Classical and Bayesian Inference of Unit Gompertz Distribution Based on Progressively Type II Censored Data

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Classical and Bayesian Inference of Unit Gompertz Distribution Based on Progressively Type II Censored Data

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Abstract

In this article, we study estimation methodologies for parameters of an unit Gompertz distribution based on two frequentist methods and Bayesian method using progressively Type II censored data. In frequentist approach, besides conventional maximum likelihood estimation, maximum product of spacing method is proposed for parameter estimation as an alternative approach to common maximum likelihood method. In order to obtain maximum likelihood estimates, we use both Newton-Raphson and stochastic expectation minimization algorithms, while for obtaining Bayes estimates for unknown parameters of the model, we have considered both traditional likelihood function as well as product of spacing function. Moreover, the approximate confidence intervals of the parameters are obtained under two the frequentist approaches and highest posterior density credible intervals of the parameters are obtained under Bayesian approaches using MCMC approach. In addition, percentile bootstrap technique is utilized to compute confidence intervals. Numerical comparisons are presented of the proposed estimators with respect to various criteria quantities using Monte Carlo simulations. Further, using different optimality criteria, an optimal censoring scheme has been suggested. Besides, one-sample and two-sample prediction problems based on observed sample and appropriate predictive intervals under Bayesian framework are discussed. Finally, to demonstrate the proposed methodology in a real-life scenario, maximum flood level data is considered to show the applicability of the proposed methods.

Keywords: Unit Gompertz distribution, maximum likelihood estimation, maximum product of spacing estimation, Bayesian estimation, progressively type II censored data, one-sample prediction, two-sample predication.

1. Introduction

The Gompertz distribution was first introduced in the year 1825 by Benjamin Gompertz and it became very popular among demographers and actuaries. This distribution is a generalization of the exponential distribution and has wide applicability in different spheres, especially in medical and actuarial studies. It possesses some relation with some well-known distributions such as exponential, double exponential, Weibull, extreme value (Gumbel distribution) or generalized logistic

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distribution (see Willekens (2001)), which makes it useful for actuarial and the medical studies. Mazucheli and Dey (2019) proposed a new transformed distribution by using the transformation $X = \exp(-Y)$, where random variable Y follows Gompertz distribution. The authors emphasized that the new transformed distribution gives a very satisfactory fit than Beta and Kumaraswamy distributions for some specific data sets. This new transformed distribution is called the Unit-Gompertz (UG) distribution. The UG distribution has the following probability density function (p.d.f.)

$$f(z; \alpha, \beta) = \alpha\beta z^{-(\beta+1)} e^{-\alpha(z^{-\beta}-1)}, \;\; 0 < z < 1, \; \alpha > 0, \; \beta > 0. \quad (1)$$

The corresponding cumulative distribution function (c.d.f.) is given by

$$F(z; \alpha, \beta) = e^{-\alpha(z^{-\beta}-1)}, x \;\; 0 < z < 1, \; \alpha > 0, \; \beta > 0. \quad (2)$$

The p.d.f. of this distribution can have variety of shapes while the hazard rate function is constant, increasing and upside-down bathtub-shaped. However, UG distribution has an edge over the Gompertz distribution because of the fact that upside-down bathtub-shaped hazard function cannot be modelled using Gompertz distribution. For more properties of UG distribution, one can refer to the works of Mazucheli and Dey (2019) and Anis and De (2020).

After the introduction of UG distribution in literature, to the best of our knowledge, very little work has been conducted by researchers on this distribution. Notable among these works are: Jha et al. (2019) considered UG distribution to estimate multicomponent stress-strength reliability based on classical and Bayesian approaches using complete sample. Kumar et al. (2020) studied classical as well as Bayesian estimation of the parameters of the model based on lower record values and inter-record times. They also obtained prediction of future record values for the UG distribution. Further, they derived single and product moments of lower record values. Jha et al. (2020) again considered this model using classical and Bayesian approaches to estimate the multicomponent stress strength reliability under progressive type II censoring samples. They also obtained bootstrap, asymptotic and highest posterior density confidence interval of reliability quantity. Recently, Arshad et al. (2021) studied this model under the framework of dual generalized order statistics. They obtained the parameters of the model using Markov Chain Monte Carlo and Lindley's approximation methods based on order statistics and lower record values.

In lifetime and reliability experiments, the necessity for censored data arises due to non- availability of complete information on failure times data for all experimental units. Although several censoring schemes have been developed over the years, yet researchers seem to prefer progressive type II censoring scheme over other conventional censoring schemes as under this scheme experimenter can reduce the total time on test and/or the number of failed items/ survival test units can be withdrawn during the experiment at different stages which is not possible in case of conventional type II censoring scheme. The progressive type II censoring can be described in the following way: suppose that n identical and independent units are put on a life test. When the first failure occurs, $X_{1:m:n}$, R_1 of the $n-1$ active units are randomly removed from the test. Similarly, when the second failure occurs, $X_{2:m:n}$, R_2 of the $n-R_1-2$ active units are randomly removed from the test, and the test continues. This is repeated m times, where $1 \leq m \leq n$ is prefixed, after which all R_m remaining surviving items are removed. Here, $\mathbf{R} = (R_1, R_2, \cdots, R_m)$, such that $m + \sum_{i=1}^{n} R_i = n$, is the progressive censoring scheme. The progressively type II order statistics generated through **R** are denoted by $\mathbf{X} = (X_{1:m:n}, X_{2:m:n}, \cdots, X_{m:m:n})$.

Independently, Cheng and Amin (1983) and Ranneby (1984) introduced the method of maximum product of spacing (PS) function as a competitive method to the method of maximum likelihood. The Maximum Product of Spacing Estimators (MPSEs) are obtained by maximizing the PS function similar to Maximum Likelihood Estimators (MLEs). Therefore, it is expected that the MPSEs retain most of the properties of the MLEs. Anatolyev and Kosenok (2005) stated that the MPSEs are more efficient than the MLEs for skewed distributions or in small sample cases for heavy tailed distributions. For some recent references, see for example Almetwally and Almongy (2019), El-Sherpieny et al. (2020) Shakhatreh et al. (2021) and Yadav et al. (2021). The general condition to obtain the MPSEs is that $f(z) > 0$; $\forall\, z \in (a, b)$ and all the sample items are i.i.d. In our case, we assume that Z follows the unit Gompertz distribution with support $(0, 1)$, which furnishes $a = 0$ and $b = 1$.

In the premise of the above, we have not come across any work related to estimation of the parameters of the UG distribution under progressive type II censoring using two frequentist and Bayesian methods. Our objectives in this study are: First, estimating the parameters of the UG distribution using two frequentist estimation approaches, namely; conventional maximum likelihood and MPS estimation methods. In addition, based on these two methods, we have obtained approximate confidence intervals (ACIs) for the parameters. Second objective is to obtain the MLEs by using both Newton-Raphson (NR) and Stochastic Expectation Maximization (SEM) algorithms. Third objective is to obtain the Bayes Estimates (BEs) of the UG distribution parameters based on Likelihood (LK) and PS functions under squared-error loss function using independent gamma priors. Furthermore, Markov Chain Monte Carlo (MCMC) techniques are considered to compute the posterior functions and consequently, Bayes estimates and associated credible intervals are computed. Fourth objective is to obtain an optimal censoring scheme using different optimality criteria. Fifth objective is obtain Bayes predictive intervals based on one-sample prediction and two-sample prediction methods. Using various choices of the effective sample size, the performance of the proposed methods is compared through a simulation study in terms of their bias, mean squared-error (MSE), sample standard error(SSE) and estimated standard error (ESE) and confidence intervals (CI) are compared in terms of their average confidence lengths (IL). A well-known data set on maximum flood level (in millions of cubic feet per second) for Susquehanna River at Harrisburg, Pennsylvania has been re-analyzed.

Rest of the paper is organised as follows: Classical point estimators based on progressive type II censored sample through LK and PS functions are investigated in Sections 2 and 3. Section 4 provides SEM algorithm to estimate MLEs. Section 5 provides asymptotic confidence intervals based on LK and PS functions. Section 6 provides parametric bootstrap confidence interval. Section 7 provides Bayes estimates as well as associated credible intervals using each of the proposed frequentist functions. The simulated results are presented in Section 8. In Section 9, optimal censoring plans are presented. Prediction problem has been considered in Section 10. An application using real dataset is provided for illustrative purposes in Section 11. Finally, we conclude the paper in Section 12.

2. Maximum Likelihood Estimators

Based on the observed sample $\mathbf{X} = (X_{1:m:n}, X_{2:m:n}, \cdots, X_{m:m:n})$ from a type II progressive censoring scheme, $\mathbf{R} = (R_1, \cdots, R_m)$, the likelihood function (LK) can be written as

$$\begin{aligned} L(\alpha, \beta|\mathbf{X}) &= A\prod_{i=1}^{m} f(x_{i:m:n}; \alpha, \beta)\left[1 - F(x_{i:m:n}; \alpha, \beta)\right]^{R_i}, \\ &= A(\alpha\beta)^m \prod_{i=1}^{m} x_{i:m:n}^{-(\beta+1)}\ e^{-\alpha(x_{i:m:n}^{-\beta}-1)} \prod_{i=1}^{m}[1 -\ e^{-\alpha(x_{i:m:n}^{-\beta}-1)}]^{R_i}, \end{aligned} \quad (3)$$

where $A = n(n-1-R_1)\cdots(n-R_1-\cdots-R_{m-1}-m+1)$. The associated log-likelihood function (without constant term) can be expressed from (3) as

$$\begin{aligned} l_n(\alpha, \beta|\mathbf{x}) &= \log(L(\alpha, \beta|\mathbf{x})) = m\log(\alpha) + m\log(\beta) - (\beta+1)\sum_{i=1}^{m}\log(x_{i:m:n}) \\ &\quad -\alpha\sum_{i=1}^{m}(x_{i:m:n}^{-\beta} - 1) + \sum_{i=1}^{m} R_i\log[1 -\ e^{-\alpha(x_{i:m:n}^{-\beta}-1)}]. \end{aligned} \quad (4)$$

Upon differentiating (4) with respect to α and β and equating to zero, the resulting equations must be satisfied to obtain the MLEs of α and β. The normal equations are

$$\frac{\partial l_n(\alpha, \beta|\mathbf{x})}{\partial\alpha} = \frac{m}{\alpha} - \sum_{i=1}^{m}(u_i - 1) + \sum_{i=1}^{m} R_i\frac{(u_i-1)v_i}{(1-v_i)} = 0, \quad (5)$$

$$\begin{aligned} \frac{\partial l_n(\alpha, \beta|\mathbf{x})}{\partial\beta} &= \frac{m}{\beta} - \sum_{i=1}^{m}\log(x_{i:m:n}) + \alpha\sum_{i=1}^{m}\ u_i\log(x_{i:m:n}) \\ &\quad -\alpha\sum_{i=1}^{m} R_i\frac{u_i\log(x_{i:m:n})\ v_i}{(1-v_i)} = 0. \end{aligned} \quad (6)$$

where

$$u_i = x_{i:m:n}^{-\beta} \quad \text{and} \quad v_i = e^{-\alpha(x_{i:m:n}^{-\beta}-1)}. \tag{7}$$

Very simple iterative procedure like bisection or Newton-Raphson method may be used to maximize equations (5) and (6) to obtain the MLEs of α and β say $\widehat{\alpha}$ and $\widehat{\beta}$, respectively.

3. Maximum Product of Spacing Estimation

Based on a progressive type II censored sample, we can write the Product of Spacing (PS) function as follows

$$\begin{aligned} S(\alpha,\beta|\mathbf{X}) &= \prod_{i=1}^{m+1}\left[F(x_{i:m:n};\alpha,\beta)-F(x_{i-1:m:n};\alpha,\beta)\right]\prod_{i=1}^{m}\left[1-F(x_{i:m:n};\alpha,\beta)\right]^{R_i}, \\ &= \prod_{i=1}^{m+1}\left[e^{-\alpha(x_{i:m:n}^{-\beta}-1)}-e^{-\alpha(x_{i-1:m:n}^{-\beta}-1)}\right]\prod_{i=1}^{m}\left[1-e^{-\alpha(x_{i:m:n}^{-\beta}-1)}\right]^{R_i} \\ &= \prod_{i=1}^{m+1}\left[v_i-v_{i-1}\right]\prod_{i=1}^{m}\left[1-v_i\right]^{R_i}, \end{aligned} \tag{8}$$

with $x_{0:m:n} = v_0 = 0$ and $x_{m+1:m:n} = v_{m+1} = 1$. The MPSEs of α and β can be obtain by maximizing (8) with respect to α and β or equivalently by maximizing the natural logarithm of the PS function in the following form

$$s_n(\alpha,\beta|\mathbf{X}) = \sum_{i=1}^{m+1}\log[v_i-v_{i-1}] + \sum_{i=1}^{m}R_i\log[1-v_i]. \tag{9}$$

The MPSEs of α and β denoted by $\tilde{\alpha}$ and $\tilde{\beta}$ can be obtained by solving the following two normal equations

$$\frac{\partial s_n(\alpha,\beta|\mathbf{X})}{\partial\alpha} = \sum_{i=1}^{m+1}\frac{-(u_i-1)v_i+(u_{i-1}-1)v_{i-1}}{(v_i-v_{i-1})} + \sum_{i=1}^{m}R_i\frac{(u_i-1)v_i}{(1-v_i)}, \tag{10}$$

and

$$\begin{aligned} \frac{\partial s_n(\alpha,\beta|\mathbf{X})}{\partial\beta} &= \alpha\sum_{i=1}^{m+1}\frac{u_i\,v_i\log(x_{i:m:n})-u_{i-1}v_{i-1}\log(x_{i-1:m:n})}{(v_i-v_{i-1})} \\ &\quad -\alpha\sum_{i=1}^{m}R_i\frac{u_i\,v_i\log(x_{i:m:n})}{(1-v_i)}. \end{aligned} \tag{11}$$

Since there are no closed from solutions for the MPSEs, therefore, one can odopt an iterative procedure to obtain MPSEs numerically from (10) and (11). Cheng and Traylor (1995) stated that the MPSEs are consistent and exhibit similar asymptotic properties to the MLEs under more general conditions. Also, the MPSEs possess the invariance principle similar to the MLEs, see for more details Coolen and Newby (1990).

4. Stochastic EM algorithm

It can be seen that in evaluating the MLEs using Newton-Raphson, one can face two difficulties. The first one is the sensitivity of the obtained estimators to the initial values of parameters and the second is the calculation of the second-order derivatives of the log-likelihood based on progressive data sometimes can be tedious. Alternatively, to find the MLEs, we propose a version of expectation-maximization (EM) algorithm which is called stochastic EM (SEM) algorithm. First, we explain the idea of EM algorithm. The EM algorithm, proposed by Dempster et al. (1977), is an iterative technique and is widely used approach for computing the maximum likelihood estimates of incomplete data or missing information problems. Here, we treat the censored observations as data missing information and apply EM algorithm to find the MLEs. The EM algorithm includes two main steps; Expectation(E)-step and Maximization(M)-step. Assume the complete lifetimes are given by $\mathbf{W} = (\mathbf{X}, \mathbf{Z})$ where $\mathbf{X} = (X_{1:m:n}, \cdots, X_{m:m:n})$ denotes the observed observations and $\mathbf{Z} = (\mathbf{Z}_1, \cdots, \mathbf{Z}_m)$ denotes the censored data where $\mathbf{Z}_j = (Z_{j1}, \cdots, Z_{jR_j})$.

Then the complete log-likelihood function based on the complete lifetimes, $\mathbf{W}$, is proportional to

$$\begin{aligned}
l_n^c(\alpha, \beta|\mathbf{W}) =& n \ln\alpha + n\ln\beta + n\alpha - (\beta+1)\sum_{i=1}^{m}\ln(x_{i:m:n}) - \alpha\sum_{i=1}^{m} x_{i:m:n}^{-\beta} \\
&- (\beta+1)\sum_{i=1}^{m}\sum_{j=1}^{R_j}\ln(z_{ij}) - \alpha\sum_{i=1}^{m}\sum_{j=1}^{R_j} z_{ij}^{-\beta}.
\end{aligned}$$

To perform the E-step of the EM algorithm, we need to compute the conditional expectation of the complete log-likelihood conditionally on the observed data $\mathbf{X}$,, using the current value $\alpha^{(k)}$ and $\beta^{(k)}$ of the parameters α and β as follows. Now

$$\begin{aligned}
E(l_n^c(\alpha,\beta|\mathbf{W})|\mathbf{X}, \alpha^{(k)}, \beta^{(k)}) =& n\ln\alpha + n\log\beta + n\alpha - (\beta+1)\sum_{i=1}^{m}\log(x_{i:m:n}) - \alpha\sum_{i=1}^{m} x_{i:m:n}^{-\beta} \\
&- (\beta+1)\sum_{i=1}^{m}\sum_{j=1}^{R_j} E(\ln(Z_{ij})|X_{i:m:n}, \alpha^{(k)}, \beta^{(k)}) - \alpha\sum_{i=1}^{m}\sum_{j=1}^{R_j} E(Z_{ij}^{-\beta}|X_{i:m:n}, \alpha^{(k)}, \beta^{(k)}).
\end{aligned} \tag{12}$$

In the M-step, we find $\alpha^{(k+1)}$ and $\beta^{(k+1)}$ which maximize the conditional expectation given in (12). This is easily achieved by solving the following likelihood equations

$$\frac{\partial E(l_n^c(\alpha, \beta|\mathbf{W})|\mathbf{X}, \alpha^{(k)}, \beta^{(k)})}{\partial\alpha} = 0$$

and

$$\frac{\partial E(l_n^c(\alpha, \beta|\mathbf{W})|\mathbf{X}, \alpha^{(k)}, \beta^{(k)})}{\partial\beta} = 0$$

or equivalently

$$\frac{n}{\alpha} = -n + \sum_{i=1}^{m} w_i^{-\beta} + \sum_{i=1}^{m}\sum_{j=1}^{R_j} E(Z_{ij}^{-\beta}|\mathbf{x}, \alpha^{(k)}, \beta^{(k)}) \tag{13}$$

$$\frac{n}{\beta} = \sum_{i=1}^{m} \ln(x_{i:m:n}) - \alpha \sum_{i=1}^{m} x_{i:m:n}^{-\beta} \ln(x_{i:m:n}) - \sum_{i=1}^{m}\sum_{j=1}^{R_j} E(\ln(Z_{ij})|x_{i:m:n}, \alpha^{(k)}, \beta^{(k)})$$
$$- \alpha \sum_{i=1}^{m}\sum_{j=1}^{R_j} E(Z_{ij}^{-\beta} \log(Z_{ij})|x_{i:m:n}, \alpha^{(k)}, \beta^{(k)}). \tag{14}$$

Observe that, the explicit expressions of the conditional expectations in (13) and (14) can be written as follows. For any function $g(z_{ij}; \alpha, \beta), i = 1, 2, \cdots, m; j = 1, \cdots, R_i$, we have

$$E(g(Z_{ij}; \alpha, \beta)|Z_{ij} > x_{i:m:n}) = \frac{\int_{x_{i:m:n}}^{1} g(y; \alpha, \beta) y^{-(\beta+1)} e^{-\alpha(y^{-\beta}-1)} dy}{\int_{x_{i:m:n}}^{1} y^{-(\beta+1)} e^{-\alpha(y^{-\beta}-1)} dy}. \tag{15}$$

It is observed that the E-step of the EM algorithm involved complex integrals which cannot be solved in a closed form. The SEM algorithm is an alternative method of the EM algorithm where the expectation in the E-step is calculated using Monte Carlo simulations. It is useful for the cases when the E-step is hard to calculate exactly. The idea of approximating the E-step in EM algorithm by the Monte-Carlo technique, was first proposed by Wei and TannerWei and Tanner (1990). As mentioned by Wang and Cheng (2010), the approximation of Wei and Tanner (1990) have more time-consuming. Later Diebolt and Celeux (1993) modified their idea by replacing the E-step with stochastic step through simulation technique. For more information about SEM, see for example, Tregouet et al. (2004), Zhang et al. (2014) and Arabi Belaghi et al. (2017).

The description of SEM method is as follows. Note that the conditional survival function of a random variable Z_{ij}, given $Z_{ij} > x_{i:m:n}$, can be computed by

$$S(Z_{ij}|Z_{ij} > x_{i:m:n}) = \frac{1 - e^{-\alpha(z_{ij}^{-\beta}-1)}}{1 - e^{-\alpha(x_{i:m:n}^{-\beta}-1)}} \tag{16}$$

We first generate independent R_i number of samples $z_{ij}, i = 1, 2, \cdots, m; j = 1, \cdots, R_i$ from the conditional survival function, (16), using the expression

$$z_{ij} = \left(1 - (1/\alpha)\log(1 - u(1 - e^{-\alpha(x_{i:m:n}^{-\beta}-1)}))\right)^{-1/\beta}, \tag{17}$$

where u is a random variate from the uniform distribution, $\mathrm{U}(0, 1)$. Upon using this simulated sample, Equations (13) and (14) reduce to

$$\frac{n}{\alpha} = -n + \sum_{i=1}^{m} x_{i:m:n}^{-\beta} + \sum_{i=1}^{m}\sum_{j=1}^{R_j} z_{ij}^{-\beta} \tag{18}$$

$$\frac{n}{\beta} = \sum_{i=1}^{m} \log(x_{i:m:n}) - \alpha\sum_{i=1}^{m} x_{i:m:n}^{-\beta}\log(x_{i:m:n}) - \sum_{i=1}^{m}\sum_{j=1}^{R_j}\log(z_{ij}) - \alpha\sum_{i=1}^{m}\sum_{j=1}^{R_j} z_{ij}^{-\beta}\log(z_{ij}). \tag{19}$$

Therefore the SEM algorithm works as follows. Set initial values of α and β as $\alpha^{(0)}$ and $\beta^{(0)}$.

Step(i) At k-th iteration, let $(\alpha^{(k)}, \beta^{(k)})$ be the estimate of (α, β).

Step(ii) Using the expression (17), simulate $z_{ij} \equiv z_{ij}(\alpha^{(k)}, \beta^{(k)}), i = 1, \cdots, m; j = 1, \cdots, R_i$, where α and β are replaced by $\alpha^{(k)}$ and $\beta^{(k)}$, respectively.

Step(iii) Compute $\alpha^{(k+1)}$ and $\beta^{(k+1)}$ using (18) and (19).

Step(iv) If $|\alpha^{(k+1)} - \alpha^{(k)}| + |\beta^{(k+1)} - \beta^{(k)}| < \epsilon$, for some pre-specified quantity ϵ, then set $\alpha^{(k+1)}$ and $\beta^{(k+1)}$, as the MLEs of α and β, otherwise, set $k = k + 1$ and go to **Step(ii)**.

5. Asymptotic Confidence Intervals (ACIs)

In this section, we construct three types of $100(1-\gamma)\%$ confidence intervals for the unknown parameters α and β. The first type of confidence interval is obtained by using the MLEs, the second by using MPSEs and the third by using parametric bootstrap method.

5.1. ACIs based on MLEs

From the log-likelihood function (4), the second order partial derivatives of $l_n(\alpha, \beta|\mathbf{X})$ can be obtained directly with respect to α and β as follows

$$\frac{\partial^2 l_n(\alpha,\beta|\mathbf{X})}{\partial\alpha^2} = -\frac{m}{\alpha^2} - \sum_{i=1}^{m} R_i \frac{(u_i-1)^2\, v_i}{(1-v_i)^2} \tag{20}$$

$$\begin{aligned}\frac{\partial^2 l_n(\alpha,\beta|\mathbf{X})}{\partial\beta^2} &= \sum_{i=1}^{m} R_i \frac{\alpha\, u_i\, v_i(\log(x_{i:m:n}))^2(1-\alpha\, v_i - u_i)}{(1-v_i)^2} - \frac{m}{\beta^2} \\ &\quad -\alpha\sum_{i=1}^{m} u_i(\log(x_{i:m:n}))^2 \end{aligned}\tag{21}$$

$$\frac{\partial^2 l_n(\alpha,\beta|\mathbf{X})}{\partial\alpha\partial\beta} = \sum_{i=1}^{m} u_i\log(x_{i:m:n}) - \sum_{i=1}^{m} R_i \frac{u_i\log(x_{i:m:n})(v_i - \alpha(u_i-1)v_i - v_i^2)}{(1-v_i)^2}. \tag{22}$$

It is observed that the asymptotic Variance-Covariance (VarCov) of the MLEs of α and β cannot be obtained in closed form because of the complicated nature of the expectations of the expressions

(20), (21) and (22). Therefore, we obtain the approximate asymptotic VarCov matrix for the MLEs by obtaining the inverse of the observed Fisher information matrix as follows

$$I(\hat{\alpha},\hat{\beta}) = \begin{bmatrix} -\frac{\partial^2 l_n(\alpha,\beta|\mathbf{X})}{\partial\alpha^2} & -\frac{\partial^2 l_n(\alpha,\beta|\mathbf{X})}{\partial\alpha\partial\beta} \\ -\frac{\partial^2 l_n(\alpha,\beta|\mathbf{X})}{\partial\beta\partial\alpha} & -\frac{\partial^2 l_n(\alpha,\beta|\mathbf{X})}{\partial\beta^2} \end{bmatrix}^{-1}_{(\alpha,\beta)=(\hat{\alpha},\hat{\beta})} = \begin{bmatrix} \widehat{var}(\alpha) & \widehat{cov}(\alpha,\beta) \\ \widehat{cov}(\beta,\alpha) & \widehat{var}(\beta) \end{bmatrix}. \tag{23}$$

Using the asymptotic properties of the MLEs, it is known that $(\hat{\alpha},\hat{\beta}) \sim N_2\Big((\alpha,\beta), I(\hat{\alpha},\hat{\beta})\Big)$, where $I(\hat{\alpha},\hat{\beta})$ is given by (23). Thus, the $(1-\gamma)\%$ ACIs of the parameters α and β can be obtained as follows

$$\hat{\alpha} \pm z_{\gamma/2}\sqrt{\widehat{var}(\alpha)} \quad \text{and} \quad \hat{\beta} \pm z_{\gamma/2}\sqrt{\widehat{var}(\beta)},$$

where $z_{\gamma/2}$ is the upper $(\gamma/2)^{th}$ percentile point of a standard normal distribution.

5.2. *ACIs based on MPSEs*

Here we construct the $(1-\gamma)\%$ ACIs of the α and β based on the asymptotic properties of the MPSEs. First, we obtain the second order partial derivatives of the logarithm of PS function, $s_n(\alpha,\beta|\mathbf{X})$, given in (9) as follow

$$\begin{aligned} \frac{\partial^2 s_n(\alpha,\beta|\mathbf{X})}{\partial\alpha^2} &= -\sum_{i=1}^{m+1}\frac{v_i v_{i-1}(u_i-u_{i-1})^2}{(v_i-v_{i-1})^2} - \sum_{i=1}^{m} R_i\frac{(u_i-1)^2 v_i}{(1-v_i)^2} \qquad (24) \\ \frac{\partial^2 s_n(\alpha,\beta|\mathbf{X})}{\partial\alpha\partial\beta} &= \sum_{i=1}^{m+1}\frac{u_i v_i\log(x_{i:m:n})(v_i-v_{i-1}+\alpha v_{i-1}(u_i-u_{i-1}))}{(v_i-v_{i-1})^2} \\ &\quad -\sum_{i=1}^{m+1}\frac{u_{i-1}v_{i-1}\log(x_{i-1:m:n})(v_i-v_{i-1}+\alpha v_i(u_i-u_{i-1}))}{(v_i-v_{i-1})^2} \\ &\quad +\sum_{i=1}^{m+1}\frac{R_i u_i v_i\log(x_{i:m:n})(v_i-1+\alpha(u_i-1))}{(1-v_i)^2} \qquad (25) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^2 s_n(\alpha,\beta|\mathbf{X})}{\partial\beta^2} &= -\alpha\sum_{i=1}^{m+1}\frac{v_i u_i(\log(x_{i:m:n}))^2(v_i-v_{i-1}+\alpha u_i\, v_{i-1})}{(v_i-v_{i-1})^2} \\ &\quad -\alpha\sum_{i=1}^{m+1}\frac{v_{i-1}u_{i-1}(\log(x_{i-1:m:n}))^2(v_{i-1}-v_i+\alpha v_i\, u_{i-1})}{(v_i-v_{i-1})^2} \\ &\quad +2\alpha\sum_{i=1}^{m+1}\frac{v_{i-1}u_{i-1}v_i u_i\log(x_{i-1:m:n})\log(x_{i:m:n})}{(v_i-v_{i-1})^2} \\ &\quad -\alpha\sum_{i=1}^{m+1}\frac{R_i u_i v_i(\log(x_{i:m:n}))^2(\alpha u_i+v_i-1)}{(1-v_i)^2}. \qquad (26) \end{aligned}$$

Due to the difficulties in obtaining the expectations of the expressions (24), (25) and (26), the asymptotic VarCov of the MPSEs of α and β cannot be obtained. Similar to the previous case in Subsection 5.1, we obtain the approximate asymptotic VarCov matrix for the MPSEs as

$$I(\tilde{\alpha}, \tilde{\beta}) = \begin{bmatrix} -\frac{\partial^2 \log S(\alpha,\beta)}{\partial \alpha^2} & -\frac{\partial^2 \log S(\alpha,\beta)}{\partial \alpha \partial \beta} \\ -\frac{\partial^2 \log S(\alpha,\beta)}{\partial \beta \partial \alpha} & -\frac{\partial^2 \log S(\alpha,\beta)}{\partial \beta^2} \end{bmatrix}^{-1}_{(\alpha,\beta)=(\tilde{\alpha},\tilde{\beta})} = \begin{bmatrix} \widetilde{var}(\alpha) & \widetilde{cov}(\alpha,\beta) \\ \widetilde{cov}(\beta,\alpha) & \widetilde{var}(\beta) \end{bmatrix}. \quad (27)$$

Based on the asymptotic properties of the MPSEs as in the case of the MLEs, (see, Cheng and Traylor (1995)) it follows that $(\tilde{\alpha}, \tilde{\beta}) \sim N_2\left((\alpha, \beta), I(\tilde{\alpha}, \tilde{\beta})\right)$, where $I(\tilde{\alpha}, \tilde{\beta})$ is given by (27). Therefore, the $(1-\gamma)\%$ ACIs of α and β are

$$\tilde{\alpha} \pm z_{\gamma/2}\sqrt{\widehat{var}(\alpha)} \quad \text{and} \quad \tilde{\beta} \pm z_{\gamma/2}\sqrt{\widehat{var}(\beta)}. \quad (28)$$

5.3. Parametric Bootstrap (Boot-p) CIs

The asymptotic normality of MLEs (or MPSEs) used to find the confidence interval is well-performed only when the sample size is large enough. Therefore, alternatively, in this subsection, the confidence intervals are computed using parametric percentile bootstrap (Boot-p) method even for small sample size. The bootsrap method, proposed by Efron and Tibshirani (1986), are widely used to estimate standard error or to build confidence intervals for the parameters as well as estimating the reliability and hazard functions. The generation of progressively type II Boot-p samples can be described as follows.

Step(1): Compute the MLEs (or MPSEs), $\hat{\alpha}$ and $\hat{\beta}$, based on the original progressively type II censored sample $\mathbf{X} = (X_{1:m:n}, \cdots, X_{m:m:n})$.

Step(2): Based on the computed MLEs (or MPSEs) in **Step(1)**, $\hat{\alpha}$ and $\hat{\beta}$, generate a progressively type II censored sample with the same censoring scheme $\mathbf{R}$ and same values of (n, m), utilizing the algorithm given in Balakrishnan and Sandhu (1995).

Step(3): Compute the MLEs (or MPSEs), $\hat{\alpha}^*$ and $\hat{\beta}^*$, based on the generated bootstrap sample in **Step(2)**.

Step(4): Repeat **Step(2)** and **Step(3)**, for B times, where B is a pre-specified quantity. Then we have two series of estimators $\hat{\alpha}^*_1, \hat{\alpha}^*_2, \cdots, \hat{\alpha}^*_B$ and $\hat{\beta}^*_1, \hat{\beta}^*_2, \cdots, \hat{\beta}^*_B$.

Next, we compute $100(1-\gamma)\%$ Boot-p confidence intervals as follows. Arrange all bootstrapped values, $\hat{\alpha}^*_1, \hat{\alpha}^*_2, \cdots, \hat{\alpha}^*_B$ and $\hat{\beta}^*_1, \hat{\beta}^*_2, \cdots, \hat{\beta}^*_B$, in ascending order and obtain $\hat{\alpha}^*_{(1)} < \hat{\alpha}^*_{(2)} < \cdots <$

$\hat{\alpha}^*_{(B)}$ and $\hat{\beta}^*_{(1)} < \hat{\beta}^*_{(2)} < \cdots < \hat{\beta}^*_{(B)}$. Then $100(1-\gamma)\%$ Boot-p confidence intervals of α and β are computed by

$$(\hat{\alpha}^*_{([B(\frac{\gamma}{2})])}, \hat{\alpha}^*_{([B(1-\frac{\gamma}{2})])}) \quad \text{and} \quad (\hat{\beta}^*_{([B(\frac{\gamma}{2})])}, \hat{\beta}^*_{([B(1-\frac{\gamma}{2})])}),$$

respectively, where $[x]$ denotes the integral part of x.

6. Bayesian Estimation

In this section, we consider Bayesian estimation of the unknown parameters of the UG distribution under progressively type II censored data. We mainly discuss the Bayes estimates and the associated credible intervals of the unknown parameter(s) based on LK and PS functions. In our Bayesian analysis, we have assumed only squared error loss function. However, other loss functions can be considered. Prior distributions play an important role for obtaining the Bayes estimates. However, there is no clear cut method to choose the best priors for the unknown model parameters for Bayesian estimation problem. In this regard, readers may refer to the works of Arnold and Press (1983). In the premise of the above arguments, we consider the piecewise independent gamma priors for the parameters of the considered model. Thus the proposed priors for the parameters α and β may be taken as:

$$\begin{aligned} \pi_1(\alpha) &\propto \alpha^{a-1}\, e^{-b\,\alpha}, \quad \alpha > 0, a, b > 0 \\ \pi_2(\beta) &\propto \beta^{c-1}\, e^{-d\,\beta}, \quad \beta > 0, c, d > 0 \end{aligned} \tag{29}$$

The hyperparameters a, b, c, d are known and non-negative. As the family of gamma distributions is highly flexible, and can provide different shapes based on parameter values and thus it can be considered as suitable priors of the model parameters. See for more details Kundu and Pradhan (2009), Dey et al. (2016b). Thus the joint prior distribution for α and β is

$$\pi(\alpha, \beta) \propto \alpha^{a-1}\beta^{c-1}e^{-(b\alpha+d\beta)}, \quad \alpha, \beta > 0. \tag{30}$$

Based on the observed sample $x_{1:m:n} < x_{2:m:n} < \cdots < x_{m:m:n}$ from a type II progressive censoring scheme, the joint posterior density using joint prior distribution defined in (30) and the traditional LK function can be written as

$$\Omega(\alpha, \beta|\mathbf{x}) \propto \alpha^{a+m-1}\beta^{c+m-1}e^{-(b\alpha+d\beta)} \prod_{i=1}^{m} z_i^{-(\beta+1)}\, e^{-\alpha(x_{i:m:n}^{-\beta}-1)} \prod_{i=1}^{m}[1-\, e^{-\alpha(x_{i:m:n}^{-\beta}-1)}]^{R_i}. \tag{31}$$

The marginal posterior probability density functions of α and β are given respectively as

$$\Omega_\alpha(\alpha|\mathbf{x}) \propto \int_0^\infty \Omega(\alpha, \beta|\mathbf{x}) d\beta. \tag{32}$$

and

$$\Omega_\beta(\beta|\mathbf{x}) \propto \int_0^\infty \Omega(\alpha, \beta|\mathbf{x}) d\alpha. \tag{33}$$

Similarly, based on the PS function, (8) and the joint prior distribution defined in (30), the joint posterior density and the marginal posterior density functions of α and β can be, respectively, expressed as

$$\begin{aligned}\Psi(\alpha, \beta|\mathbf{x}) &\propto \alpha^{a-1}\beta^{c-1}e^{-(b\alpha+d\beta)} \prod_{i=1}^{m+1} [e^{-\alpha(x_{i:m:n}^{-\beta}-1)} - e^{-\alpha(x_{i-1:m:n}^{-\beta}-1)}] \\ &\times \prod_{i=1}^{m}[1 - e^{-\alpha(x_{i:m:n}^{-\beta}-1)}]^{R_i} \qquad (34)\\ \Psi_\alpha(\alpha|\mathbf{x}) &\propto \int_0^\infty \Psi(\alpha, \beta|\mathbf{x}) d\beta \qquad (35)\\ \Psi_\beta(\beta|\mathbf{x}) &\propto \int_0^\infty \Psi(\alpha, \beta|\mathbf{x}) d\alpha. \qquad (36)\end{aligned}$$

Suppose $\hat{\omega}$ is an estimator of parameter ω, we propose to use squared-error (SE) loss function which is defined as

$$\ell(\omega, \hat{\omega}) = (\hat{\omega} - \omega)^2. \tag{37}$$

From (37), the Bayes estimate is given by the posterior mean of ω. For arbitrary function $\varphi(\alpha, \beta)$, the Bayes estimator, namely $\hat{\varphi}_B$ (or $\tilde{\varphi}_B$), is the expectation of the posterior distribution under squared error loss, which is given by

$$\hat{\varphi}_B = \frac{\int_0^\infty \int_0^\infty \varphi(\alpha, \beta)\Omega(\alpha, \beta|\mathbf{x}) d\alpha d\beta}{\int_0^\infty \int_0^\infty \Omega(\alpha, \beta|\mathbf{x}) d\alpha d\beta}, \tag{38}$$

using the LK function and

$$\tilde{\varphi}_B = \frac{\int_0^\infty \int_0^\infty \varphi(\alpha, \beta)\Psi(\alpha, \beta|\mathbf{x}) d\alpha d\beta}{\int_0^\infty \int_0^\infty \Psi(\alpha, \beta|\mathbf{x}) d\alpha d\beta}, \tag{39}$$

using the PS function.

Next, we construct the credible intervals for α and β. Let θ denote the parameter α or the parameter β. After obtaining the marginal posterior distribution of θ, a symmetric $100(1-\gamma)\%$ Bayes credible interval of θ, denoted by $[L_\theta, U_\theta]$, can be obtained by solving the following equations

$$P(\theta > U_\theta) = \int_{U_\theta}^\infty \Omega_\theta(\theta|\mathbf{x})(\text{ or } \Psi_\theta(\theta|\mathbf{x}))d\theta = \gamma/2 \tag{40}$$

and

$$P(\theta < L_\theta) = \int_0^{L_\theta} \Omega_\theta(\theta|\mathbf{x})(\text{ or } \Psi_\theta(\theta|\mathbf{x}))d\theta = \gamma/2. \tag{41}$$

We need to apply suitable numerical method to compute the above intervals.

6.1. Bayesian Inference using MCMC Approach

It is clear that the Bayes estimates of the unknown parameters α and β do not possess closed forms. Therefore, instead of applying numerical methods, we utilize Markov Chain Monte-Carlo (MCMC) approach to obtain the Bayes estimates and construct credible intervals of the unknown parameters. The advantage of using MCMC is its ability to construct Highest Posterior Density (HPD) credible intervals for α and β. We first draw random samples from the posterior density function, $\pi(\alpha, \beta|\mathbf{x})$ and then, we use the simulated values to compute the Bayes estimates and HPDs of α and β. Here $\pi(\alpha, \beta|\mathbf{x})$ denotes $\Omega(\alpha, \beta|\mathbf{x})$ given in (31) or $\Psi(\alpha, \beta|\mathbf{x})$ given in (34). Since the posterior density function $\pi(\alpha, \beta|\mathbf{x})$ can not be simulated easily, we utilize a random walk Metropolis-Hastings (MH) algorithm to generate samples from it. The purpose of using the random-walk MH algorithm is due to its flexibility in generating random samples from any proposal distribution, especially when the conditional posterior distributions of the parameters are unknown distributions. For our case, we propose a bivariate normal density $N_2((\log(\alpha), \log(\beta)), \hat{\Sigma})$, where $\hat{\Sigma}$ is the estimated variance-covariance matrix. These samples are utilized to obtain the Bayes estimates and construct credible intervals for the parameters as given in following algorithm.

Step. 1 Start with an initial values (α, β) as (α_0, β_0), and set $j = 1$.

Step. 2 Set $(\alpha, \beta) = (\alpha_{j-1}, \beta_{j-1})$.

Step. 3 Generate a new candidate parameter values (δ_1, δ_2) from bivariate normal distribution $N_2((\log(\alpha), \log(\beta)), \hat{\Sigma})$.

Step. 4 Set $\alpha^* = \exp(\delta_1)$ and $\beta^* = \exp(\delta_2)$.

Step. 5 Calculate $d = \min\{1, \frac{\pi(\alpha^*,\beta^*|\mathbf{x})\alpha\beta}{\pi(\alpha,\beta|\mathbf{x})\alpha^*\beta^*}\}$

Step. 6 Generate a random number u from uniform $(0, 1)$.

Step. 7 If $u \leq d$ accept (α^*, β^*) else retain (α, β).

Step. 8 If $j < M$, where M is a pre-specified value, set $j = j + 1$ and go to **Step. 2**, otherwise stop.

The retained sample values, $(\alpha_1, \beta_1), \cdots, (\alpha_M, \beta_M)$ is a random sample from the posterior density $\pi(\alpha, \beta|\mathbf{x})$. To reduce the effects of the initial values on the simulated samples, we discard some of the initial $N < M$ number of samples (burn-in). Now, using Monte-Carlo integration technique, the Bayes estimates of α and β under squared error loss function can be obtained as

$$\hat{\alpha}_B(\text{ or } \tilde{\alpha}_B) = \frac{1}{M-N}\sum_{j=N+1}^{M}\alpha_j,$$
$$\hat{\beta}_B(\text{ or } \tilde{\alpha}_B) = \frac{1}{M-N}\sum_{j=N+1}^{M}\beta_j.$$

For constructing HPD credible intervals of α and β, we use the method proposed by Chen and Shao (1999) as follows. Let $\alpha_{(N+1)} < \alpha_{(N+2)} < \cdots < \alpha_{(M)}$ and $\beta_{(N+1)} < \beta_{(N+2)} < \cdots < \beta_{(M)}$ be the ordered values of α_j and β_j for $j = N+1, \cdots, M$. Consider the following 100(1- γ)% credible intervals of α and β

$$(\alpha_{(j)}, \alpha_{(j+[(1-\gamma)M])}) \text{ and } (\beta_{(j)}, \beta_{(j+[(1-\gamma)M])}), j = N+1, \cdots, [\gamma M],$$

where $[x]$ denotes the integral part of x. Thus the HPD credible interval of α (or β) can be derived by choosing the credible interval which has the shortest length.

6.2. Selecting hyper-parameter values

Now, we consider the issue of how to select the hyper-parameter values in order to compute Bayes estimators with respect to informative priors instead of non-informative priors. Following Dey et al. (2016a) and Singh and Tripathi (2018), the selection of hyperparameters values can be done based on the past available data. With this respect, suppose that we have K number of samples available from UG(α, β) distribution and let the associated MLEs of (α, β) based on these samples be $(\hat{\alpha}^j, \hat{\beta}^j), j = 1, 2\cdots, K$. By equating the mean and variance of the MLEs with the mean and variance of the priors, we obtain the values of hyperparameters. In specific, since we have adopted gamma priors, $\pi(\theta) \propto \theta^{h_1-1}e^{-h_2\theta}$, then we have

$$\frac{1}{K}\sum_{i=1}^{K}\hat{\theta}^j = h_1/h_2 \quad \text{and} \quad \frac{1}{K-1}\sum_{i=1}^{K}(\hat{\theta}^j - \frac{1}{K}\sum_{i=1}^{K}\hat{\theta}^j)^2 = h_1/h_2^2.$$

Here (θ, h_1, h_2) represents (α, a, b) or (β, c, d). By solving the above equations, the estimated hyper-parameters are computed as

$$h_1 = \frac{(\frac{1}{K}\sum_{i=1}^{K}\hat{\theta}^j)^2}{\frac{1}{K-1}\sum_{i=1}^{K}(\hat{\theta}^j - \frac{1}{K}\sum_{i=1}^{K}\hat{\theta}^j)^2}$$

and

$$h_2 = \frac{\frac{1}{K}\sum_{i=1}^{K}\hat{\theta}^j}{\frac{1}{K-1}\sum_{i=1}^{K}(\hat{\theta}^j - \frac{1}{K}\sum_{i=1}^{K}\hat{\theta}^j)^2}.$$

7. Simulation

Here, a simulation study is conducted in order to explore the performance of the classical and Bayes estimates. The configurations of the sample size, n, and inspection times, m are adopted as $(25, 10)$, $(50, 20)$ and $(100, 30)$. The censoring schemes are considered as scheme I= $(0 * (m-1), n-m)$ and scheme II= $((n-m)/2, 0 * (m-2), (n-m)/2)$ when $n-m$ is an even number and scheme II= $((n-m+1)/2, 0 * (m-2), (n-m-1)/2)$ when $n-m$ is an odd number, where we abbreviated for example, $(1, 1, 0, 0, 1)$ by $(1 * 2, 0 * 2, 1)$. In all cases we have taken the true values of the parameters as $(\alpha, \beta) = (1.25, 1.5)$ and $(0.5, 0.75)$. The size of the bootstrap samples is taken to be 5000. The progressive type II censored samples are generated by utilizing the algorithm proposed by Balakrishnan and Sandhu (1995) as follows.

(1) Generate m independent Uniform $U(0, 1)$ observations $W_1, W_2, \cdots, W_m$.

(2) Set $V_i = W_i^{1/(i+R_m+R_{m-1}+\cdots+R_{m-i+1})}$ for $i = 1, 2, \cdots, m$.

(3) Set $U_i = 1 - V_m V_{m-1} \cdots V_{m-i+1}$ for $i = 1, 2, \cdots, m$. Then $U_1, U_2, \cdots, U_m$ is the required progressive type II censored sample from the Uniform $U(0, 1)$ distribution.

(3) Finally, set $X_i = F^{-1}(U_i)$ for $i = 1, 2, \cdots, m$ where F^{-1} is the inverse cdf of the UG distribution. Then $X_1, X_2, \cdots, X_m$ is the required progressive type II censored sample from the UG distribution.

For the classical estimation, we estimate the unknown parameters using the MLE, MPS and SEM methods. For each of these methods, we have computed the absolute average bias (Bias) , the sample standard deviation (SSE), the estimated standard deviation (ESE) using the observed information matrix based on the LK (or PS) function, the mean square error (MSE). Moreover, we have evaluated 95% Wald's confidence intervals using the observed information matrix based on the LK (or PS) function (CI), the Boot-p based on LK function (MBT) and Boot-p based of PS function (PBT).

For the Bayes estimations, we obtain the Bayes estimate by numerically solving (38) (BSL), the Bayes estimate by numerically solving (39) (BSP), MCMC based on LK fuction (MCL) and MCMC based on PS function (MCP). For each of these methods, we compute the absolute average bias (Bias), mean square error (MSE), credible intervals (CI) by numerically solving (40) and (41) and HPD intervals using the generated MCMC samples based on LK (or PS) function. For a given sample, the random-walk MH algorithm is used to generate MCMC samples of size $M = 20100$ with burn-in period of $N = 100$. The process for the both types of estimation (classical and Bayesian) is replicated 1000 times. The results of classical estimation are reported in Tables 1-3 and the results of Bayes estimation are reported in Tables 4-5.

From Table 1, it is observed that the Bias for all the estimators, in general, are reasonably small which indicates that the estimated values are close to the true parameter values. As expected, the Bias, MSE, SSE an ESE of all estimators are decreasing when sample sizes are increasing for all the cases. However, the MPS method presents less bias estimates than the MLE and SEM for the all cases. In addition, the SEM algorithm performs better than MLE based on this aspect. Clearly, the MSE of MPS and SEM is less than that of MLE. Moreover, MPS presents higher MSE as compared with SEM. The SSE for the SEM method are less than that of MLE and MPS methods while the ESE of the MLE is the while the ESE of the MLE is less that than of MPS and SEM. With respect to 95% confidence interval, from Table 2 and Table 3, the confidence intervals constructed by Boot-p methods based on LK function (MBT) have the smaller. In addition, the length of the confidence intervals is decreasing when the value of sample size is increasing. Hence, the performance of the MLEs are satisfactory in terms of the biases and standard errors of the estimates. For the Bayesian estimates, the Bias of all proposed methods are also reasonably small. It is clear that the Bias, RMSE and the length of the credible/HPD intervals are decreasing when the value of sample size is increasing.

8. Optimal censoring

In life-testing experiment, it is usually taken a fixed and pre-specified censoring scheme. However, choosing the optimum censoring scheme from a set of all possible schemes can be of important concerns to the estimation problem as it may lead to efficient estimates for parameters.

In this section, instead of considering pre-specified inspection censoring scheme, we investigate by using different techniques for selecting the the optimum censoring scheme under the progressive type II censored data. The problem of identifying the optimal censoring scheme for different distributions has received considerable attention in the statistical literature. See for example, Abouammoh and Alshingiti (2009), Pradhan and Kundu (2013), Sultan et al. (2014), Dube et al. (2016), Sen et al. (2018) and Ashour et al. (2020). The first two criteria, proposed in this section, for selecting the optimal censoring scheme depend on the comparison of the Fisher information matrices (or equivalently of the variance-covraince (VarCov) matrix) of the MLEs of the unknown parameters as follow.

Criterion(I): Minimizing the determinant of VarCov matrix of the MLEs.

Criterion(II): Minimizing the trace of VarCov matrix of the MLEs.

In the case of one parameter distributions criteria I and II are quite effective. However, if we have multi-parameter distributions, these criteria are not scale invariant (see, Gupta and Kundu (2006)). So we have to use other criteria which are scale invariant. The other two criteria, we adopted here, are based on the comparison of the precisions of the logarithm of MLE for p-th quantile of the UG

			$\alpha=1.25$				$\beta=1.5$			
(n,m)	scheme		Bias	MSE	SSE	ESE	Bias	MSE	SSE	ESE
$(25,10)$	I	MLE	0.911	0.894	0.571	0.828	0.880	2.960	1.090	0.828
		MPS	0.760	0.537	0.716	1.473	0.663	0.864	0.601	1.473
		SEM	0.820	0.641	0.488	0.916	0.708	0.813	0.505	0.916
	II	MLE	0.439	0.512	0.565	0.835	0.928	2.136	1.130	0.835
		MPS	0.201	0.501	0.701	1.745	0.110	0.378	0.607	1.745
		SEM	0.335	0.357	0.495	0.969	0.524	0.541	0.517	0.969
$(50,20)$	I	MLE	0.876	0.779	0.120	0.264	0.726	2.656	0.574	0.260
		MPS	0.665	0.452	0.130	0.497	0.642	0.708	0.261	0.497
		SEM	0.741	0.553	0.081	0.320	0.695	0.800	0.179	0.320
	II	MLE	0.274	0.297	0.471	0.695	0.453	0.627	0.649	0.695
		MPS	0.193	0.408	0.620	1.397	0.103	0.245	0.495	1.397
		SEM	0.246	0.255	0.441	0.731	0.360	0.341	0.460	0.731
$(100,30)$	I	MLE	0.728	0.542	0.114	0.250	0.696	1.111	0.345	0.250
		MPS	0.564	0.335	0.127	0.432	0.630	0.460	0.250	0.432
		SEM	0.681	0.469	0.076	0.277	0.656	0.761	0.166	0.277
	II	MLE	0.238	0.200	0.378	0.542	0.315	0.308	0.457	0.542
		MPS	0.185	0.275	0.518	0.955	0.101	0.168	0.410	0.955
		SEM	0.229	0.187	0.366	0.554	0.289	0.239	0.394	0.554
			$\alpha=0.5$				$\beta=0.75$			
$(25,10)$	I	MLE	0.171	0.073	0.209	0.285	0.308	0.236	0.376	0.285
		MPS	0.103	0.173	0.403	0.736	0.050	0.108	0.325	0.736
		SEM	0.170	0.072	0.207	0.287	0.289	0.186	0.321	0.287
	II	MLE	0.139	0.074	0.234	0.325	0.275	0.222	0.383	0.355
		MPS	0.153	0.220	0.444	0.955	0.044	0.124	0.350	0.955
		SEM	0.137	0.073	0.233	0.333	0.258	0.182	0.340	0.333
$(50,20)$	I	MLE	0.121	0.050	0.189	0.281	0.165	0.078	0.225	0.281
		MPS	0.083	0.113	0.326	0.458	0.020	0.047	0.217	0.458
		SEM	0.121	0.050	0.189	0.284	0.164	0.077	0.224	0.284
	II	MLE	0.111	0.042	0.173	0.242	0.154	0.072	0.220	0.242
		MPS	0.059	0.080	0.277	0.540	0.014	0.045	0.212	0.540
		SEM	0.111	0.042	0.173	0.244	0.154	0.072	0.219	0.244
$(100,30)$	I	MLE	0.094	0.030	0.146	0.195	0.115	0.041	0.168	0.195
		MPS	0.041	0.049	0.218	0.342	0.007	0.028	0.166	0.342
		SEM	0.094	0.030	0.147	0.195	0.115	0.041	0.168	0.195
	II	MLE	0.076	0.028	0.148	0.203	0.092	0.032	0.153	0.203
		MPS	0.056	0.060	0.236	0.364	0.012	0.025	0.156	0.364
		SEM	0.076	0.028	0.149	0.204	0.093	0.032	0.153	0.204

Table 1: Simulation results of classical estimation methods. MLE: maximum likelihood estimator; MPS: maximum product of spacing estimator; SEM: stochastic EM; Bias: absolute average bias; SSE: sample standard error; ESE: estimated standard error; MSE: mean square error.

			α		β	
(n,m)			CI	IL	CI	IL
$(25,10)$	I	MLE	(0,2.449)	2.449	(0.790,4.066)	3.275
		MPS	(0,4.772)	4.772	(0,5.021)	6.841
		SEM	(0,2.815)	2.815	(0.124,3.924)	3.800
		MBT	(0.253,2.047)	1.794	(1.250,3.000)	1.750
		PBT	(0.282,2.499)	2.217	(0.661,2.804)	2.143
	II	MLE	(0,2.461)	2.461	(0.758,4.001)	3.244
		MPS	(0,4.296)	4.296	(0,4.450)	4.450
		SEM	(0,2.725)	2.725	(0.213,3.802)	3.590
		MBT	(0.259,1.987)	1.728	(1.269,3.000)	1.731
		PBT	(0.310,2.499)	2.188	(0.704,2.818)	2.114
$(50,20)$	I	MLE	(0,0.813)	0.813	(1.587,3.465)	1.878
		MPS	(0,1.559)	1.559	(1.339,3.286)	1.947
		SEM	(0,1.136)	1.136	(1.808,3.063)	1.255
		MBT	(0.267,1.022)	0.755	(1.835,3.000)	1.165
		PBT	(0.315,2.336)	2.021	(0.970,2.828)	1.858
	II	MLE	(0,0.863)	0.863	(0.431,1.378)	0.947
		MPS	(0,1.618)	1.618	(0,1.818)	1.818
		SEM	(0,0.868)	0.868	(0.425,1.383)	0.957
		MBT	(0.093,0.844)	0.750	(0.601,1.426)	0.826
		PBT	(0.130,0.963)	0.834	(0.485,1.503)	1.018
$(100,30)$	I	MLE	(0.032,1.013)	0.981	(2.006,2.987)	0.981
		MPS	(0,1.533)	1.694	(1.283,2.977)	1.694
		SEM	(0.025,1.113)	1.088	(1.813,2.900)	1.088
		MBT	(0.287,1.172)	0.885	(1.636,3.000)	1.364
		PBT	(0.352,2.122)	1.770	(1.068,2.818)	1.750
	II	MLE	(0,2.074)	2.074	(0.753,2.877)	2.123
		MPS	(0,3.208)	3.208	(0,3.388)	3.388
		SEM	(0,2.106)	2.106	(0.704,2.875)	2.172
		MBT	(0.366,2.110)	1.744	(1.119,2.879)	1.760
		PBT	(0.526,2.417)	1.891	(0.871,2.414)	1.542

Table 2: Simulation results of classical estimation methods for $\alpha = 1.25$ and $\beta = 1.5$.. MLE: maximum likelihood estimator; MPS: maximum product of spacing estimator; SEM: stochastic EM; MBT: Boot-p based on LK function; PBT: Boot-p based on PS function; CI: 95% confidence interval and IL: interval length.

			α		β	
(n,m)			CI	IL	CI	IL
$(25, 10)$	I	MLE	(0,0.887)	0.887	(0.499,1.617)	1.118
		MPS	(0,2.046)	2.046	(0,2.242)	2.242
		SEM	(0,0.893)	0.893	(0.476,1.603)	1.127
		MBT	(0.023,0.783)	0.761	(0.660,1.500)	0.840
		PBT	(0,0.965)	0.965	(0.463,1.608)	1.145
	II	MLE	(0,0.998)	0.998	(0.387,1.662)	1.274
		MPS	(0,2.526)	2.526	(0,2.667)	2.667
		SEM	(0,1.016)	1.016	(0.355,1.661)	1.306
		MBT	(0.039,0.821)	0.782	(0.636,1.498)	0.862
		PBT	(0.040,0.962)	0.923	(0.459,1.626)	1.167
$(50, 20)$	I	MLE	(0,0.929)	0.929	(0.364,1.466)	1.101
		MPS	(0,3.440)	3.440	(0,3.609)	3.609
		SEM	(0,0.936)	0.936	(0.358,1.471)	1.113
		MBT	(0.088,0.877)	0.789	(0.583,1.444)	0.861
		PBT	(0.131,0.982)	0.852	(0.470,1.522)	1.052
	II	MLE	(0,0.863)	0.863	(0.431,1.378)	0.947
		MPS	(0,1.618)	1.618	(0,1.818)	1.818
		SEM	(0,0.868)	0.868	(0.425,1.383)	0.957
		MBT	(0.093,0.844)	0.750	(0.601,1.426)	0.826
		PBT	(0.130,0.963)	0.834	(0.485,1.503)	1.018
$(100, 30)$	I	MLE	(0.024,0.787)	0.763	(0.484,1.246)	0.763
		MPS	(0,1.210)	1.210	(0.087,1.426)	1.339
		SEM	(0.024,0.788)	0.764	(0.483,1.247)	0.764
		MBT	(0.013,0.773)	0.760	(0.615,1.356)	0.741
		PBT	(0.087,0.948)	0.861	(0.519,1.457)	0.938
	II	MLE	(0.027,0.821)	0.795	(0.445,1.240)	0.795
		MPS	(0,1.279)	1.279	(0.025,1.451)	1.427
		SEM	(0.023,0.825)	0.801	(0.442,1.243)	0.801
		MBT	(0.142,0.838)	0.697	(0.594,1.285)	0.691
		PBT	(0.216,0.961)	0.745	(0.508,1.438)	0.930

Table 3: Simulation results of classical estimation methods for $\alpha = 0.50$ and $\beta = 0.75.$. MLE: maximum likelihood estimator; MPS: maximum product of spacing estimator; SEM: stochastic EM; MBT: Boot-p based on LK function; PBT: Boot-p based on PS function; CI: 95% confidence interval and IL: interval length.

(n,m)			α Bias	α MSE	α CI/HPD	α IL	β Bias	β MSE	β CI/HPD	β IL
$(25,10)$	I	BSL	0.723	0.533	(0.440,1.865)	1.425	0.638	0.426	(1.015,2.673)	1.659
		BSP	0.188	0.064	(0.636,2.299)	1.663	0.054	0.027	(0.949,2.126)	1.177
		MCL	0.577	0.477	(0.357,1.900)	1.543	0.483	0.298	(0.952,2.356)	1.404
		MCP	0.966	0.794	(0.567,4.879)	4.312	0.233	0.337	(0.747,1.816)	1.068
	II	BSL	0.419	0.060	(0.502,2.019)	1.517	0.355	0.166	(1.010,2.300)	1.290
		BSP	0.098	0.057	(0.577,2.195)	1.618	0.042	0.026	(0.949,2.128)	1.180
		MCL	0.339	0.375	(0.466,2.065)	1.599	0.584	0.291	(0.973,2.440)	1.468
		MCP	0.652	0.891	(0.520,3.592)	3.072	0.214	0.325	(0.780,1.818)	1.038
$(50,20)$	I	BSL	0.435	0.210	(0.356,1.567)	1.211	0.309	0.124	(1.137,2.612)	1.476
		BSP	0.094	0.054	(0.679,2.355)	1.676	0.031	0.024	(0.951,2.091)	1.140
		MCL	0.399	0.439	(0.336,1.462)	1.126	0.276	0.122	(1.110,2.479)	1.368
		MCP	0.374	0.642	(0.628,2.122)	1.494	0.023	0.119	(0.957,2.020)	1.064
	II	BSL	0.136	0.052	(0.551,1.967)	1.416	0.141	0.041	(1.075,2.317)	1.242
		BSP	0.069	0.040	(0.678,2.264)	1.586	0.029	0.020	(0.977,2.064)	1.087
		MCL	0.234	0.351	(0.400,1.754)	1.354	0.221	0.197	(1.056,2.425)	1.369
		MCP	0.561	0.606	(0.692,3.059)	2.367	0.099	0.118	(0.869,1.841)	1.028
$(100,30)$	I	BSL	0.253	0.084	(0.227,1.041)	0.814	0.122	0.058	(1.386,2.687)	1.301
		BSP	0.053	0.034	(0.720,2.303)	1.583	0.006	0.017	(0.994,2.071)	1.076
		MCL	0.183	0.375	(0.274,1.154)	0.880	0.128	0.090	(1.291,2.602)	1.311
		MCP	0.124	0.571	(0.691,2.176)	1.486	0.007	0.036	(0.994,2.012)	1.018
	II	BSL	0.129	0.020	(0.394,1.535)	1.141	0.490	0.020	(1.207,2.399)	1.192
		BSP	0.027	0.018	(0.714,2.312)	1.598	0.010	0.017	(0.994,2.067)	1.072
		MCL	0.177	0.284	(0.274,1.154)	0.880	0.284	0.090	(1.291,2.502)	1.211
		MCP	0.124	0.571	(0.690,2.175)	1.485	0.007	0.036	(0.994,2.012)	1.018

Table 4: Simulation results of Bayesian methods for $\alpha = 1.25$ and $\beta = 1.5$. BSL: Bayes estimate by numerically solving (38); BSP: Bayes estimate by numerically solving (39); MCL: Bayes estimate using MCMC based on LK function; MCP: Bayes estimate using MCMC based on PS function; Bias: absolute average bias; MSE: mean square error; CI: 95% credible interval by numerically solving by solving (40) and (41); HPD: highest posterior density using MCMC samples based on LK (or PS) function; IL: interval length.

			α				β			
(n,m)			Bias	MSE	CI/HPD	IL	Bias	MSE	CI/HPD	IL
$(25, 10)$	I	BSL	0.177	0.031	(0.219,0.686)	0.467	0.085	0.009	(0.561,0.989)	0.427
		BSP	0.018	0.004	(0.287,0.814)	0.527	0.012	0.002	(0.560,0.937)	0.377
		MCL	0.235	0.056	(0.186,0.649)	0.462	0.136	0.032	(0.588,1.026)	0.438
		MCP	0.022	0.027	(0.261,0.261)	0.553	0.006	0.010	(0.547,0.947)	0.399
	II	BSL	0.128	0.018	(0.256,0.768)	0.512	0.076	0.008	(0.563,0.986)	0.423
		BSP	0.019	0.004	(0.281,0.817)	0.536	0.012	0.002	(0.559,0.938)	0.379
		MCL	0.143	0.010	(0.218,0.731)	0.513	0.094	0.009	(0.585,1.022)	0.437
		MCP	0.026	0.033	(0.253,0.253)	0.573	0.011	0.012	(0.541,0.942)	0.401
$(50, 20)$	I	BSL	0.120	0.016	(0.201,0.601)	0.400	0.065	0.006	(0.620,1.019)	0.399
		BSP	0.017	0.004	(0.299,0.803)	0.504	0.009	0.002	(0.571,0.928)	0.357
		MCL	0.135	0.021	(0.181,0.569)	0.388	0.078	0.029	(0.627,1.038)	0.411
		MCP	0.014	0.005	(0.284,0.284)	0.483	0.005	0.002	(0.579,0.937)	0.358
	II	BSL	0.050	0.006	(0.249,0.726)	0.477	0.045	0.004	(0.608,1.022)	0.414
		BSP	0.018	0.004	(0.296,0.817)	0.521	0.011	0.002	(0.569,0.931)	0.362
		MCL	0.057	0.007	(0.232,0.676)	0.445	0.057	0.005	(0.614,1.007)	0.392
		MCP	0.012	0.008	(0.277,0.277)	0.496	0.001	0.002	(0.574,0.933)	0.359
$(100, 30)$	I	BSL	0.088	0.009	(0.163,0.527)	0.363	0.025	0.008	(0.752,1.079)	0.327
		BSP	0.013	0.004	(0.311,0.799)	0.488	0.004	0.002	(0.586,0.928)	0.342
		MCL	0.094	0.016	(0.122,0.427)	0.305	0.051	0.020	(0.702,1.107)	0.405
		MCP	0.010	0.005	(0.296,0.296)	0.476	0.003	0.002	(0.581,0.919)	0.338
	II	BSL	0.026	0.004	(0.199,0.598)	0.399	0.024	0.002	(0.664,1.068)	0.404
		BSP	0.010	0.004	(0.310,0.800)	0.490	0.003	0.002	(0.586,0.926)	0.340
		MCL	0.037	0.003	(0.180,0.555)	0.375	0.048	0.001	(0.750,1.059)	0.309
		MCP	0.010	0.004	(0.292,0.292)	0.478	0.001	0.002	(0.582,0.921)	0.338

Table 5: Simulation results of Bayesian methods for $\alpha = 0.50$ and $\beta = 0.75$. BSL: Bayes estimate by numerically solving (38); BSP: Bayes estimate by numerically solving (39); MCL: Bayes estimate using MCMC based on LK function; MCP: Bayes estimate using MCMC based on PS function; Bias: absolute average bias; MSE: mean square error; CI: 95% credible interval by numerically solving by solving (40) and (41); HPD: highest posterior density using MCMC samples based on LK (or PS) function;IL: interval length.

distribution. For the UG distribution, the logarithm for p-th quantile is given by

$$\log(T_p) = (-1/\beta)\log(1 - \log(p)/\alpha), \;\; 0 < p < 1.$$

Then the asymptotic variance of $\log(\hat{T}_p)$, the MLE of $\log(T_p)$ based on the censoring scheme $(R_1, \cdots, R_m)$, can be computed by using delta method as

$$Var(\log(\hat{T}_p)) = G^T(\hat{\alpha}, \hat{\beta})\text{VarCov}(\hat{\alpha}, \hat{\beta})G(\hat{\alpha}, \hat{\beta}),$$

where

$$G^T(\hat{\alpha}, \hat{\beta}) = \left(\frac{\partial \log(T_p)}{\partial \alpha}, \frac{\partial \log(T_p)}{\partial \beta}\right)\Big|_{\alpha=\hat{\alpha},\beta=\hat{\beta}} = \left(-\frac{\log(p)}{\hat{\alpha}^2\hat{\beta}(1 - \log(p)/\hat{\alpha})}, \frac{\log(1 - \log(p)/\hat{\alpha})}{\hat{\beta}^2}\right)$$

and

$$\text{VarCov}(\hat{\alpha}, \hat{\beta}) = \begin{pmatrix} \text{Var}(\hat{\alpha}) & \text{Cov}(\hat{\alpha}, \hat{\beta}) \\ \text{Cov}(\hat{\beta}, \hat{\alpha}) & \text{Var}(\hat{\beta}) \end{pmatrix}.$$

Following Gupta and Kundu (2006), for a given censoring scheme $(R_1, \cdots, R_m)$, we define the following information measure

$$I_S(R_1, \cdots, R_m) = \int_0^1 \text{Var}(\log(\hat{T}_p))W(p)dp,$$

where $W(p) \geq 0$ is a non-negative weight function such that $\int_0^1 W(p)dp = 1$. Hence, we propose the following criteria

Criterion(III): Minimizing the variance of the p-th quantile estimator, T_p for $p = 0.5$

Criterion(IV): Minimizing the variance of the p-th quantile estimator, T_p for $p = 0.9$

Criterion(V): Minimizing the information measure $I_S(R_1, \cdots, R_m)$ for $w(p) = 1$

Criterion(VI): Minimizing the information measure $I_S(R_1, \cdots, R_m)$ for $w(p) = 2p$.

Table 6 presents the results of optimal censoring based on MLEs and MPSs with respect to Criteria I-VI for $(\alpha, \beta) = (1.25, 1.5), (0.5, 0.75), m = 5$ and $n = 10, 15, 20$. From Table 6, we observed that the censoring scheme $(0 * m, n - m)$ is the most preferred one among the other schemes based on all criteria. Moreover, the reported censoring schemes are almost the same under the criteria V and VI for the MLE with $n = 10, 15$ and for the MPS with $n = 10, 20$.

9. Bayesian Prediction

The prediction of the censored or future observation(s) based on current sample (also known as informative sample) is important practical problem in applied statistics. For more information and recent references about this problem, see the articles Al-Hussaini (1999), Mousa and Jaheen (2002), Balakrishnan et al. (2010), Dey and Dey (2014), AL-Hussaini et al. (2015), Dey

		$(\alpha,\beta)=(1.25,1.5)$				$(\alpha,\beta)=(0.50,0.75)$			
		MLE		MPS		MLE		MPS	
n	Crit	Optimum	Value	Optimum	Value	Optimum	Value	Optimum	Value
10	I	(0*2,3,2,0)	0.0226	(0,1,0*2,4)	0.2361	(0*2,1,4,0)	0.0005	(0*4,5)	0.0042
	II	(2,0*3,3)	5.2522	(1,0*3,4)	7.2320	(1,0*3,4)	0.4981	(0*4,5)	0.7586
	III	(0*4,5)	0.0068	(0*4,5)	0.0101	(0*4,5)	0.0612	(0*4,5)	0.0972
	IV	(0*4,5)	0.0048	(0*4,5)	0.0035	(1,0*3,4)	0.0774	(0*4,5)	0.0662
	V	(0*4,5)	0.0062	(0*4,5)	0.0833	(0*2,1,0,4)	0.0617	(2,0*3,3)	0.4540
	VI	(0*4,5)	0.0058	(0*4,5)	0.0311	(0*2,1,0,4)	0.0666	(2,0*3,3)	0.1913
15	I	(0*2,10,0*2)	0.0158	(0,2*2,1,5)	0.1516	(0,1,0,9,0)	0.0003	(0*4,10)	0.0027
	II	(5,0*3,5)	3.5804	(0,2*2,1,5)	6.1574	(6,0*3,4)	0.3915	(4,0*3,6)	0.6698
	III	(0*4,10)	0.0068	(0*4,10)	0.0098	(0*4,10)	0.0583	(0*4,10)	0.0972
	IV	(3,0*3,7)	0.0042	(6,0*3,4)	0.0028	(4,0*3,6)	0.0726	(5,0*3,5)	0.0591
	V	(0*4,10)	0.0058	(2,0*3,8)	0.0738	(0*4,10)	0.0563	(4,0*3,6)	0.3566
	VI	(0*4,10)	0.0056	(5,0*3,5)	0.0262	(0*4,10)	0.0636	(7,0*3,3)	0.1430
20	I	(0*4,15)	0.0120	(1,3,1,2,8)	0.0964	(0*3,15,0)	0.0003	(1,2,0,1,11)	0.0020
	II	(10,0*3,5)	2.8574	(5,3,0,2,5)	4.8190	(10,0*3,5)	0.3332	(1,2,0,1,11)	0.6054
	III	(0*4,15)	0.0070	(2,0*3,13)	0.0100	(0*4,15)	0.0607	(1,2,0,1,11)	0.0976
	IV	(8,0*3,7)	0.0041	(3,0,2,1,9)	0.0029	(9,0*3,6)	0.0659	(3,2,7,1,2)	0.0482
	V	(0*4,15)	0.0058	(8,0*3,7)	0.0528	(1,0*3,14)	0.0559	(10,0*3,5)	0.2922
	VI	(4,0*2,1,10)	0.0057	(8,0*3,7)	0.0184	(2,1,0*2,12)	0.0639	(10,0*3,5)	0.1114

Table 6: The optimal censoring scheme (Optimum) with its value (Value) based on MLEs and MPSs for the criteria I-VI when $m = 5$ and $n = 10, 15, 20$.

et al. (2016b) and Kotb and Raqab (2019). Our main objective in this section is to investigate the predictive estimate and the predicative interval of the j-th order statistic $T = X_{j:R_k}$, $j = 1, 2, \cdots, R_k; k = 1, 2, \cdots, m$, based on progressively type II censored sample with respect one-sample and two-sample prediction problems.

9.1. One-sample prediction problem

Let $\mathbf{X} = (X_{1:m:n}, X_{2:m:n}, \cdots, X_{m:m:n})$ be a progressively type II censored sample with progressive censoring scheme $\mathbf{R} = (R_1, \cdots, R_m)$. To simplified our notation, let $X_{k:m:n} = X_k$ and $x_{k:m:n} = x_k$. Let $T = X_{j:R_k}$ denote the j-the order statistic out of R_k removed unites at stage k, for $j = 1, \cdots, R_k$ and $k = 1, 2 \cdots, m$. We assume that the predicted value of the j-th order statistic, $X_{j:R_k}$, should be take a value in the range $[x_k, x_{k+1}]$. Then the conditional distribution of T given $\mathbf{X} = \mathbf{x}$ is given by

$$\begin{aligned} f_{T|\mathbf{X}}(t|\mathbf{x}) =& f_{T|X_k}(t|x_k < t < x_{k+1}) \\ =& j\binom{R_k}{j}\frac{(F(t)-F(x_k))^{j-1}(F(x_{k+1})-F(t))^{R_k-j}f(t)}{(F(x_{k+1})-F(x_k))^{R_k}}, \quad x_k < t < x_{k+1}, \end{aligned}$$

where $j = 1, 2, \cdots, R_k$ and $k = 1, 2, \cdots, m$. Here, the p.d.f. f and the c.d.f. F are given in (1) and (2), respectively. Upon substituting (1) and (2) and using binomial and negative binomial expansions in the above expression, we get, for $0 < x_k < t < x_{k+1} < 1$,

$$\begin{aligned} f_{T|\mathbf{X}}(t|\mathbf{x}) =& \alpha\beta j\binom{R_k}{j}\frac{(e^{-\alpha(t^{-\beta}-1)} - e^{-\alpha(x_k^{-\beta}-1)})^{j-1}}{(e^{-\alpha(x_{k+1}^{-\beta}-1)} - e^{-\alpha(x_k^{-\beta}-1)})^{R_k}} \\ & \times (e^{-\alpha(x_{k+1}^{-\beta}-1)} - e^{-\alpha(t^{-\beta}-1)})^{R_k-j} t^{-(\beta+1)} e^{-\alpha(t^{-\beta}-1)} \\ =& \alpha\beta j\binom{R_k}{j}\sum_{i=0}^{R_k-j}\sum_{r=0}^{j+i-1}\sum_{u=0}^{\infty}\binom{R_k-j}{i}\binom{j+i-1}{r}\binom{i+j+u-1}{u} \\ & \times (-1)^{i-r-1} t^{-(\beta+1)} e^{-\alpha((r+1)t^{-\beta}-(u+r-1)x_k^{-\beta}+ux_{k+1}^{-\beta}-2)}. \end{aligned}$$

Let $\pi(\alpha, \beta|\mathbf{X})$ be the joint posterior density $\Omega(\alpha, \beta|\mathbf{X})$ given in (31) or the joint posterior density $\Psi(\alpha, \beta|\mathbf{X})$ given in (34). Then the Bayes prediction density function of T given $\mathbf{X} = \mathbf{x}$, can be computed by taking the expectation of the conditional distribution of T given $\mathbf{X} = \mathbf{x}$, $f_{T|\mathbf{X}}$, with respect to the joint posterior density, $\pi(\alpha, \beta|\mathbf{X})$, as follows.

$$\begin{aligned} f^*_{T|\mathbf{X}}(t|\mathbf{x}) =& E_{Posterior}(f_{T|\mathbf{X}}(t|\mathbf{x})) \\ =& \int_0^\infty\int_0^\infty f_{T|\mathbf{X}}(t|\mathbf{x})\pi(\alpha, \beta|\mathbf{x})d\alpha d\beta \\ =& j\binom{R_k}{j}\sum_{i=0}^{R_k-j}\sum_{r=0}^{j+i-1}\sum_{u=0}^{\infty}\binom{R_k-j}{i}\binom{j+i-1}{r}\binom{i+j+u-1}{u}(-1)^{i-r-1} \\ & \times \int_0^\infty\int_0^\infty \alpha\beta t^{-(\beta+1)} e^{-\alpha((r+1)t^{-\beta}-(u+r-1)x_k^{-\beta}+ux_{k+1}^{-\beta}-2)}\pi(\alpha, \beta|\mathbf{x})d\alpha d\beta. \end{aligned} \tag{42}$$

From (42), the Bayes predication of $T = X_{j:R_k}, j = 1, 2, \cdots, R_k$, given $\mathbf{X} = \mathbf{x}$, can be obtained by

$$E(T|\mathbf{X} = \mathbf{x}) = \int_{x_k}^{x_{k+1}} t f^*_{T|\mathbf{X}}(t|\mathbf{x}) dt. \tag{43}$$

Consequently, the predictive survival function of $T = X_{j:R_k}$ given $\mathbf{X} = \mathbf{x}$ is obtained as

$$\begin{aligned}
S^*_{T|\mathbf{X}=\mathbf{x}}(t|\mathbf{x}) =& \Pr(T > t|\mathbf{x}) \\
=& \int_t^{x_{k+1}} f^*_{T|\mathbf{X}}(w|\mathbf{x}) dw \\
=& j\binom{R_k}{j} \sum_{i=0}^{R_k-j} \sum_{r=0}^{j+i-1} \sum_{u=0}^{\infty} \binom{R_k - j}{i} \binom{j+i-1}{r} \binom{i+j+u-1}{u} (-1)^{i-r-1} \\
& \times \int_0^{\infty}\int_0^{\infty}\int_t^{x_{k+1}} \alpha\beta w^{-(\beta+1)} e^{-\alpha(r+1)w^{-\beta}} dw e^{-\alpha((u+r-1)x_k^{-\beta} + u x_{k+1}^{-\beta} - 2)} \pi(\alpha, \beta|\mathbf{x}) d\alpha d\beta \\
=& j\binom{R_k}{j} \sum_{i=0}^{R_k-j} \sum_{r=0}^{j+i-1} \sum_{u=0}^{\infty} \binom{R_k - j}{i} \binom{j+i-1}{r} \binom{i+j+u-1}{u} \frac{(-1)^{i-r-1}}{r+1} \\
& \times \int_0^{\infty}\int_0^{\infty} \left(e^{-\alpha(r+1)x_{k+1}^{-\beta}} - e^{-\alpha(r+1)t^{-\beta}}\right) e^{-\alpha((u+r-1)x_k^{-\beta} + u x_{k+1}^{-\beta} - 2)} \pi(\alpha, \beta|\mathbf{x}) d\alpha d\beta.
\end{aligned}$$

Therefore the $100(1-\gamma)\%$ Bayes predictive interval of $T = X_{j:R_k}$ can be computed by solving the following equations for the lower bound, L, and upper bound , U, as

$$S^*_{T|\mathbf{X}}(L|\mathbf{x}) = \frac{1+\gamma}{2} \text{ and } S^*_{T|\mathbf{X}}(U|\mathbf{x}) = \frac{1-\gamma}{2}.$$

9.2. Two-sample Bayesian predication

Let $\mathbf{Y} = (Y_1, Y_2, \cdots, Y_N)$ be an unobserved independent ordered sample of size N from the same population of the informative sample. This sample will be referred to as the future sample. Our aim here is to predict the j-th ordered statistic in the future sample, Y_j, based on the progressively type II censored informative sample. The density function, Y_j, is given by

$$f_{Y_j|\mathbf{X}}(y|\mathbf{x}, \alpha, \beta) = f_{Y_j}(y|\alpha, \beta) = j\binom{N}{j} (F(y|\alpha, \beta))^{j-1} (1 - F(y|\alpha, \beta))^{N-j} f(y|\alpha, \beta). \tag{44}$$

Upon substituting (1) and (2) and using binomial expansion in the above expression, we get

$$\begin{aligned}
f_{Y_j|\mathbf{X}}(y|\mathbf{x}, \alpha, \beta) =& \alpha\beta j \binom{N}{j} e^{-\alpha(j-1)(y^{-\beta}-1)} (1 - e^{-\alpha(y^{-\beta}-1)})^{N-j} y^{-(\beta+1)} e^{-\alpha(y^{-\beta}-1)} \\
=& \alpha\beta j \sum_{u=0}^{N-j} \binom{N}{j} \binom{N-j}{u} (-1)^{N-j-u} y^{-(\beta+1)} e^{-\alpha(N-u)(y^{-\beta}-1)}.
\end{aligned} \tag{45}$$

Then the Bayes predictive density function of Y_j given $\mathbf{x}$ is given by

$$\begin{aligned} f^*_{Y_j|\mathbf{X}}(y|\mathbf{x}) =& E_{Posterior}(f_{Y_j|\mathbf{X}}(y|\mathbf{x})) \\ =& j\sum_{u=0}^{N-j}\binom{N}{j}\binom{N-j}{u}(-1)^{N-j-u}\int_0^\infty\int_0^\infty \alpha\beta y^{-(\beta+1)}e^{-\alpha(N-u)(y^{-\beta}-1)}\pi(\alpha,\beta|\mathbf{X})d\alpha d\beta. \end{aligned} \tag{46}$$

Consequently, the predictive survival function of Y_j given $\mathbf{x}$, $S^*_{Y_j|\mathbf{X}}$, is obtained as

$$\begin{aligned} S^*_{Y_j|\mathbf{X}}(t|\mathbf{x}) =& \Pr(Y_j > t|\mathbf{x}) \\ =& \int_t^\infty f^*_{Y_j|\mathbf{X}}(y|\mathbf{x})dy \\ =& j\sum_{u=0}^{N-j}\binom{N}{j}\binom{N-j}{u}(-1)^{N-j-u} \\ &\times\int_0^\infty\int_0^\infty \alpha\beta\int_t^1 y^{-(\beta+1)}e^{-\alpha(N-u)(y^{-\beta}-1)}dy\pi(\alpha,\beta|\mathbf{X})d\alpha d\beta, \\ =& j\sum_{u=0}^{N-j}\binom{N}{j}\binom{N-j}{u}\frac{(-1)^{N-j-u}}{N-u}\int_0^\infty\int_0^\infty\left[1-e^{-\alpha(N-u)(t^{-\beta}-1)}\right]\pi(\alpha,\beta|\mathbf{X})d\alpha d\beta. \end{aligned}$$

Similarly, the $100(1-\gamma)\%$ Bayes predictive interval of Y_j given $\mathbf{x}$ can be computed by solving the following equations for the lower bound, L and upper bound U

$$S^*_{Y_j|\mathbf{X}}(L|\mathbf{x}) = \frac{1+\gamma}{2} \quad \text{and} \quad S^*_{Y_j|\mathbf{X}}(U|\mathbf{x}) = \frac{1-\gamma}{2}.$$

The solutions of the above equations can not be obtained analytically so we apply a numerical technique for solving them simultaneously.

10. Real data analysis

In this section, we analyze a data set as a real life application of the UG distribution. The data set represents 20 observations of the maximum flood level (in millions of cubic feet per second) for Susquehanna River at Harrisburg, Pennsylvania and is reported in Dumonceaux and Antle (1973).
Data Set:

$$0.26, 0.27, 0.30, 0.32, 0.32, 0.34, 0.38, 0.38, 0.39, 0.40,$$

$$0.41, 0.42, 0.42, 0.42, 0.45, 0.48, 0.49, 0.61, 0.65, 0.74.$$

By Mazucheli and Dey (2019), the GU distribution fits the real data set in comparison to the beta, Kumaraswamy and McDonal distributions. The MLEs (standard errors) of the completed data set of α and β are $0.02(0.02)$ and $4.14(0.74)$, respectively. For estimating the unknown parameters,

Method	Scheme	R	Generated sample
LK	I	(0*3,1*2,0,1,0,2,5)	0.26,0.27,0.30,0.32,0.34,0.38,0.39,0.41,0.42,0.45
	II	(5,0,2,0,2,0*4,1)	0.26,0.38,0.38,0.41,0.42,0.45,0.48,0.49,0.61,0.65
	II	(1,0*5,1*2,2,5)	0.26,0.30,0.32,0.32,0.34,0.38,0.38,0.40,0.42,0.45
	IV	(0,1,2,4,2,0,1,0*3)	0.26,0.27,0.32,0.38,0.42,0.45,0.48,0.61,0.65,0.74
	V	(0*3,1*2,0,1,0,2,5)	0.26,0.27,0.30,0.32,0.34,0.38,0.39,0.41,0.42,0.45
	VI	(0*3,1*2,0,1,0,2,5)	0.26,0.27,0.30,0.32,0.34,0.38,0.39,0.41,0.42,0.45
PS	I	(1*2,0*3,1,0*2,1,6)	0.26,0.30,0.32,0.34,0.38,0.38,0.40,0.41,0.42,0.42
	II	(0,3,0*2,2,0,2,0,1,2)	0.26,0.27,0.34,0.38,0.38,0.41,0.42,0.45,0.48,0.61
	II	(0*5,4,0,1,0,5)	0.26,0.27,0.30,0.32,0.32,0.34,0.41,0.42,0.42,0.45
	IV	(3,0*2,1,2,0*2,3,1,0)	0.26,0.32,0.34,0.38,0.39,0.42,0.42,0.42,0.61,0.74
	V	(3,0*2,2,1*2,0*3,3)	0.26,0.32,0.34,0.38,0.40,0.42,0.42,0.45,0.48,0.49
	VI	(3,0*2,2,1*2,0*3,3)	0.26,0.32,0.34,0.38,0.40,0.42,0.42,0.45,0.48,0.49

Table 7: The generated progressively type II censored sample from the real data set based on schemes I-VI using the likelihood (LK) and product of spacing (PS) functions.

we have considered 6 different censoring schemes corresponding to the six optimal criteria presented in Section 8. The progressively censored samples are generated from the completed set by considering $m = 10$ and using all 6 censoring schemes and the generated samples are reported in Table 7. The results of classical and Bayesian estimations are presented in Table 8 and 9. In Table 8 we report the estimates (Est.), estimated standard error (ESE), 95% confidence interval (CI) and the length of these intervals (IL) of the parameters α and β using MLE, MPS and SEM methods. Moreover, 95% percentile bootstrap confidence interval (CIB) and their lengths (ILB) for the two parameters using MLE and MPS methods are also included. In Table 9, we report Bayes estimates (Est.), 95% credible/HPD intervals (CI/HPD) and their lengths (IL) of the two parameters using numerical calculations (DIR) and MCMC methods based on the likelihood (LK) and product of spacing (PS) methods. From the reported values in Tables 8 and 9, it can seen that, in terms of confidence/ceridable intervals, the censoring scheme III has lower interval lengths for the classical estimation while the censoring scheme II has the lower interval lengths for Bayesian estimation.

With respect to the prediction problem, the predictive values, $\tilde{x}_{j:R_k}$ or Y_j, and the 95% prediction intervals, (L,U), and their lengths (IL) using one-sample and two-sample techniques for the first three optimal censoring schemes, I, II and III, with $N = 20$ and $m = 10$ are reported in Table 10 and Table 11. From Table 11, it can be seen that, for a future sample of size 10, the length of the predication intervals become wider with the increase in j for $j < 10$ and for all the censoring schemes.

11. Concluding remarks

In this article, statistical inference of the unknown parameters of UG distribution based on progressively type II censoring scheme is considered. The MLEs and MPSEs as well as associated

		α						β					
		Est	ESE	CI	IL	CIB	ILB	Est	ESE	CI	IL	CIB	ILB
MLE	I	0.05	0.01	(0,0.15)	0.15	(0,0.21)	0.21	3.19	0.75	(1.50,4.88)	3.39	(2.00,6.72)	4.72
	II	0.09	0.01	(0,0.25)	0.25	(0,0.44)	0.43	3	0.69	(1.35,4.62)	3.27	(1.71,6.19)	4.48
	III	0.04	0.01	(0,0.12)	0.12	(0,0.16)	0.16	3.48	0.82	(2,5.06)	3.06	(2.27,6.17)	3.90
	IV	0.17	0.03	(0,0.51)	0.51	(0,0.80)	0.79	2.29	0.63	(0.74,3.84)	3.10	(1.21,5.20)	3.99
	V	0.05	0.01	(0,0.15)	0.15	(0,0.21)	0.21	3.19	0.75	(1.50,4.88)	3.39	(2.03,6.72)	4.69
	VI	0.05	0.01	(0,0.15)	0.15	(0,0.21)	0.21	3.19	0.75	(1.50,4.88)	3.39	(2.03,6.67)	4.64
MPS	I	0.08	0.01	(0,0.28)	0.28	(0,0.93)	0.93	2.71	0.95	(0.80,4.62)	3.82	(1.19,4.83)	3.64
	II	0.25	0.07	(0,0.76)	0.76	(0.04,2.68)	2.64	1.88	0.65	(0.30,3.46)	3.16	(0.48,3.57)	3.08
	III	0.14	0.02	(0,0.24)	0.24	(0.01,0.82)	0.81	2.25	0.81	(0.89,4.02)	3.13	(1,4.07)	3.07
	IV	0.31	0.13	(0,1.02)	1.02	(0.04,4.11)	4.07	1.74	0.79	(0,3.49)	3.49	(0.34,3.41)	3.07
	V	0.11	0.01	(0,0.35)	0.35	(0.01,0.93)	0.91	2.56	0.79	(0.82,4.30)	3.48	(1,4.51)	3.51
	VI	0.11	0.01	(0,0.35)	0.35	(0.01,0.94)	0.93	2.56	0.79	(0.82,4.30)	3.48	(0.98,4.47)	3.49
SEM	I	0.05	0.05	(0,0.15)	0.15	-	-	3.20	0.85	(1.52,4.87)	3.35	-	-
	II	0.09	0.09	(0,0.27)	0.27	-	-	2.97	0.86	(1.28,4.66)	3.38	-	-
	III	0.04	0.04	(0,0.11)	0.11	-	-	3.47	0.84	(1.82,5.11)	3.30	-	-
	IV	0.17	0.15	(0,0.47)	0.47	-	-	2.31	0.74	(0.87,3.75)	2.88	-	-
	V	0.05	0.05	(0,0.14)	0.14	-	-	3.22	0.80	(1.65,4.79)	3.14	-	-
	VI	0.05	0.05	(0,0.16)	0.16	-	-	3.18	0.92	(1.37,4.99)	3.62	-	-

Table 8: Classical estimation of the real data set based on censoring schemes I-VI using the likelihood (LK) and the product of spacing (PS) functions

		LK						PS					
		α			β			α			β		
		Est	CI/HPD	IL	Est	CI/HPD	IL	Est	CI/HPD	IL	Est	CI/HPD	IL
DIR	I	0.02	(0,0.05)	0.05	3.94	(3.14,4.83)	1.69	0.02	(0,0.05)	0.05	4.34	(3.33,5.28)	1.95
	II	0.03	(0,0.06)	0.06	4.01	(3.23,4.86)	1.63	0.03	(0,0.06)	0.06	3.96	(3.17,4.85)	1.68
	III	0.02	(0,0.05)	0.05	4.04	(3.22,4.97)	1.75	0.02	(0,0.05)	0.05	3.95	(3.15,4.91)	1.76
	IV	0.03	(0,0.06)	0.06	3.77	(3.04,4.58)	1.54	0.03	(0,0.06)	0.06	4.22	(3.33,5.25)	1.93
	V	0.02	(0,0.05)	0.05	3.94	(3.14,4.83)	1.69	0.02	(0,0.05)	0.05	4.36	(3.43,5.43)	1.99
	VI	0.02	(0,0.05)	0.05	3.94	(3.14,4.83)	1.69	0.02	(0,0.05)	0.05	4.36	(3.43,5.43)	1.99
MCMC	I	0.03	(0.02,0.04)	0.02	3.51	(3.09,3.89)	0.80	0.05	(0.03,0.08)	0.05	4.21	(3.26,5.76)	2.50
	II	0.05	(0.03,0.06)	0.03	3.54	(3.01,3.81)	0.80	0.03	(0.01,0.04)	0.03	3.96	(3.32,4.66)	1.34
	III	0.05	(0.04,0.06)	0.02	3.29	(2.88,3.67)	0.79	0.04	(0.02,0.07)	0.05	4.29	(2.73,5.81)	3.08
	IV	0.03	(0.01,0.05)	0.04	3.86	(3.18,4.59)	1.41	0.03	(0.01,0.05)	0.04	4.09	(3.27,4.85)	1.58
	V	0.03	(0.02,0.05)	0.03	3.61	(3.09,4.10)	1.02	0.04	(0.02,0.08)	0.06	3.86	(3.13,4.59)	1.46
	VI	0.03	(0.03,0.04)	0.01	3.50	(3.11,3.91)	0.80	0.04	(0.02,0.06)	0.04	3.79	(3.15,4.42)	1.27

Table 9: Bayesian estimation of the real data set based on censoring schemes I-VI using the likelihood (LK) and the product of spacing (PS) functions

	LK						PS					
	k	R_k	j	$\tilde{x}_{j:R_k}$	(L,U)	IL	k	R_k	j	$\tilde{x}_{j:R_k}$	(L,U)	IL
I	4	1	1	0.330	(0.320,0.340)	0.020	1	1	1	0.281	(0.263,0.300)	0.037
	5	1	1	0.360	(0.339,0.380)	0.041	2	1	1	0.310	(0.300,0.320)	0.012
	7	1	1	0.400	(0.390,0.410)	0.020	6	1	1	0.390	(0.380,0.400)	0.012
	9	2	1	0.430	(0.419,0.445)	0.026	9	1	1	0.415	(0.408,0.420)	0.012
	9	2	2	0.440	(0.425,0.450)	0.025	10	6	1	0.477	(0.420,0.617)	0.197
	10	5	1	0.510	(0.454,0.660)	0.206	10	6	2	0.537	(0.434,0.719)	0.285
	10	5	2	0.577	(0.464,0.769)	0.305	10	6	3	0.604	(0.461,0.810)	0.349
	10	5	3	0.654	(0.498,0.866)	0.368	10	6	4	0.680	(0.499,0.890)	0.391
	10	5	4	0.745	(0.546,0.948)	0.402	10	6	5	0.768	(0.555,0.958)	0.403
	10	5	5	0.857	(0.626,0.994)	0.368	10	6	6	0.872	(0.643,0.995)	0.352
II	1	5	1	0.289	(0.263,0.339)	0.076	2	3	1	0.290	(0.271,0.320)	0.049
	1	5	2	0.312	(0.270,0.355)	0.085	2	3	2	0.307	(0.275,0.335)	0.060
	1	5	3	0.331	(0.285,0.370)	0.085	2	3	3	0.324	(0.292,0.340)	0.048
	1	5	4	0.348	(0.303,0.375)	0.072	5	2	1	0.390	(0.381,0.405)	0.024
	1	5	5	0.365	(0.330,0.380)	0.050	5	2	2	0.400	(0.385,0.410)	0.025
	3	2	1	0.390	(0.379,0.405)	0.026	7	2	1	0.430	(0.422,0.445)	0.023
	3	2	2	0.400	(0.385,0.410)	0.025	7	2	2	0.440	(0.425,0.450)	0.025
	5	2	1	0.430	(0.422,0.445)	0.023	9	1	1	0.543	(0.484,0.605)	0.121
	5	2	2	0.440	(0.425,0.450)	0.025	10	2	1	0.718	(0.613,0.919)	0.306
	10	1	1	0.804	(0.658,0.988)	0.330	10	2	2	0.846	(0.655,0.993)	0.338
III	1	1	1	0.281	(0.260,0.300)	0.040	6	4	1	0.354	(0.344,0.381)	0.037
	7	1	1	0.390	(0.380,0.400)	0.020	6	4	2	0.368	(0.345,0.399)	0.054
	8	1	1	0.410	(0.401,0.420)	0.019	6	4	3	0.382	(0.354,0.405)	0.051
	9	2	1	0.430	(0.423,0.445)	0.022	6	4	4	0.396	(0.368,0.410)	0.042
	9	2	2	0.440	(0.425,0.450)	0.025	8	1	1	0.415	(0.409,0.420)	0.011
	10	5	1	0.510	(0.453,0.659)	0.206	10	5	1	0.512	(0.454,0.664)	0.210
	10	5	2	0.576	(0.465,0.771)	0.306	10	5	2	0.581	(0.466,0.774)	0.308
	10	5	3	0.653	(0.495,0.869)	0.374	10	5	3	0.659	(0.501,0.873)	0.372
	10	5	4	0.744	(0.548,0.949)	0.401	10	5	4	0.751	(0.549,0.951)	0.402
	10	5	5	0.856	(0.624,0.994)	0.370	10	5	5	0.862	(0.630,0.995)	0.365

Table 10: One-sample predictive estimates of the real data set , $\tilde{x}_{j:R_k}$, 95% prediction intervals, (L,U) and their lengths, IL, using likelihood (LK) and product of spacing (PS) functions for the first three optimal censoring schemes, I, II and III.

	LK				PS			
	j	Y_j	(L,U)	IL	j	Y_j	(L,U)	IL
I	1	0.271	(0.151,0.414)	0.263	1	0.264	(0.141,0.421)	0.281
	2	0.330	(0.205,0.490)	0.285	2	0.327	(0.191,0.501)	0.310
	3	0.380	(0.243,0.554)	0.311	3	0.380	(0.234,0.571)	0.337
	4	0.428	(0.279,0.624)	0.344	4	0.431	(0.270,0.638)	0.367
	5	0.479	(0.315,0.690)	0.375	5	0.484	(0.307,0.701)	0.394
	6	0.534	(0.351,0.754)	0.403	6	0.542	(0.346,0.770)	0.424
	7	0.595	(0.395,0.823)	0.429	7	0.605	(0.393,0.840)	0.447
	8	0.666	(0.444,0.896)	0.452	8	0.678	(0.446,0.905)	0.459
	9	0.752	(0.504,0.955)	0.451	9	0.763	(0.511,0.962)	0.451
	10	0.859	(0.598,0.995)	0.397	10	0.867	(0.609,0.995)	0.386
II	1	0.310	(0.177,0.468)	0.291	1	0.273	(0.141,0.441)	0.300
	2	0.375	(0.237,0.546)	0.309	2	0.341	(0.198,0.523)	0.325
	3	0.429	(0.279,0.613)	0.334	3	0.397	(0.239,0.595)	0.355
	4	0.480	(0.317,0.677)	0.360	4	0.451	(0.283,0.659)	0.377
	5	0.532	(0.355,0.741)	0.385	5	0.506	(0.321,0.728)	0.407
	6	0.586	(0.398,0.802)	0.404	6	0.565	(0.364,0.792)	0.428
	7	0.646	(0.443,0.860)	0.417	7	0.629	(0.414,0.859)	0.445
	8	0.714	(0.492,0.914)	0.422	8	0.701	(0.467,0.917)	0.450
	9	0.792	(0.561,0.968)	0.407	9	0.784	(0.538,0.963)	0.425
	10	0.884	(0.657,0.995)	0.338	10	0.881	(0.642,0.995)	0.353
III	1	0.276	(0.156,0.418)	0.262	1	0.257	(0.133,0.412)	0.279
	2	0.335	(0.208,0.494)	0.287	2	0.304	(0.186,0.497)	0.311
	3	0.384	(0.248,0.557)	0.309	3	0.374	(0.227,0.568)	0.341
	4	0.432	(0.287,0.625)	0.338	4	0.426	(0.266,0.636)	0.370
	5	0.482	(0.322,0.691)	0.369	5	0.481	(0.304,0.700)	0.396
	6	0.537	(0.356,0.760)	0.404	6	0.539	(0.343,0.769)	0.426
	7	0.597	(0.399,0.824)	0.425	7	0.603	(0.388,0.840)	0.452
	8	0.668	(0.449,0.896)	0.447	8	0.677	(0.442,0.905)	0.464
	9	0.753	(0.507,0.955)	0.448	9	0.763	(0.508,0.962)	0.454
	10	0.859	(0.601,0.995)	0.394	10	0.868	(0.608,0.995)	0.387

Table 11: Two-sample predictive estimates for the future sample of the real data set, Y_j, 95% prediction intervals, (L,U) and their lengths (IL) using likelihood (LK) and product of spacing (PS) functions for the first three optimal censoring schemes, I, II and III.

ACIs are obtained. MLEs are obtained by using both Newton-Raphson and stochastic expectation minimization (SEM) algorithms. The Bayes estimates based on likelihood and product of spacings functions are developed using gamma informative priors relative to squared-error loss function. Random-walk Metropolis-Hastings algorithm has been used to obtain the point Bayes estimates and their associated HPD intervals. Simulation results showed that the MPS method performs better than ML and SEM methods in estimating the unknown parameters in terms of bias, while SEM algorithm performs better than MLE. With respect to 95% confidence interval, Boot-p methods based on LK function (MBT) have the smaller CI. Bayes estimates obtained by PS method out performs LK method in terms of both bias and MSE, while the length of the HPD intervals based on LK function are shorter than PS function. In regard to optimal censoring, the censoring scheme $(0 * m, n - m)$ is the most preferred one among the other schemes based on all criteria. However, the considered censoring schemes are almost the same under the criteria V and VI for the MLE with $n = 10, 15$ and for the MPS with $n = 10, 20$. In real data analysis, we observed that confidence/ceridable intervals based on censoring scheme III has the lower interval lengths for the classical estimation while the censoring scheme II has the lower interval lengths for Bayesian estimation. Finally, the Bayesian approach according to both the traditional likelihood and product of spacing functions to estimate the parameters of the UG distribution under progressive type II censoring is recommended. We hope that the methodologies proposed in this work will be useful to applied statisticians. It will be interesting to study the methods of estimation under hybrid censored data. The work is in progress and it will be reported later.

Conflict of interest

No potential conflict of interest was reported by the authors.

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