The Symmetric Kazdan–warner Problem and Applications

João Marcos do Ó
Federal University of Paraíba
Leonardo Francisco Cavenaghi (leonardofcavenaghi@gmail.com)
State University of Campinas
Llohann Dallagnol Sperança
Federal University of São Paulo

Research Article

Keywords: Prescribing Scalar Curvature, Yamabe problem, Kazdan-Warner problem, Elliptic PDE's, Exotic Manifolds, Cheeger deformations, group symmetry

Posted Date: June 23rd, 2022

DOI: https://doi.org/10.21203/rs.3.rs-1769624/v1

THE SYMMETRIC KAZDAN–WARNER PROBLEM AND APPLICATIONS

LEONARDO F. CAVENAGHI, JOÃO MARCOS DO Ó,
AND LLOHANN D. SPERANÇA

Abstract. After R. Schoen completed the solution of the Yamabe problem, compact manifolds could be categorized in three classes, depending on whether they admit a metric with positive, non-negative, or only negative scalar curvature. Here we follow Yamabe's first attempt to solve his problem through variational methods and provide an analogous equivalent classification for manifolds equipped with actions by non-discrete compact Lie groups. Moreover, we apply the method and the results to: classify total spaces of fiber bundles with compact structure groups (concerning scalar curvature), to conclude density results and compare realizable scalar curvature functions between some exotic manifolds and their standard counterpart. We also provide an extended range of prescribed scalar curvature functions of warped products, especially with Calabi–Yau manifolds.

1. Declaration

Declaration and Statements Our manuscript has no associated data.

2. Introduction

As exemplified by the Bonnet–Meyers Theorem, Differentiable Sphere Theorem ([Bre10]) and Poincaré Conjecture ([Per02, Per03a, Per03b]), to assume the existence of a special type of geometry is a consistent way to understand a manifold's topology. The converse question, however,

Given a fixed manifold, which are its admissible geometries?

is widely open. The existence of manifolds such as the very interesting *exotic spheres*, firstly introduced By J. Milnor in [Mil56], adds to the relevance of the question.

On the other hand, as a generalization of the classification of closed surfaces, Yamabe [Yam60] proposed the existence of metrics of constant scalar curvature on manifolds. Once found a mistake in Yamabe's proof, the problem came to be known as the *Yamabe problem.* A complete solution came later, through the works of N. Trudinger [Tru68], T. Aubin [Aub76]

2000 *Mathematics Subject Classification.* 53C18,53C20,53C21.

Key words and phrases. Prescribing Scalar Curvature, Yamabe problem, Kazdan-Warner problem, Elliptic PDE's, Exotic Manifolds, Cheeger deformations, group symmetry.

and R. Schoen [Sch84], R. Schoen and S-T Yau [SY79a, SY79b, SY81]. By combining it with Kazdan–Warner [KW75a, KW75b], one can now state Yamabe's classification in the following way. Every n-dimensional closed manifold M with $n \geq 3$ falls in one of the following classes:

- ($\mathscr{P}$) Every function is the scalar curvature of a Riemannian metric on M;
- ($\mathscr{Z}$) A function is the scalar curvature of a Riemannian metric on M if and only if it is the zero function or it is negative somewhere;
- ($\mathscr{N}$) A function is the scalar curvature of a Riemannian metric on M if and only if it is negative somewhere.

On the one hand, it is known that every class is nonempty and admits varieties of manifolds. On the other hand, curiously, all obstructions for the realizability of scalar curvature functions vanish in the presence of non-abelian symmetry (see Lawson–Yau [LY74]). However, in the realm of G-invariant metrics, one only knows that the scalar curvature must be basic, i.e., invariant by the group action and hence constant along orbits; but it is unknown how large the class of realizable ones is.

Here we give a complete answer to this question by providing a classification analogous to the Yamabe one in the G-invariant setting.

Theorem A. *Let (M, G) be a closed smooth manifold with dimension $n \geq 3$ equipped with an effective action by a non-discrete compact Lie group G. Then, (M, G) belongs to one of the following classes:*

- ($\mathscr{P}^G$) *Every basic function is the scalar curvature of a G-invariant metric on M;*
- ($\mathscr{Z}^G$) *A basic function is the scalar curvature of a G-invariant metric on M if and only if it is the zero function or it is negative somewhere;*
- ($\mathscr{N}^G$) *A basic function is the scalar curvature of a G-invariant metric on M if and only if it is negative somewhere.*

Moreover, if G has a non-abelian Lie algebra, $(M, G) \in \mathscr{P}^G$.

We also observe that the respective classes do depend on both the manifolds and the equipped action, as L. Bergery [Ber83] provides examples of manifolds in $\mathscr{P}$ that do not admit a G-invariant metric as in $\mathscr{P}^G$.

The proof of Theorem A is divided into two main results (Theorems B and C) and an auxiliary result (Theorem E). The main results are G-invariant versions of the solution of the Yamabe problem and of Kazdan–Warner [KW75a, Theorem A], respectively.

Theorem B. *Let (M, G) be a closed smooth manifold with dimension $n \geq 3$ equipped with an action by a non-discrete compact Lie group G. Then,*

1. *every (M, G) admits a G-invariant metric with constant negative scalar curvature;*
2. *if (M, G) admits a G-invariant metric with non-negative scalar curvature, then it admits a metric with zero scalar curvature;*

(3) *if (M,G) admits a G-invariant metric with a non-vanishing non-negative scalar curvature, then it admits a metric with positive constant scalar curvature.*

The desired metric on Theorem B is obtained by a G-invariant conformal change of the original metric, relating it to Yamabe's problem.

Yamabe's first attempt to prove the existence of constant scalar curvature in a fixed conformal class dates back to 1960. The proof presented an error, as pointed out by Trudinger [Tru68], due to the lack of compactness caused by a critical exponent in the Rellich–Kondrachov Theorem. By combining the works of Aubin and Schoen [Aub76, Sch84], which used Green functions among other techniques, a complete solution was given in 1984.

Surprisingly, the proof of Theorem B takes a complete turnaround and follows along the lines of Yamabe's original attempt via variational methods. The merit and novelty of the present paper is to show how it is possible to variationally approach PDE problems associated with basic quantities by restricting the analysis to *slices*. This dimension reduction improves the classical Rellich–Kondrachov compactness embedding, naturally handling the critical exponent (see Theorem 3.5). In particular, we provide a simpler proof for Yamabe's problem in the large class of invariant metrics. Related works that inspired the discussion can also be seen in [Heb90, HebVa93, HebVa97].

The second main result places Kazdan–Warner [KW75a, Theorem A] in the G-invariant setting. We observe that G can be discrete in this theorem.

Theorem C. *Let (M,G,g) be a closed Riemannian manifold where G is a compact Lie group acting by isometries. Suppose $f : M \to \mathbb{R}$ is basic and $c > 0$ is a constant such that*

$$c \min_M f < \mathrm{scal}_g(x) < c \max_M f \tag{1}$$

for every $x \in M$. Then, there is a G-invariant metric on M whose scalar curvature is f.

From Kazdan–Warner [KW75a], one knows that the metric $\tilde{g}$ realizing $\mathrm{scal}_{\tilde{g}} = f$ is arbitrarily close to the orbit of g by the diffeomorphism group. Using this fact, we conclude:

Corollary D. *Let (M,G) be a closed smooth manifold equipped with the action of a compact Lie group G. If $M \in \mathscr{A} \cap \mathscr{A}^G$, $\mathscr{A} \in \{\mathscr{P}, \mathscr{Z}\}$, then any smooth function which is the scalar curvature of a metric on M is realized by a metric arbitrarily L_p-close to a G-invariant metric, for $1 \leq p < \infty$.*

In order to refine Theorem A, we study some very classical metric deformations. A deformation commonly used in the construction of metrics with positive/non-negative sectional curvature is the *Cheeger deformation* (we refer to [Zil, M8̈7] or [CeSS18] for details). To prove Theorem E below, we base on the improved version of Lawson–Yau Theorem from [CeSS18], which guarantees that every G-invariant metric g develops positive scalar curvature after Cheeger deformation, as long as G has a non-abelian Lie algebra. Theorem E implies the last statement in Theorem A.

Theorem E. *Suppose that $(M,G) \notin \mathscr{P}^G$ where M, G are compact. Then G has an abelian Lie algebra and there is no G-invariant metric on M such that the induced metric on the manifold part of M/G has positive scalar curvature.*

We refer to [Wie16] for results on the existence of positive scalar curvature under abelian symmetry.

A more detailed study of the evolution of the scalar curvature under Cheeger deformation improves Corollary D in interesting cases. Recall that an action is called semi-free if every isotropy group is discrete.

Theorem F. *Suppose that (M,g) is a compact Riemannian manifold and G is a compact Lie group acting semi-freely by isometries in M. Assume that G has a non-abelian Lie algebra and that $f : M \to \mathbb{R}$ is smooth. Then there is a metric $\tilde{\mathrm{g}}$ satisfying $\mathrm{scal}_{\tilde{\mathrm{g}}} = f$ which is arbitrarily L_p-close to a Cheeger deformation g_t of g, for $1 \leq p < \infty$, up to diffeomorphism.*

This is equivalent to say that, fixed a G-invariant metric g, the set of scalar curvatures realized by the orbits $\mathrm{Diff}(M)\cdot\{g_t \mid t \in (1,\infty)\}$ is dense in the set of smooth functions. For instance, Grove–Ziller [GZ00] and Goette–Kerin–Shankar [GKS20] construct several interesting 7-manifolds with semi-free S^3-actions. Theorem F applies directly to them.

A precursor of the Cheeger deformation is the *canonical variation*. By combining it with Theorem A and [Ber83], a result in the realm of (locally trivial) fiber bundles arises in a natural way: there exists strong restrictions whenever the total space do not admit a metric with positive scalar curvature.

Corollary G. *Suppose that $\pi : F \hookrightarrow \overline{M} \to M$ is a compact fiber bundle with compact structure group (acting effectively) G. If $\overline{M} \notin \mathscr{P}$ then $M \notin \mathscr{P}$ and $(F,G) \notin \mathscr{P}^G$. In particular, G has an abelian Lie algebra.*

In the context of a fiber bundle $\pi : \overline{M} \to M$, a partial conformal change, called *general vertical warping*, is in place. Given a basic function $\phi : \overline{M} \to \mathbb{R}$, the general vertical warping defined by ϕ is defined by multiplying by $e^{2\phi}$ the component of the metric tangent to the fibers. Common examples of fiber bundles are vector bundles and warped products, which are often non-compact. On the other hand, the problem of prescribing scalar curvature of complete metrics on non-compact manifolds is still open.

Here, the variational method used to prove Theorem B is applied to extend Ehrlich–Yoon-Tae–Kim [EYTK96]. In particular, we realize a great range of (basic) scalar curvature functions as warped products, including the case of non-compact fibers, while there the authors focus on prescribing constant scalar curvature.

Theorem H. *Let (M, h) be a closed n-dimensional Riemannian manifold and F be a k-dimensional manifold with a (complete) metric g_F with constant scalar curvature c. Given a basic smooth function $f : M \times F \to \mathbb{R}$, assume one of the following conditions:*

(1) $c > 0$ *and* $\mathrm{scal}_h > 0$*;*
(2) $c \leq 0$ *and* $f : M \times F \to \mathbb{R}$ *satisfies the condition:*

$$-\left(\frac{k+1}{8k}\right)(f - \mathrm{scal}_h) + \left(\frac{(k+1)^2}{8(k-1)k}\right) c\,(\mathrm{vol}(M))^{\frac{2}{k-1}} \geq 0.$$

Then,

(a) if $|c| > 0$ there are $\mu_1, \mu_2 > 0$ and a (complete) warped product metric $\bar{\mathrm{g}}$ on $(M \times F, \mu_1 \mathrm{h} \times \mu_2\, g_F)$ such that $\mathrm{scal}_{\bar{\mathrm{g}}}(x, y) = f(x)$. In particular, if M admits a metric with constant scalar curvature c', $M \times F$ admits a warped product metric with scalar curvature c'.
(b) If $c = 0$ there exists $A \geq 0$ such that the previous conclusion holds to $f + A$.

A family of manifolds fundamental to the study of scalar and Ricci curvature is of *Calabi–Yau* manifolds. We call a compact Kähler manifold manifold Calabi–Yau if its first Chern class vanishes, or, equivalently, if it admits a Ricci flat Kähler metric. For simplicity, we call such a metric as Calabi–Yau. To the authors' knowledge, Theorem H is the first result on prescribing scalar curvature functions on non-compact fiber bundles over Calabi–Yau manifolds. On the other hand, Tosatti–Zhang [TZ14] proved that homolorphic submersions from compact Calabi–Yau manifolds are trivial, up to covering. We combine Theorem H with this result of Tosatti–Zhang to obtain:

Corollary I. *Suppose $\pi : X \to Y$ is a holomorphic submersion with connected fiber F satisfying one of the following:*

(1) X, Y are projective manifolds with X Calabi–Yau;
(2) X, Y are compact Kähler manifolds; Y, F are Calabi–Yau; and either $b_1(F) = 0$ or $b_1(Y) = 0$ and F is a torus.

If $f : Y \to \mathbb{R}$ is a scalar curvature function in Y, then there is a metric $\tilde{\mathrm{g}}$ on X whose scalar curvature is $f \circ \pi$. Moreover, if g is a Calabi–Yau metric on Y, $\tilde{\mathrm{g}}$ can be chosen so that π is a Riemannian submersion over g.

In order to illustrate the theory, in section 6 we present families of examples of exotic manifolds closely related to their standard counterpart. Specifically, we recall the manifolds in [GZ00, Dur01, DPR10, Spe16, CS18, GKS20], which are realized as $\star$-*diagrams*. That is, a cross-diagram

$$\begin{tikzcd}
& G \arrow[d, dotted, no head, "\bullet"] & \\
G \arrow[r, dotted, no head, "\star"] & P \arrow[r, "\pi'"] \arrow[d, "\pi"] & M' \\
& M &
\end{tikzcd} \tag{2}$$

where P is a smooth manifold admitting two free commuting G-actions, $\bullet$ and $\star$, whose quotients we denote by M and M', respectively. Observe that the $G \times G$-action on P descends to G-actions on both M and M'. For brevity, we denote the $\star$-diagram (39) by $M \leftarrow P \rightarrow M'$. Examples of manifolds realized as the M'-term in $\star$-diagrams are Kervaire, and exotic 7-, 8- and 10-spheres, their connected sums with various other manifolds, bundles, and candidates for 'exotic' homogeneous manifolds (see Section 6 for explicit constructions).

We justify our interest on such diagrams in a natural way: the first construction of exotic manifolds, due to Milnor [Mil56], makes use of the classical *Reeb's Theorem* to show that certain 7-dimensional total spaces of sphere bundles are homeomorphic to a standard sphere. Moreover, one can recover the smooth structure of a manifold through its space of smooth function (see, for example, [MS74, Problem 1-C]). However, in the presence of a $\star$-diagram, the set of basic functions of M and M' are naturally identified, since they are naturally identified with the space of $G \times G$-invariant functions on P, proving that the set of basic functions does not recover (M, G).

Seeing $G \times G$-invariant function $f : P \to \mathbb{R}$ as basic function on both M and M', Corollary G gives:

Corollary J. *Consider a $\star$-diagram $M \leftarrow P \rightarrow M'$ with G connected. Then, a basic function on M is the scalar curvature of a G-invariant metric on M if and only if it is the scalar curvature of a G-invariant metric on M' if and only if it is the scalar curvature of a $G \times G$-invariant metric on P.*

Further examples and applications of the theory can be found in section 6.1.

The paper is organized as follows: in section 3.2 we deal with the Kazdan–Warner problem on Riemannian manifolds with isometric actions. Namely, we start with the proof of Theorem C and then approach the Yamabe problem in the G-invariant setting (section 3.3), proving Theorem B. Such a combination leads to the proof of Theorem A.

Section 4 is devoted to a deeper study of the relation between scalar curvature and the group of isometries. We combine classical methods such as Cheeger deformations and Canonical Variations to prove Theorems E, F and Corollaries D, G.

In Section 5 we apply the variational method again to prove Theorem H. Section 6 is devoted to the construction of several examples for which the developed theory is applicable. In section 6.1 we prove Corollary I. Corollary J and its applications are proved in section 6.2.

3. Prescribing G-invariant scalar curvature

3.1. **General facts about Sobolev Spaces of G-invariant functions.**

Let (M, g) be a closed Riemannian manifold with an isometric effective action by a compact Lie group G. Let Q denote a bi-invariant Riemannian metric on G. Recall that the G-action defines a smooth map

$$\mu : G \times M \to M$$

$$\mu(g, x) := gx.$$

For each point $x \in M$, the map $\mu_x(g) = gx$ induces a diffeomorphism of G/G_x onto the orbit Gx, where G_x is the isotropy subgroup at x. Moreover, the bi-invariant metric Q induces an orthogonal decomposition $\mathfrak{g} = \mathfrak{g}_x \oplus \mathfrak{m}_x$, where $\mathfrak{g}_x$ is the Lie algebra of G_x. In particular, $\mathfrak{m}_x$ is isomorphic to T_xGx, the isomorphism being induced by computing action fields: given an element $U \in \mathfrak{m}_x$, if exp denotes the exponential map between $\mathfrak{g}$ and G and e denotes the neutral element on G, then the vectors

$$U_x^* := d(\mu_x)_e(U) = \frac{d}{dt}\Big|_{t=0} \exp(tU)\, x.$$

generate T_xGx.

Let $W^{k,p}(M), k \geq 0, p \in [1, \infty)$ denote the Sobolev space of M with respect to any invariant metric g. We reinforce that since M is assumed to be closed, hence compact, we do not need to fix some specific metric since any metric induces equivalent spaces. Once fixed a Haar measure in G, any element $u \in W^{k,p}(M)$ can be averaged along G in order to produce a basic function $\bar{u}$, i.e., $\bar{u}(gx) = \bar{u}(x)$ for almost every $x \in M$ and every $g \in G$. We denote the set of such elements by $W_G^{k,p}(M)$. Denote $C_G^{\infty}(M)$ as the set of smooth basic functions in (M, G). As in [Heb96, Lemma 5.4, p. 91], it can be shown that $\overline{C_G^{\infty}(M;\mathbb{R})}^{\|\cdot\|_{W^{k,p}(M)}} = W_G^{k,p}(M)$. Moreover, $W_G^{0,p}(M) = L_G^p(M)$. Throughout this paper, unless explicitly contrarily said, we shall assume that G is a compact Lie group with positive dimension, i.e, it is not discrete.

Let $\pi_E : E \to M$ be any smooth Euclidean vector bundle over M. It is possible to consider the Sobolev Space of G-invariant local sections of π_E. More precisely, if $s : U \subset M \to E$ is a smooth local section, where U is an open set in M, we say that s is G-invariant if $GU \subseteq U$ and $s(gx) = s(x)\ \forall g \in G,\ \forall x \in U$. We denote the the set of these elements by $C_G^{\infty}(M; E)$ and define

$$W_G^{k,p}(M; E) := \overline{C_G^{\infty}(M; E)}^{\|\cdot\|_{k,p}},$$

where

$$\|s\|_{k,p} := \left(\sum_{j=0}^{k} |\nabla_E^j s|^p_{\langle,\rangle} \right)^{1/p},$$

$\langle,\rangle$ is the family of Euclidean inner products on the fibers and ∇_E a compatible connection on E.

In section 3.2 we will consider $E = S^2TM \to M$, i.e, the vector bundle over M whose the fiber at x consists of the vector space of symmetric bilinear forms defined in T_xM, and the respective Sobolev space $W^{2,p}_G(M; S^2TM)$.

3.2. **A direct approach to prescribing G-invariant scalar curvature.**

One of the main results in the problem of prescribing curvature is

Theorem 3.1 ([KW75a], Theorem A)**.** *Let (M, g) be a closed Riemannian manifold. Then any smooth function $f : M \to \mathbb{R}$ satisfying*

$$\min_M cf < \mathrm{scal}_g < \max_M cf,$$

for some $c > 0$, is the scalar curvature of a Riemannian metric on M.

In this section we prove Theorem C, which is a natural generalization for Kazdan–Warner's result in the G-invariant setting. For convenience, we restate it below.

Theorem C. *Let (M, g) be a closed Riemannian manifold with an isometric effective action by a compact Lie group G. Then any smooth basic function $f : M \to \mathbb{R}$ satisfying*

$$\min_M cf < \mathrm{scal}_g < \max_M cf, \tag{3}$$

for some $c > 0$, is the scalar curvature of a G-invariant Riemannian metric on M.

Let (M^n, g) be a closed Riemannian manifold with an isometric effective action by a compact Lie group G. Denote by $W^{2,p}_G(M; S^2TM)$ the Sobolev space of symmetric bilinear forms in M that are invariant by the G-action (recall the discussion in section 3.1).

Let $K : M \to \mathbb{R}$ be a smooth G-invariant function. Following Kazdan–Warner, finding a G-invariant metric g whose scalar curvature is K consists of solving a quasilinear PDE that we simply write as

$$F(g) = K.$$

Here we start with G-invariant metric g and the operator F is G-equivariant. Moreover, the linearization A of F at g,

$$Ah := \frac{d}{dt}\Big|_{t=0} F(g + th) =: F'(g)(h),$$

is given in coordinates by

$$Ah = -\Delta_g h^i_i + h^{ij}_{;ij} - h_{ij}R_{ij},$$

where R_{ij} are the coordinates of the Ricci tensor of g. It's formal L^2_G-adjoint is given by

$$A^*u = -(\Delta_g u)\, g + \text{Hess } u - u\text{Ric}(g) \tag{4}$$

We remark that, as in [KW75a], a straightforward computation shows that the symbol of A^* is injective.

The proof of Theorem C follows easily from the following Lemmas (compare with Lemmas 1, 2 and the Approximation Lemma in [KW75a]):

Lemma 3.2. *Let (M^n, g) be a closed manifold with an isometric effective action by a compact Lie group G. Let $A = F'(g)$. Then* $\ker A^* = \{0\}$ *except possibly if $F(g)$ is either a positive constant or $F(g) = 0$ and* $\text{Ric}(g) = 0$. *Moreover, if* $\ker A^* = \{0\}$ *then the map $A : W^{2,p}_G(M; S^2TM) \to L^p_G(M;\mathbb{R}),\ p > 1$, is a surjection.*

Proof. Our proof follows very closely the proof of [FM74, Theorem 3, p. 480].

It is easy to see that the image of $A\Big|_{W^{2,p}_G(M;S^2TM)} \subset L^p_G(M;\mathbb{R})$. Assume first that $F(g) = 0$ and that $\text{Ric}(g) \neq 0$ and let u be such that $A^*u = 0$. Taking the trace on equation (4) one has

$$(n-1)(-\Delta_g u) = F(g)u = 0. \tag{5}$$

Hence, by the Hopf Lemma it follows that $u =$ constant. But equation (4) implies then that $u\,\text{Ric}(g) = 0$, what is a contradiction except if $u \equiv 0$.

Assume now that $F(g)$ is not a positive constant and take u such that $A^*u = 0$. Taking the divergence on equation (4) gives:

$$\frac{1}{2} u dF(g) = 0 \tag{6}$$

where we have used that

$$\delta\,\text{Ric}(g) = -\frac{1}{2} dF(g),$$
$$\delta\,\text{Hess } u - \mathrm{d}\Delta_g u + g\,(du, \text{Ric}(g)) = 0,$$

where δ denotes the divergence operator.

Suppose by contradiction that $u \not\equiv 0$. If u never vanishes, then $dF(g) = 0$ and hence $F(g) =$ constant. According to the first equality on (5), it follows that u is an eigenvector of $-\Delta_g$. Therefore $F(g) \geq 0$. So by hypotheses, it must follow that $F(g) = 0$. But equation (4) implies, in this case, that $\text{Ric}(g) = 0$, which is again a contradiction. Therefore, u must vanishes at a point x.

If $du(x) = 0$, by considering a geodesic γ starting at x, the real valued function $h(s) = u \circ \gamma(s)$ satisfies a linear second order ODE with initial conditions $u(0) = 0, u'(0) = g\,(\nabla u, \gamma'(0)) = du(x)(\gamma'(0)) = 0$. Hence, $h \equiv 0$ and, since M is assumed to be complete, $u \equiv 0$. Therefore, $du(x) \neq 0$ and 0 is a regular value of u. In this case, (6) implies that $dF(g) = 0$ on an open dense set, what is again a contradiction with the first equality in (5).

For the second claim, note that $\ker AA^* = \ker A^* = \{0\}$. Therefore, according to [BE69, Lemma 4.4, p. 383], the composition operator $AA^* : W^{4,p}_G(M;\mathbb{R}) \to L^p_G(M;\mathbb{R})$ is elliptic. Moreover, according to [BE69, Theorem 4.1, p. 383], it is surjective and consequently invertible, so it follows that A is surjective. □

Assume that $p > n = \dim M$. Then the Sobolev Embedding Theorem implies that any $g \in W^{2,p}_G(M;S^2TM)$ is continuously differentiable. Denote by S^2_+TM the vector bundle over M whose fiber at $x \in M$ consists of the vector space of symmetric bilinear positive definite forms in T_xM. Since F is quasi-linear it follows that $F : W^{2,p}_G(M;S^2_+TM) \to L^p_G(M;\mathbb{R})$ is continuous and that F' is continuous in the sense that: given any $\epsilon > 0$ there exists $\delta > 0$ such that if $\| g - \tilde{g}\|_{2,p} < \delta$ then

$$\|F'(g)h - F'(\tilde{g})h\|_p < \epsilon\|h\|_{2,p}.$$

That is, F is Fréchet differentiable.

Lemma 3.3. *Let $p > n$ and $A = F'(g)$ and $f \in L^p_G(M;\mathbb{R})$. Assume that A^* is injective. Then $F : W^{2,p}_G(M;S^2_+TM) \to L^p_G(M;\mathbb{R})$ is locally surjective.*

Proof. This is done by the means of the classical *Inverse Function Theorem* on Banach spaces. Take $U \subset W^{4,p}_G(M)$, a small G-invariant neighbourhood of 0, such that $V := A^*(U)$ satisfies $g+V \subset W^{1,2}_G(M;S^2_+TM)$. That is, $g+V$ consists ofly of positive definite symmetric tensors. Define the quasilinear elliptic operator

$$Q(u) := F(g+A^*u).$$

Note that Q satisfies $Q'(0) = AA^*$, which is an elliptic operator with trivial kernel, being therefore invertible (see [BE69, Theorem 4.1, p. 383]). Moreover, since $\ker A^* = \{0\}$ then A is surjective (Lemma 3.2). Therefore, the linear operator AA^* is bounded with bounded inverse. The Inverse Function Theorem then implies that $Q(U)$ contains a G-invariant neighbourhood $\widetilde{U}$ of $Q(0) = F(g)$ where $Q|_{\widetilde{U}}$ is invertible, concluding the proof. □

Lemma 3.4 (Approximation Lemma)**.** *Let (M,g) be a closed Riemannian manifold with an isometric action by a compact Lie group G. Let $f,h : M \to \mathbb{R}$ be continuous basic L_p functions in M such that* $\min_M f \leq g \leq \max_M f$. *Then, for any $\epsilon > 0$ and $1 \leq p < \infty$, there exist a G-equivariant diffeomorphism $\widetilde{\Phi} : M \to M$ such that*

$$\|f \circ \widetilde{\Phi} - h\|_p < \epsilon.$$

Sketch of the proof. We follow closely to Kazdan–Warner original proof ([KW75b, Theorem 2.1]) by observing that the original construction can be done in the manifold part of M/G through flows of vector fields, and so they can be lifted to M via *basic horizontal fields*. With this in mind, we only sketch the proof adding the missing elements to it.

Fix $\delta > 0$ throughout. There exists an open invariant subset $\tilde{E} \subset M$ such that

$$\int_{M\setminus\tilde{E}} |f(x) - g(x)|^p \,\mathrm{d}x \leq \delta.$$

Moreover, $\tilde{E}$ can be chosen to contain a tubular neighborhood of the singular stratum. Therefore, $\tilde{E}$ is contained in the regular stratum and $\tilde{E}, M\setminus\tilde{E}$ are G-invariant subsets. This observation allows us to work in the manifold part of the quotient M/G and use the same steps as Kazdan–Warner. Denote the regular stratum of M by M^{reg} and the manifold part of M/G by $M^{reg}/G = M^*$. Recall that M^* is a convex open dense subset of M/G. Write $E = \tilde{E}/G$ and $\bar{f}, \bar{g}$ for the respective functions defined on M/G.

Choose a triangulation $\{\Delta_i\}$ of some subset $M^* \supseteq M_1 \supseteq E$ such that g is arbitrarily close to a constant function in every triangle Δ_i. That is, we can find a triangulation $\{\Delta_i\}$ of M_1 such that

$$\sup_{x,y\in\Delta_i} |\bar{g}(x) - \bar{g}(y)| < \delta.$$

Now, for each i, fix a point b_i in the interior of Δ_i and take a collection of pairwise different points $\{x_i\} \subset M/G$ such that $|\bar{f}(x_i) - \bar{g}(b_i)| < \delta$. Such collection exists since the required condition on f and g is open and the image of g is contained in the image of f.

We then decompose Φ into two diffeomorphisms $\Phi = \Phi_2 \circ \Phi_1$. We require that Φ fix $M/G \setminus E$ and that $\Phi_1(b_i) = x_i$. It is well-known that such a diffeomorphism can be realized as the time-1 flow of a vector field. We include a construction for completeness sake. Given a pair x_i, b_i, choose an embedding $\phi : S^1 \to E$ of the unit circle $S^1 \subset \mathbb{C}$ such that $\phi(1) = b_1$ and $\phi(-1) = x_1$. The map $z \mapsto \mathrm{d}\phi_z(iz)$ defines a vector field along $\phi(S^1)$ which can be extended as a vector field that vanishes outside an arbitrarily thin tubular neighborhood of $\phi(S^1)$. Note that the time-π flow of this vector field interchanges the position of b_i and x_i, as desired. Moreover, we can assume that no other b_j, x_j is contained in the support of the field, allowing us to construct Φ_1 as the composition of a finite number of time-π flows.

Now, choose an open neighborhood Ω of the $(n-1)$-skeleton $\cup_i \partial\Delta_i$ such that

$$\int_{\Omega} |\bar{f} \circ \Phi_1(x) - \bar{g}(x)|^p \,\mathrm{d}x < \delta.$$

Next, for every b_i, choose an open neighbourhood V_i such that $\overline{V}_i \subset M_1 \setminus \Omega$ and $|\bar{f} \circ \Phi_1(b_i) - \bar{f} \circ \Phi_1(y)| < \delta$. Construct Φ_2 in such a way that it is the identity in some neighborhood Ω_1 satisfying

$$\cup_i \partial\Delta_i \subset \Omega_1 \subset \overline{\Omega}_1 \subset \Omega,$$

and $\Phi_2(\Delta_i \setminus \Omega_1) \subset V_i$. Such a diffeomorphism can be easily realized as a flow by identifying the interior of Δ_i with an n-ball B^n. For instance, one can consider some sort of radial contraction which is the identity near the boundary and choose Ω_1 accordingly.

The arguments above together with the last computation in the proof of [KW75b, Theorem 2.1] shows that, in each Δ_i, we can choose Φ such that $\bar{f}\circ\Phi$ is L^p-close to the constant function $\bar{g}(b_i)$. The Lemma now follows by recalling that one can always lift vector fields from M^* to M^{reg} in a special way: for every point $p\in M^{reg}$ and vector $\bar{v}\in TM^*$, there is a unique $v\in(T_pGp)^\perp$ that projects to $\bar{v}$. A vector field in M^{reg} which is the result of such a lift is called a *basic horizontal fields*. Its flow is known to be G-equivariant, concluding the proof. □

We now complete the proof of Theorem C.

Proof of Theorem C. Assume that scal_g satisfies inequality (3). If needed, in order to fulfill the hypotheses of Lemma 3.2, make a G-invariant perturbation of g in a way that the inequality (3) still holds. According to Lemma 3.4, there is a G-equivariant diffeomorphism Φ such $f\circ\Phi$ is arbitrarily close to $F(g)$ in the L^p-norm. Lemma 3.3 implies that there is a G-invariant Riemannian metric $\widetilde{g}$ such that $\mathrm{scal}_{\widetilde{g}}=f\circ\Phi$. A direct computation shows that the G-invariant metric $\Phi^{-1}(c\widetilde{g})$ has scalar curvature equals to f. The regularity theory implies that, provided $p>n$, $\Phi^{-1}(c\widetilde{g})$ is smooth since $\widetilde{g}=g+A^*u$ for some smooth G-invariant function $u:M\to\mathbb{R}$. □

3.3. **A variational approach to the G-invariant Kazdan-Warner problem: the case of conformally equivalent metrics.** Throughout this section, let (M^n,g), $n\geq 3$, be a closed Riemannian manifold with an isometric effective action by a compact non-discrete Lie group G.

Given any $c\in\mathbb{R}$, we search for smooth positive G-invariant solutions to the following PDE, obtained by a suitable conformal change of g:

$$4b_n\Delta_g u-\mathrm{scal}_g u+cu^{\gamma_n}=0, \tag{7}$$

where $\gamma_n:=\dfrac{n+2}{n-2}$ and $b_n:=\dfrac{n-1}{n-2}$.

In contrast to Kazdan–Warner ([KW75a]), we proceed by variational methods. Namely, consider the following functional

$$J(u)=2b_n\int_M|\nabla u|_g^2+\frac{1}{2}\int_M\mathrm{scal}_g u^2-\frac{c}{2^*}\int_M u^{2^*}$$

defined in $W^{1,2}(M)$, where $2^*:=2n/(n-2)$. Note that J is well defined and of class C^1 according to the Sobolev Embedding Theorem. Moreover, since J is G-invariant, it admits a restriction to $W^{1,2}_G(M)$. The presence of an isometric actions makes it possible to deal with equation (7) by variational methods. This is justified since as a consequence of the classical Slice Theorem, [AB15, Theorem 3.49, p.65], every orbit in the regular stratum, which is an open dense set with full measure, has positive constant dimension.

Denote by $d:=\max\{0,\min_{x\in M}\dim Gx\}$. According to Theorem [Heb96, Theorem 5.6, p. 92], it follows that

(i) if $n-d\leq p$, then for any real number $q\geq 1$ one has that $W^{1,p}_G(M)$ compactly embeds in $L^p(M)$;

(ii) if $n-d>p$, then for any real number $1\leq q\leq \dfrac{(n-d)q}{(n-d-q)}$ one has that $W^{1,p}_G(M)$ embeds in $L^q_G(M)$ and this embedding is compact provided that $q<\dfrac{(n-d)p}{(n-d-p)}$.

One then concludes that (i) and (ii) imply (compare with [Heb96, Corollary 5.7, p. 93], [HebVa93, HebVa97]):

Theorem 3.5 (Larger embeddings)**.** *Let (M,g) be a closed Riemannian manifold with an isometric effective action by a compact non-discrete Lie group G. Then for any $p\in[1,n)$ there is $p_0>\dfrac{np}{(n-p)}=:p^*$ such that $W^{1,p}_G(M)$ embeds compactly in $L^q_G(M)$ for every $1\leq q\leq p_0$.*

Here we prove Theorem B after proving several minor results. We start by observing that:

(a) if $\mathrm{scal}_g\equiv 0$ then the PDE (7) is reduced to $4b_n\Delta_g u=-cu^{\gamma_n}$. Therefore, if one takes $c\neq 0$, the only possible solution to it is $u\equiv 0$. Similarly, if $c=0$ the only solution is $u\equiv$ constant;

(b) on the other hand, by assuming $c>0$ one can multiply (7) by u^{-1} and integrate it by parts to get
$$4b_n\int_M\frac{1}{u^2}|\nabla u|^2_g-\int_M\mathrm{scal}_g+\int_M cu^{2^*}=0.$$
Thus concluding that $\int_M\mathrm{scal}_g\geq 0$;

(c) finally, if $c=0$ then (7) is reduced to
$$4b_n\Delta_g u=\mathrm{scal}_g u.$$
Therefore, by multiplying it by u^{-1} and integrating one concludes that $\int_M\mathrm{scal}_g\geq 0$.

Considering the aforementioned obstructions we prove:

Lemma 3.6. *Assume that* scal_g *is a continuous non-zero function such that* $\min\mathrm{scal}_g\geq 0$ *and let $c>0$. Given $\epsilon>0$, consider the set*
$$\mathbf{M}_G:=\left\{u\in W^{1,2}_G(M):u\geq 0,\ \frac{c}{2^*}\int_M u^{2^*}=\epsilon\right\}.$$
For
$$J(u)=2b_n\int_M|\nabla u|^2_g+\frac{1}{2}\int_M\mathrm{scal}_g u^2-\frac{c}{2^*}\int_M u^{2^*},$$
it holds that

(1) $J|_{\mathbf{M}_G}$ is coercive;
(2) $\mathbf{M}_G$ is weakly closed and $J|_{\mathbf{M}_G}$ is weakly lower semicotinuous.

Proof. (1) Note that

$$J(u) \geq 2b_n \int_M |\nabla u|_g^2 + \frac{\min_M \mathrm{scal}_g}{2} \int_M u^2 - \frac{c}{2^*} \int_M u^{2^*} \tag{8}$$

$$= 2b_n \int_M |\nabla u|_g^2 + \frac{\min_M \mathrm{scal}_g}{2} \int_M u^2 - \epsilon. \tag{9}$$

Therefore, according to the Poincaré inequality, since $\min_M \mathrm{scal}_g \geq 0$, $J(u) \to \infty$ if $\|u\|_{W_G^{1,2}(M)} \to \infty$.

(2) Assume that $\mathbf{M}_G \supset \{u_m\} \rightharpoonup u \in W^{1,2}(M)$. Once $\{u_m\} \subset W_G^{1,2}(M)$ and this is a Banach space (so it is convex and closed, then weakly closed) one has that $u \in W_G^{1,2}(M)$. According to Theorem 3.5 one has the compact embedding of $W_G^{1,2}(M)$ in $L^{2^*}(M)$, from where it follows that $c\int_M u^{2^*} = \epsilon 2^*$. Moreover, the sequence $\{u_m\}$ has a pointwise convergent subsequence, so that $u \geq 0$ almost everywhere. Therefore, $u \in \mathbf{M}_G$ and $\mathbf{M}_G$ is weakly closed.

On the other hand, the weak convergence implies that

$$\liminf_{m\to\infty} \|u_m\|_{1,2} \geq \|u\|_{1,2}.$$

Moreover, since $\int_M u_m^2 \to \int_M u^2$ is a convergent sequence, one has

$$\liminf_{m\to\infty} \|u_m\|_{1,2}^2 = \int_M u^2 + \liminf_{m\to\infty} \int_M |\nabla u_m|^2.$$

Hence

$$\liminf_{m\to\infty} \int_M |\nabla u_m|^2 \geq \int_M |\nabla u|^2.$$

Since scal_g is continuous, one concludes that

$$\liminf_{m\to\infty} J(u_m) \geq 2b_n \int_M |\nabla u|^2 + \frac{1}{2} \int_M \mathrm{scal}_g u^2 - \frac{c}{2^*} \int_M u^{2^*}. \qquad \square$$

It then follows that the restriction $J\Big|_{\mathbf{M}_G}$ has a minimum $u \in \mathbf{M}_G$. We will show that has a symmetric minimum point. Note that if $v \in W_G^{1,2}(M)$, the Lagrange Multiplier Theorem states that there is $\lambda \in \mathbb{R}$ such that

$$J'(u)(v) = 4b_n \int_M \langle \nabla u, \nabla v \rangle + \int_M \mathrm{scal}_g uv = (1+\lambda)c \int_M u^{\gamma_n} v.$$

Therefore, u is a weak solution of (7) with c replaced by $c' = (1+\lambda)c$. On the other hand, by computing $J'(u)(u)$, we conclude that $1+\lambda > 0$, thus $c' > 0$.

Moreover, due to the regularity theory of elliptic PDE's one concludes that the solution u is smooth as long as scal_g is smooth. In fact, note that the non-linearity on the PDE (7) corresponds to the term u^{γ_n}. Since the function $F : x \to x^{\gamma_n}$ is of class C^1 and the solution u has finite essential supremum, then $F(u) \in W^{1,2}(M)$. An iterative application of Theorem 3.58 in [Aub98, p. 87] implies the result. The maximum principle [Aub98, Proposition 3.75, p.98] implies that the obtained solution is positive. A simple rescaling of the resulting metric (corresponding to $u \mapsto u + a$, for some a) provides the right constant scalar curvature metric. We thus have proved that:

Proposition 3.7. *Let (M^n, g), $n \geq 3$ be a closed Riemannian manifold with an isometric effective action by a compact non-discrete Lie group G. If $\mathrm{scal}_g \geq 0$ and $\mathrm{scal}_g \not\equiv 0$ then any $c > 0$ is the scalar curvature of a G-invariant metric.*

As an immediate consequence one gets:

Corollary 3.8. *Let $(M^n, g), n \geq 3$ be closed Riemannian manifold with an effective isometric action by a compact non-discrete Lie group G. Assume that M does not carry a G-invariant metric of positive scalar curvature. Then, if $\mathrm{scal}_g \geq 0$ necessarily $\mathrm{scal}_g \equiv 0$.*

We proceed to the proof of item (1) in Theorem B.

Proposition 3.9. *Let (M^n, g), $n \geq 3$ be a closed Riemannian manifold with an isometric effective action by a compact non-discrete Lie group G. Then there is $c \geq 0$ such that for any $c' \geq c$ there exists a G-invariant Riemannian metric g such that $\mathrm{scal}_g = -c'$.*

Proof. Take $c \geq 0$ such that

$$\left(\frac{2^*}{2}\right) \min_M \mathrm{scal}_g \mathrm{vol}(M)^{1-2^*/2} + c \geq 0,$$

and consider the functional

$$J(u) = 2b_n \int_M |\nabla u|_g^2 + \frac{1}{2} \int_M \mathrm{scal}_g u^2 + \frac{c}{2^*} \int_M u^{2^*}.$$

Hölder inequality implies that

$$\left(\int_M u^2\right) \leq \mathrm{vol}(M)^{1-2/2^*} \left(\int_M u^{2^*}\right)^{2/2^*}. \tag{10}$$

We now consider two separate cases. Depending if $\min \mathrm{scal}_g \leq 0$ or $\min \mathrm{scal}_g > 0$, respectively, we have

$$J(u) \geq 2b_n \int_M |\nabla u|_g^2 + \frac{1}{2} \min_M \mathrm{scal}_g \mathrm{vol}(M)^{1-2/2^*} \left(\int_M u^{2^*}\right)^{2/2^*} + \frac{c}{2^*} \int_M u^{2^*};$$

$$J(u) \geq 2b_n \int_M |\nabla u|_g^2 + \frac{c}{2^*} \int_M u^{2^*}.$$

To show that J is coercive, observe that if $\int_M u^{2^*} \to \infty$ then $\left(\int_M u^{2^*}\right)^{2/2^*} < \int_M u^{2^*}$. Hence,

$$J(u) \geq 2b_n \int_M |\nabla u|_g^2 + \left(\frac{1}{2}\min_M \mathrm{scal}_g \mathrm{vol}(M)^{1-2/2^*} + \frac{c}{2^*}\right)\int_M u^{2^*};$$
$$J(u) \geq 2b_n \int_M |\nabla u|_g^2 + \frac{c}{2^*}\int_M u^{2^*}.$$

from where it follows that J is coercive if $\frac{2^*}{2}\min_M \mathrm{scal}_g \mathrm{vol}(M)^{1-2/2^*} + c > 0$ or $c > 0$, respectively. Whenever one of these terms vanish we can proceed in a similar fashion as in item 1. in Lemma 3.6.

Following further the proof of Lemma 3.6, we can show that J has a critical point in $W^{1,2}_G$. The Principle of Symmetric Criticality of Palais [Pal79, p. 23] implies that this is a weak solution for the PDE (7). Combining Theorem 3.58 on [Aub98, p. 87] and the maximum principle [Aub98, Proposition 3.75, p.98] we conclude that such weak solution is smooth and positive. □

We thus conclude the proof of Theorem B: items (1), (2) and (3) follows from Proposition 3.9, Corollary 3.8 and Proposition 3.7, respectively.

We finish this section by proving Theorem A.

Proof of Theorem A. We first observe that Theorem C implies that a manifold admitting a G-invariant metric with constant scalar curvature c guarantees that every basic function that is negative somewhere (resp. positive) is a scalar curvature of (M, G), if $c < 0$ (resp. $c > 0$). Therefore, Proposition 3.9 shows that every (M, G) admits any scalar curvature function that is negative somewhere. Moreover, if a non-vanishing non-negative scalar is admissible, Proposition 3.7 shows that $(M, G) \in \mathscr{P}^G$.

The last assertion in the Theorem follows from a slightly improved version of Theorem A in [LY74], proved in [CeSS18], and Proposition 3.7:

Theorem 3.10 (Cavenaghi–e Silva–Sperança)**.** *Let (M, g) be a compact Riemannian manifold with an effective isometric G-action, where G is a compact non-abelin Lie group. Then g develops positive scalar curvature after a finite Cheeger deformation.*

□

4. Improved curvature estimates

The proof of Theorem 3.10, together with Theorem 3.7, provide a better view on the variety of functions that are realizable as scalar curvature functions of (G-invariant) metrics. Specifically, the proof of Theorem 3.10 gives a pinching estimate for a G-invariant metric after a long Cheeger deformation, which is described only in terms of orbit types and isotropy representations. On the other hand, a similar behaviour is found after a canonical deformation. We study both cases in this section.

4.1. **Scalar curvature Pinching under Cheeger deformations.** Given a singular point $x \in M$, its isotropy representation $\tilde{\rho}_x : G_x \to O(T_xM)$, naturally acts on the space normal to the orbit. Fixed $X \in \nu_x G_x$, the differential of $\tilde{\rho}_x$, computed at the identity, defines a map

$$\begin{aligned} \rho_X : \mathfrak{g}_x &\to \nu_x Gx \\ U &\mapsto (d\tilde{\rho}_x)_e(U)X. \end{aligned}$$

Following [CeSS18], we call an element in the image of ρ_X as a *fake horizontal vector with respect to* X, and denote $\rho_X(\mathfrak{g}_x) = \mathcal{H}_X$. We (abuse notation and) denote by ρ_X^{-1} the inverse of ρ_X restricted to $\mathcal{H}_X$.

If $\mathfrak{g}$ is non-abelian, we can describe the scalar curvature pinching of a long-time Cheeger deformation by using the following parameters:

$$\overline{\mathrm{scal}}_{G/G_x} = \frac{1}{4}\sum_{i,j=1}^{k} \|[v_i, v_j]\|_g^2, \tag{11}$$

$$\xi(\rho_x) = \sum_{i,j=1}^{n-k} \frac{\|e_j^{\mathcal{H}_{e_i}}\|_g^4}{\|\rho_{e_i}^{-1}(e_j^{\mathcal{H}_{e_i}})\|_Q^2} \tag{12}$$

where $\{e_i\}$ and $\{v_i\}$ are g- and Q-orthonormal basis for $\nu_x Gx$ and $\mathfrak{m}_x$, respectively.

Proposition 4.1. *Suppose* $\mathfrak{g}$ *non-abelian and let* g_t *be the Cheeger deformation of* g. *Then,*

$$\lim_{t\to\infty} \frac{\max_{x\in M} \mathrm{scal}_{g_t}(x)}{\min_{x\in M} \mathrm{scal}_{g_t}(x)} = \frac{\max_{x\in M}\{\overline{\mathrm{scal}}_{G/G_x} + 3\xi(\rho_x)\}}{\min_{x\in M}\{\overline{\mathrm{scal}}_{G/G_x} + 3\xi(\rho_x)\}}.$$

Proof. Given a G-invariant metric g, the scalar curvature of its Cheeger deformation was computed in [CeSS18]:

$$\begin{aligned} \mathrm{scal}_{g_t}(p) = \sum_{i,j=1}^{n} K_g(C_t^{1/2}e_i, C_t^{1/2}e_j) + \sum_{i,j=1}^{n} z_t(C_t^{1/2}e_i, C_t^{1/2}e_j) \\ + \sum_{i,j=1}^{k} \frac{\lambda_i\lambda_j t^3}{(1+t\lambda_i)(1+t\lambda_j)}\frac{1}{4}\|[v_i, v_j]\|_Q^2. \end{aligned} \tag{13}$$

Some definitions are required:

(1) Given $x \in M$, let $P : \mathfrak{m}_x \to \mathfrak{m}_x$ is the unique endomorphism that satisfies $Q(PU, V) = g(U^*, V^*)$ for every $U, V \in \mathfrak{m}_x$. Consider $\{\lambda_i\}$ as the set of eigenvalues of P;
(2) $C_t : T_xM \to T_xM$ is defined by $C_t|_{\nu_x Gx} = 1$ and $C_tU^* = (1 + tP)^{-1}(U^*)$.

z_t, on its turn, is quadratic in each entry and is described in [CeSS18, Lemma 3.1.1]. We gather the relevant information below:

Lemma 4.2. *For every* $\overline{X} = X + U^*$, $\overline{Y} = Y + V^*$, $X, Y \in \nu_x Gx$, z_t *satisfies*

$$z_t(\overline{X},\overline{Y}) = 3t \max_{\substack{Z\in\mathfrak{g},\\ \|Z\|_Q=1}} \frac{\{\mathrm{d}w_Z(\overline{X},\overline{Y}) + \frac{t}{2}Q([PU,PV],Z)\}^2}{t\, g(Z^*,Z^*)+1}, \tag{14}$$

where $w_Z(\overline{X}) = \frac{1}{2}\, g(\overline{X}, Z^*)$ *is a one-form in* TM *satisfying*

$$\mathrm{d}w_Z(V^*,X) = \frac{1}{2} X\, g(V^*,Z^*), \tag{15}$$

$$\mathrm{d}w_Z(Y,X) = g(\nabla_X Y, Z^*), \tag{16}$$

$$\mathrm{d}w_Z(U^*,V^*) = Q([PU,V] + [U,PV] - P[U,V], Z). \tag{17}$$

Moreover, if $U \in \mathcal{H}_X$,

$$\mathrm{d}w_Z(U,X) = g(U, \nabla_X Z^*). \tag{18}$$

One concludes that the second and third summands in (13) are either zero or grow like t. It is sufficient to study these terms to prove the Theorem since every other term is bounded. Moreover,

$$\lim_{t\to\infty} \frac{1}{t} K_g(C_t^{1/2} e_i, C_t^{1/2} e_j) = 0 \tag{19}$$

$$\lim_{t\to\infty} \frac{1}{t} \sum_{i,j=1}^{k} \frac{\lambda_i\lambda_j t^3}{(1+t\lambda_i)(1+t\lambda_j)} \frac{1}{4} \|[v_i,v_j]\|_Q^2 = \overline{\mathrm{scal}}_{G/G_x}; \tag{20}$$

$$\lim_{t\to\infty} \frac{1}{t} \sum_{i,j=1}^{n} z_t(C_t^{1/2} e_i, C_t^{1/2} e_j) = \lim_{t\to\infty} \frac{1}{t} \sum_{i,j=1}^{n-k} z_t(e_i,e_j). \tag{21}$$

It is just left to study the last term. Using Lemma 4.2, one concludes that every term in (21) with either $i > n-k$ or $j > n-k$ is bounded, since $C_t^{1/2}|_{TGx} = (1+tP)^{-1/2}$ goes to zero as $t^{-1/2}$. To analyze the case $i, j \leq n-k$, we use the compactness of the unit ball $\|Z\|_Q = 1$, and take a fixed Z such that

$$\lim_{t\to\infty} \tfrac{1}{t} z_t(e_i,e_j) = 3 \lim_{t\to\infty} \frac{\mathrm{d}w_Z(e_i,e_j)^2}{tg(Z^*,Z^*)+1}.$$

One readily sees that this limit is zero unless $g(Z^*,Z^*) = 0$. On the other hand, if $Z \in \mathfrak{g}_x$, by observing that

$$\nabla_{e_j} Z^*(x) = (\mathrm{d}\rho_x)_e(Z)(e_j) = \rho_{e_j}(Z),$$

Lemma 4.2 guarantees that $\mathrm{d}w_Z(e_i,e_j) = \mathrm{d}w_Z(e_i^{\mathcal{H}_{e_j}}, e_j)$. In this case, Z can be taken as $\rho_{e_j}^{-1}(e_i^{\mathcal{H}_X})/\|\rho_{e_j}^{-1}(e_i^{\mathcal{H}_{e_j}})\|$, which necessarily realizes the maximum and shows that

$$\sum_{i,j=1}^{n-k} \lim_{t\to\infty} \tfrac{1}{t} z_t(e_i,e_j) = 3\xi(\rho_x). \qquad \square$$

Theorem F now follows directly from Proposition 4.1.

Proof of Theorem F. Since a semi-free action has no singular orbits, the ξ-term is not present. Moreover, since the regular part of M is convex (see [AB15, Theorem 3.49, p. 65]), every pair of isotropy Lie subalgebra are conjugate to each other. Thus, we conclude that $\overline{\mathrm{scal}}_{G/G_x}$ is constant with respect to x, making the limit in Proposition 4.1 equal to 1. Theorem F now follows from Theorem A in [KW75a] and the paragraph before Corollary D. $\square$

We now complete the proof of Theorem E by considering $\mathfrak{g}$ abelian.

Proof of Theorem E. As in the proof of Lemma 3.4, we work in the regular part of the M/G. Denote by $\pi : M \to M/G$ the quotient map and note that the restriction of π to the regular stratum defines a smooth principal bundle with image a smooth manifold. Denote by $\tilde{g}$ the Riemannian metric onduced by this restriction of π to its image and assume that $\bar{g}$ has positive scalar curvature. In particular $\mathrm{scal}_{\tilde{g}}$ has a positive lower bound in any compact set inside the regular stratum. Since Lemma 4.2 guarantees that $\lim_{t\to\infty} \mathrm{scal}_{g_t}(p) = +\infty$ at non-regular points, it is sufficient to prove that $\mathrm{scal}_{g_t}(p)$ converges to $\mathrm{scal}_{\tilde{g}}$ in the regular stratum.

To this aim, assume $\mathfrak{g}$ abelian and recall that the respective equations for the Ricci curvature of g_t are, as in [CeSS18, Lemma 2.6],

$$\mathrm{Ric}_{g_t}(C_t^{1/2}V^*) = \lim_{t\to\infty}\sum_{i=1}^{n} K_g(C_t^{1/2}e_i, C_t^{1/2}V^*) + \sum_{i,j=1}^{n} z_t(C_t^{1/2}e_i, C_t^{1/2}V^*),$$

$$\mathrm{Ric}_{g_t}(X) = \sum_{i=1}^{n} K_g(C_t^{1/2}e_i, C_t^{1/2}X) + \sum_{i,j=1}^{n} z_t(C_t^{1/2}e_i, C_t^{1/2}X),$$

where $V \in \mathfrak{g}$ and $X \in (T_pGp)^{\perp}$.

One the one hand, observe that $C_t^{1/2}V^* \to 0$ for every $V \in \mathfrak{g}$ and z_t is bounded uniformly on t at points in the regular stratum (Lemma 4.2). Thus, $\mathrm{Ric}_{g_t}(V^*) \to 0$ whenever $\mathfrak{g}$ is abelian.

On the other hand, for p in the regular stratum, [CeSS18, Lemma 4.2] or [SW15, Proposition 4.3] implies that

$$\lim_{t\to\infty} \mathrm{Ric}_{g_t}(X) = \mathrm{Ric}_{\bar{g}}(\mathrm{d}\pi X)$$

for every $X \in (T_pGp)^{\perp}$. Since $C_t^{1/2}X = X$, we conclude that $\mathrm{scal}_{g_t}(p)$ converges to $\mathrm{scal}_{\bar{g}}(\pi(p))$, as desired. $\square$

4.2. **Canonical variations on Fiber Bundles.** Let $F \hookrightarrow (\overline{M}, g') \stackrel{\pi}{\to} (M, h)$ be a fiber bundle with compact structure group G. From Theorem B, there is a G-invariant metric g_F on F with constant scalar curvature c. Therefore, there is a metric g on $\overline{M}$, where π is a Riemannian submersion over (M, h) and every fiber is isometric to (F, g_F) (this is a standard construction using associated bundles. See, for example, Proposition 2.7.1

in [GW09], for details). Moreover, the fibers of $\pi : (\overline{M}, g) \to (M, h)$ can be taken as totally geodesic.

Corollary G is proved by using a widely-known deformation called *canonical variation.* To simplify the text, we denote

$$\mathcal{V} = \ker \mathrm{d}\pi, \qquad \mathcal{H} = (\ker \mathrm{d}\pi)^{\perp}.$$

Given $s > 0$, the canonical variation $\widetilde{g}$ of g is defined as:

$$\widetilde{g} = g\Big|_{\mathcal{H}\times\mathcal{H}} + s\, g\Big|_{\mathcal{V}\times\mathcal{V}}.$$

Since the fibers are contracted, it is expected that the positive curvature of the fibers increases the overall curvature, somehow. It is easy to see that this expected behaviour is the right one and this is the first step towards Theorem G. The remaining is a direct application of a result in [Ber83] and Theorem E.

Lemma 4.3. *Suppose $F \hookrightarrow \overline{M} \to M$ is a fibre bundle with compact structure group G. If $(F, G) \in \mathscr{P}^G$, then $\overline{M} \in \mathscr{P}$.*

Proof. We first recall the curvature formulas in [Bes87].

Proposition 4.4. *Let $\pi : (F, g_F) \hookrightarrow (\overline{M}, g) \to (M, h)$ be a Riemannian submersion with totally geodesic fibers. Denote by $K_{\widetilde{g}}, K_g, K_h, K_{g_F}$ the non-reduced sectional curvatures of $\widetilde{g}, g, h$ and g_F, respectively. If $X, Y, Z \in \mathcal{H}$ and $V, W \in \mathcal{V}$, then*

$$\begin{aligned} K_{\widetilde{g}}(X, Y) &= K_g(\mathrm{d}\pi X, \mathrm{d}\pi Y)(1 - s) + sK_g(X, Y), \\ K_{\widetilde{g}}(X, V) &= s^2 K_g(X, V), \\ K_{\widetilde{g}}(V, W) &= sK_{g_F}(V, W). \end{aligned}$$

Fix $x \in M$ and let $\{e_i\}_{i=1}^n$ be an orthonormal basis for $\mathcal{H}_x$ and $\{e_j\}_{j=n+1}^{n+k}$ be an orthonormal basis for $\mathcal{V}_x$. Note that $\{e_i\}_{i=1}^n \cup \{s^{-\frac{1}{2}} e_j\}_{j=n+1}^{n+k}$ is a $\widetilde{g}$-orthonormal basis for T_xM. Using Proposition 4.4, we obtain:

$$\mathrm{scal}_{\widetilde{g}}(x) = \mathrm{scal}_{\widetilde{g}}{}^{\mathcal{H}}(x) + 2s \sum_{j=n+1}^{n+k} \sum_{i=1}^{n} K_g(e_i, e_j) + s^{-1}\, \mathrm{scal}_{g_F}(x),$$

where

$$\mathrm{scal}_{\widetilde{g}}^{\mathcal{H}}(x) = \mathrm{scal}_h(x)(1 - s) + s \sum_{i,j=1}^{n} K_{\widetilde{g}}(e_i, e_j).$$

Therefore, if $c = \mathrm{scal}_{g_F} > 0$, $\mathrm{scal}_{\widetilde{g}} > 0$ for s sufficiently small. □

Proof of Theorem G. Consider a fibre bundle $\pi : \overline{M} \to M$ with compact structure group G and suppose that $\overline{M} \notin \mathscr{P}$. From Lemma 4.3, we conclude that $(F, G) \notin \mathscr{P}^G$ therefore, G is a compact group whose identity component is a torus T.

Recall that π can be seen as a bundle associated to a principal G-bundle $P \to M$. More specifically, $\overline{M}$ is diffeomorphic to the twisted product

$P\times_G F$. On the other hand, a result of Bergery [Ber83] shows that a manifold N with a free T-action admits an invariant metric with positive scalar curvature if and only if the quotient N/T does. We conclude that $\overline{M} = P\times_T F$ admits a metric with positive scalar curvature if and only if $P \times F$ admits a T-invariant metric with positive scalar curvature. On the other hand, if $M = P/T$ would have a metric with positive scalar curvature, P would have a T-invariant metric with positive scalar curvature. Therefore, $P \times F$ would have a $T \times T$-invariant metric with positive scalar curvature. Since $\overline{M} \notin \mathscr{P}$, we conclude that no metric on M or T-invariant metric on (F, T) has positive scalar curvature. ☐

5. Prescribing scalar curvature via general vertical warpings

Our goal in this section is to study how to prescribe scalar curvature in fiber bundles which are, essentially, locally isometric to products. To this aim, we resort to a natural generalization of the canonical variation, called *general vertical warping*. Given a Riemannain submersion $\pi : (\overline{M}, g) \to (M, h)$ and a smooth function $\phi : M \to \mathbb{R}$, we define the metric g_ϕ along the same lines as the canonical variation:

$$g_\phi = g\Big|_{\mathcal{H}\times\mathcal{H}} + e^{2\phi}\, g\Big|_{\mathcal{V}\times\mathcal{V}}.$$

The submersion $\pi : (\overline{M}, g_\phi) \to (M, h)$ is Riemannian and its fibers are $e^{2\phi}$-homotetic to the original fibers in $(\overline{M}, g)$. The main result in this section is Proposition 5.1, from which Theorem H follows.

Define

$$b_k := \frac{k+1}{8k},\ c_k := \frac{(k+1)^2}{8(k-1)k},\ \theta_k := 2\frac{k-1}{k+1}. \tag{22}$$

Remark. *In what follows, the superscripts denote the dimension of the underlined manifolds.*

Proposition 5.1. *Let $\pi : F^k \hookrightarrow (\overline{M}^{n+k}, g) \to (M^n, h)$ be a Riemannian submersion from a closed oriented manifold $\overline{M}$. Assume that $(\overline{M}, g)$ is locally isometric to a product whose fibers have constant scalar curvature $c \in \mathbb{R}$.*

1. *If $c > 0$ and $\min_{\overline{M}} \mathrm{scal}_h > 0$ then any basic smooth function is the scalar curvature of some general vertical warping metric;*
2. *Assume $c \leq 0$ and suppose that a basic function $f : \overline{M} \to \mathbb{R}$ satisfies the condition:*

$$-b_k\,(f - \mathrm{scal}_h) + c_k c\,(\mathrm{vol}(M))^{2/\theta_k - 1} \geq 0. \tag{23}$$

Then, if $|c| > 0$, f is the scalar curvature of some general vertical warping metric. In contrast to it, if $c = 0$, then there exists some $A \geq 0$ such that $f + A$ is the scalar curvature of a general vertical warping metric.

Before proceeding to the proof, we observe that the scope of application of Proposition 5.1 is broader than local products of the form $M\times F$. Indeed, every warped product $(\overline{M},g)=(M\times F, h\times_{\phi} g_f)$ admits a metric where it is locally a product. Lemma 5.1 also holds for submersions which are locally warped products (see section 6 for details.)

We now proceed to the proof of Proposition 5.1. We divide it in two cases:

5.1. **The case** $c\leq 0$**.** To begin with, assume that (M^n,h) is a closed Riemannian manifold and (F,g_F) is a Riemannian manifold with constant scalar curvature $c\leq 0$. Let $f: M\times F\to\mathbb{R}$ be a smooth basic function. We start by discussing the proof of Theorem 5.1 to the case where $\overline{M}$ is a Riemannian product. To do so, we apply standard variational methods to ensure the existence of positive solutions for the following PDE:

$$\Delta_h u+\frac{k+1}{4k}(f-\mathrm{scal}_h)u-\frac{k+1}{4k}u^{\frac{k-3}{k+1}}c=0. \tag{24}$$

Equation (24) is obtained from a general vertical warping through Lemma 5.2 (see [CS21, Lemma 1, p. 14] for the proof).

Lemma 5.2. *Let (M,h) and (F,g_F) be Riemannian manifolds and $\phi: M\to\mathbb{R}$ be a smooth function. Denote by g the warped metric on $M\times_{e^{2\phi}}F$. Then the scalar curvature of g is given by*

$$\mathrm{scal}_g=\mathrm{scal}_h+e^{-2\phi}\mathrm{scal}_{g_F}-k(k-1)|\nabla\phi|_h^2-2k|\nabla\phi|_h^2-2k\Delta_h\phi. \tag{25}$$

Remark. *Note that since* $\mathrm{scal}_{g_F}=c$*, equation* (25) *implies that* scal_g *is basic.*

Given a basic function $f\in C^\infty(M\times F;\mathbb{R})$, consider the PDE

$$\mathrm{scal}_h+e^{-2\phi}\mathrm{scal}_{g_F}-f=k(k-1)|\nabla\phi|_h^2+2k\left\{|\nabla\phi|_h^2+\Delta_h\phi\right\}. \tag{26}$$

The solution ϕ is such that f is the scalar curvature of the warped metric $\widetilde{g}=h+e^{2\phi}g_F$. By setting $\phi=\log u^{\frac{2}{k+1}}$ a direct computation shows that equation (26) is reduced to equation (24).

To show that the PDE (24) has a positive solution, assume that f,scal_h are continuous functions. Define the following functional $J: W^{1,2}(M)\to\mathbb{R}$ by

$$J(u)=\frac{1}{2}\int_M|\nabla u|_h^2-b_k\int_M(f-\mathrm{scal}_h)\,u^2+c_k\int_M cu^{\theta_k}. \tag{27}$$

Note that J is well defined and according to the classical Sobolev Embedding Theorem it is of class C^1.

5.1.1. *The case* $c<0$. Assume that $c<0$. We will obtain the desired solution u to equation (24) as a minimum for J restricted to the set $\mathbf{M}:=\{u\in W^{1,2}(M): u\geq 0,\ \int_M u^{\theta_k}=1\}$. This is the content of Lemma 5.3.

Lemma 5.3. *Define* $\mathbf{M} := \{u \in W^{1,2}(M) : u \geq 0, \int_M u^{\theta_k} = 1\}$. *Assume that* f, scal_h *are continuous functions, that* $c < 0$ *and that the following inequality holds*

$$-b_k \left(f - \mathrm{scal}_h\right) + c_k c \left(\mathrm{vol}(M)\right)^{2/\theta_k - 1} \geq 0. \tag{28}$$

Then,

(1) There is a constant $c_0 > 0$ *such that* $J\big|_{\mathbf{M}} > c_0$,
(2) $J\big|_{\mathbf{M}}$ *is coercive,*
(3) $\mathbf{M}$ *is weakly closed and* $J\big|_{\mathbf{M}}$ *is weakly lower semi-continuous.*

Before proving Lemma 5.3, let us show how Proposition 5.1 follows from it.

Lemma 5.3 ensures that $J\big|_{\mathbf{M}}$ has a minimum point in $\mathbf{M}$. Therefore, given any function $v \in W^{1,2}(M)$ tangent to $\mathbf{M}$, one has that either

$$0 = \frac{d}{dt}\Big|_{t=0} J(u + tv) = \int_M \left(\Delta_h u + \frac{k+1}{4k}(f - \mathrm{scal}_h)u - \frac{k+1}{4k} c u^{k-3/k+1}\right) v,$$

from where it follows that u is a weak solution to (24); or we have a Lagrange Multiplier problem. We omit the first case since it is subcase of the latter (when $\Lambda = 0$ below). Therefore, we suppose that u is such that there exists $\Lambda \in \mathbb{R}$ satisfying

$$\int_M \langle \nabla u, \nabla v\rangle - 2b_k \int_M (f - \mathrm{scal}_h) u v + c c_k \theta_k \int_M u^{\theta_k - 1} v = \theta_k \Lambda \left(\int_M u^{\theta_k - 1} v\right), \tag{29}$$

$\forall v \in W^{1,2}(M)$. We claim that $\Lambda \geq 0$. By taking $v = u$, we have

$$\int_M |\nabla u|^2 - 2b_k \int_M (f - \mathrm{scal}_h) u^2 + c c_k \theta_k \int_M u^{\theta_k} = \theta_k \Lambda \int_M u^{\theta_k} \tag{30}$$

However, sine $\theta_k \leq 2$, Hölder inequality implies

$$\int_M u^{\theta_k} = \left(\int_M u^{\theta_k}\right)^{2/\theta_k} \leq \left(\mathrm{vol}(M)\right)^{2/\theta_k - 1} \int_M u^2.$$

Therefore,

$$\begin{aligned} \theta_k \Lambda \int_M u^{\theta_k} &\geq \int_M |\nabla u|^2 + 2\int_M \left(-b_k (f - \mathrm{scal}_h) + \frac{\theta_k}{2} c c_k (\mathrm{vol}(M))^{2/\theta_k - 1}\right) u^2 \\ &\geq 2\int_M \left(-b_k (f - \mathrm{scal}_h) + c c_k (\mathrm{vol}(M))^{2/\theta_k - 1}\right) u^2 \geq 0. \end{aligned} \tag{31}$$

On the other hand, by multiplying both sides of (29) by $\zeta > 0$, we have

$$\int_M \langle \nabla(\zeta u), \nabla v\rangle - 2b_k \int_M (f - \mathrm{scal}_h)(\zeta u) v + \zeta^{-\theta_k + 2}(c c_k - \Lambda)\theta_k \int_M (\zeta u)^{\theta_k - 1} v = 0,$$

$\forall v \in W^{1,2}(M)$. Since $\Lambda \geq 0$, we can find ζ such that $\zeta^{-\theta_k + 2}(c c_k - \Lambda) = c c_k$, concluding that $\tilde{u} = \zeta u$ is a weak solution for equation (24).

To finish the argument, note that if f and scal_h are smooth functions, the classical regularity theory for elliptic PDE's (see [Aub98, Theorem 3.58, p. 87]) guarantees that the solution u is smooth. Finally, we observe that the solution u is positive. Otherwise, a Maximum Principle argument as in [Aub98, Proposition 3.75, p.98] guarantees that $u \equiv 0$.

We thus have proved that any basic smooth function $f : M \times F \to \mathbb{R}$ satisfying (23) can be realized as the scalar curvature of a warped product metric, thus concluding item (2) of Proposition 5.1 in the case of Riemannian products. Next we prove Lemma 5.3.

Proof of Lemma 5.3. It is easy to see that $\mathbf{M} \neq \emptyset$. Moreover, in analogy to (31), we have

(32)
$$\begin{aligned} J(u) &\geq \frac{1}{2}\int_M |\nabla u|^2 + \int_M \left(-b_k(f - \mathrm{scal}_h) + cc_k(\mathrm{vol}(M))^{2/\theta_k - 1}\right) u^2 \\ &\geq \int_M \left(\frac{\lambda_1}{2} - b_k(f - \mathrm{scal}_h) + cc_k(\mathrm{vol}(M))^{2/\theta_k - 1}\right) u^2 - \frac{\lambda_1}{2\mathrm{vol}(M)}\left(\int_M u\right)^2, \end{aligned}$$

where λ_1 is the first eigenvalue of Δ. On the other hand, Hölder inequality gives:

$$\int_M u \leq \mathrm{vol}(M)\left(\int_M u^{\theta_k}\right)^{1/\theta_k} = \mathrm{vol}(M),$$

giving an explicit bound for the last term above. We thus conclude that $J(u)$ has a lower bound, since $-b_k(f - \mathrm{scal}_h) + cc_k(\mathrm{vol}(M))^{2/\theta_k - 1} \geq 0$ by hypothesis. Likewise, coercivity follows directly from (32) and the mentioned hypothesis.

To see that J is weakly lower semi-continuous note that according to the Rellich–Kondrachov's Theorem, the Sobolev space $W^{1,2}(M)$ compactly embeds in $L^2(M)$. Therefore, take $\{u_m\} \subset \mathbf{M}$ weakly converging to $u \in H^1(B)$. Since $\mathbf{M}$ is weakly closed it follows that $u \in \mathbf{M}$. Moreover, since $u_m \to u \in L^2(M)$; M is compact; scal_h, scal_{g_F} and f are continuous functions and

$$\int_M |\nabla u|_h^2 \leq \liminf_{m\to\infty} \int_M |\nabla u_m|_h^2,$$

we conclude that

$$\begin{aligned} \liminf_{m\to\infty} J(u_m) &= \liminf_{m\to\infty}\left(\int_M |\nabla u_m|_h^2 - b_k\int_M (f - \mathrm{scal}_h)u_m^2 + c_k\int_M c u_m^{\theta_k}\right) \\ &\geq \int_M |\nabla u|_h^2 - b_k\int_M (f - \mathrm{scal}_h)u^2 + c_k\int_M c u^{\theta_k} = J(u). \qquad \square \end{aligned}$$

5.1.2. *The case $c = 0$.* We finally deal with the case $c = 0$. We only sketch it since it follows closely to the proof of Lemma 5.3 as we shall make it clear.

In this case the functional J is reduced to

$$J(u) = \frac{1}{2}\int_M |\nabla u|_h^2 - b_k \int_M (f - \mathrm{scal}_h)u^2 \tag{33}$$

To this case we can simply search for a minimum to J restricted to the set

$$\mathbf{M} = \left\{u \in W^{1,2}(M) : \int_M u^2 = 1\right\}.$$

Since the Hölder inequality implies that for any $u \in \mathbf{M}$ we have $(\int_M u)^{\frac{1}{2}} \leq C\left(\int u^2\right) = C$, it is immediate to see that the Poincaré inequality implies that J has a lower-bound restricted to $\mathbf{M}$. Moreover, under the assumption that $f - \mathrm{scal}_h \geq 0$ the coercivity also follows. Also, the same routine argument used in Lemma 5.3 implies that $\mathbf{M}$ is weakly-closed. Therefore, J has a critical point in $\mathbf{M}$.

Once more, given any function $v \in W^{1,2}(M)$ tangent to $\mathbf{M}$, one has that either

$$0 = \frac{d}{dt}\Big|_{t=0} J(u+tv) = \int_M \left(\Delta_h u + \frac{k+1}{4k}(f - \mathrm{scal}_h)u - \frac{k+1}{4k}cu^{k-3/k+1}\right)v,$$

from where it follows that u is a weak solution to (24); or we have a Lagrange Multiplier problem. Again we omit the first case since it is subcase of the latter.

The Lagrange Multiplier problem can be read as

$$\int_M \langle \nabla u, \nabla v\rangle - 2b_k \int_M (f - \mathrm{scal}_h)uv = 2\Lambda \int_M uv, \tag{34}$$

for every $v \in W^{1,2}(M)$. Making $v = u$ we see that since $-2b_k \max_M(f - \mathrm{scal}_h) \geq 0$ and $\int_M u^2 = 1$ then $\Lambda \geq 0$.

Now let us rewrite equation (34) as

$$\int_M \langle \nabla u, \nabla v\rangle - 2b_k \int_M (f - \mathrm{scal}_h + \Lambda b_k^{-1})uv = 0. \tag{35}$$

Note that equation (35) is the weak formulation of the warped product approach to prescribing the function $f + \Lambda b_k^{-1}$, what concludes the result defining $A := +\Lambda b_k^{-1}$.

We finish this subsection by sketching the proof of the general case of item (2) of Proposition 5.1 – it follows via straightforward modifications of the previous arguments. We start by recalling the following result (see [CS21, Lemma 3, p. 14], for a proof).

Lemma 5.4. *Let $F^k \hookrightarrow (\overline{M}^{n+k}, \mathrm{g}) \xrightarrow{\pi} (M^n, h)$ be a Riemannian submersion. Let $\widetilde{\mathrm{g}}$ be a general vertical warping metric on M via the function $u^{\frac{4}{k+1}}$, where $u : M \to \mathbb{R}$ is a smooth basic function. Then the scalar curvature of $\widetilde{\mathrm{g}}$ is* *given by*

$$\operatorname{scal}_{\widetilde{g}} = \operatorname{scal}_g - \frac{4k}{k+1}u^{-1}\Delta_h u + (4+2(k-1))\frac{2}{k+1}u^{-1}du(H) + \left(u^{-\frac{4}{k+1}} - 1\right)\operatorname{scal}_{g_F} - \left(1 - u^{\frac{4}{k+1}}\right)\|A\|^2, \tag{36}$$

where $\|A\|^2$ *is a term that depends on the non-integrability of the horizontal distribution* $\mathcal{H} = (\ker \mathrm{d}\pi)^\perp$*. In particular,* $\|A\|^2 = 0$ *if* π *is locally a warped product.*

Moreover:

Lemma 5.5. *Let* $F^k \hookrightarrow (\overline{M}^{n+k}, g) \xrightarrow{\pi} (M^n, h)$ *be a Riemnnian submersion which is locally isometric to a product. If* Δ_g *denotes the Laplace operator on the metric g, then the restriction of* Δ_g *to basic functions defines a strongly elliptic operator.*

Proof. Let $u : \overline{M} \to \mathbb{R}$ be a basic function. Once $\overline{M}$ is compact, it is possible to choose a collection of open sets $\{W_l\} \subset \overline{M}$ trivializing the submersion π in the following sense (see [Her60])

$$W_l = U_l \times F,\ U_l \subset U.$$

Once u is a basic function,

$$u|_{W_l}(x) = u(x_1, x_2) = u(x_1, x_2'),\ \forall x_2, x_2' \in F,\ \forall x_1 \in U_l.$$

If $\{\psi_l\}$ denotes a partition of unity subordinated to $\{U_l\}$, then $u = \sum_l \psi_l u$ and there is a well defined injection

$$\zeta : W^{1,2}(\overline{M}) \to W^{1,2}(M)$$

$$u \mapsto v,$$

where $v = \sum_l v_l,\ v_l(x_1) = \psi_l(x_1)u(x_1, x_2),\ \forall x_1 \in U_l$.

Since the submersion is locally a product, it follows that its fibers are totally geodesic, so $H = 0$. Hence, by identifying $\zeta u = u$ one has that $\Delta_h u = \Delta_g u$ once $\Delta_h u = \Delta_g u - du(H) = \Delta_g u$ (see [GW09, Section 2.1.4, p.53]). □

To prove Proposition 5.1, item (2), we observe that since the Riemannian submersion π is locally isometric to a product, its fibers are totally geodesic submanifolds and its horizontal distribution is integrable. In particular, $H \equiv 0$ and $\|A\| = 0$. Hence, according to Lemmas 5.4 and 5.5, given a basic function $f : M \to \mathbb{R}$, we need to study the following elliptic problem

$$\left(\frac{k+1}{4k}\right)u(f - \operatorname{scal}_g) = -\Delta_h u + \left(\frac{k+1}{4k}\right)\left(u^{\frac{k-3}{k+1}} - u\right)c, \tag{37}$$

where $0 \geq c = \operatorname{scal}_{g_F}$.

The proof of item (2) on Proposition 5.1 follows by adapting the proof of Lemma 5.3 to solve the PDE (37). This is done by searching for a minimum point in the set

$$\mathbf{M}_{\mathcal{F}} := \{u \in W^{1,2}(\overline{M}) : u \geq 0,\ u \text{ is basic and } \int_M u^{\theta_k} = 1\}.$$

We use the subscript $\mathcal{F}$ in order to emphasize that we are considering classes of elements whose function representatives are constant along the foliation induced by the fibers F.

5.2. **The case $c > 0$.** We consider this case separately since we need a different version of Lemma 5.3. Our result is a natural generalization of [EYTK96, Theorem 4.10, p. 253]. We first remark that this case suffers no obstruction as it is substantiated by the fact that a product of a manifold with positive scalar curvature always admits positive scalar curvature (compare this fact with the mentioned Kazdan–Warner tricotomy).

Following the notation established in section 5.1 and motivated by Lemma 5.5, we consider the following functional of class C^1:

$$J(u) = \frac{1}{2}\int_M |\nabla u|^2 + 2b_k \int_M \left\{ (\mathrm{scal}_g - c - f)\,\frac{u^2}{2} + c\theta_k^{-1} u^{\theta_k} \right\} \tag{38}$$

defined in

$$\mathbf{M}_{\mathcal{F}} := \{u \in W^{1,2}(\overline{M}) : u \geq 0,\ u \text{ is basic and } 2b_k c\theta_k^{-1} \int_M u^{\theta_k} = 1\}.$$

By following arguments analogous to the ones in the proof of Lemma 5.3 it can be shown that if

$$\min_M(\mathrm{scal}_g - c) \geq \max_M f,$$

then $J|_{\mathbf{M}_{\mathcal{F}}}$ has a minimum point that is a positive smooth solution to equation (37). The item is proved since we can always divide f by an arbitrary positive constant and then scale the resulting metric. ☐

6. Examples

We finish the paper by presenting examples of manifolds with the desired symmetric properties. The intention is to present to the reader how vast the variety of such examples can be. We begin with examples where Theorem H and Proposition 5.1 can be applied. Then we follow with standard and less-standard constructions of G-manifolds.

6.1. **Polar foliations, toric symmetry and Calabi–Yau bundles.** Observe that a manifold is locally a metric product if it admits some cover which splits as a metric product. Specifically, $(\overline{M}, g)$ is locally isometric to a product if there are (M, h) and (F, g_F) such that $\overline{M}$ is isometric to the quotient of $(M\times F, h \times g_T)$ by a subgroup $\Gamma < \pi(\overline{M})$ acting as deck transformations.

Although the condition of being locally isometric to a product is quite restrictive, we can slightly weaken it by considering manifolds which are locally warped products. A manifold is locally a warped product if there is a function ψ such that the general vertical warping induced by ψ, $\widetilde{\mathrm{g}}$, makes $(\overline{M}, \widetilde{\mathrm{g}})$ locally a metric product.

To characterize the latter, we recall to a result in Gromoll–Walschap [GW09].

Proposition 6.1 (Proposition 2.2.1, [GW09]). *Let $\pi : \overline{M}^{n+k} \to M^n$ be a Riemannian submersion. Then π is locally a warped product if and only if*

1. *the distribution $\mathcal{H} = (\ker \mathrm{d}\pi)^\perp$ is integrable;*
2. *the fibers are totally umbilic submanifolds of $\overline{M}$;*
3. *π is isoparametric; i.e., the mean curvature form κ (dual to the mean curvature vector field) is basic.*

Since the metric on the factor (M, h) is preserved by this construction, Proposition 5.1 can be applied to manifolds satisfying the conditions in Proposition 6.1.

6.1.1. *Calabi–Yau bundles.* We proceed with a brief discussion about prescribing scalar curvature on some Calabi–Yau bundles. We first observe that realizing basic scalar functions on a trivial bundle $Y \times F \to Y$ follows from previous works: assume that F has a metric g_F with constant scalar curvature c. If $f : Y \to \mathbb{R}$ is a smooth function, consider a metric g on Y that realizes $\mathrm{scal}_g = f - c$. Then, the scalar curvature of the product $\tilde{g} = g \times g_F$ satisfy $\mathrm{scal}_{\tilde{g}}(x, y) = f(x)$. The originality of Theorem H and Proposition 5.1 is the possibility to fix and preserve a initial metric g on the base Y.

An interesting application revolves around Calabi–Yau manifolds. Specifically, we apply Proposition 5.1 to the following result of Tosatti–Zhang:

Theorem 6.2 (Theorems 1.2 and 1.3 in [TZ14]). *Suppose $\pi : X \to Y$ is a holomorphic submersion with connected fiber F satisfying one of the following:*

1. *X, Y are projective manifolds with X Calabi–Yau;*
2. *X, Y are compact Kähler manifolds; Y, F are Calabi–Yau; and either $b_1(F) = 0$ or $b_1(Y) = 0$ and F is a torus.*

Then Y is Calabi–Yau and there is a finite unramified cover $p : \widetilde{Y} \to Y$ such that the pullback bundle $\tilde{\pi} : p^*X \to \widetilde{Y}$ is holomorphically trivial.

In the present context, a *finite unramified covering map* means a smooth covering map with a finite number of sheets. Theorem 6.2 is proved in the holomorphic context, without considering any compatibility between the metrics and the submersion. We observe here that the holomorphic submersion in the Theorem can be taken as a Riemannian one while choosing a Calabi–Yau metric g_Y on Y.

To this aim, note that, since Y is Calabi–Yau, its fundamental group is abelian, therefore every covering is regular. In particular, $p : \widetilde{Y} \to Y$ is a principal bundle with a finite principal group $\Gamma = \pi_1(Y)/\pi_1(\widetilde{Y})$. Since Γ is finite, there is a Γ-invariant metric g_F on F, therefore there is a metric g on X such that both $\tilde{\pi} : (p^*X, p^*g_Y \times g_F) \to (X, g)$ and $\pi : (X, g) \to (Y, g_Y)$ are Riemannian submersions. We thus conclude that g is locally a metric product and Theorem H applies. The proof of Corollary I is concluded by observing that the construction carries over with any arbitrary metric on Y.

6.1.2. *Toric symmetry.* Theorem A states that, if G is non-abelian, every G-invariant function can be realized as the scalar curvature of a G-invariant metric on a compact (M, G). On the other hand, there is a wealth of manifolds with abelian symmetry as it is illustrated by the following Theorem of Corro–Galaz-García ([CGG20, Theorem A]):

Theorem 6.3 (Corro–Galaz-García)**.** *For each integer $n \geq 1$, the following holds:*

(1) *There exist infinitely many diffeomorphism types of closed simply-connected smooth $(n+4)$-manifolds M with a T^n-invariant Riemannian metric with positive Ricci curvature;*
(2) *The manifolds M realize infinitely many spin and non-spin diffeomorphism types;*
(3) *Each manifold M supports a smooth, effective action of a torus T^{n+2} extending the isometric T^n-action in item (i).*

In particular,

Corollary 6.4. *For each integer $n \geq 1$, $\mathscr{P}^G$ contains infinitely many diffeomorphism types of both spin and non-spin closed simply-connected smooth $(n+4)$-manifolds M with abelian symmetry.*

We observe that the quotient M/G in Theorem 6.3 are connected sums of finitely many copies of products of 2-spheres and complex projective planes, therefore these manifolds admit metrics of positive scalar curvature.

On the other hand, a manifold with toric symmetry which does not lie in $\mathscr{P}^G$ can be constructed as follows: take an enlargeable manifold M_1 and consider its product with a torus, $M = M_1 \times T$. From the discussion following Theorem H in the Introduction, we conclude that $(M, T) \notin \mathscr{P}^T$. More generally, Bergery [Ber83] shows that a compact manifold with a free T-action (M, T) do not belong to $\mathscr{P}^T$ if and only if it is a T-bundle over a manifold in the complement of $\mathscr{P}$. More interestingly, [Ber83] presented examples of manifolds $(M, T) \in (\mathscr{P}^T)^c \cap \mathscr{P}$, where $(\mathscr{P}^T)^c$ stands for the complement of $\mathscr{P}^T$.

Based on the information above, we explicitly leave open the questions on the structure of manifolds in $(\mathscr{P}^T)^c \cap \mathscr{P}$; and if there is a manifold $(M, T) \notin \mathscr{P}^T$ whose quotient M/T admits positive scalar curvature (in a generalized sense).

6.2. **Equivariant constructions and examples.** In [Mil56], John Milnor introduced the first examples of *exotic manifolds*, which were topologically equivalent to spheres, but not diffeomorphic. Since then, lots of new exotic spaces were produced. For instance, there are uncountable many pairwise non-diffeomorphic structures on $\mathbb{R}^4$ (see [Tau87]) as exotic manifolds not bounding spin manifolds [Hit74], exotic projective spaces and some of their connected sums [CS18].

On the other hand, Carlos Durán [Dur01] explored symmetries to produce an exotic sphere out of its classical counterpart. It was promptly generalized in [DPR10, Spe16, CS18, CS19] in a procedure that enjoys a rough dictionary between invariant objects on them. More concretely, consider a compact connected principal bundle $G \hookrightarrow P \to M$ with a principal action $\bullet$. Assume that there is another free action on P that commutes with $\bullet$, which we denote by $\star$. These make P a $G \times G$-manifold resulting in a $\star$-*diagram*:

$$
\begin{tikzcd}
 & G \arrow[d, dotted, "\bullet"] & \\
G \arrow[r, dotted, "\star"] & P \arrow[r, "\pi'"] \arrow[d, "\pi"] & M' \\
 & M &
\end{tikzcd}
\tag{39}
$$

Here M, M' are the quotients of P by the $\bullet$- and $\star$-actions, respectively. Since the two actions commute, they can be put together to define a $G\times G$-action on P. Moreover, $\bullet$ descends to and action on M' and $\star$ to an action in M. We simply denote the orbit spaces associated to the descended actions by M'/G and M/G, respectively. Both orbit spaces are *canonically* identified with $P/(G\times G)$, in particular, they are naturally identified between themselves.

This section is dedicated to explore the relation between the manifolds M and M' and to recall some of the examples in literature. We begin by proving Corollary J, which follows from the lemma below and Corollary G.

Proof of Corollary J. Let $M \leftarrow P \rightarrow M'$ be a $\star$-diagram and $p \in P$. Denote $\pi(p) = x$ and $\pi'(p) = x'$. First of all, a straightforward calculation shows that the isotropy groups G_x and $G_{x'}$ are isomorphic. Therefore, if G acts effectively on M, so it does on M'. Therefore, if G has a non-abelian Lie algebra, then $(M, G), (M, G') \in \mathscr{P}^G$. If G is abelian, then it is a torus. Therefore, we can apply [Wie16, Theorem 2.2] to conclude that P admits a $G \times G$-invariant metric with positive scalar curvature if and only if both M and M' admit G-invariant metrics with positive scalar curvature, as desired. $\square$

We finish by sketching concrete examples where Corollary J can be applied.

Example 1 (The Gromoll–Meyer exotic sphere)**.** This construction first appeared in [GM72] and was first put in a $\star$-diagram in [Dur01] (see also [CS19]). Consider the compact Lie group

$$Sp(2) = \left\{ \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in S^7 \times S^7 \;\middle|\; a\bar{b} + c\bar{d} = 0 \right\}, \tag{40}$$

where $a, b, c, d \in \mathbb{H}$ are quaternions, equipped with their usual conjugation and multiplication and norm. The projection $\pi : Sp(2) \to S^7$ of an element to its first row defines a principal S^3-bundle with principal action:

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \bar{q} = \begin{pmatrix} a & c\overline{q} \\ b & d\overline{q} \end{pmatrix}. \tag{41}$$

Gromoll–Meyer [GM74] introduced the $\star$-action

$$q \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} qa\overline{q} & qc \\ qb\overline{q} & qd \end{pmatrix}, \tag{42}$$

whose quotient is an exotic 7-sphere. It all fits in the following diagram

$$\begin{tikzcd}
 & S^3 \arrow[d, dash, dotted, "\bullet"] & \\
S^3 \arrow[r, dash, dotted, "\star"] & Sp(2) \arrow[r, "\pi'"] \arrow[d, "\pi"] & \Sigma^7_{GM} \\
 & S^7 &
\end{tikzcd} \tag{43}$$

which is a building block for several examples. If fact, if $\phi : N \to M$ is a G-equivariant function, the whole diagram can be pulled back, producing a new quotient $(\phi^* P)/G = N'$. The pull-back construction was applied in [Spe16, CS18] to obtain the following examples:

(Σ^7_k): consider $\phi : S^7 \to S^7$ as the octonionic kth fold power. Then the corresponding $\star$-diagram $S^7 \leftarrow \phi^* Sp(2) \to (S^7)'$ yields $(S^7)'$ diffeomorphic to the connected sum of k times Σ^7_{GM} ;

(Σ^8): there is a S^3-equivariant suspension $\eta : S^8 \to S^7$ of the Hopf map $S^3 \to S^2$ whose quotient $(S^8)' = \eta^* Sp(2)/S^3$ is the only exotic 8-sphere;

(Σ^{10}): there is a S^3-equivariant suspension $\theta : S^{10} \to S^7$ of a generator of $\pi_6 S^3$ whose induced $\star$-quotient $(S^{10})'$ is a generator of the index 2 subgroup os homotopy 10-spheres that bound spin manifolds;

(Σ^{4n+1}): the frame bundle $\mathrm{pr}_n : SO(2n{+}2) \to S^{2n+1}$ can be also seem as a $\star$-diagram: one can endow $SO(2n{+}2)$ with both the right and left multiplication by $SO(2n{+}1)$. In this case, $M = M' = S^{2n+1}$. However, there is a pull-back map $J\tau : S^{4n+1} \to S^{2n+1}$, whose $\star$-diagram $S^{4n+1} \leftarrow (J\tau)^* SO(2n{+}2) \to (S^{4n+1})'$ has $(S^{4n+1})'$ diffeomorphic to a Kervaire sphere. Moreover, one can 'reduce'

$G = SO(2n{+}1)$ in to either $U(n)$ or $Sp(n)$ (supposing n odd for the last).

Another class of examples are the manifolds constructed in Grove–Ziller [GZ00]. Given integers $p_+, q_+, p_-, q_- \equiv 1 \mod 4$, [GZ00] produces a cohomogeneity-one manifold $(P^{10}_{p_-,q_-,p_+,q_+}, (S^3)^3)$. We further assume that $\gcd(p_-, q_-) = \gcd(p_+, q_+) = 1$. Then one can observe that the subactions of $S^3_\bullet = S^3 \times \{1\} \times \{1\}$ and of $S^3_\star = \{1\} \times \Delta S^3$ are commuting and free, where ΔS^3 is the diagonal in $S^3 \times S^3$. They fit in the diagram

(44)
$$\begin{tikzcd}
 & S^3 \arrow[d, dotted, no head, "\bullet"] & \\
S^3 \arrow[r, dotted, no head, "\star"] & P^{10}_{p_-,q_-,p_+,q_+} \arrow[r, "\pi'"] \arrow[d, "\pi"] & M'_{p_-,q_-,p_+,q_+} \\
 & M_{p_-,q_-,p_+,q_+} &
\end{tikzcd}$$

Here, M'_{p_-,q_-,p_+,q_+} is the S^3-bundle over S^4 classified by the transition function $\alpha : S^3 \to SO(4)$ defined by $\alpha(x)v = x^k v x^l$, where $k = (p_-^2 - p_+^2)/8$ and $l = -(q_-^2 - q_+^2)/8$. However, we find ourselves in a curious situation here since the 'exotic' manifold M' is topologically better recognized then M itself: the only descriptions the literature presents for M are its cohomogeneity-one diagram and its appearance in the family P^7_C in [Hoe10]. Maybe the $\star$-diagram together with the techniques in [CS18] could be used to determine something about its structure, compared to the one in M'_{p_-,q_-,p_+,q_+}.

Example 2 (Gluing and connected sums)**.** Consider W_1, W_2 manifolds with boundaries and equipped with G-actions. Assume that $f : \partial W_1 \to \partial W_2$ is an equivariant diffeomorphism. Then one can produce a new manifold $W = W_1 \cup_f W_2$, by gluing W_1, W_2 via f. W thus inherits a natural smooth G-action whose restrictions to $W_1, W_2 \subset W$ coincide with the original actions.

Interesting examples arise in the following way: let $(M_1, G), (M_2, G)$ be closed manifolds with G-actions. Suppose that $x_i \in M_i$, $i = 1, 2$ have the same *orbit type*, that is, G_{x_1}, G_{x_2} are subgroups in the same conjugacy class and their isotropy representations are equivalent. Then one can remove small tubular neighborhoods of the orbits Gx_1, Gx_2 and glue the boudaries together. In particular, if x_1, x_2 are fixed points with equivalent isotropy representations, one can perform a connected sum.

More generally, one can consider the case where (M, G) admits an equivariant embedding of $(S^k \times D^{l+1}, G)$, where G acts on $S^k \times D^{l+1}$ is equipped with a linear action. In this case, one can perform surgery along ψ. That is,

At this point $\star$-diagrams becomes quite useful, since corresponding orbits in M, M' often have the same orbit type, as the next lemma points out.

Lemma 6.5. *Let $M \leftarrow P \rightarrow M'$ be a $\star$-diagram and $p \in P$. Then, there is a group isomorphism $\phi : G_{\pi(p)} \rightarrow G_{\pi'(p)}$ and a linear isomorphism ψ such that $\rho_{\pi(p)} = \psi\rho_{\pi'(p)}\phi$.*

Proof. Denote $\pi(p) = x$ and $\pi'(p) = x'$. For simplicity, we only prove the last assertion since the existence of ϕ follows by a direct computation.

Note that $\mathrm{d}\pi_p, \mathrm{d}\pi'$ define isomorphisms between the normal spaces:

$$\frac{T_xM}{T_xGx} \xleftarrow{\overline{\mathrm{d}\pi}} \frac{T_pP}{T_p(G\times G)p} \xrightarrow{\overline{\mathrm{d}\pi'}} \frac{T_{x'}M'}{T_{x'}Gx'}.$$

Moreover, since π, π' commute with the (respective complementary) actions,

$$\rho_x(h)\overline{\mathrm{d}\pi}(\overline{v}) \hookleftarrow \rho_p(h,\phi(h))\overline{v} \mapsto \rho_{x'}(\phi(h))\overline{\mathrm{d}\pi'}(\overline{v}),$$

inducing the desired identification. ☐

For instance, the isomorphism ϕ in the examples (Σ^7_k)-(Σ^{4n+1}) are all of the form $\phi(h) = ghg^{-1}$ for $g \in G$ only depending on p. In such cases, every pair of points x, x', $\pi^{-1}(x) \cap (\pi')^{-1}(x') \neq \emptyset$, have the same orbit type. We claim that several surgeries can be done equivariantly on the manifolds resulting on these examples. Moreover, such surgeries can be done keeping the $\star$-diagram apparatus. We give more details on the (Σ^7_k)-case in order to illustrate the assertion.

In this case, S^3 acts on S^7 as $q(a,b)^T = (qa\bar{q}, qb\bar{q})^T$. This action is inherited from the representation $\tilde{\rho} : S^3 \rightarrow SO(8)$ defined by $2\rho_0 \oplus 2\rho_1$, where ρ_0 and ρ_1 stands for the trivial representation and the representation defined by the composition of the double-cover $S^3 \rightarrow SO(3)$ and the standard action of $SO(3)$ in $\mathbb{R}^3$. That is, $\tilde{\rho}$ is the double suspension of the bi-axial action of $SO(3)$ in $\mathbb{R}^6$, up to a double-cover.

Note that $(a,b)^T$ is a fixed point of $\tilde{\rho}$ whenever $a, b \in \mathbb{R}$ and consider another manifold (M^7, S^3) with a fixed point p whose isotropy representation is $\rho_0 \oplus 2\rho_1$. One can produce a standard degree-one equivariant map $\phi : M^7 \rightarrow S^7$ by 'wrapping' S^7 with an open ball centered at the fixed point, and sending the remaining of M to the antipodal of $\phi(p)$. As in [CS18, Theorem 4.1], the induced $\star$-diagram results in $M \leftarrow \phi^*P \rightarrow M\#\Sigma^7_k$.

To proceed with the surgery process, note that (S^7, S^3) admits the equivariant submanifolds below. We omit the explicit embeddings and use the representation instead of G in the notation (M, G), in order to present a more explicit information.

$$\begin{aligned}(S^1,2\rho_0)\times(D^6,2\rho_1)&=\{(a,b)^T\in S^7\mid(\mathrm{Re}(a),\mathrm{Re}(b))\neq(0,0)\};\\(S^2,\rho_1)\times(D^5,2\rho_0\oplus\rho_1)&=\{(a,b)^T\in S^7\mid\mathrm{Im}(a)\neq 0\};\\(S^3,\rho_0\oplus\rho_1)\times(D^4,\rho_0\oplus\rho_1)&=\{(a,b)^T\in S^7\mid a\neq 0\};\\(S^4,2\rho_0\oplus\rho_1)\times(D^3,\rho_1)&=\{(a,b)^T\in S^7\mid(a,\mathrm{Re}(b))\neq(0,0)\};\\(S^5,2\rho_1)\times(D^2,2\rho_0)&=\{(a,b)^T\in S^7\mid(\mathrm{Im}(a),\mathrm{Im}(b))\neq(0,0)\};\\(S^6,\rho_0\oplus 2\rho_1)\times(D^1,\rho_0)&=\{(a,b)^T\in S^7\mid(\mathrm{Im}(a),b)\neq(0,0)\}.\end{aligned}$$

Save $S^1\times D^6$ and $S^4\times D^3$, every submanifold above can be chosen to lie in an arbitrarily small region of a fixed $\mathrm{Re}(a)$. In particular, arbitrarily many of these surgeries can be performed. Moreover, we conclude that such surgeries can be performed preserving infinitely many fixed points. Therefore, the degree-one map mentioned above can be considered, producing a $\star$-diagram over the new manifold M. Although the connected sum applied to this context seems *ad-hoc*, the resulting manifold $(\phi^*P)/G = M\#\Sigma$ is the same manifold one obtains by performing the same surgeries on Σ. The construction is part of more general framework, which we state without a proof.

Proposition 6.6. *Suppose that p is a fixed point in (M,G) and $M\leftarrow P\rightarrow M'$ is a $\star$-diagram. Suppose that there is an equivariant embedding $\psi:(S^k\times D^{l+1},G)\to(M,G)$, where G acts linearly in both S^k and D^{l+1}, and whose image lies in a sufficiently small neighborhood of p. Then, there is a $\star$-diagram over the resulting surgery, $\widetilde{M}$:*

$$\widetilde{M}\leftarrow\widetilde{P}\rightarrow\widetilde{M}'$$

where $\widetilde{M}=(M\setminus\psi(S^k\times D^{l+1})\cup D^{k+1}\times S^l$. Moreover, there is an equivariant embedding $\psi':(S^k\times D^{l+1},G)\to(M',G)$, with the same action on $S^k\times D^{l+1}$ such that

$$\widetilde{M}'=(M'\setminus\psi'(S^k\times D^{l+1})\cup D^{k+1}\times S^l.$$

We further recall that $\star$-diagrams were originally used in [CS18] to produce positive Ricci curvature in M'. Indeed, they provide a natural setting to apply Serle–Wilhelm's positive Ricci lifting theorem [SW15, Theorem A]:

Theorem 6.7 (Theorem 6.6, [CS18])**.** *Let $M\leftarrow P\rightarrow M'$ be a $\star$-diagram with both G and M compact. Suppose that (M,G) has a regular orbit with finite fundamental group and that M admits a G-invariant metric whose induced quotient metric satisfy $\mathrm{Ric}_{M/G}\geq 1$. Then, both M and M' have a G-invariant metric of positive Ricci curvature.*

Example 3 (More connected sums)**.** A list of manifolds whose fixed points have isotropy representations isomorphic to the ones of the examples (Σ^7_k)-(Σ^{4n+1}) are found in [CS18]. We compile it here, in order to reinforce Proposition 6.6.

Proposition 6.8 (Cavenaghi–Sperança)**.** *The following manifolds have fixed points whose isotropy representations are isomorphic to the ones in* (Σ^7_k)-(Σ^{4n+1})*:*

- (*i*) (Σ^7_k)*: any 3-sphere bundle over* S^4 *;*
- (*ii*) (Σ^8)*: every 3-sphere bundle over* S^5 *or a 4-sphere bundle over* S^4*;*
- (*iii*) (Σ^{10})*:*
  - (*a*) $M^8 \times S^2$ *with* M^8 *as in item (ii);*
  - (*b*) *any 3-sphere bundle over* S^7*, 5-sphere bundle over* S^5 *or 6-sphere bundle over* S^4*;*
- (*iv*) $(\Sigma^{4m+1}, U(n))$*:*
  - (*a*) *a sphere bundle* $S^{2m} \hookrightarrow M^{4m+1} \to S^{2m+1}$ *associated to any multiple of* $O(2m{+}1) \hookrightarrow O(2m{+}2) \to S^{2m+1}$*, the frame bundle of* S^{2m+1}
  - (*b*) *a* $\mathbb{C}\mathrm{P}^m$*-bundle* $\mathbb{C}\mathrm{P}^m \hookrightarrow M^{4m+1} \to S^{2m+1}$ *associated to any multiple of the bundle of unitary frames* $U(m) \hookrightarrow U(m+1) \to S^{2m+1}$
  - (*c*) $M^{4m+1} = \frac{U(m+2)}{SU(2)\times U(m)}$
- (*v*) $(\Sigma^{8r+5}, Sp(r))$*:* $M \times N^{5-k}$*, where* N *is any manifold and*
  - (*a*) $S^{4r+k-1} \hookrightarrow M^{8r+k} \to S^{4r+1}$ *is the* k*-th suspension of the unitary tangent* $S^{4r-1} \hookrightarrow T_1 S^{4r+1} \to S^{4r+1}$*,*
  - (*b*) $k = 1$ *and* $\mathbb{H}\mathrm{P}^m \hookrightarrow M^{8m+1} \to S^{4m+1}$ *is the* $\mathbb{H}\mathrm{P}^m$*-bundle associated to any multiple of* $Sp(m) \hookrightarrow Sp(m+1) \to S^{4m+1}$
  - (*c*) $k = 0$ *and* $M = \frac{Sp(m+2)}{Sp(2)\times Sp(m)}$
  - (*d*) $k = 1$ *and* $M = M^{8m+1}$ *is as in item (iv)*

Acknowledgments

Part of this work was developed during the PhD of the first named author, funded by Fundação de Amparo à Pesquisa do Estado de São Paulo #2017/24680-1, and between his former postdoctoral position at the Federal University of Paraíba (UFPB) and his current position at the University of Fribourg supported in part by the SNSF-Project 200020E_193062 and the DFG-Priority programme SPP 2026. He also thanks Gustavo Ramos and Daniel Fadel for fruitful conversations on the subject.

The second named author is funded by CNPq grant 305726/2017-0.

References

[AB15] M. M. Alexandrino and R. G. Bettiol. *Lie Groups and Geometric Aspects of Isometric Actions*. Springer International Publishing, 2015.

[Aub76] T. Aubin. Equations différentielles non linéaires et problème de yamabe concernant la courbure scalaire. *J. Math. Pures Appl*, 55:269–296, 1976.

[Aub76] T. Aubin. Espaces de Sobolev sur les variétés Riemanniennes. *Bull. Sci. Math*, 100:149–173, 1976.

[Aub98] T. Aubin. *Some Nonlinear Problems in Riemannian Geometry*. Springer Monographs in Mathematics. Springer Berlin Heidelberg, 1998.

[BCD93] P. Buser, B. Colbois, and J. Dodziuk. Tubes and eigenvalues for negatively curved manifolds. *The Journal of Geometric Analysis*, 3(1):1–26, 1993.

[BE69] M. Berger and D. Ebin. Some decompositions of the space of symmetric tensors on a Riemannian manifold. *Journal of Differential Geometry*, 3(3-4):379 – 392, 1969.

[Ber83] L. Bergery. Scalar curvature and isometry group. *Spectra of Riemannian Manifolds, Tokyo*, pages 9–28, 1983.

[Bes87] A.L. Besse. *Einstein Manifolds:*. Classics in mathematics. Springer, 1987.

[Bre10] S. Brendle. *Ricci Flow and the Sphere Theorem*. Graduate studies in mathematics. American Mathematical Society, 2010.

[CeSS18] L. Cavenaghi, R. J. M. e Silva, and L. D. Sperança. Positive Ricci curvature through cheeger deformation, 2018.

[CGG20] D. Corro and F. Galaz-García. Positive Ricci curvature on simply-connected manifolds with cohomogeneity-two torus actions. *Proceedings of the American Mathematical Society*, 148:3087–3097, 2020.

[CRD84] I. Chavel, B. Randol, and J. Dodziuk. *Eigenvalues in Riemannian Geometry*. ISSN. Elsevier Science, 1984.

[CS18] L. F. Cavenaghi and L. D. Sperança. On the geometry of some equivariantly related manifolds. *International Mathematics Research Notices*, page rny268, 2018.

[CS19] L. Cavenaghi and L. Sperança. Positive Ricci curvature on fiber bundles with compact structure group. 2019.

[CS21] L. F. Cavenaghi and L. D. Sperança. On prescribing scalar curvature on bundles and applications. 2021.

[DPR10] C. Durán, T. Püttmann, and A. Rigas. An infinite family of Gromoll-Meyer spheres. *Archiv der Mathematik*, 95:269–282, 2010.

[Dur01] C. Durán. Pointed wiedersehen metrics on exotic spheres and diffeomorphisms of s6. *Geometriae Dedicata*, 88(1-3):199–210, 2001.

[EYTK96] P. Ehrlich, J. Yoon-Tae, and S-B. Kim. Constant scalar curvatures on warped product manifolds. *Tsukuba J. Math.*, 20(1):239–256, 06 1996.

[FM74] A. Fischer and J. Marsden. Manifolds of Riemannian metrics with prescribed scalar curvature. *Bulletin of the American Mathematical Society*, 80(3):479 – 484, 1974.

[GKS20] S. Goette, M. Kerin, and K. Shankar. Highly connected 7-manifolds and non-negative sectional curvature. *Annals of Mathematics*, 191(3):829–892, 2020.

[GM72] D. Gromoll and W. Meyer. An exotic sphere with nonnegative curvature. *Annals of Mathematics*, 96:413–443, 1972.

[GM74] D. Gromoll and W. Meyer. An exotic sphere with nonnegative sectional curvature. *Annals of Mathematics*, pages 401–406, 1974.

[GW09] D. Gromoll and G. Walshap. *Metric Foliations and Curvature*. Birkhäuser Verlag, Basel, 2009.

[GZ00] K. Grove and W. Ziller. Curvature and symmetry of Milnor spheres. *Annals of Mathematics*, 152:331–367, 2000.

[Heb96] E. Hebey. *Sobolev Spaces on Riemannian Manifolds*. Number Nº 1635 in Lecture Notes in Mathematics. Springer, 1996.

[Heb90] E. Hebey. *Changements de métriques conformes sur la sphère - Le problème de Nirenberg. Bulletin des Sciences Mathématiques*, 114:215–242, 1990

[HebVa93] E. Hebey and M. Vaugon *Le problème de Yamabe équivariant. Bulletin des Sciences Mathématiques*, 117:241–286, 1993

[HebVa97] E. Hebey and M. Vaugon *Sobolev spaces in the presence of symmetries. Bulletin des Sciences Mathématiques*, 76:859–881, 1997

[Her60] R. Hermann. A sufficient condition that a mapping of Riemannian manifolds be a fibre bundle. 1960.

[Hit74] N. Hitchin. Harmonic spinors. *Advances in Mathematics*, 14(1):1–55, 1974.

[Hoe10] Corey A Hoelscher. Classification of cohomogeneity one manifolds in low dimensions. *Pacific journal of mathematics*, 246(1):129–185, 2010.

[KW75a] J. L. Kazdan and F. W. Warner. A direct approach to the determination of gaussian and scalar curvature functions. *Inventiones mathematicae*, (28):227–230, 1975.

[KW75b] J. L. Kazdan and F. W. Warner. Existence and conformal deformation of metrics with prescribed Gaussian and scalar curvatures. *Annals of Mathematics*, 101(2):317–331, 1975.

[LY74] H. Lawson and S-T. Yau. Scalar curvature, non-abelian group actions, and the degree of symmetry of exotic spheres. *Comm. Math. Helv.*, 49:232–244, 1974.

[M8̈7] M. Müter. *Krummüngserhöhende deformationen mittels gruppenaktionen*. PhD thesis, Westfälischen Wilhelms-Universität Münster, 1987.

[Mil56] J. Milnor. On manifolds homeomorphic to the 7-sphere. *Annals of Mathematics*, 64:399–405, 1956.

[MS74] J. Milnor and J. Stasheff. *Characteristic Classes*. Annals of mathematics studies. Princeton University Press, 1974.

[Pal79] R. Palais. The principle of symmetric criticality. *Communications in Mathematical Physics*, 69(1):19–30, 1979.

[Per02] G. Perelman. The entropy formula for the ricci flow and its geometric applications, 2002.

[Per03a] G. Perelman. Finite extinction time for the solutions to the Ricci flow on certain three-manifolds, July 2003.

[Per03b] G. Perelman. Ricci flow with surgery on three-manifolds, March 2003.

[Sch84] R. Schoen. Conformal deformation of a Riemannian metric to constant scalar curvature. *Journal of Differential Geometry*, 20(2):479 – 495, 1984.

[Spe16] L. Sperança. Pulling back the gromoll-meyer construction and models of exotic spheres. *Proceedings of the American Mathematical Society*, 144(7):3181–3196, 2016.

[SW15] C. Searle and F. Wilhelm. How to lift positive Ricci curvature. *Geometry & Topology*, 19(3):1409–1475, 2015.

[SY79a] R. Schoen and S-T. Yau. On the proof of the positive mass conjecture in general relativity. *Communications in Mathematical Physics*, 65(1):45 – 76, 1979.

[SY79b] R. Schoen and S-T. Yau. Proof of the Positive-Action Conjecture in Quantum Relativity. *Physical Review Letters*, 42:547–548, Feb 1979.

[SY81] R. Schoen and S-T. Yau. Proof of the positive mass theorem. II. *Communications in Mathematical Physics*, 79(2):231 – 260, 1981.

[Tal] G. Talenti. Best constant in Sobolev inequality. *Ann. Mat. PuraAppl.*, 110:353 – 372, 1976.

[Tau87] C. Taubes. Gauge theory on asymptotically periodic 4-manifolds. *Journal of Differential Geometry*, 25(3):363 – 430, 1987.

[Tru68] N. Trudinger. Remarks concerning the conformal deformation of Riemannian structures on compact manifolds. *Annali della Scuola Normale Superiore di Pisa - Classe di Scienze*, Ser. 3, 22(2):265–274, 1968.

[TZ14] V. Tosatti and Y. Zhang. Triviality of fibered Calabi–Yau manifolds without singular fibers. *Mathematical Research Letters*, 21:905–918, 10 2014.

[Wie16] M. Wiemeler. Circle actions and scalar curvature. *Transactions of the American Mathematical Society*, 368(4):2939–2966, 2016.

[Yam60] H. Yamabe. On a deformation of Riemannian structures on compact manifolds. *Osaka Mathematical Journal*, 12(1):21 – 37, 1960.

[Zil] W. Ziller. http://www.math.upenn.edu/ wziller/research.html.

Department of Mathematics – University of Fribourg, Ch. du musée 23, CH-1700 Fribourg, Switzerland
Email address: leonardofcavenaghi@gmail.com, leonardo.cavenaghi@unifr.ch

Departamento de Matemática – Universidade Federal da Paraíba - Campus 1, 1º floor - Lot. Cidade Universitaria 58051-900, João Pessoa, PB, Brazil
Email address: jmbo@pq.cnpq.br

Instituto de Ciência e Tecnologia – Unifesp, Avenida Cesare Mansueto Giulio Lattes, 1201, 12247-014, São José dos Campos, SP, Brazil
Email address: lsperanca@gmail.com