^{\mathscr{A}_0})$ for almost all t. Now, the fact that $((T(t))_{t\geq 0}, B, K)$ is L^p-regular, we have $\Phi_t^{T,B} u \in D(K_\Lambda^A)$ for a.e $t \geq 0$. This implies in turn that $\Phi_t^{\mathscr{T}_0,\mathscr{B}} u \in D(\mathscr{P}_\Lambda^{\mathscr{A}_0})$ for all $u \in L^p(\mathbb{R}^+, U)$ and for almost all t and that $\left\| \mathscr{P}_\Lambda^{\mathscr{A}_0} \Phi_\cdot^{\mathscr{T}_0,\mathscr{B}} u \right\|_{L^p([0,\tau_0];\mathscr{X})} \leq \|u\|_{L^p([0,\tau_0];U)}$ due to the expressions of $\Phi_t^{\mathscr{T}_0,\mathscr{B}} u$ and $\mathscr{P}_\Lambda^{\mathscr{A}_0}$ together with $\mathscr{P}_\Lambda^{\mathscr{A}_0} = \mathscr{P}_\Lambda^{\mathscr{A}^V}$ on $D(\mathscr{P}_\Lambda^{\mathscr{A}_0})$ (see [38, Theorem 3.2], [53, Corollary 2.11]) respectively. Now thanks to [36, Theorem 3.1], we deduce that $\mathscr{B} \in \mathscr{L}(U, \mathscr{X}_{-1}^{\mathscr{A}^V})$ and that $\mathscr{B}$ is finite-time L^p-admissible for $(\widetilde{\mathscr{T}}^V(t))_{t\geq 0}$.

(iii) By the statement (ii), $\mathscr{P}$ is finite-time L^p-admissible for $(\mathscr{T}_0(t))_{t\geq 0}$ which in turn implies that $\mathscr{P}$ is finite-time L^p-admissible for $(\widetilde{\mathscr{T}}^V(t))_{t\geq 0}$ (see [38, Theorem 3.1 (i)]). Now, let us define

$$Z(t) = \int_0^t \widetilde{\mathscr{T}}_{-1}^V(t-s)\mathscr{B}u(s)ds. \tag{22}$$

From above, we have $Z(s) \in D(\mathscr{P}_\Lambda^{\mathscr{A}_0})$ for almost all t and Z satisfies with $x_0 = 0$

$$Z(t) = \int_0^t \mathscr{T}_0(t-s)\mathscr{P}_\Lambda^{\mathscr{A}_0} Z(s)ds + \int_0^t \mathscr{T}_{0,-1}(t-s)\mathscr{B}u(s)ds. \tag{23}$$

Thus, to prove the L^p-regularity of $((\mathscr{T}^V(t))_{t\geq 0}, \mathscr{B}, \mathscr{P})$ it suffices only to show that

$$\forall t \geq 0, \exists \kappa_t > 0 \; \|\mathscr{P}_\Lambda^{\mathscr{A}_0} Z\|_{L^p([0,t],\mathscr{X})} \leq \kappa_t \|u\|_{L^p([0,t],U)}, \forall u \in L^p(0,t;U).$$

Applying $\mathscr{P}_\Lambda^{\mathscr{A}_0}$ to both side of the identity (23), we get

$$\mathscr{P}_\Lambda^{\mathscr{A}_0} Z(t) = \mathscr{P}_\Lambda^{\mathscr{A}_0} \int_0^t \mathscr{T}_0(t-s)\mathscr{P}_\Lambda^{\mathscr{A}_0} Z(s)ds + \mathscr{P}_\Lambda^{\mathscr{A}_0} \int_0^t \mathscr{T}_{0,-1}(t-s)\mathscr{B}u(s)ds.$$

Equivalently, $(I - \mathbb{F}^{\mathscr{T}_0,I,\mathscr{P}})(\mathscr{P}_\Lambda^{\mathscr{A}_0} Z)(t) = (\mathbb{F}^{\mathscr{T}_0,\mathscr{B},\mathscr{P}} u)(t)$. The fact that $\mathscr{P}$ is finite-time L^p-admissible for $(\mathscr{T}^V(t))_{t\geq 0}$, it is well-know that the triple $((\mathscr{T}_0(t))_{t\geq 0}, I, \mathscr{P})$ is always L^p-regular and the operator $(I - \mathbb{F}^{\mathscr{T}_0,I,\mathscr{P}}) : L^p(0,t;U) \to L^p(0,t;\mathscr{X})$ is boundedly invertible. This trivially implies that $\mathscr{P}_\Lambda^{\mathscr{A}_0} Z(t) = (I - \mathbb{F}^{\mathscr{T}_0,I,\mathscr{P}})^{-1}(\mathbb{F}^{\mathscr{T}_0,\mathscr{B},\mathscr{P}} u)(t)$, and the L^p- regularity of $((\widetilde{\mathscr{T}}^V(t))_{t\geq 0}, \mathscr{B}, \mathscr{P})$ is trivially deduced from that of $((\mathscr{T}_0(t))_{t\geq 0}, \mathscr{B}, \mathscr{P})$.

As promised, we present the following result.

Proposition 4 *Under* (**H3**) *the nsVIDP* (4) *has a unique resolvent operator satisfying* (**S4**).

Proof Existence. From Lemma 2(i), the operator $\mathscr{A}^V$ generates the C_0-semigroup $(\widetilde{\mathscr{T}}^V(t))_{t\geq 0}$ on $\mathscr{X}$. Thanks to [37, Theorem 2.1], $\widetilde{\mathscr{T}}^V(s)z_0 \in D(\mathscr{P}_\Lambda^{\mathscr{A}_0})$ for all $z_0 \in \mathscr{X}$ and $s-$a.e. and we have

$$\widetilde{\mathscr{T}}^V(t)z_0 = \mathscr{T}_0(t)z_0 + \int_0^t \mathscr{T}_0(t-s)\mathscr{P}_\Lambda^{\mathscr{A}_0}\widetilde{\mathscr{T}}^V(s)z_0 ds, \tag{24}$$

$$\widetilde{\mathscr{T}}^V(t)z_0 = \mathscr{T}_0(t)z_0 + \int_0^t \widetilde{\mathscr{T}}^V(t-s)\mathscr{P}_\Lambda^{\mathscr{A}_0}\mathscr{T}_0(s)z_0 ds. \tag{25}$$

The semigroup $(\widetilde{\mathscr{T}}^V(t))_{t\geq 0}$ can be written $\widetilde{\mathscr{T}}^V(t) = \begin{pmatrix} \widetilde{\mathscr{T}}_{11}^V(t) & \widetilde{\mathscr{T}}_{12}^V(t) \\ \widetilde{\mathscr{T}}_{21}^V(t) & \widetilde{\mathscr{T}}_{22}^V(t) \end{pmatrix}$. Hence, for $x_0 \in D(A)$, identity (4.9) with $z_0 = \begin{pmatrix} x_0 \\ 0 \end{pmatrix}$ yields

$$\widetilde{\mathscr{T}}_{11}^V(t)x_0 = T(t)x_0 + \int_0^t T(t-s)\delta_0 \widetilde{\mathscr{T}}_{21}^V(s)x_0 ds, \tag{26}$$

$$\widetilde{\mathscr{T}}_{21}^V(t)x_0 = \int_0^t W(t-s)k(\cdot)K\widetilde{\mathscr{T}}_{11}^V(s)x_0 ds. \tag{27}$$

Thus, we deduce that

$$(\widetilde{\mathscr{T}}_{21}^V(t)x_0)(0) = \int_0^t k(t-s)K\widetilde{\mathscr{T}}_{11}^V(s)x_0 ds.$$

Using (4.10), we get

$$\widetilde{\mathscr{T}}_{11}^V(t)x_0 = T(t)x_0 + \int_0^t T(t-s)\int_0^s k(s-\sigma)K\widetilde{\mathscr{T}}_{11}^V(\sigma)x_0 d\sigma ds.$$

Hence, the differentiation of the above equality shows that $(\widetilde{\mathscr{T}}_{11}^V(t))_{t\geq 0}$ satisfies the first resolvent equation in (S3). Now, due to the fact that $\mathscr{A}^V$ generates the C_0-semigroup $(\widetilde{\mathscr{T}}^V(t))_{t\geq 0}$ we obtain for $x_0 \in D(A)$

$$\widetilde{\mathscr{T}}^V(t)\begin{pmatrix} x_0 \\ 0 \end{pmatrix} = \begin{pmatrix} x_0 \\ 0 \end{pmatrix} + \int_0^t \widetilde{\mathscr{T}}^V(s)\begin{pmatrix} A & \delta_0 \\ \mathscr{K} & d/ds \end{pmatrix}\begin{pmatrix} x_0 \\ 0 \end{pmatrix} ds.$$

This implies

$$\begin{aligned}\widetilde{\mathscr{T}}_{11}^{V}(t)x_0 &= x_0 + \int_0^t \widetilde{\mathscr{T}}_{11}^{V}(s)Ax_0 ds + \int_0^t (\widetilde{\mathscr{T}}_{12}^{V}\mathscr{K})(s)x_0 ds \\ &= x_0 + \int_0^t \widetilde{\mathscr{T}}_{11}^{V}(s)Ax_0 ds + \int_0^t \int_0^s (\widetilde{\mathscr{T}}_{11}^{V}(s-\sigma)k(\sigma)K)x_0 d\sigma ds.\end{aligned}$$

Differentiating now the above identity, we deduce that $\widetilde{\mathscr{T}}_{11}^{V}(t)$ satisfies the second resolvent equation in **(S3)**. Moreover, some well-known properties of the semigroups, trivially imply **(S1), (S2)** and **(S4)** for $(\widetilde{\mathscr{T}}_{11}^{V}(t))_{t\geq 0}$. Consequently, $(S(t) := \widetilde{\mathscr{T}}_{11}^{V}(t))_{t\geq 0}$ is exactly the resolvent operator associated to the uncontrolled system (16). Uniqueness follows from [27, Theorem 1].

Now, due to the Lemma 2, the controlled nsVIDP (16) can be embedded into linear control one on the larger product space $\mathscr{X}$. The idea is that the L^p-admissibility of control operators of (16) can be inherited from that of the controlled CP (18) and the obtained main result is presented in Theorem 5 below. In Hilbert setting, this embedding technique has been used in [46, section 3] to study finite-time L^2-admissibility of finite-rank observation (hence for finite-rank control by duality) operators of sVIDPs (4) with $k(\cdot) \in W^{1,2}(\mathbb{R}^+)$ in Hilbert setting. Likewise, for the sVIPs (5) with some specific kernel functions $a(\cdot)$, the authors in [45] linked infinite-time L^2-admissibility of observation with the infinite-time L^2-admissibility for controlled CP (18). But, Jung [48] was the first who studied the notion of L^2-admissibility of control operators for a class of sVIPs (5), and in particular presented some results linking his (strong) notion of finite-time L^2-admissibility with finite-time L^2-admissibility for the controlled CP (18).

The following theorem, which is interesting for itself, is also of such type. The first reaction of the reader to this results might be to think that this is obvious, but this is not the case contrary to the observation problem.

Theorem 3 *Let* $(\mathbf{H3})-(\mathbf{H4})$ *hold. If* $((T(t))_{t\geq 0}, B, K)$ *is* L^p*-regular then the unique mild solution of controlled nsVIDP (16) is given by the first coordinate of the solution of the linear control system*

$$\begin{cases}\dot{\mathscr{Z}}(t) = \mathscr{A}_{-1}^{V}\mathscr{Z}(t) + \mathscr{B}u(t), \ t > 0,, \\ \mathscr{Z}(0) = (x_0, 0) \in \mathscr{X}.\end{cases} \tag{28}$$

In particular, B is uniformly L^p*-admissible for* $(S(t))_{t\geq 0}$.

Remark 2 For sVIDEs with $k(\cdot) \in W^{1,1}(\mathbb{R}^+, X)$ such that $a = 1 + 1 * k$ is a *creep* function and finite-time L^1- admissible control operator B for $(T(t))_{t\geq 0}$, the finite-time L^1-admissibility of A for $(T(t))_{t\geq 0}$, and hence the L^1-regularity of $((T(t))_{t\geq 0}, B, A)$ are not necessary to conclude that B is finite-time L^1-admissible for $(S(t))_{t\geq 0}$. Indeed, first in [9] it was proved that a control operator B is finite-time L^1-admissible for $(T(t))_{t\geq 0}$ if and only if $Range(B) \subset F_{-1}(T)$ (equivalently $B \in \mathscr{L}(U, F_{-1}(T))$ where $F_{-1}(T)$ is the extrapolated Favard class of $(T_{-1}(t))_{t\geq 0}$. Second, in [10] the authors introduced the Favard class $F(S)$ associated to $(S(t))_{t\geq 0}$ (see [10, Definition 4.1]) and it is clear that $F_{-1}(T) \subset F_{-1}(S)$ and by virtue of [10, Corollary 5.4] B is finite-time L^1- admissible for $(S(t))_{t\geq 0}$.

Proof (of Theorem 3) Assume that $((T(t))_{t\geq 0}, B, K)$ is L^p-regular, according to Lemma 2(ii), the control operator $\mathscr{B}$ is finite-time L^p-admissible for $(\widetilde{\mathscr{T}}^V(t))_{t\geq 0}$. Consequently, the controlled CP (28) admits a unique $\mathscr{X}$-valued mild solution given by

$$\mathscr{X}(t) = \widetilde{\mathscr{T}}^V(t)Z_0 + \int_0^t \widetilde{\mathscr{T}}^V_{-1}(t-s)\mathscr{B}u(s)ds. \tag{29}$$

By making a straightforward computation, $\mathscr{X}(\cdot)$ also satisfies the following variation of parameters formula

$$\mathscr{X}(t) = \mathscr{T}_0(t)\mathscr{X}_0 + \int_0^t \mathscr{T}_0(t-s)\mathscr{P}^{\mathscr{A}_0}_\Lambda \mathscr{X}(s)ds + \int_0^t \mathscr{T}_{0,-1}(t-s)\mathscr{B}u(s)ds. \tag{30}$$

Now, let $\mathscr{X}(t) = (x(t), g(t))^T$ be the above solution. Take $\lambda > 0$ larger enough and $u \in L^p_\lambda(\mathbb{R}^+;U) := \{v \in L^p_{loc}(\mathbb{R}^+;U) / \int_0^\infty e^{-p\lambda t}\|v(t)\|^p_U < \infty\}$, then according to [69] $\mathscr{X} \in L^p_\lambda(\mathbb{R}^+;\mathscr{X})$. Hence, applying the Laplace transform to both sides of the identities (29) and (30) respectively, we get the two following expressions of $(\widehat{x}(\lambda), \widehat{g}(\lambda))^T$

$$\begin{pmatrix}\widehat{x}(\lambda)\\ \widehat{g}(\lambda)\end{pmatrix} = \begin{cases} \begin{pmatrix}R(\lambda)x_0\\ 0\end{pmatrix} + \begin{pmatrix}R(\lambda,A) & 0\\ 0 & R(\lambda,\frac{d}{ds})\end{pmatrix}\begin{pmatrix}0 & \delta_0\\ k(\cdot)K^A_\Lambda & 0\end{pmatrix}\begin{pmatrix}\widehat{x}(\lambda)\\ \widehat{g}(\lambda)\end{pmatrix} \\ \qquad + \begin{pmatrix}R(\lambda,A_{-1}) & 0\\ 0 & R(\lambda,(\frac{d}{ds})_{-1})\end{pmatrix}\begin{pmatrix}B\widehat{u}(\lambda)\\ 0\end{pmatrix} \\ R(\lambda,\mathscr{A}^V)\begin{pmatrix}x_0\\ 0\end{pmatrix} + R(\lambda,\mathscr{A}^V_{-1})\begin{pmatrix}B\widehat{u}(\lambda)\\ 0\end{pmatrix}. \end{cases},$$

To prove the finite-time L^p-admissibility of B, we will exploit the above expressions of $(\widehat{x}(\lambda), \widehat{g}(\lambda))^T$. Thus, from the first expression, it is apparent that

$$\begin{cases} \widehat{x}(\lambda) = \begin{cases} R(\lambda,A)(x_0 + \delta_0\widehat{g}(\lambda)) + R(\lambda,A_{-1})B\widehat{u}(\lambda) \\ H(\lambda)x_0 + (R(\lambda,\mathscr{A}^V_{-1}))_{11}B\widehat{u}(\lambda) \end{cases} \\ \widehat{g}(\lambda) = R(\lambda,\frac{d}{ds})k(\cdot)K^A_\Lambda\widehat{x}(\lambda), \end{cases}$$

where $R(\lambda,\mathscr{A}^V_{-1})) = ((R(\lambda,\mathscr{A}^V_{-1}))_{ij})_{1\leq i,j\leq 2}$.
By plugging $\widehat{g}(\lambda)$ in the first identity of $\widehat{x}(\lambda)$ we get

$$\begin{aligned}\widehat{x}(\lambda) &= R(\lambda,A)(x_0 + \delta_0\big(R(\lambda,\frac{d}{ds})k(\cdot)K^A_\Lambda\widehat{x}(\lambda)\big)) + R(\lambda,A_{-1})B\widehat{u}(\lambda),\\ &= R(\lambda,A)(x_0 + \widehat{k}(\lambda)K^A_\Lambda\widehat{x}(\lambda)) + R(\lambda,A_{-1})B\widehat{u}(\lambda).\end{aligned}$$

Therefore, $R(\lambda,A)x_0 + R(\lambda,A_{-1})B\widehat{u}(\lambda) = \big(I - R(\lambda,A)\widehat{k}(\lambda)K^A_\Lambda\big)\widehat{x}(\lambda)$.
By (**H4**) and according to Proposition 3(ii), $\overline{K}$ commutes with $R(\lambda,A_{-1})$. This easily implies that $(\overline{K})_{|D(K^A_\Lambda)}$, the restriction of $\overline{K}$ to $D(K^A_\Lambda)$, coincide with K^A_Λ. Substituting this into the above equality, we get $R(\lambda,A)x_0 + R(\lambda,A_{-1})B\widehat{u}(\lambda) = \big(I - R(\lambda,A_{-1})\widehat{k}(\lambda)\overline{K}\big)\widehat{x}(\lambda)$, which implies that $(\lambda - A_{-1} - \widehat{k}(\lambda)\overline{K})\widehat{x}(\lambda) = x_0 + B\widehat{u}(\lambda)$ in X^A_{-1}. Thanks to Proposition 3, the operator $(\lambda - A_{-1} - \widehat{k}(\lambda)\overline{K})$ with domain X is invertible in X^A_{-1}. Hence $\widehat{x}(\lambda) = H(\lambda)x_0 + H_{-1}(\lambda)B\widehat{u}(\lambda)$ and therefore, the uniqueness of Laplace transform imply that $x(t)$ is none other than $S(t)x_0 + (S_{-1} * Bu)(t)$ according to Propositions 3 and 1. Furthermore, from the

expression $\widehat{x}(\lambda) = H(\lambda)x_0 + (R(\lambda, \mathscr{A}_{-1}^V))_{11}B\widehat{u}(\lambda)$ we deduce that $(R(\lambda, \mathscr{A}_{-1}^V))_{11}B\widehat{u}(\lambda) = H_{-1}(\lambda)B\widehat{u}(\lambda)$ and that $((\widetilde{\mathscr{T}}^V(t))_{-1})_{11} = S_{-1}(t)$. Finally, the finite-time L^p-admissibility of $\mathscr{B}$ with respect to $(\widetilde{\mathscr{T}}^V(t))_{t\geq 0}$ shows that B is uniformly L^p-admissible for $(S(t))_{t\geq 0}$, in particular Theorem (2)(i) is proved.

Corollary 1 *Let the conditions of Theorem 3 are fulfilled. If* $\omega_0(\mathscr{A}^V) < 0$ *then* B *is* L^p*-admissible for* $(S(t))_{t\geq 0}$.

Proof Thanks to Lemma 2(ii), the control operator $\mathscr{B}$ is finite-time L^p-admissible for $(\widetilde{\mathscr{T}}^V(t))_{t\geq 0}$ and hence L^p-admissible due to $\omega_0(\mathscr{A}^V) < 0$. Finally, Theorem 3 ends the proof.

Remark 3 Generally, $\omega_0(\mathscr{A}^V) < 0$ does not necessarily imply $\omega_0(A) < 0$ and hence in Corollary 1 B is not necessarily L^p-admissible for $(T(t))_{t\geq 0}$ (see the example in [14, Remark 3.9]).

As for the control operators, and due to the expression of $(\widetilde{\mathscr{T}}^V(t))_{t\geq 0}$ (see the proof of Proposition 4 for its definition), it is clear that one can obtain finite-time or infinite-time L^p-admissibility of observation operator C for $(S(t))_{t\geq 0}$ from that of $\mathscr{C}$ (see (31) for its definition) for $(\widetilde{\mathscr{T}}^V(t))_{t\geq 0}$. Recall, that for some classes of kernel functions $a(\cdot)$ and $k(\cdot)$, the authors in [45,34,16] related the L^2-admissibility of the observation operators for sVIEs (5) and sVIDPs (4)) from the L^2-admissibility of the observation operators for CP (20). In the same direction, the following proposition establish a links between the L^p-admissibility of observation operators for the nsVIDPs (16) for $p \in (1,\infty)$, and for sVIDPs (4) with $p = 1$, and the CP (20). Moreover, in Lemma 6, we will prove that the so-called p-Weiss-conjectures for the CPs (20) and the nsVIDPs (16) and (19) are equivalent.

Proposition 5 *Let* (**H3**) *holds. Then the following assertions hold.*

(i) *The operator* C *is finite-time* L^p*-admissible for* $(T(t))_{t\geq 0}$ *if and only if* C *is finite-time (hence uniformly)* L^p*-admissible for* $(S(t))_{t\geq 0}$.
(ii) *If* $\omega_0(\mathscr{A}^V) < 0$, *then* C *is* L^p*-admissible for* $(S(t))_{t\geq 0}$ *if the operator* C *is finite-time* L^p*-admissible for* $(T(t))_{t\geq 0}$.
(iii) *The operator* C *is finite-time* L^p*-admissible for* $(S(t))_{t\geq 0}$ *if and only if there exist constant* M *and* $\omega \in \mathbb{R}$ *such that*

$$\int_0^t \|CS(s)x_0\|_Y^p ds \leq M^p e^{\omega t}\|x_0\|_X^p \text{ for all } x_0 \in D(A) \ \ t \geq 0.$$

Proof (i) Let us consider the operator $\mathscr{C} \in \mathscr{L}(D(\mathscr{A}_0), \mathscr{X})$ given by

$$\mathscr{C}\begin{pmatrix} x \\ f \end{pmatrix} = Cx, \ \ \begin{pmatrix} x \\ f \end{pmatrix} \in D(\mathscr{A}). \tag{31}$$

Assume that C is finite-time L^p-admissible $(T(t))_{t\geq 0}$, then it is clear that $\mathscr{C}$ is finite-time L^p-admissible for $(\mathscr{T}_0(t))_{t\geq 0}$. By virtue of Lemma 2(i), the operator $\mathscr{P}$ is finite-time L^p-admissible for $(\mathscr{T}_0(t))_{t\geq 0}$, thus $\mathscr{C}$ is finite-time L^p-admissible for $(\widetilde{\mathscr{T}}^V(t))_{t\geq 0}$ accordingly to [38, Theorem 3.1(i)]. The fact that $(\widetilde{\mathscr{T}}^V(t))_{t\geq 0}$ is given by the expression (15) implies

$$\begin{aligned}\int_0^t \|CS(s)x\|_Y^p ds &= \int_0^t \|\mathscr{C}\widetilde{\mathscr{T}}^V(s)\begin{pmatrix} x \\ 0 \end{pmatrix}\|_Y^p ds \\ &\leq M_t^p \|x\|_X^p,\end{aligned} \tag{32}$$

for all $t \geq 0$, and hence C is uniformly L^p-admissible for $(S(t))_{t\geq 0}$.
Claim. Assume that C is L^p-admissible for $(S(t))_{t\geq 0}$ in time τ_0, then for all $f \in W^{1,p}(\mathbb{R}^+, X)$, we have

$$(S*f)(\tau_0) \in D(A) \ \text{ and } \ \int_0^{\tau_0} \|C(S*f)(s)\|_Y^p ds \leq \xi \|f\|_{L^p([0,\tau_0];X)}, \quad \xi > 0.$$

Since $D(\mathscr{A}_0) = D(\mathscr{A}^V)$, then we have $\widetilde{\mathscr{T}}^V(\tau_0)D(\mathscr{A}_0) \subset D(\mathscr{A}_0)$ and by using once gain the expression of $(\widetilde{\mathscr{T}}^V(t))_{t\geq 0}$, we conclude that $S(\tau_0)x + (S*f)(\tau_0) \in D(A)$ for all $x \in D(A)$ and $f \in W^{1,p}(\mathbb{R}^+, X)$, which gives the result since $S(\tau_0)D(A) \subset D(A)$. Now the claim follows by arguing as in the proof of [37, Proposition 3.3].
Back now to finish the proof of (i). Suppose now that C is finite-time L^p-admissible for $(S(t))_{t\geq 0}$, then there exist $\tau_0 > 0$ and $M_{\tau_0} > 0$ such that

$$\begin{aligned}\int_0^{\tau_0} \|\mathscr{C}\widetilde{\mathscr{T}}^V(s)\begin{pmatrix} x \\ f \end{pmatrix}\|_Y^p ds &= \int_0^{\tau_0} \|CS(s)x\|_Y^p ds + \int_0^{\tau_0} \|C(S*f)(t)\|_Y^p dt \\ &\leq M_{\tau_0}(\|x\|_X^p + \|f\|^p_{L^p([0,\tau_0];X)}).\end{aligned}$$

Thus, $\mathscr{C}$ is L^p-admissible for $(\widetilde{\mathscr{T}}^V(t))_{t\geq 0}$ in time τ_0. Keeping in mind that $\mathscr{P}$ is finite-time L^p-admissible for $(\mathscr{T}_0(t))_{t\geq 0}$, and using once again [38, Theorem 3.1(i)], we deduce that $\mathscr{C}$ is finite-time L^p-admissible for $(\mathscr{T}_0(t))_{t\geq 0}$ which in turn implies that C is finite-time L^p-admissible for $(T(t))_{t\geq 0}$ in time τ_0.
(ii) If C is finite-time L^p-admissible for $(T(t))_{t\geq 0}$ then the statement (i) shows that C is uniformly L^p-admissible for $(S(t))_{t\geq 0}$. Now, thanks to the Claim, $\mathscr{C}$ is finite-time L^p-admissible for $(\widetilde{\mathscr{T}}^V(t))_{t\geq 0}$ and hence it is L^p-admissible for $(\widetilde{\mathscr{T}}^V(t))_{t\geq 0}$ due to $\omega_0(\mathscr{A}^V) < 0$ as mentioned before. Now, L^p-admissibility of $\mathscr{C}$ for $(\widetilde{\mathscr{T}}^V(t))_{t\geq 0}$ combined with (32) end the proof.
(iii) Necessity is trivial. Now, if C is finite-time L^p-admissible for $(S(t))_{t\geq 0}$ then by inspecting the proof of (i) we deduce that C is finite-time L^p-admissible for $(\widetilde{\mathscr{T}}^V(t))_{t\geq 0}$. Hence, sufficiency comes directly from (32) and by applying [46, Lemma 4.2] to $\mathscr{C}$. This concludes the proof.

Remark 4 (1) As mentioned before, the property **(P1)** is true for the CP (20), thus it is so for nsVIDPs (4) according to Proposition 5(i).
(2) The Proposition 5(ii) remains true, in particular, if $\omega_0(\mathscr{A}^V_{|\mathscr{M}}) < 0$ where $\mathscr{A}^V_{|\mathscr{M}}$ is the generator of the semigroup $\widetilde{\mathscr{T}}^V(t))_{|\mathscr{M}}$ with $\mathscr{M} = X \times M$ where M is a closed $W(t)$-invariant subspace of $L^p(\mathbb{R}^+, X)$.
(3) The statement (i) together with (ii) of Proposition 5 imply that the property **(P2)** for the observation operators for nsVIDPs (4) is true if $\omega_0(\mathscr{A}^V) < 0$, in particular if $\omega_0(\mathscr{A}^V_{|\mathscr{M}}) < 0$, which is a strong condition since it implies $\omega_0(S) < 0$ via (15).
(4) In [16, Corrolary 2.8], the author presented conditions under which the sVIDPs (4) with $p = 2$ and $k(t) = \alpha e^{-\beta t}$ ($\alpha < 0, \beta > 0$ and $\alpha + \beta > 0$) and A generates a contraction operator semigroup on Hilbert space X, is such that $\omega_0(\mathscr{A}^V_{|\mathscr{M}}) < 0$ with $M = \{e^{-\beta\cdot}x : x \in X\}$ and proved that a finite-rank observation operator C is L^2-admissible for sVIDP (4) if C is L^2-admissible for the CP (20). Thus, Proposition 5(ii) combined with (2) may be viewed as a version of the above result for a class of nsVIDP (4). Further, for $p = 1$ and $\alpha \neq 0$ and A has the finite-time L^1-maximal regularity, in particular $(T(t))_{t\geq 0}$ is norm continuous (even analytic), then with the conditions of [14, Corollary 3.5] we obtain once again $\omega_0(\mathscr{A}^V_{|\mathscr{M}}) < 0$. Last, it is mentioned above that finite time L^1-maximal

regularity and finite-time L^1-admissibility of A are equivalent. Thus, every observation operator C is finite-time L^1-admissible for $(T(t))_{t\geq 0}$, and hence for $(S(t))_{t\geq 0}$ according to Proposition 5(i). By virtue of (2), we conclude that C is L^1-admissible for $(S(t))_{t\geq 0}$. Notice that C is not necessarily L^1-admissible for $(T(t))_{t\geq 0}$ unless $(T(t))_{t\geq 0}$ is exponentially stable or A has the L^1-maximal regularity see [49, Theorem 3.6].

Nowadays, it is well-known that the L^p-admissibility of C for $(T(t))_{t\geq 0}$ implies a resolvent condition, that is there exist constant $M>0$ and $\omega>\omega_0(T)$ such that

$$\|CR(\lambda,A)\| \leq \frac{M}{(\mathrm{Re}\lambda-\omega)^{1/p'}},\ \lambda\in\mathbb{C}_\omega, \tag{33}$$

where p' is the conjugate of $p\in[1,\infty]$, see e.g. [68,61]. For $p=1$, (33) says exactly that C is an observation operator for A.

For the observed CP (20), it was conjectured in [68] that if (33) holds then C is finite-time L^p-admissible for $(T(t))_{t\geq 0}$. This we shall refer as the *p-Weiss conjecture*. Early, and for $p=2$ counterexamples where U is not a Hilbert space were mentioned in [66]. However, the question for Hilbert spaces U and X was the starting point of intensive research around it, see [43] for surveys. Although even in this case, counterexamples were found [16,17,41], there are situations with positive answers, the case of contraction semigroups and $U=\mathbb{C}$ [44]. For bounded analytic semigroups on Banach spaces and for any U, the author in [50] characterized when the 2-Weiss conjecture is true. For $p\in[1,\infty)$ and more general spaces, the p-Weiss conjecture has also been studied in the past, see e.g. [8], [32] (for the particular case of analytic semigroups). For the somewhat 'exotic' case $p=\infty$, result in that direction is in [7] which implies that the ∞-Weiss conjecture for any reflexive Banach space X holds if and only if A_{-1} itself is L^∞-admissible. However, and very recently, the case $p=\infty$ has attracted some attention in [47] for bounded analytic semigroups, showing that it is true by drawing a link to the notion of maximal regularity.

By virtue of Proposition 4, the nsVIDPs (4) with (**H3**) admits a resolvent operators $(S(t))_{t\geq 0}$ satisfying **(S4)**. Thus integrating $CS(\cdot)x_0$ against an exponential function and according to Proposition 1, yields a more checkable necessary condition for L^p-admissibility of C: there exist $M>0$ and $\omega>\omega_0(S)$ such that

$$\left\|C(\lambda I-A-\widehat{k}(\lambda)K))^{-1}\right\|_{\mathscr{L}(Y)} \leq \frac{M}{(\mathrm{Re}\lambda-\omega)^{1/p'}},\ \lambda\in\mathbb{C}_\omega. \tag{34}$$

Analogously, it is natural to ask whether (34) would be a sufficient condition for L^p-admissibility (henceforth called from now the p-Weiss conjecture for convenience) for nsVIDPs (4). To the best of the authors' knowledge, the only partial affirmative answer to 2-Weiss conjecture for sVIDPs (4) was given in [46] for contraction semigroups, where $k(\cdot)\in W^{1,2}(\mathbb{R}^+)$ and X is a Hilbert space and $Y=\mathbb{C}$. For the sVIPs (5) with some *creep* kernel functions $a(\cdot)$ (which are somehow related to sVIDPs (4)) on the space X (Hilbert, reflexive,..), the 1-Weiss conjecture was proved very recently [10].

Now, Proposition 5(i) leads to the following result extending the well-known one for CPs.

Corollary 2 *Let* (**H3**) *holds. Assume that one of the following statements holds:*

- *(a) $p=2$ and Y is finite-dimensional, X is a Hilbert space and A generates a contraction semigroup;*

- *(b) $p=2$ and X is a Hilbert space and A generates a normal, analytic semigroup;*
- *(c) $p\in[1,\infty)$ and A generates a bounded analytic semigroup and $(-A)^{1/p}$ is finite-time L^p-admissible observation operator for $(T(t))_{t\geq 0}$.*

If C satisfies (33) *then C is uniformly L^p-admissible for $(S(t))_{t\geq 0}$.*

Proof For the sVIDPs (4) with kernel functions $k(\cdot)\in W^{1,2}(\mathbb{R}^+)$, it is mentioned above that under (a) the result was obtained in [46]. Under the assumptions of the corollary, estimation (33) implies that C is a finite-time L^p-admissible observation operator for $(T(t))_{t\geq 0}$, see [40], [44], [50], [8]. Thus the result is a direct consequence of Proposition 5(i).

In Proposition 6(i) below, we present a stronger version of Corollary 2. Furthermore, we give the explicit expression for the Λ-extension of the operator $\mathscr{C}$ with respect to $\mathscr{A}^V$ which is useful when we want to represent the extended input-output operator $\mathbb{F}^{\widetilde{\mathscr{T}}^V,\mathscr{B},\mathscr{C}}$ associated to the triple $((\widetilde{\mathscr{T}}^V(t))_{t\geq 0},\mathscr{B},\mathscr{C})$.

Proposition 6 *Let* (**H3**) *holds. Then the following assertions hold:*

(i) *$(T(t))_{t\geq 0}$ has the p-Weiss conjecture if and only if $(S(t))_{t\geq 0}$ is so.*
(ii) *If $p\geq 2$ and C satisfies the resolvent estimate (33) with respect to A, and $\mathscr{C}$ is the operator defined by (31) then*

$$\mathscr{C}_\Lambda^{\mathscr{A}^V}\begin{pmatrix} x\\ f\end{pmatrix}=C_\Lambda^A x \text{ for } \begin{pmatrix} x\\ f\end{pmatrix}\in D(\mathscr{C}_\Lambda^{\mathscr{A}^V})=D(C_\Lambda^A)\times L^p(\mathbb{R}^+,X).$$

(iii) *If $p\geq 2$ and C is L^p-admissible for $(S(t))_{t\geq 0}$ in time τ_0, then $S(t)x\in D(C_\Lambda^A)$ for any $x\in X$ and a.e. $t\in[0,\tau_0]$ and $(\mathbb{L}_{\tau_0}x)(t)=C_\Lambda^A S(t)x$ for a.e. $t\in[0,\tau_0]$.*

Proof (i) First, let us consider $\beta_0>0$ such that $\|\widehat{k}(\lambda)KR(\lambda,A)\|<\dfrac{1}{2}$ for $\lambda\in\mathbb{C}_{\beta_0}$ and ω_0^V:=max$(\beta_0,\omega_0(T),\omega_0(S))$. We now assume that $(S(t))_{t\geq 0}$ satisfies the p-Weiss conjecture and let C be an observation operator such that

$$\|CR(\lambda,A)x\|_Y\leq \frac{M\|x\|_X}{(\mathrm{Re}\lambda-\omega_0^V)^{1/p'}}\ (M>0,\lambda\in\mathbb{C}_{\omega_0^V}).$$

From above and for $\lambda\in\mathbb{C}_{\omega_0^V}$ and $x\in X$, we get

$$\begin{aligned}\|CH(\lambda)x\|_Y&\leq\|CR(\lambda,A)(I-\widehat{k}(\lambda)KR(\lambda,A))^{-1}x\|_Y\\&\leq\frac{2M\|x\|_X}{(\mathrm{Re}\lambda-\omega_0^V)^{1/p'}}.\end{aligned}$$

Therefore, C is finite-time L^p-admissible for $(S(t))_{t\geq 0}$ which in turn implies that C is finite-time L^p-admissible for $(T(t))_{t\geq 0}$ according to Proposition 5(i).
Conversely, assume now that $(T(t))_{t\geq 0}$ satisfies the p-Weiss conjecture and C is such that

$$\|CH(\lambda,A)x\|_Y\leq \frac{M\|x\|_X}{(\mathrm{Re}\lambda-\omega_0^V)^{1/p'}}\ (M>0,\lambda\in\mathbb{C}_{\omega_0^V}).$$

A direct computation allows us to obtain $\|CR(\lambda,A)x\|_Y\leq\frac{3M\|x\|_X}{(\mathrm{Re}\lambda-\omega_0^V)^{1/p'}}$, thereby, C is finite-time L^p-admissible for $(T(t))_{t\geq 0}$ which in turn implies that C is finite-time L^p-admissible

for $(S(t))_{t\geq 0}$ according to Proposition 5(i).
(ii) Let $\lambda > 0$ be large enough, and $\begin{pmatrix} x \\ f \end{pmatrix} \in \mathscr{X}$. By virtue of Lemma 1, we get

$$\lambda \mathscr{C} R(\lambda I, \mathscr{A}^V) \begin{pmatrix} x \\ f \end{pmatrix} = \lambda CH(\lambda)x + \lambda CH(\lambda)\widehat{f}(\lambda).$$

Moreover, we have

$$\begin{aligned}\lambda CH(\lambda)x &= \lambda CR(\lambda,A)(I-\widehat{k}(\lambda)KR(\lambda,A))^{-1} \\ &= \lambda^{1/p'}CR(\lambda,A)\widehat{k}(\lambda)K\lambda^{1/p}R(\lambda,A)(I-\widehat{k}(\lambda)KR(\lambda,A))^{-1}x + \lambda CR(\lambda,A)x.\end{aligned}$$

The estimate (33) and the $L^{p'}$-admissibility of identity observation operator for $(T(t))_{t\geq 0}$ involve that $\lambda^{1/p'}CR(\lambda,A)$ and $\lambda^{1/p}R(\lambda,A)$ are uniformly bounded, likewise for $\big(I-\widehat{k}(\lambda)KR(\lambda,A)\big)^{-1}$. Finally, Riemann-Lebesgue lemma implies that the first term on the right in the above equality goes to 0 as $\lambda \to +\infty$. Now, for $f \in W^{1,p}(\mathbb{R}^+,X)$, we have $\begin{pmatrix} 0 \\ f \end{pmatrix} \in D(\mathscr{A}^V)$ and using $D(\mathscr{A}^V) \subset D(\mathscr{C}_\Lambda^{\mathscr{A}^V})$ and $\mathscr{C}^{\mathscr{A}^V}_{\Lambda|D(\mathscr{A}^V)} = \mathscr{C}$, we get

$$\begin{aligned}\lim_{\lambda\to+\infty} \lambda \mathscr{C}_\Lambda^{\mathscr{A}^V} R(\lambda I, \mathscr{A}^V) \begin{pmatrix} 0 \\ f \end{pmatrix} &= \lim_{\lambda\to+\infty} \lambda CH(\lambda)\widehat{f}(\lambda), \\ &= \mathscr{C}\begin{pmatrix} 0 \\ f \end{pmatrix}, \\ &= 0.\end{aligned}$$

From the proof of the statement (i) $\lambda^{1/p'}CH(\lambda)$ is uniformly bounded for sufficiently large λ, and Cauchy-Schwartz inequality yields $\lambda^{1/p}\|\widehat{f}(\lambda)\|_X \leq \dfrac{1}{2^{1/p'}\lambda^{1/p'-1/p}}\|f\|_{L^p(\mathbb{R}^+,X)}$. The fact that $\lambda^{1/p'}CR(\lambda,A)$ is uniformly bounded for a larger λ, if we rewrite $\lambda CH(\lambda)\widehat{f}(\lambda) = \lambda^{1/p'}CH(\lambda)\lambda^{1/p}\widehat{f}(\lambda)$ then the above inequality combined with the uniform boundedness principal yields $\lim_{\lambda\to+\infty} \lambda CH(\lambda)\widehat{f}(\lambda) = 0$ for all $f \in L^p(\mathbb{R}^+,X)$ and therefore the statement (ii) holds.
(iii) Taking into account that $\mathscr{C}$ is finite-time L^p-admissible for $(\mathscr{T}^V(t))_{t\geq 0}$ (see Theorem 5), [62, Proposition 4.2] together with [67, Theorem 4.5] assert that $\mathscr{T}^V(t)\begin{pmatrix} x \\ 0 \end{pmatrix} \in D(\mathscr{C}_\Lambda^{\mathscr{A}^V})$ for all $x \in X$ and for almost all t. Appealing now to statement (ii) and keeping in mind that $(\mathscr{T}^V(t))_{t\geq 0}$ is given by (15) yield $S(t)x \in D(C_\Lambda^A)$ for almost all t and $(\mathbb{L}_\tau x)(t) = C_\Lambda^A S(t)x$. for a.e. $t \in [0,\tau]$.

Leaning on Lemmas 2 and 6 and Theorem 3, we are now in a position to complete the proof of our main result.

Proof (of Theorem 2). From Lemma 2, the triple $((\mathscr{T}_0(t))_{t\geq 0}, \mathscr{B}, \mathscr{P})$ is L^p-regular and that $((\mathscr{T}_0(t))_{t\geq 0}, \mathscr{B}, \mathscr{C})$ is so (simple to verify). By virtue of [54, Theorem 3.4], we deduce that the triple $((\widetilde{\mathscr{T}}^V(t))_{t\geq 0}, \mathscr{B}, \mathscr{C})$ is L^p-regular and its associated extended input-output $\mathbb{F}^{\widetilde{\mathscr{T}}^V,\mathscr{B},\mathscr{C}}$ is given by

$$\mathbb{F}^{\widetilde{\mathscr{T}}^V,\mathscr{B},\mathscr{C}} = \mathbb{F}^{\widetilde{\mathscr{T}}^V,I,\mathscr{C}}\mathbb{F}^{\mathscr{T}_0,\mathscr{B},\mathscr{P}} + \mathbb{F}^{\mathscr{T}_0,\mathscr{B},\mathscr{C}}.$$

Now, as the triple $((\widetilde{\mathscr{T}}^V(t))_{t\geq 0}, \mathscr{B}, \mathscr{C})$ is L^p-regular, we have $\Phi_t^{\widetilde{\mathscr{T}}^V,\mathscr{B}} u \in D(\mathscr{C}_\Lambda^{\mathscr{A}^V})$ and $(\mathbb{F}^{\widetilde{\mathscr{T}}^V,\mathscr{B},\mathscr{C}} u)(t) = \mathscr{C}_\Lambda^{\mathscr{A}^V} \Phi_t^{\widetilde{\mathscr{T}}^V,\mathscr{B}} u$ for any $u(\cdot) \in L^p([0,\tau],U)$ and a.e. $t \in [0,\tau]$. Combining now Theorem 3 and Lemma 6(ii), we deduce that $\int_0^t S_{-1}(t-s)Bu(s)\,ds \in D(C_\Lambda^A)$ and $(\mathbb{F}^{\widetilde{\mathscr{T}}^V,\mathscr{B},\mathscr{C}} u)(t) = C_\Lambda^A \int_0^t S_{-1}(t-s)Bu(s)\,ds$ for almost all t, in particular the triple $((S(t))_{t\geq 0}, B, C)$ is L^p-regular. Combining all this with Proposition 6(ii), we deduce that the output function (19) is given by

$$\begin{aligned} y(t) &= \mathscr{C}_\Lambda^{\mathscr{A}^V} \mathscr{X}(t) \ t-\text{a.e}, \\ &= C_\Lambda^A x(t). \end{aligned}$$

As $x(t)$ is given by (17), using once again the Proposition (6)(ii) and the regularity of $((S(t))_{t\geq 0}, B, C)$, we deduce that for all $x_0 \in X, T > 0$ and $u \in L^p([0,T];U)$ there exists $C_T > 0$ such that

$$\|x(T)\|_X^p + \|y\|_{L^p([0,T];Y)}^p \leq C_T\big(\|x_0\|_X^p + \|u\|_{L^p([0,T];U)}^p\big).$$

Now thanks to [69, Theorem 1.1], for $s \in \mathbb{C}_\beta$ with β sufficiently big real, the transfer function G of the input-output regular linear system

$$\begin{cases} \dot{\mathscr{X}}(t) = \mathscr{A}_{-1}^V \mathscr{X}(t) + \mathscr{B}u(t)\ t > 0 \ \ (\text{in } X_{-1}^A), \\ \mathscr{X}(0) = (x_0, 0) \in \mathscr{X} \\ y(t) = \mathscr{C}_\Lambda^{\mathscr{A}^V} \mathscr{X}(t) \ \ t-\text{a.e}, \end{cases} \tag{35}$$

is given explicitly by

$$\begin{aligned} G(s) &= \mathscr{C}_\Lambda^{\mathscr{A}^V} (sI - \mathscr{A}_{-1}^V)^{-1} \mathscr{B} \\ &= C_\Lambda^A \big(sI - A_{-1} - \widehat{k}(s)\overline{K}\big)^{-1} B, \end{aligned}$$

equivalently, for any $u \in L_\beta^p(\mathbb{R}^+;U)$ we have $y \in L_\beta^p(\mathbb{R}^+;Y)$ and their Laplace transforms are related by $\widehat{y}(s) = G(s)\widehat{u}(s)$.

5 Example

We consider the equation of the heat propagation in a rod of length π involving some space spectral fractional Laplacian as the memory term, with Neumann boundary control at the left end and the Dirichlet boundary condition at the right end respectively, modelled by the following equation

$$\begin{cases} \dfrac{\partial w}{\partial t}(\xi,t) = \Delta w(\xi,t) + \displaystyle\int_0^t k_\alpha(t-s)(-\Delta)^\gamma x(\xi,s)ds,\ t > 0,\ \xi \in (0,\pi), \\ \dfrac{\partial w}{\partial t}(0,t) = u(t),\ w(\pi,t) = 0, & t > 0, \\ w(\xi,0) = w_0(x), & \xi \in (0,\pi), \end{cases} \tag{36}$$

with $\gamma \in (0,\frac{1}{2})$, the standard kernel function $k_\alpha(t) := \mathbf{1}_{[0,1]} \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ $\alpha \in (\frac{1}{2},\frac{3}{2}]$ and Γ denotes the gamma-function. Notice, that for such α, it is easy to see that $k_\alpha(\cdot) \notin W^{1,2}(\mathbb{R}^+)$.
The system (36) is augmented with the Dirichlet boundary observation at the left end

$$y(t) = w(0,t),\ t > 0. \tag{37}$$

In order to adopt the previous abstract results and to rewrite (36)-(37) in an abstract form, we first consider $X = L^2[0,\pi]$ as the state space and define the unbounded linear operator A on X as follows::

$$A = \Delta, \quad D(A) = \left\{ f \in H^2(0,\pi) : f'(0) = f(\pi) = 0 \right\},$$

where Δ is the famous *Laplacian* operator defined on $H^2(0,\pi)$.
It is well-known that A is self-adjoint and generates the following analytic C_0-semigroup on X given by $T(t)f = \sum_{k\in\mathbb{N}} e^{-\lambda_k t}\langle f, \Phi_k\rangle \Phi_k \ \forall t \geq 0,\ f \in X$, where $\lambda_k = (k-\frac{1}{2})^2$ is the set of eigenvalues of $-A$ with the corresponding orthonormal basis in X defined by $\Phi_k(\xi) = \sqrt{\frac{2}{\pi}}\cos[(k-\frac{1}{2})\xi]$ (see [64]). As we are in the Hilbert setting, the extrapolation space X_{-1}^A of X associated with A is defined as a completion of X for the norm $\|f\|_{-1} = (\sum_{k\in\mathbb{N}} \frac{1}{\lambda_k}\langle f, \Phi_k\rangle^2)^{\frac{1}{2}}$ for all $f \in X$ $(0 \in \rho(A))$ and it is isomorph to the topological dual space $(D(A))'$.
The input and output spaces are the same $U = Y = \mathbb{R}$ and $\gamma \in (0, \frac{1}{2})$. For $\gamma \in [0,\infty)$, we define the fractional derivative operator $(-A)^\gamma$ as follows:

$$\begin{cases} D((-A)^\gamma) = \left\{ f \in X / f = \sum\limits_{k\in\mathbb{N}} \langle f, \Phi_k\rangle \Phi_k \text{ with } \sum\limits_{k\in\mathbb{N}} \lambda_k^{2\gamma}\langle f, \Phi_k\rangle^2 \right\} \\ (-A)^\gamma f = \sum\limits_{k\in\mathbb{N}} \lambda_k^\gamma \langle f, \Phi_k\rangle \Phi_k,\ \forall f = \sum\limits_{k\in\mathbb{N}} \langle f, \Phi_k\rangle \Phi_k, \end{cases}$$

and in this case, the extension $\overline{(-A)}^\gamma$ given by Proposition 3(i) can be expressed as:

$$\overline{(-A)}^\gamma f = \sum_{k\in\mathbb{N}} \lambda_k^\gamma \langle f, \Phi_k\rangle \Phi_k \text{ for all } f \in X,$$

which is unbounded on X and it is bounded from X to X_{-1}^A.
The system (36)-(37) without memory term has been studied in detail in [64] (see e.g [4]) and it is transformed to the system (2), where it is proved that the control operator $B = -A_{-1}\mathscr{D}_0 \in \mathscr{L}(U, (D(A))')$, $(\mathscr{D}_0 u)(x) = (x-\pi)\cdot u$, is finite-time L^2-admissible operator for $(T(t))_{t\geq 0}$ [64] and the observation operator $C = -B^*$ is given by $Cf = f(0)$ for $f \in D(A)$.

Now, the system (36)-(37) can be reads as the abstract controlled-observed nsVIDP

$$\begin{cases} \dot{x}(t) = Ax(t) + \int_0^t k_\alpha(t-s)(-A)^\gamma x(s)ds + Bu(t),\ t > 0, \\ y(t) = Cx(t), x(0) = w_0 \in X. \end{cases} \tag{38}$$

which leads to the following result:

Proposition 7 (i) *For all $w_0 \in L^2[0,\pi]$ and $u \in L^2_{loc}(\mathbb{R}^+)$, the system* (38) *has a unique mild solution $x(\cdot)$ and the function $w(\cdot,\cdot)$ defined by $w(\xi,t) = x(t)(\xi)$ for $t \geq 0$ and $\xi \in [0,\pi]$ is the unique solution of the equation* (36)*. Moreover, $w(\cdot,t) \in D(C_\Lambda^A)$ t-a.e. and the output function in* (37) *satisfies $y(\cdot) \in L^2_{loc}(\mathbb{R}^+)$ and $y(t) = C_\Lambda^A w(\cdot,t)$, and for any $T > 0$ there exists $C_T > 0$, independent of (w_0,u) such that*

$$\|w(\cdot,t)\|^2_{L^2[0,\pi]} + \|y\|^2_{L^2[0,T]} \leq C_T(\|w_0\|^2_{L^2[0,\pi]} + \|u\|^2_{L^2[0,\pi]}).$$

(ii) *For any $s > 0$ large enough, the transfer function G of* (36)-(37) *is given by*

$$G(s) = \Big(\frac{\tanh(\sqrt{s}\pi)}{\sqrt{s}} - \sqrt{\frac{2}{\pi}}\sum_{j\geq 1}\sum_{k\in\mathbb{N}}\Big[e^{-js}\Big(\sum_{q\in\mathbb{N}}\frac{s^q}{\Gamma(\alpha+q+1)}\Big)^j\Big]\frac{\lambda_k^{j\gamma}}{(s+\lambda_k)^{j+1}}\Big).$$

Proof (i) It is clear that $(\mathbf{H3})-(\mathbf{H4})$ are satisfied. Now, thanks to the analyticity of $(T(t))_{t\geq 0}$, the operator $(-A)^{\gamma}_{|D(A)}$ is a finite-time L^2-admissible observation for $(T(t))_{t\geq 0}$, and that A has the finite-time L^2-maximal regularity, which implies in turns that that $((T(t))_{t\geq 0}, B, (-A)^{\gamma})$ generates a L^2-regular system (see [1]). Furthermore, the L^2-regularity of $((T(t)_{t\geq 0}), B, C)$ has been recently established in [4]). Consequently, the conditions of Theorem 2 are fulfilled and this ends the proof.
(ii) It is well-known that the control operator B can be rewritten as $B = (sI - A_{-1})\mathscr{D}_s$ and for any $s \in \rho(A)$, where the Dirichlet operator $\mathscr{D}_s$ $(s > 0)$ is given by $(\mathscr{D}_s v)(\xi) = \frac{\sinh(\sqrt{s}(\xi-\pi))}{\sqrt{s}\cosh(\sqrt{s}\pi)} v$ for all $\xi \in [0,\pi], v \in \mathbb{R}$. Hence, by virtue of Theorem 2(ii), for any $s > 0$ large enough and $v \in \mathbb{R}$, we have

$$G(s)v = C^A_\Lambda \left(I - \widehat{k}_\alpha(s) R(s, A_{-1}) \overline{(-A)}^{\gamma}\right)^{-1} \mathscr{D}_s v.$$

The fact that $R(s,A_{-1})$ commutes with $\overline{(-A)^{\gamma}}$ on X and for $s > 0$ large enough, $\|\widehat{k}_\alpha(s)(-A)^{\gamma} R(s,A)\| < 1$, so the fractional powers properties implies that

$$G(s)v = C^A_\Lambda \Big(\sum_{j\in\mathbb{N}} \widehat{k}^j_\alpha(s)(-A)^{j\gamma} R^j(s,A)\Big)\mathscr{D}_s v.$$

Using once again the commutativity of both $R(s,A)$ with $(-A)^{\gamma}$ on $D(A)$ and $R(s,A_{-1})$ with $\overline{(-A)^{\gamma}}$ on X respectively, it follows that for all $j \geq 2$, $R(s,A)$ and $(-A)^{j\gamma}$ commutes on $D(A^{j-1})$ and $R^{j-1}(s,A)\mathscr{D}_s u \in D((-A)^{j\gamma})$ $(j\gamma < j-1)$. Consequently,

$$\sum_{j\in\mathbb{N}} \widehat{k}^j_\alpha(s)(-A)^{j\gamma} R^j(s,A) = I + \widehat{k}_\alpha(s)(-A)^{\gamma} R(s,A) + R(s,A) \sum_{j\geq 2} \widehat{k}^j_\alpha(s)(-A)^{j\gamma} R^{j-1}(s,A).$$

Now, from the regularity's characterization of $((T(t))_{t\geq 0}, B, (-A)^{\gamma})$ (see Section 4), we have $\mathscr{D}_s u \in D((-A)^{\gamma}_\Lambda)$. Moreover, the closedness of $(-A)^{\gamma}$ implies $D((-A)^{\gamma}_\Lambda) = D((-A)^{\gamma})$ and $(-A)^{\gamma}_\Lambda f = (-A)^{\gamma} f$ for all $f \in D((-A)^{\gamma})$. Thus, commutativity of $R(s,A_{-1})$ and $\overline{(-A)^{\gamma}}$ on X implies that $(-A)^{\gamma} R(s,A)\mathscr{D}_s v = R(s,A)(-A)^{\gamma}\mathscr{D}_s v \in D(A)$. As $C^A_{\Lambda\,|D(A)} = C$, we obtain

$$G(s)v = C^A_\Lambda \mathscr{D}_s v + \Big(\sum_{j\in\mathbb{N}^*} \widehat{k}^j_\alpha(s)(-A)^{j\gamma} R^j(s,A)\mathscr{D}_s v\Big)(0).$$

From [39, Lemma 3.6]), we easily get $C^A_\Lambda \mathscr{D}_s v = \frac{\tanh(\sqrt{s}\pi)}{\sqrt{s}} v$. Now, for $j \in \mathbb{N}^*$, a direct computation yields

$$\begin{aligned}(-A)^{j\gamma} R^j(s,A)\mathscr{D}_s v &= \sum_{k\in\mathbb{N}} \lambda_k^{j\gamma} \langle R^j(s,A)\mathscr{D}_s, \Phi_k\rangle \Phi_k v \\ &= \sum_{k\in\mathbb{N}} \frac{\lambda_k^{j\gamma}}{(s+\lambda_k)^j} \langle \mathscr{D}_s, \Phi_k\rangle \Phi_k v.\end{aligned}$$

A straightforward computation shows that $\langle \mathscr{D}_s, \Phi_k\rangle = -\frac{\sqrt{2}}{\sqrt{\pi}(\lambda_k+s)}$ and $\widehat{k}_\alpha(s) = \frac{\gamma(\alpha,s)}{s^\alpha \Gamma(\alpha)}$, where $\gamma(\alpha,s)$ is the lower incomplete gamma function given by $\gamma(\alpha,s) = \int_0^s t^{\alpha-1} e^{-t} dt$ and satisfies $\gamma(\alpha,s) = s^\alpha \Gamma(\alpha) e^{-s} \sum_{q\in\mathbb{N}} \frac{s^q}{\Gamma(\alpha+q+1)}$. Finally, we get the wanted expression

$$G(s) = \Big(\frac{\tanh(\sqrt{s}\pi)}{\sqrt{s}} - \sqrt{\frac{2}{\pi}} \sum_{j\in\mathbb{N}^*} \sum_{k\in\mathbb{N}} \Big[e^{-js}\Big(\sum_{q\in\mathbb{N}} \frac{s^q}{\Gamma(\alpha+q+1)}\Big)^j\Big] \frac{\lambda_k^{j\gamma}}{(s+\lambda_k)^{j+1}}\Big),$$

and the proof is complete.

CRediT authorship contribution statement:
Hamid Bounit: Investigation, Writing - reviewing and editing.
Mounir Tismane: Conceptualization Methodology, Writing- original draft, Formal analysis, Investigation, Supervision, Validation.

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