Approximate Stochastic Response of Hysteretic System With Fractional Element and Subjected to Combined Stochastic and Periodic Excitation

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Research Article

Keywords: Bouc-Wen model, fractional derivative, statistical linearization, harmonic balance, combined excitation

Posted Date: February 10th, 2021

DOI: https://doi.org/10.21203/rs.3.rs-181868/v1

Version of Record: A version of this preprint was published at Nonlinear Dynamics on November 10th, 2021. See the published version at https://doi.org/10.1007/s11071-021-07014-w.

Noname manuscript No.
(will be inserted by the editor)

Approximate stochastic response of hysteretic system with fractional element and subjected to combined stochastic and periodic excitation

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Received: date / Accepted: date

Abstract A method based on statistical linearization is proposed, for determining response of the single-degree-of-freedom (SDOF) hysteretic system endowed with fractional derivatives and subjected to combined periodic and white/colored excitation. The method is developed by decomposing the system response into a combination of a periodic and of a zero-mean stochastic components. In this regard, first, the equation of motion is cast into two sets of coupled fractional-order non-linear differential equations with unknown deterministic and stochastic response components. Next, the harmonic balance method and the statistical linearization for the fractional-order deterministic and stochastic subsystems are used, to obtain the Fourier coefficients of the deterministic component and the variance of the stochastic component, respectively. This yields two sets of coupled non-linear algebraic equations which can be solved by appropriate standard numerical method. Pertinent numerical examples, including both softening and hardening Bouc-Wen hysteretic system endowed with different fractional-orders, are used to demonstrate the applicability and accuracy of the proposed method.

Keywords Bouc-Wen model · fractional derivative · statistical linearization · harmonic balance · combined excitation

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1 Introduction

Many mechanical and structural systems subjected to severe dynamic loads exhibit hysteretic behavior[1]. The hysteretic restoring force not only depends on the instantaneous system response, but also depends on the response history. Therefore, compared to the zero- memory non-linear elements with elastic behavior, the hysteretic force-displacement relationships exhibit multi-valued loops [2]. To describe the history-depended non-linear behavior of the hysteretic systems, various parametric models have been proposed. A dry friction based hysteretic bilinear model can be found in Ref. [3, 4]. Proposed by Bouc[5] and improved by Wen[6], the Bouc-Wen hysteretic model and its extensions [7] has gained wide popularity among researchers, due to its versatility and simplicity; Also see a comprehensive literature review in [8] and a book [9] devoted to various application of this celebrated model.

A proper hysteretic model can represent various of engineering structural/material hysteresis behaviors. In this regard, the Bouc-Wen model belongs to a class of smooth analytical models described by differential equations[10]. This yields a mathematical based framework to develop analytical knowledge on relevant properties, which can be used for multi-purposes in engineering applications. In this regard, Chassiakos[11] et al. developed an on-line identification method for hysteretic systems using the Bouc-Wen model. Later on, the Bouc-Wen model was utilized for representing force-displacement relationship of MR dampers [12], lead-core rubber isolators [13] and steel dampers [14] and for describing constitutive law of smart materials[15].

Recently, a useful mathematical tools named fractional-order calculus has been widely used in various engineering applications, especially for mechanical modeling of visco-elastic materials[16]. In this context, one may argue that if visco-elastic relaxation test of the material is well fitted by a power law decay, then the fractional constitutive law in Caputo's derivative sense naturally appears [17]. Compared to the standard linear solid model, the fractional derivative model can describe the broad-band of visco-elastic materials with small number of parameters [18]. Further, fractional derivatives has proved particularly useful in the fields of structural engineering, where visco-elastic materials are used in vibration mitigation devices for structures. Certain examples are related to visco-elastic dampers for connecting shear walls [19] and adjacent stories [20] of buildings. In this regards, the response determination of a structural dynamic systems endowed with fractional derivatives and subjected to random excitation arise naturally; see [21] for example.

For the engineering applications where the visco-elastic controlled/isolated structures or/and the involved visco-elastic dampers/isolators behave hysteretically, one need to study a hysteretic dynamic model with fractional derivative. This kind of model has a rate-dependent force- displacement relationship, which can be used for modeling visco-elastic controlled/isolated structures subjected to server external excitation. Very recently, Kang [22] et al. presented a new fractional-order normalized Bouc-Wen model to describe the asymmetric and rate-dependent hysteresis nonlinearity of piezoelectric actuators.

From the preceding literature review one may conclude that very few researchers focus on the response of a hysteretic dynamic system endowed with integer-/fractional- damping element subject to purely periodic or stochastic excitation. However, for some engineering applications, structures/systems are subjected to combined periodic and stochastic excitation. For example, aircrafts with rotation mechanism[23] are often subjected to the mixture of colored stochastic and harmonic excitation. Other examples include response of a gear system [24, 25] and airfoil model [26] under stochastic excitation with uncertain disturbance, and so forth. More recently, non-linear systems subjected to combined harmonic and stochastic excitation attract increasing interest of researchers to account a more realistic operation condition in energy harvest application[27, 28]. The preceding literature review necessitate the following research regarding the response determination of a hysteretic systems endowed with fractional element and subjected to combined periodic and stochastic excitation.

In this paper, a method based on statistical linearization(SL) [29] is proposed, for determining response of the single-degree-of-freedom (SDOF) hysteretic system endowed with fractional element and subjected to combined periodic and white/colored excitation. It can be regarded as an extension of the recent developed method for response determination of a integer-order hysteretic dynamic system subjected to combined harmonic and white noise excitation [30]. This is achieved based on an assumption that the system response can be decomposed into a summation of deterministic and stochastic component[31]. Accordingly, the original fractional order equation of motion can be cast into a sets of two coupled fractional-order differential equations in terms of the deterministic component, and variance of the stochastic component, respectively. The harmonic balance method and the SL method are then utilized for these sub-equations of motion, leading to a set of two non-linear algebraic equations. These non-linear algebraic equations can be solved simultaneously using standard numerical schemes, such as Newton's iteration method. The applicability and accuracy of the proposed method is demonstrated by comparing with pertinent Monte Carlo simulations.

2 Mathematical formulation

Consider a SDOF hysteretic system endowed with fractional element and subjected to a combined periodic and stochastic excitation

$$m\ddot{x}\left(t\right)+c\mathrm{D}_{\mathrm{C}}^{q}x\left(t\right)+\alpha kx\left(t\right)+\left(1-\alpha\right)kz\left(t\right)=w\left(t\right)+f\left(t\right), \tag{1}$$

where $x\left(t\right)$ and $\ddot{x}\left(t\right)$ is the displacement and acceleration of the system, respectively; m, c and k is the coefficient for mass, damping and stiffness, respectively; $D_{\mathrm{C}}\left(\cdot\right)$ represents the fractional order in Caputo's sense

$$\mathrm{D}_{\mathrm{C}}^{q}\left[x\left(t\right)\right]=\frac{1}{\Gamma\left(1-q\right)}\frac{\mathrm{d}}{\mathrm{d}t}\int_{0}^{t}\frac{\dot{x}\left(\tau\right)}{\left(t-\tau\right)^{q}}\mathrm{d}\tau; \tag{2}$$

The superscript q denotes the order of fractional derivative; α is usually referred to as the 'rigidity ratio'; $w\left(t\right)$ is a white/colored excitation with power spectrum density $S_{w}\left(\omega,t\right)$; $f\left(t\right)$ is a deterministic periodic excitation, written in a form of Fourier series

$$f\left(t\right)=\sum_{n=1}^{N}\left[A_{n}\cos\omega_{n}t+B_{n}\sin\omega_{n}t\right] \tag{3}$$

with A_n and B_n being the Fourier coefficients of the nth harmonic term; $z(t)$ is the hysteretic displacement governed in a differential form

$$\dot{z} = g_z(z, \dot{x}). \tag{4}$$

As an archetypal hysteretic model, the Bouc-Wen model has been widely used in structural engineering applications. In this context, consider

$$\dot{z}(t) = \dot{x}\left[A - |z|^n (\gamma \mathrm{sgn}(\dot{x})\,\mathrm{sgn}(z) + \beta)\right], \tag{5}$$

where A, n, γ and β are the Bouc-Wen hysteretic parameters. In particular, when $n = 1$ Eq.(5) becomes

$$\dot{z}(t) = A\dot{x} - \gamma z|\dot{x}| - \beta\dot{x}|z|. \tag{6}$$

Note the damping coefficient $c = 2\zeta m\omega_n^{2-q}$ to ensure a consistent dimension of the fractional-order damping coefficient with the one of the integer-order derivative. $\omega_n = \sqrt{k/m}$ is the natural frequency and ζ is the damping ratio.

Assume next the steady-state response of the system, governed by Eq.(1), can be cast into a combination of a deterministic and of a stochastic component. That is

$$x(t) = \hat{x}(t) + \mu_x(t), \tag{7a}$$
$$z(t) = \hat{z}(t) + \mu_z(t), \tag{7b}$$

where $\hat{x}(t)$ and $\hat{z}(t)$ are the zero-mean stochastic processes; $\mu_x(t)$ and $\mu_z(t)$ are the deterministic mean processes which can be further written as a Fourier series

$$\mu_x(t) = \sum_{n=1}^{N} (C_n \cos\omega_n t + D_n \sin\omega_n t), \tag{8a}$$

$$\mu_z(t) = \sum_{n=1}^{N} (U_n \cos\omega_n t + V_n \sin\omega_n t), \tag{8b}$$

where C_n, D_n, V_n, D_n are the Fourier coefficients of response. Substituting Eq.(7) into the equation of motion shown in Eq.(1) yields

$$\begin{aligned} m\left(\mu_{\ddot{x}} + \ddot{\hat{x}}\right) + c\mathrm{D}_{\mathrm{L}}^{q}(\mu_x + \hat{x}) + \alpha k(\mu_x + \hat{x}) + \\ (1-\alpha)k(\mu_z + \hat{z}) = w(t) + f(t). \end{aligned} \tag{9}$$

Taking mathematical expection on both sides of Eq.(9) leads to

$$m\mu_{\ddot{x}}(t) + c\mathrm{D}_{\mathrm{C}}^{q}\mu_x(t) + \alpha k\mu_x(t) + (1-\alpha)k\mu_z(t) = f(t). \tag{10}$$

Substracting Eq.(10) from Eq.(9) yields

$$m\ddot{\hat{x}}(t) + c\mathrm{D}_{\mathrm{C}}^{q}\hat{x}(t) + \alpha k\hat{x}(t) + (1-\alpha)k\hat{z}(t) = w(t). \tag{11}$$

Similary, substituting Eq.(7) into the auxiliary first-order differential equation governing the hysteretic term shown in Eq.(6), yields

$$\begin{aligned} \dot{\hat{z}} + \mu_{\dot{z}} = A\left(\dot{\hat{x}} + \mu_{\dot{x}}\right) - \gamma(\mu_z + \hat{z})\left|\dot{\hat{x}} + \mu_{\dot{x}}\right| - \\ \beta\left(\dot{\hat{x}} + \mu_{\dot{x}}\right)|\mu_z + \hat{z}|. \end{aligned} \tag{12}$$

Taking mathematical expectation on both sides of Eq.(12) leads to

$$\mu_{\dot{z}} = A\mu_{\dot{x}} - \gamma\mathrm{E}[z|\dot{x}|] - \beta\mathrm{E}[\dot{x}|z|]. \tag{13}$$

Substituting Eq.(13) form Eq.(12) yields

$$\dot{\hat{z}}(t) = A\dot{\hat{x}} - \gamma z|\dot{x}| - \beta\dot{x}|z| + (\gamma\mathrm{E}[z|\dot{x}|] + \beta\mathrm{E}[\dot{x}|z|]). \tag{14}$$

Note that these two sets of differential equations (Eqs. (10)-(11) and Eqs. (13)-(14)) are coupled, because the expectations of the non-linear functions include both μ_x, μ_z and $\sigma_{\dot{\hat{x}}}$, $\sigma_{\hat{z}}$. Therefore, a simultaneous solution of them should be adopted. In the following part, the harmonic balance method is utilized for solving the Fourier coefficients of the deterministic component, while the statistical linearization method is used for the variance/covariance of the stochastic component.

3 Harmonic balance method for the deterministic component

An assumption often used in the statistical linearization is the Gaussian distribution of response. In this regard, the expectation in Eqs.(13) and (14) can be evaluated in an approximate closed-form [30]. That is

$$E[z|\dot{x}|] = \sqrt{\frac{2}{\pi}}\left(\rho\mu_{\dot{x}}\sigma_{\hat{z}} + \mu_z\sigma_{\dot{\hat{x}}} + \frac{\mu_z\mu_{\dot{x}}^2}{2\sigma_{\dot{\hat{x}}}}\right), \tag{15a}$$

$$E[\dot{x}|z|] = \sqrt{\frac{2}{\pi}}\left(\rho\mu_z\sigma_{\dot{\hat{x}}} + \mu_{\dot{x}}\sigma_{\hat{z}} + \frac{\mu_z\mu_{\dot{x}}^2}{2\sigma_z}\right), \tag{15b}$$

where $\sigma_{\dot{\hat{x}}}$ and $\sigma_{\hat{z}}$ are standard deviation of $\dot{\hat{x}}$ and $\hat{z}$, respectively; ρ is the correlation coefficient of $\dot{x}$ and z.

Taking fractional-, first- and second- order derivative on both sides of Eq.(8a) yields

$$D_C^q\mu_x(t) = \sum_{n=1}^{N}\left[C_n\omega_n^q\cos\left(\omega_n t + \frac{\pi q}{2}\right) + D_n\omega_n^q\sin\left(\omega_n t + \frac{\pi q}{2}\right)\right], \tag{16a}$$

$$\mu_{\dot{x}}(t) = \sum_{n=1}^{N}\left[-C_n\omega_n\sin\omega_n t + D_n\omega_n\cos\omega_n t\right], \tag{16b}$$

and

$$\mu_{\ddot{x}}(t) = \sum_{n=1}^{N} \left[-C_n\omega_n^2 \cos\omega_n t + D_n\omega_n^2 \sin\omega_n t\right]. \tag{16c}$$

Substituting Eqs. (3), (8a), (8b), (16a) and (16c) into Eq.(10) and taking harmonic balance yields

$$-mC_n\omega_n^2 + c\omega_n^q \left(C_n \cos\frac{\pi q}{2} + D_n \sin\frac{\pi q}{2}\right) + \alpha k C_n + (1-\alpha) k U_n = A_n, \tag{17a}$$

$$-mD_n\omega_n^2 + c\omega_n^q \left(D_n \cos\frac{\pi q}{2} - C_n \sin\frac{\pi q}{2}\right) + \alpha k D_n + (1-\alpha) k V_n = B_n \tag{17b}$$

with $n = 1, 2, \cdots, N$. Similarly, substituting Eq.(15) into Eq.(13) leads to

$$\mu_{\dot{z}} = \mu_{\dot{x}}[A - \sqrt{\frac{2}{\pi}}\sigma_{\hat{\dot{z}}}(\rho\gamma + \beta)] - \sqrt{\frac{2}{\pi}}\sigma_{\hat{\dot{x}}}(\gamma + \rho\beta)\mu_z - \sqrt{\frac{2}{\pi}}\left(\frac{\gamma}{2\sigma_{\hat{\dot{x}}}}\mu_z\mu_{\dot{x}}^2 + \frac{\beta}{2\sigma_{\hat{\dot{z}}}}\mu_{\dot{x}}\mu_z^2\right). \tag{18}$$

Substituting Eqs.(16b) and (8b) into Eq.(18) leads to an equation with both sides being summation of weighted harmonics. For multi-harmonic representation of $\mu_{\dot{x}}$ and μ_z, one may refer to [32] for an analytical solution of the weighting factors. For a special case when the periodic excitation is mono-harmonic, one may adopt an approximation that the response $\mu_{\dot{x}}$ and μ_z is also mono-harmonic. In this regard, consider

$$f(t) = F_0 \cos\omega_0 t, \tag{19a}$$

$$\mu_x(t) = C_0 \cos\omega_0 t + D_0 \sin\omega_0 t, \tag{19b}$$

and

$$\mu_z(t) = U_0 \cos\omega_0 t + V_0 \sin\omega_0 t \tag{19c}$$

for an illustrative example, where U_0 and V_0, and C_0 and D_0 are Fourier coefficients for μ_z and μ_x, respectively. Eq. (18) reduces to

$$\begin{aligned} &-U_0\omega_0 \sin\omega_0 t + V_0\omega_0 \cos\omega_0 t = \\ &(-C_0\omega_0 \sin\omega_0 t + D_0\omega_0 \cos\omega_0 t)\left[A - \sqrt{\frac{2}{\pi}}\sigma_{\hat{\dot{z}}}(\rho\gamma + \beta)\right] - \\ &\sqrt{\frac{2}{\pi}}\sigma_{\hat{\dot{x}}}(\gamma + \beta\rho)(U_0 \cos\omega_0 t + V_0 \sin\omega_0 t) - \\ &\sqrt{\frac{2}{\pi}}\frac{\gamma}{x\sigma_{\hat{\dot{x}}}}(M \cos\omega_0 t + N \sin\omega_0 t) - \\ &\sqrt{\frac{2}{\pi}}\frac{\beta}{2\sigma_z}(P \cos\omega_0 t + Q \sin\omega_0 t) \end{aligned} \tag{20}$$

The weighting factors

$$M = \frac{U_0C_0^2\omega_0^2}{4} - \frac{V_0C_0D_0\omega_0^2}{2} + \frac{3D_0^2U_0\omega_0^2}{4}, \tag{21a}$$

$$N = \frac{V_0D_0^2\omega_0^2}{4} - \frac{U_0C_0D_0\omega_0^2}{2} + \frac{3C_0^2V_0\omega_0^2}{4}, \tag{21b}$$

$$P = -\frac{C_0U_0V_0\omega_0}{2} + \frac{3D_0U_0^2\omega_0}{4} + \frac{D_0V_0^2\omega_0}{4}, \tag{21c}$$

and

$$Q = \frac{D_0U_0V_0\omega_0}{2} - \frac{3C_0V_0^2\omega_0}{4} - \frac{C_0U_0^2\omega_0}{4} \tag{21d}$$

are derived from the Fourier series of $\mu_z\mu_{\dot{x}}^2$ and $\mu_{\dot{x}}\mu_z^2$. That is

$$\mu_z\mu_{\dot{x}}^2 = M\cos\omega_0 t + N\sin\omega_0 t + (\cdots)\cos 3\omega_0 t + (\cdots)\sin 3\omega_0 t, \tag{22a}$$

$$\mu_{\dot{x}}\mu_z^2 = P\cos\omega_0 t + Q\sin\omega_0 t + (\cdots)\cos 3\omega_0 t + (\cdots)\sin 3\omega_0 t, \tag{22b}$$

where coefficients of frequency $3\omega_0$ are omitted. Further taking harmonic balance on both sides of Eq.(20) gives

$$-V_0\omega_0 + D_0\omega_0\left[A - \sqrt{\frac{2}{\pi}}\sigma_{\hat{\dot{z}}}(\rho\gamma + \beta)\right] - \sqrt{\frac{2}{\pi}}\sigma_{\hat{\dot{x}}}(\rho + \rho\beta)U_0 - \sqrt{\frac{1}{2\pi}}\left(\frac{\gamma}{\sigma_{\hat{\dot{x}}}}M + \frac{\beta}{\sigma_{\hat{\dot{z}}}}P\right) = 0, \tag{23a}$$

$$-U_0\omega_0 + C_0\omega_0\left[A - \sqrt{\frac{2}{\pi}}\sigma_{\hat{\dot{z}}}(\rho\gamma + \beta)\right] - \sqrt{\frac{2}{\pi}}\sigma_{\hat{\dot{x}}}(\rho + \rho\beta)V_0 - \sqrt{\frac{1}{2\pi}}\left(\frac{\gamma}{\sigma_{\hat{\dot{x}}}}N + \frac{\beta}{\sigma_{\hat{\dot{z}}}}Q\right) = 0. \tag{23b}$$

Combining Eqs.(17a)-(17b) and Eqs.(23a)-(23b) yields a solution of U_0, V_0, C_0 and D_0 in terms of $\sigma_{\hat{\dot{x}}}$, $\sigma_{\hat{\dot{z}}}$ and ρ. However, $\sigma_{\hat{\dot{x}}}$, $\sigma_{\hat{\dot{z}}}$ and ρ are unknowns. Therefore, more relationships between the unknown Fourier coefficients and response second moments are need to supplement the set of non-linear algebraic equations shown in Eq.(17a)-(17b) and Eq.(23a)-(23b). This can be achieved by the statistical linearization method, shown in the following section.

4 Statistical linearization for the stochastic component

Let $\boldsymbol{q} = \{\hat{x}, \hat{z}\}^{\mathrm{T}}$, then the equation of motion of the zero mean response $\hat{x}$ and $\hat{z}$ shown in Eqs.(11) and (14) can be cast into

$$\boldsymbol{M}\ddot{\boldsymbol{q}}(t)+\boldsymbol{C}\dot{\boldsymbol{q}}(t)+\boldsymbol{C}_{\boldsymbol{q}}\mathrm{D}_{\mathrm{C}}^{q}\boldsymbol{q}(t)+\boldsymbol{K}\boldsymbol{q}(t)+\boldsymbol{\Phi}(\boldsymbol{q},\dot{\boldsymbol{q}})=\boldsymbol{Q}(t), \tag{24}$$

where

$$\boldsymbol{M}=\begin{bmatrix} m & 0\\ 0 & 0\end{bmatrix}, \boldsymbol{C}=\begin{bmatrix} 0 & 0\\ 0 & 1\end{bmatrix}, \boldsymbol{C}_{\boldsymbol{q}}=\begin{bmatrix} c & 0\\ 0 & 0\end{bmatrix}, \boldsymbol{Q}=\begin{Bmatrix} w(t)\\ 0\end{Bmatrix}$$
$$\boldsymbol{K}=\begin{bmatrix} \alpha k & (1-\alpha)k\\ 0 & 0\end{bmatrix}$$

are the mass, integer order damping, fractional damping and stiffness matrix, respectively; $\boldsymbol{\Phi} = [\phi_1, \phi_2]^{\mathrm{T}}$ with $\phi_1 = 0$ and

$$\begin{aligned}\phi_2 =& -A\dot{\hat{x}}+\gamma(\hat{z}+\mu_z)\left|\dot{\hat{x}}+\mu_{\dot{x}}\right|+\\ &\beta\left(\dot{\hat{x}}+\mu_{\dot{x}}\right)|\hat{z}+\mu_z|-\gamma\mathrm{E}[z|\dot{x}|]+\beta\mathrm{E}[\dot{x}|z|]\end{aligned} \tag{25}$$

is the non-linear term.

Next, according to the statistical linearization method, Eq.(24) can be equivalently written as [33]

$$\begin{aligned}&\boldsymbol{M}\ddot{\boldsymbol{q}}(t)+(\boldsymbol{C}+\boldsymbol{C}_{\mathbf{e}})\dot{\boldsymbol{q}}(t)+\boldsymbol{C}_{\boldsymbol{q}}\mathrm{D}_{\mathrm{C}}^{q}\boldsymbol{q}(t)+\\ &\qquad(\boldsymbol{K}+\boldsymbol{K}_{\mathbf{e}})\boldsymbol{q}(t)=\boldsymbol{Q}(t),\end{aligned} \tag{26}$$

where

$$\boldsymbol{C}_{\mathbf{e}}=\begin{bmatrix} 0 & 0\\ c_{21}^{\mathrm{e}} & 0\end{bmatrix}=\mathrm{E}\left[\frac{\partial\boldsymbol{\Phi}}{\partial\dot{\boldsymbol{q}}^{\mathrm{T}}}\right], \boldsymbol{K}_{\mathbf{e}}=\begin{bmatrix} 0 & 0\\ 0 & k_{22}^{\mathrm{e}}\end{bmatrix}=\mathrm{E}\left[\frac{\partial\boldsymbol{\Phi}}{\partial\boldsymbol{q}^{\mathrm{T}}}\right],$$

with

$$c_{\mathrm{e}}=c_{21}^{\mathrm{e}}=-\mathrm{E}\left[\frac{\partial g}{\partial\dot{q}_1}\right], k_{\mathrm{e}}=k_{22}^{\mathrm{e}}=-\mathrm{E}\left[\frac{\partial g}{\partial q_2}\right]$$

and

$$g(\dot{q}_1, q_2)=-\phi_2$$

Further manipulation yields

$$c_{\mathrm{e}}=-\mathrm{E}\left[\frac{\partial g}{\partial\dot{q}_1}\right]=-A+\gamma\mathrm{E}[z\,\mathrm{sgn}(\dot{x})]+\beta\mathrm{E}[|z|], \tag{27a}$$

$$k_{\mathrm{e}}=-\mathrm{E}\left[\frac{\partial g}{\partial q_2}\right]=\gamma\mathrm{E}[|\dot{x}|]+\beta\mathrm{E}[\dot{x}\,\mathrm{sgn}(z)]. \tag{27b}$$

The expectation of non-linear functions in Eq.(27a)-(27b) can be approximated as [30]

$$\mathrm{E}[z\,\mathrm{sgn}(\dot{x})]=\sqrt{\frac{2}{\pi}}\rho\sigma_z\left(1-\frac{\mu_{\dot{x}}^2}{2\sigma_{\dot{\hat{x}}}^2}\right)+\sqrt{\frac{2}{\pi}}\frac{\mu_z\mu_{\dot{x}}}{\sigma_{\dot{\hat{x}}}}, \tag{28a}$$

$$\mathrm{E}[\dot{x}\,\mathrm{sgn}(z)]=\sqrt{\frac{2}{\pi}}\rho\sigma_{\dot{\hat{x}}}\left(1-\frac{\mu_z^2}{2\sigma_{\hat{z}}^2}\right)+\sqrt{\frac{2}{\pi}}\frac{\mu_z\mu_{\dot{x}}}{\sigma_{\hat{z}}}, \tag{28b}$$

$$\mathrm{E}[|z|]=\sqrt{\frac{2}{\pi}}\sigma_{\hat{z}}\left(1-\frac{\mu_z^2}{2\sigma_{\hat{z}}^2}\right)+\sqrt{\frac{2}{\pi}}\frac{\mu_z^2}{\sigma_{\hat{z}}}, \tag{28c}$$

$$\mathrm{E}[|\dot{x}|]=\sqrt{\frac{2}{\pi}}\sigma_{\dot{\hat{x}}}\left(1-\frac{\mu_{\dot{x}}^2}{2\sigma_{\dot{\hat{x}}^2}}\right)+\sqrt{\frac{2}{\pi}}\frac{\mu_{\dot{x}}^2}{\sigma_{\dot{\hat{x}}}}. \tag{28d}$$

Combining Eqs.(28a)-(28d) with Eqs.(27a)-(27b) one may find that the equivalent parameters of the system are time varying with frequency ω_0, for the mean process $\mu_{\dot{x}}$ and μ_z are included. Note that the standard deviations σ_z and $\sigma_{\dot{\hat{x}}}$ are cyclostationary when $t \to \infty$. Therefore, the fast varying content can be eliminated by averaging over the fundamental period $T_0 = 2\pi/\omega_0$. That is

$$\bar{c}_{\mathrm{e}}=\frac{\gamma}{T_0}\int_0^{T_0}\mathrm{E}[z\,\mathrm{sgn}(\dot{x})]\,\mathrm{d}t+\frac{\beta}{T_0}\int_0^{T_0}\mathrm{E}[|z|]\,\mathrm{d}t-A, \tag{29a}$$

$$\bar{k}_{\mathrm{e}}=\frac{\gamma}{T_0}\int_0^{T_0}\mathrm{E}[|\dot{x}|]\,\mathrm{d}t+\frac{\beta}{T_0}\int_0^{T_0}\mathrm{E}[\dot{x}\,\mathrm{sgn}(z)]\,\mathrm{d}t, \tag{29b}$$

where the deterministic averaging on the right hand side of Eq.(29a)-(29b) yields the following approximations [30]

$$\begin{aligned}\mathrm{E}[z\,\mathrm{sgn}(\dot{x})]\approx&\sqrt{\frac{2}{\pi}}\rho\sigma_{\hat{z}}\left[1-\frac{(C_0^2+D_0^2)\,\omega_0^2}{4\sigma_{\dot{\hat{x}}}^2}\right]+\\ &\sqrt{\frac{2}{\pi}}\frac{(D_0U_0-C_0V_0)\,\omega_0}{2\sigma_{\dot{\hat{x}}}},\end{aligned} \tag{30a}$$

$$\begin{aligned}\mathrm{E}[\dot{x}\,\mathrm{sgn}(z)]\approx&\sqrt{\frac{2}{\pi}}\rho\sigma_{\dot{\hat{x}}}\left[1-\frac{U_0^2+V_0^2}{4\sigma_{\hat{z}}^2}\right]+\\ &\sqrt{\frac{2}{\pi}}\frac{(D_0U_0-C_0V_0)\,\omega_0}{2\sigma_z},\end{aligned} \tag{30b}$$

$$\mathrm{E}[|z|]\approx\sqrt{\frac{2}{\pi}}\sigma_{\hat{z}}\left[1-\frac{U_0^2+V_0^2}{4\sigma_{\hat{z}}^2}\right], \tag{30c}$$

$$\mathrm{E}[|\dot{x}|]\approx\sqrt{\frac{2}{\pi}}\sigma_{\dot{\hat{x}}}\left[1+\frac{(C_0^2+D_0^2)\,\omega_0^2}{4\sigma_{\dot{\hat{x}}^2}}\right]. \tag{30d}$$

Note that the approximate equivalent parameters $\bar{c}_{\mathrm{e}}$ and $\bar{k}_{\mathrm{e}}$ depend on seven unknowns $C_0, D_0, U_0, V_0, \sigma_{\dot{\hat{x}}}, \sigma_{\hat{z}}$ and ρ.

Further, the supplemented relationship between the unknown response statistics $\sigma_{\dot{\hat{x}}}, \sigma_{\hat{z}}$ and ρ, and the Fourier coefficients of the deterministic response component, can be obtained by the standard linear random vibration theory for linear dynamic systems with fractional elements [34]. In this regard,

$$\boldsymbol{S_q}(\omega) = \boldsymbol{H}(\omega)\, \boldsymbol{S_Q}(\omega)\, \boldsymbol{H}^{\mathrm{T}}(\omega), \tag{31}$$

where

$$\boldsymbol{S_Q}(\omega) = \begin{bmatrix} S_w(\omega) & 0 \\ 0 & 0 \end{bmatrix}, \boldsymbol{S_q} = \begin{bmatrix} S_{\hat{x}\hat{x}} & S_{\hat{x}\hat{z}} \\ S_{\hat{z}\hat{x}} & S_{\hat{z}\hat{z}} \end{bmatrix}$$

is the two-sided PSD matrix of $\boldsymbol{Q}$ and $\boldsymbol{q}$, respectively;

$$\boldsymbol{H}(\omega) = 1/\left(-\omega^2 \boldsymbol{M} + \mathrm{i}\omega\,(\boldsymbol{C} + \boldsymbol{C}_e) + (\boldsymbol{K} + \boldsymbol{K}_e) + (\mathrm{i}\omega)^q\, \boldsymbol{C_q}\right)$$

is the frequency response matrix. $S(\omega)$ is the PSD of the stochastic excitation component; $S_{\hat{x}\hat{x}}$, $S_{\hat{x}\hat{z}} = S_{\hat{z}\hat{x}}$ and $S_{\hat{z}\hat{z}}$ are the two-sides auto-PSD of $\hat{x}$, the two-sides cross-PSD of $(\hat{x}, \hat{z})$, and the two-sides auto-PSD of $\hat{z}$. Therefore,

$$\sigma_{\dot{\hat{x}}}^2 = \int_{-\infty}^{\infty} \omega^2 S_{\hat{x}\hat{x}}(\omega)\, \mathrm{d}\omega, \tag{32a}$$

$$\sigma_{\hat{z}}^2 = \int_{-\infty}^{\infty} S_{\hat{z}\hat{z}}(\omega)\, \mathrm{d}\omega, \tag{32b}$$

$$\rho\sigma_{\dot{\hat{x}}}\sigma_{\hat{z}} = -\int_{-\infty}^{\infty} \mathrm{i}\omega S_{\hat{x}\hat{z}}(\omega)\, \mathrm{d}\omega. \tag{32c}$$

5 Implementation procedures

To summarize, the harmonic balance method for the deterministic response component leads to four indeterminate non-linear algebraic equations (17a)-(17b) and (23a)-(23b). These equations include seven unknowns $\left(C_0, D_0, U_0, V_0, \sigma_{\dot{\hat{x}}}, \sigma_{\hat{z}}, \rho\right)$ that only four equations do not yields an unique solution. Besides, Eqs.(32a)-(32c) establish supplementary relationships between the unknown response Fourier coefficients (C_0, D_0, U_0, V_0) and the response statistics $\left(\sigma_{\dot{\hat{x}}}, \sigma_{\hat{z}}, \rho\right)$. Therefore, the procedures of the proposed method is the following:

1. Consider the corresponding fractional-order linear system $(F_0 = 0, \alpha = 0)$ subjected to the stochastic excitation. Set the initial guess of the response variances equal to those of the fractional-order linear system. Set the zero-mean hysteretic displacement $\hat{z} = \hat{x}$, and thus, $\rho = 0$.
2. Solve Eq.(17a)-(17b) and Eq.(23a)-(23b) simultaneously to obtain the first guess of C_0, D_0, U_0 and V_0, using standard numerical schemes for non-linear algebraic equations. Newton's iterative method is used in this paper.
3. Use Eq.(27a)-(27b) and Eq.(30a)-Eq.(30d) to get c_{e} and k_{e}. Use Eq.(31) and Eq.(32a)-(32c) to update $\sigma_{\dot{\hat{x}}}, \sigma_{\hat{z}}$ and ρ.
4. Repeat Step 1 to Step 3 until certain convergence criterion is achieved.

For applying Newton's iterative method in Step 2, one needs to solve the following matrix equation for the unknown Fourier coefficients of the harmonic component. That is

$$\boldsymbol{K}\left(\boldsymbol{\alpha}^{(i)}\right) + \boldsymbol{J}\left(\boldsymbol{\alpha}^{(i)}\right)\left(\boldsymbol{\alpha}^{(i+1)} - \boldsymbol{\alpha}^{(i)}\right) = 0 \tag{33}$$

where the superscript (i) denotes the ith iteration step; $\boldsymbol{\alpha}$ is the vector of the unknowns $\boldsymbol{\alpha} = [C_0, D_0, U_0, V_0]^{\mathrm{T}}$; The colum vector $\boldsymbol{K}$ with four entries can be written as

$$\boldsymbol{K}_1(\boldsymbol{\alpha}) = -mC_0\omega_0^2 + c\omega_0^q\left(C_0 \cos\frac{\pi q}{2} + D_0 \sin\frac{\pi q}{2}\right) + \alpha k C_0 + (1-\alpha)\, k U_0 \tag{34a}$$

$$\boldsymbol{K}_2(\boldsymbol{\alpha}) = -mD_0\omega_0^2 + c\omega_0^q\left(D_0 \cos\frac{\pi q}{2} - C_0 \sin\frac{\pi q}{2}\right) + \alpha k D_0 + (1-\alpha)\, k V_0 \tag{34b}$$

$$\boldsymbol{K}_3(\boldsymbol{\alpha}) = -V_0\omega_0 + D_0\omega_0\left[A - \sqrt{\frac{2}{\pi}}\sigma_{\hat{z}}(\rho\gamma + \beta)\right] - \sqrt{\frac{2}{\pi}}\sigma_{\dot{\hat{x}}}(\rho + \rho\beta)\, U_0 - \sqrt{\frac{1}{2\pi}}\left(\frac{\gamma}{\sigma_{\dot{\hat{x}}}}M + \frac{\beta}{\sigma_{\hat{z}}}P\right) \tag{34c}$$

$$\boldsymbol{K}_4(\boldsymbol{\alpha}) = -U_0\omega_0 + C_0\omega_0\left[A - \sqrt{\frac{2}{\pi}}\sigma_{\hat{z}}(\rho\gamma + \beta)\right] - \sqrt{\frac{2}{\pi}}\sigma_{\dot{\hat{x}}}(\rho + \rho\beta)\, V_0 - \sqrt{\frac{1}{2\pi}}\left(\frac{\gamma}{\sigma_{\dot{\hat{x}}}}N + \frac{\beta}{\sigma_{\hat{z}}}Q\right) \tag{34d}$$

the Jacobin matrix is

$$\boldsymbol{J} = \frac{\partial \boldsymbol{K}}{\partial \boldsymbol{\alpha}^{\mathrm{T}}} \tag{35}$$

6 Numerical example

A Bouc-Wen hysteretic model endowed with fractional element and subjected to combined harmonic and stochastic excitation (white/colored) is considered, where the parameters of the corresponding integer-order linear system $m = 1, \omega_n = \sqrt{\frac{k}{m}} = 1, \zeta = \frac{c}{2m\omega_n} = 0.1$ are used.

6.1 White noise as the stochastic excitation

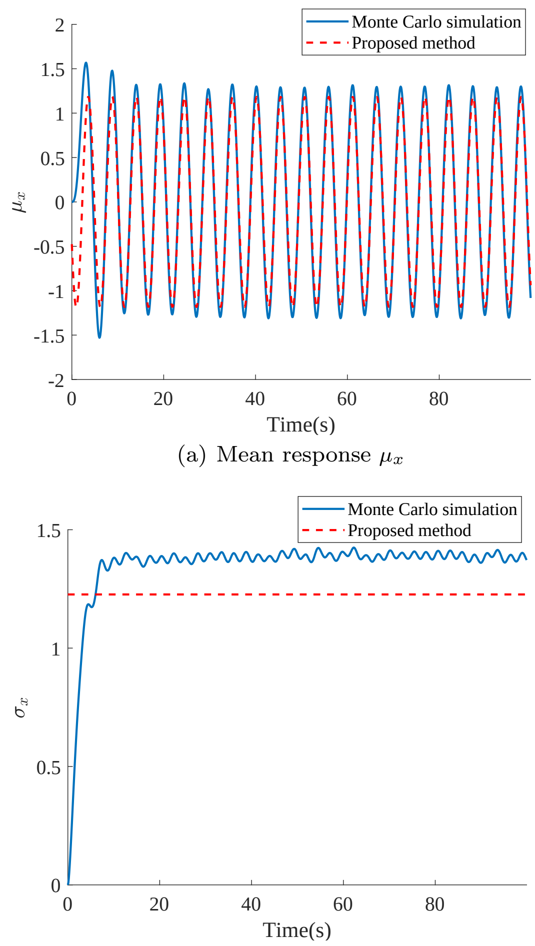
The strength of the white noise excitation is chosen as $w(t) = \sqrt{4\zeta} f_n(t)$, where $f_n(t)$ has 'unit strength', i.e.,

$\mathrm{E}\left[f_n(t) f_n(t+\tau)\right]=\delta(\tau)$. Dividing both sides of Eq. (1) by m yields

$$\begin{aligned} &\ddot{x}(t)+2\zeta\omega_n^{2-q}\mathrm{D}_{\mathrm{C}}^{q}\left[x(t)\right]+\alpha\omega_n^2 x(t)+\\ &\qquad(1-\alpha)\,\omega_n^2 z(t)=w(t)+F_0\sin\omega_0 t \end{aligned} \tag{36}$$

When $\alpha=1, F_0=0, q=1$ the fractional-order hysteretic system reduces to a special case of the linear system with integer-order derivative and subjected to stochastic excitation only. The displacement standard deviation of the reduced linear system is $\sigma_{\hat{x}}=\sigma_x=1$.

In this illustrative case, to demonstrate the applicability of the proposed method, the amplitude and the frequency of the harmonic excitation is chosen as $F_0=1$ and $\omega_0=1.2$, respectively. Both softening and hardening Bouc-Wen systems with fractional order $q=0.5$ are considered herein. For the softening Bouc-Wen system, the hysteretic parameters are $A=1, \gamma=0.5, \beta=0.5, n=1, \alpha=0.1$, whereas for the hardening Bouc-Wen system, all the other hysteretic parameters are the same as in the softening case except $\beta=-0.35, \gamma=0.65$. The response comparison shown in Figs.1(a)-1(b) for the softening Bouc-Wen model verifies the accuracy of the proposed method. Specifically, the deterministic component time history and the stochastic component standard deviation of the displacement $x(t)$, obtained by the proposed method is in good agreement with the pertinent Monte Carlo estimates over 10,000 samples. Besides, it seems that the proposed method yields a stationary standard deviation of the stochastic component, vis-a-vis the MC estimates gives a cyclostationary standard deviation. This is caused by the coupling effect between the deterministic and random response component, shown in Eqs. (28a)-(28d). These equations indicate that the equivalent parameters, and thus, the standard deviation of the stochastic component, depend on deterministic component. However, in the proposed analytical method, the oscillation of the equivalent parameters, and thus, of the cyclostationary variance, is eliminated by averaging over one period for further simplification (see Eqs. (29a)-(29b)). Therefore, the response variance during the cyclostationary phase obtained by the Monte Carlo data is averaged for the ensuing error analysis. Specifically, for the deterministic response $\mu_x(t)$ shown in Fig. 1(a), the response amplitudes obtained by the proposed method is around 8.89% less than the Monte Carlo estimates, whereas for the stationary standard deviation shown in Fig.1(b), the analytical result is about 11.68% less the MC data. Further MC investigations show that the amplitude of harmonic-like cyclostationary part of the standard deviation depends on the amplitude and the frequency of the harmonic excitation. Besides, it seems that the oscillation frequency of the cyclostationary standard deviation is twice as higher as the excitation frequency. This conclusion is reasonable because the equivalent stiffness, shown in Eq. (27b), depends on quadratic terms of the deterministic response component.

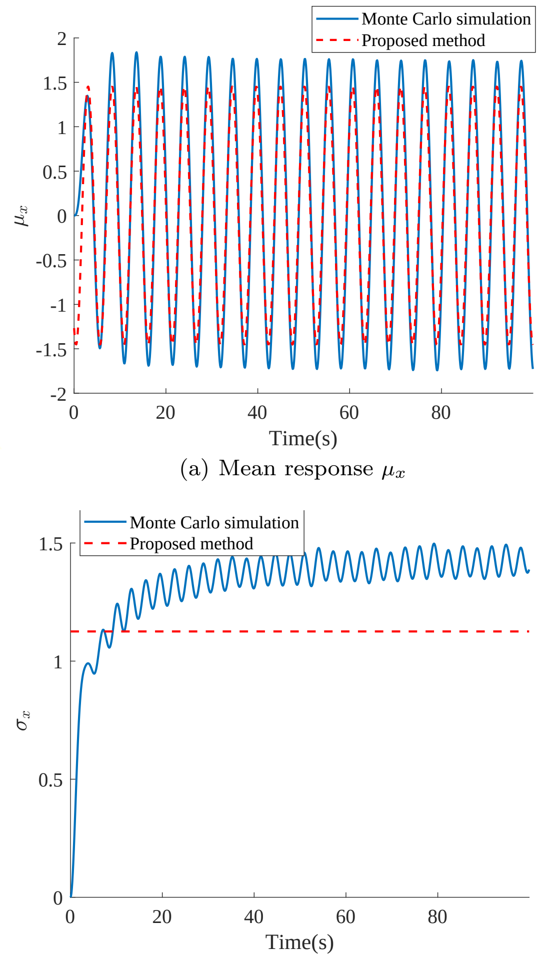


(a) Mean response μ_x

(b) Std $\sigma_{\hat{x}}$

Fig. 1 Displacement of the softening Bouc-Wen system subjected to combined excitation.

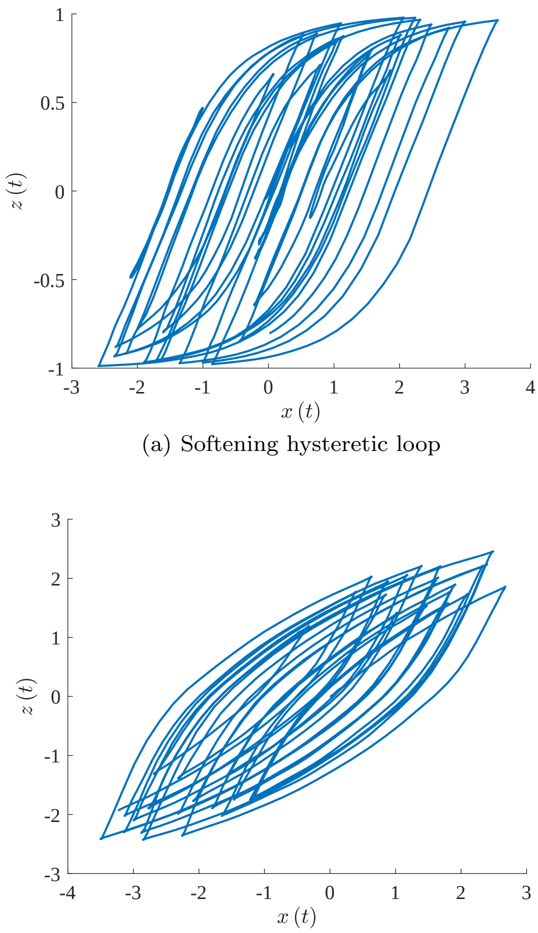
Numerical investigations, as shown in Figs. 2(a)-2(b), demonstrate the applicability of the proposed method for the considered hardening Bouc-Wen model endowed with fractional element. Error analysis suggests satisfactory accuracy of the proposed method. Specifically, for the displacement response $x(t)$, the amplitude of the deterministic component and the standard deviation of the stochastic component, obtained by the proposed method, is 16.55% and 10.5% lower than the corresponding MC estimates, respectively. All the above errors are within the reasonable range of the classic statistical linearization method reported by previous researches[33].



(a) Mean response μ_x

(b) Std $\sigma_{\hat{x}}$

Fig. 2 Displacement of the hardening Bouc-Wen system subjected to combined excitation.



(a) Softening hysteretic loop

(b) Hardening hysteretic loop

Fig. 3 Hysteretic loops of the Bouc-Wen dynamic system subjected to combined harmonic and stochastic excitation.

Sample hysteretic loops of the considered softening and hardening Bouc-Wen system endowed with fractional element under combined harmonic and white noise excitation is shown in Figs.3(a)-3(b), respectively. It is seen that for both circumstances, the hysteretic loops exhibit history-depended strong non-linearity. Besides, compared to the hysteretic loops under harmonic excitation, the white noise disturbance renders the loop center move around the equilibrium position.

The ensuing analyses further investigate the applicability of the proposed method under other harmonic excitation with different frequencies. Consider the softening Bouc-Wen model with the same parameters as in Fig. 3(a). Fig.4 shows the response standard deviation $(\sigma_{\hat{x}}, \sigma_{\dot{\hat{x}}}, \sigma_{\hat{z}})$ versus excitation frequency curve when the harmonic excitation amplitude is $F_0 = 1$. It seems that the displacement standard deviation obtained by the proposed method agrees well with the Monte Carlo estimates over 10,000 sample responses.

The amplitudes of the deterministic response component $(\mu_x, \mu_{\dot{x}}, \mu_z)$ for different excitation frequencies, obtained by the proposed method and the pertinent Monte Carlo simulation, are compared in Fig.5 when $F_0 = 1$. It seems that under the considered cases, the amplitudes of the deterministic component obtained by the proposed method are in good agreement with the Monte Carlo estimates over 10,000 sample responses. Further investigations on the applicability and accuracy of the proposed method for response of the hardening Bouc-Wen model further validate the versatility of this method. These results are not included herein for they indicate similar conclusions.

Investigate next the applicability of the proposed method for the considered softening/hardening Bouc-Wen systems with different fractional orders. The amplitude and frequency of harmonic excitation are selected to be $F_0 = 1$ and $\omega_0 = 1.2$, while both softening and hardening system with the same hysteretic param-

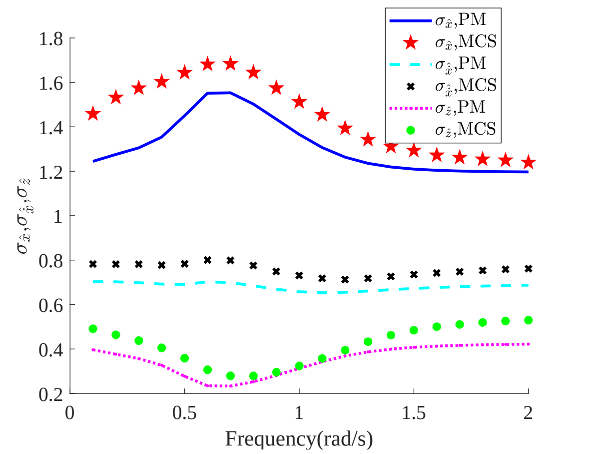


Fig. 4 Standard deviation of the stochastic response component of a softening Bouc-Wen system subjected to combined stochastic excitation and harmonic excitation with different frequencies

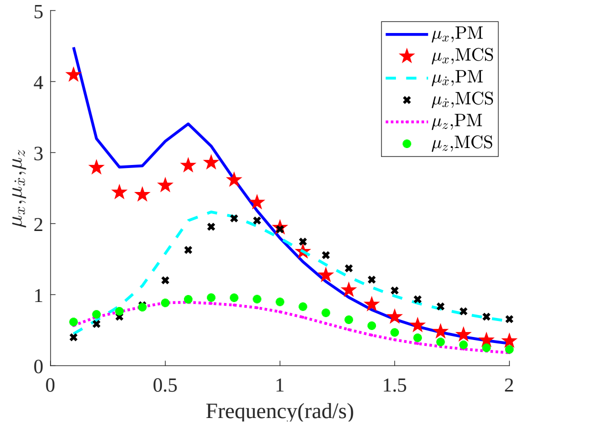
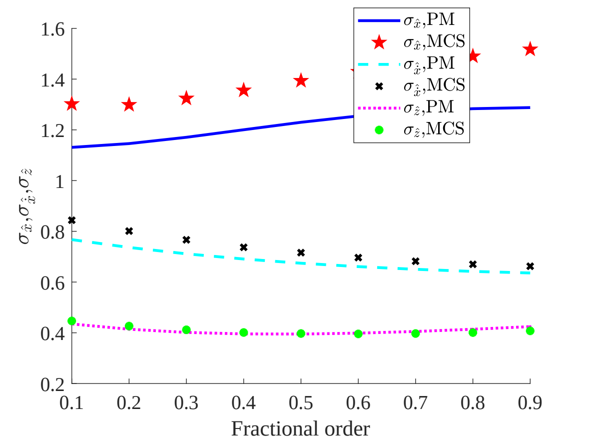
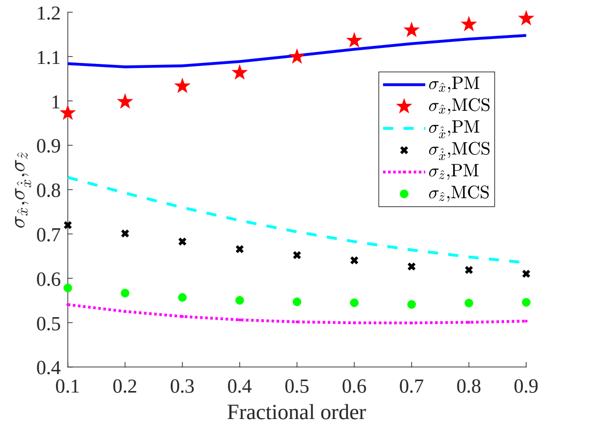


Fig. 5 Amplitude of the deterministic response component of a softening Bouc-Wen system subjected to combined stochastic excitation and harmonic excitation with different frequencies



(a) softening system



(b) hardening system

Fig. 6 Standard deviation of the stochastic response component of softening and hardening Bouc-Wen systems subjected to combined stochastic excitation and harmonic excitation with different fractional orders

eters as in Figs. 3(a)-3(b) are considered herein for illustration. Figs. 6(a)-6(b) show the response standard deviation $\left(\sigma_{\hat{x}}, \sigma_{\dot{\hat{x}}}, \sigma_{\hat{z}}\right)$ versus fractional order curve for the softening and for the hardening system, respectively. It seems that for the considered cases with fractional order $q \in [0.1, 0.9]$, the response statistical characteristics obtained by the proposed method agrees quite well with the pertinent Monte Carlo estimates. Comparisons regarding the fractional-order versus deterministic response amplitude curve is not included here for they indicate similar conclusions.

6.2 Colored noise as the stochastic excitation

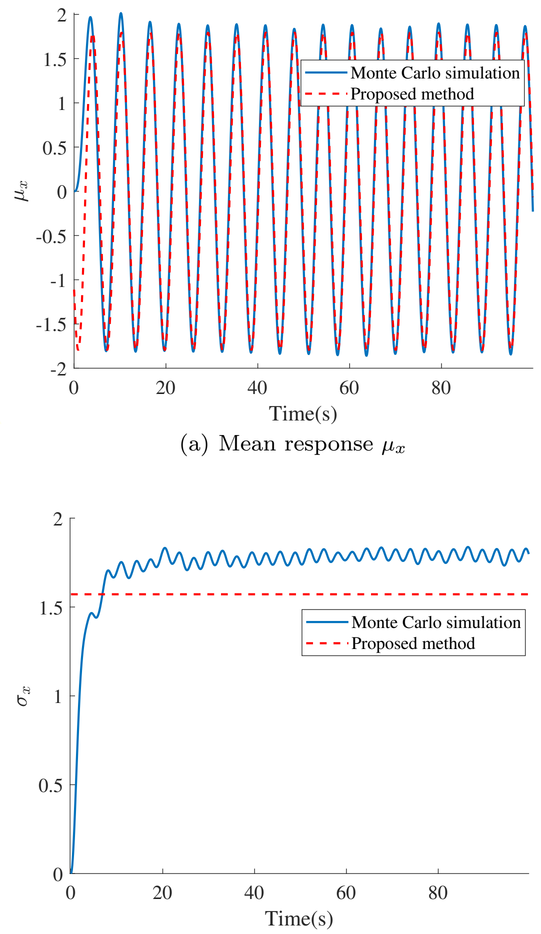
The proposed method can be extended for response determination of a hysteretic systems endowed with fractional element and subjected to combined periodic and colored noise, readily. Consider the equation of motion shown in Eq. (36), with $w(t)$ being a zero-mean colored noise with

$$S_w(\omega) = \frac{1 + 4\xi_{\mathrm{g}}^2 \frac{\omega^2}{\omega_{\mathrm{g}}^2}}{\left(1 - \frac{\omega^2}{\omega_{\mathrm{g}}^2}\right)^2 + 4\xi_{\mathrm{g}}^2 \frac{\omega^2}{\omega_{\mathrm{g}}^2}} \cdot S_0 \tag{37}$$

where $\zeta_{\mathrm{g}} = 0.2$ and $\omega_{\mathrm{g}} = 2$ is the damping ratio and natural frequency for the filter; $S_0 = 2\zeta/\pi$ is the power spectral density for the white noise to be filtered. In this illustrative case, to demonstrate the applicability of the proposed method, the amplitude and the frequency of the harmonic excitation is chosen as $F_0 = 1$ and $\omega_0 = 1$, respectively.

Consider both a softening and a hardening Bouc-Wen system. For the softening Bouc-Wen system, the hysteretic parameters are $A = 1$, $\gamma = 0.5$, $\beta = 0.5$,

$n = 1$, $\alpha = 0.1$, whereas for hardening Bouc-Wen system, all the other hysteretic parameters are the same as in the softening case except $\beta = -0.1$, $\gamma = 0.8$. The response comparison shown in Figs.7(a)-7(b) for the softening Bouc-Wen system verifies the accuracy of the proposed method. Specifically, for the deterministic harmonic response $x(t)$ shown in Fig. 7(a), the amplitude of the deterministic response component, obtained by the proposed method is around 3.83% less than the Monte Carlo estimates. For the stationary standard deviation of the stochastic response component calculated by the proposed method, shown in Fig.7(b), is about 11.63% less than the time average of the cyclostationary standard deviation obtained by the Monte Carlo simulation. Further MC investigations show that the amplitude of the harmonic-like cyclostationary part of the standard deviation depends on the amplitude and the frequency of the harmonic excitation.

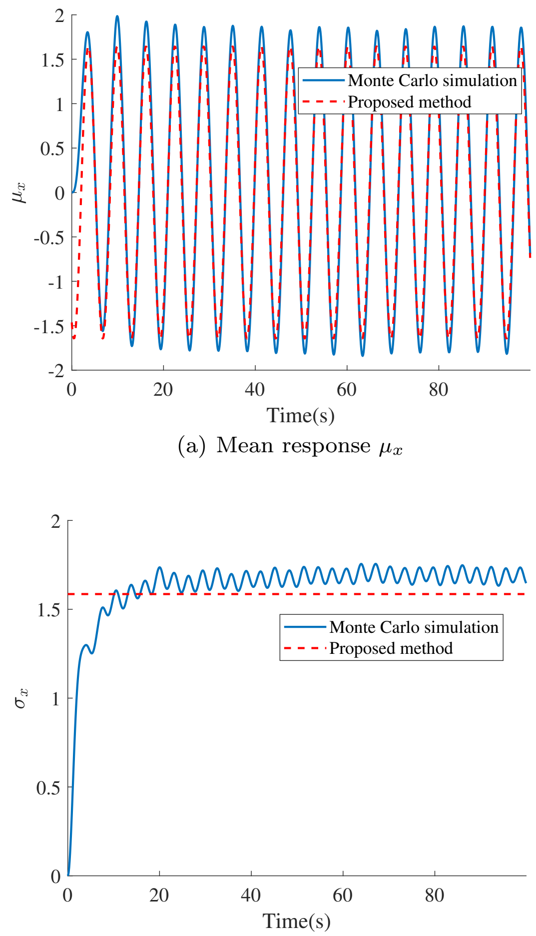


(a) Mean response μ_x

(b) Std $\sigma_{\hat{x}}$

Fig. 7 Displacement of the softening Bouc-Wen system subjected to combined excitation.

Numerical investigations, as shown in Figs. 8(a)-8(b), demonstrate the applicability of the proposed method for the considered hardening Bouc-Wen system endowed with fractional elements. Error analysis suggests satisfactory accuracy of the proposed method. Specifically, for the displacement response $x(t)$, the amplitude of the deterministic component and the standard deviation of the stochastic component, obtained by the proposed method, is 11.32% and 5.91% less than the corresponding MC estimates, respectively. All the above errors are within the reasonable range of the classic statistical linearization method.



(a) Mean response μ_x

(b) Std $\sigma_{\hat{x}}$

Fig. 8 Displacement of the hardening Bouc-Wen system subjected to combined excitation.

The same conclusion as in the white noise case can be obtained, that the sample hysteretic loops of the considered hysteretic system under combined harmonic and colored noise excitation, exhibit history depended strong non-linearity. Besides, compared to the hysteretic loops under harmonic excitation, the colored noise disturbance renders the loop center move around the equilibrium position.

The ensuing analysis further investigates the applicability of the proposed method under other harmonic excitations with different frequencies. Consider the softening Bouc-Wen model with the same parameters as in Figs. 7(a)-7(b). Fig. 9 shows the response standard deviation $\left(\sigma_{\hat{x}}, \sigma_{\dot{\hat{x}}}, \sigma_{\hat{z}}\right)$ versus excitation frequency curve, when the harmonic excitation amplitude is $F_0 = 1$. It seems that the proposed linearization method exhibits reasonable accuracy ($< 18\%$) over all the considered excitation frequencies for the derived displacement standard deviation compared with the Monte Carlo estimates.

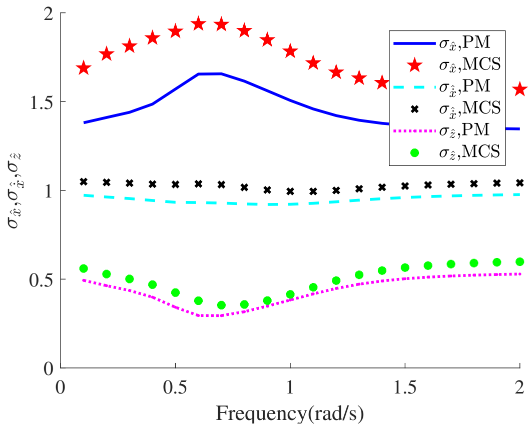


Fig. 9 Standard deviation of the stochastic response component of a softening Bouc-Wen system subjected to combined stochastic excitation and harmonic excitation with different frequencies

The amplitudes of the deterministic response component for the harmonic excitation with different frequencies, obtained by the proposed method and estimated by the pertinent Monte Carlo simulation, are compared in Figs. 10 when $F_0 = 1$. It seems that under the considered cases, the amplitudes of the deterministic response component versus harmonic excitation frequency curve, can be well predicted by the proposed analytical method.

Investigations on the applicability and accuracy of the proposed method regarding the amplitude-frequency curve of the deterministic responses of the considered hardening Bouc-Wen system, further validate the similar conclusions as in the softening Bouc-Wen case, which are not included herein for the limited space. All the preceding numerical examples yield results with reasonable accuracy when the relevant assumptions [30] are not violated.

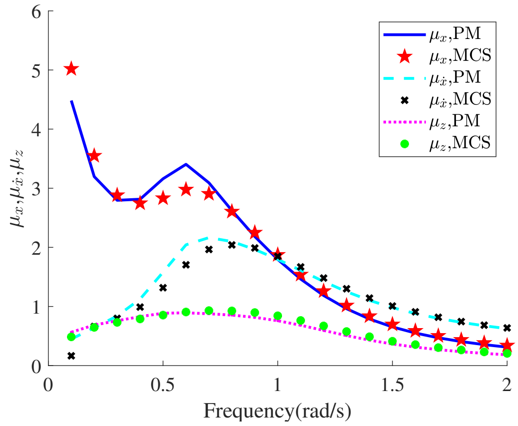
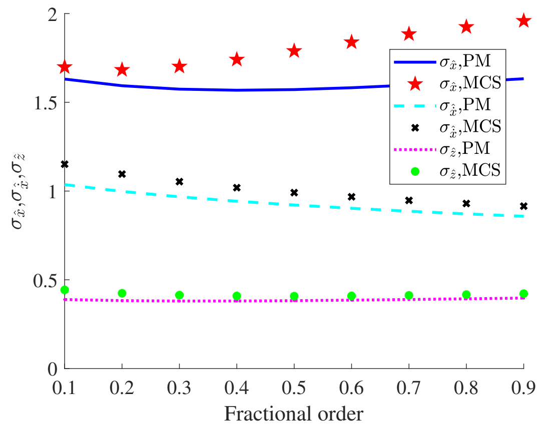


Fig. 10 Amplitude of the deterministic response component of a softening Bouc-Wen system subjected to combined stochastic excitation and harmonic excitation with different frequencies

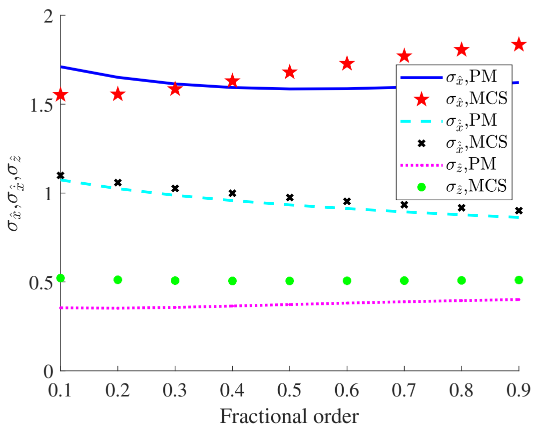
The last analysis investigates the applicability of the proposed method for the considered Bouc-Wen systems endowed with different fractional orders. The amplitude and frequency of harmonic excitation are selected to be $F_0 = 1$ and $\omega_0 = 1$, while both softening and hardening system with the same parameters as in Figs. 7(a)-7(b) and 8(a)-8(b), respectively, are considered herein for illustration. Figs. 11(a)-11(b) show the response standard deviation $\left(\sigma_{\hat{x}}, \sigma_{\dot{\hat{x}}}, \sigma_{\hat{z}}\right)$ versus fractional order curve for softening and hardening system, respectively. It seems that for the considered cases, the response statistical characteristics obtained by the proposed method show reasonable accuracy when compared with the pertinent Monte Carlo estimates.

7 Concluding remarks

A statistical linearization method has been proposed for the response determination of a SDOF hysteretic system endowed with fractional element and subjected to combined periodic and white/colored excitation. This has been achieved by decomposing the total stochastic response into a summation of deterministic and zero-mean stochastic component. By doing so, the differential equation governing the total response has been decomposed into two sets of differential equations, governing the deterministic response component and the zero-mean stochastic response component, respectively. Next, the harmonic balance method has been utilized to establish a relationship between the Fourier coefficients of the periodic excitation and those of the deterministic response component, whereas the statistical linearization has been used for the second moments of the stochastic response component. However, the de-



(a) softening system



(b) hardening system

Fig. 11 Standard deviation of the stochastic response component of softening and hardening Bouc-Wen systems subjected to combined stochastic excitation and harmonic excitation with different fractional orders

rived non-linear algebraic equations obtained by the two techniques are coupled, and thus standard numerical iteration method has been adopted to obtain the response quantities numerically and simultaneously. Finally, the pertinent Monte Carlo simulation has been used to demonstrate the applicability and accuracy of the proposed method. Specifically, a softening and hardening Bouc-Wen hysteretic system endowed with fractional element and subjected to combined excitation has been used as numerical examples for illustration. It has been found that the accuracy of the proposed method is comparable with the well established statistical linearization method for a integer-order Bouc-Wen system subjected to stochastic excitation only. It should be noted that the proposed method can be served as an beneficial attempt toward efficient and versatile solution for the considered problem. Further investigation should focus on the extension of the method for multi-degree-of-freedom fractional-order systems with other type of hysteretic models and subjected to nonstationary colored excitation.

Acknowledgment

This work was supported by the National Natural Science Foundation of China (Grant no. 52078399). The first author would like to thank the Chinese Scholarship Council (CSC) for financial support (File no. 201706955030) during his visit to Rice University as a visiting scholar. The first author would like to greatly thank Professor Pol D. Spanos at Rice University for the discussing the statistical linearization method for fractional-order systems.

Declarations

- Funding: All sources of funding for the research reported have been declared.
- Competing of interest: The authors declare that they have no competing interests.
- Code availability: All codes relevant to the present work are available upon request.
- Authors' contributions: KF conceived the study, derived the theoretical formulation and finalized the manuscript. HRJ composed the relevant codes, drafted the manuscript. ZYJ participated in coding, and drafting the manuscript.
- Availability of data and material: Not applicable.

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Figures

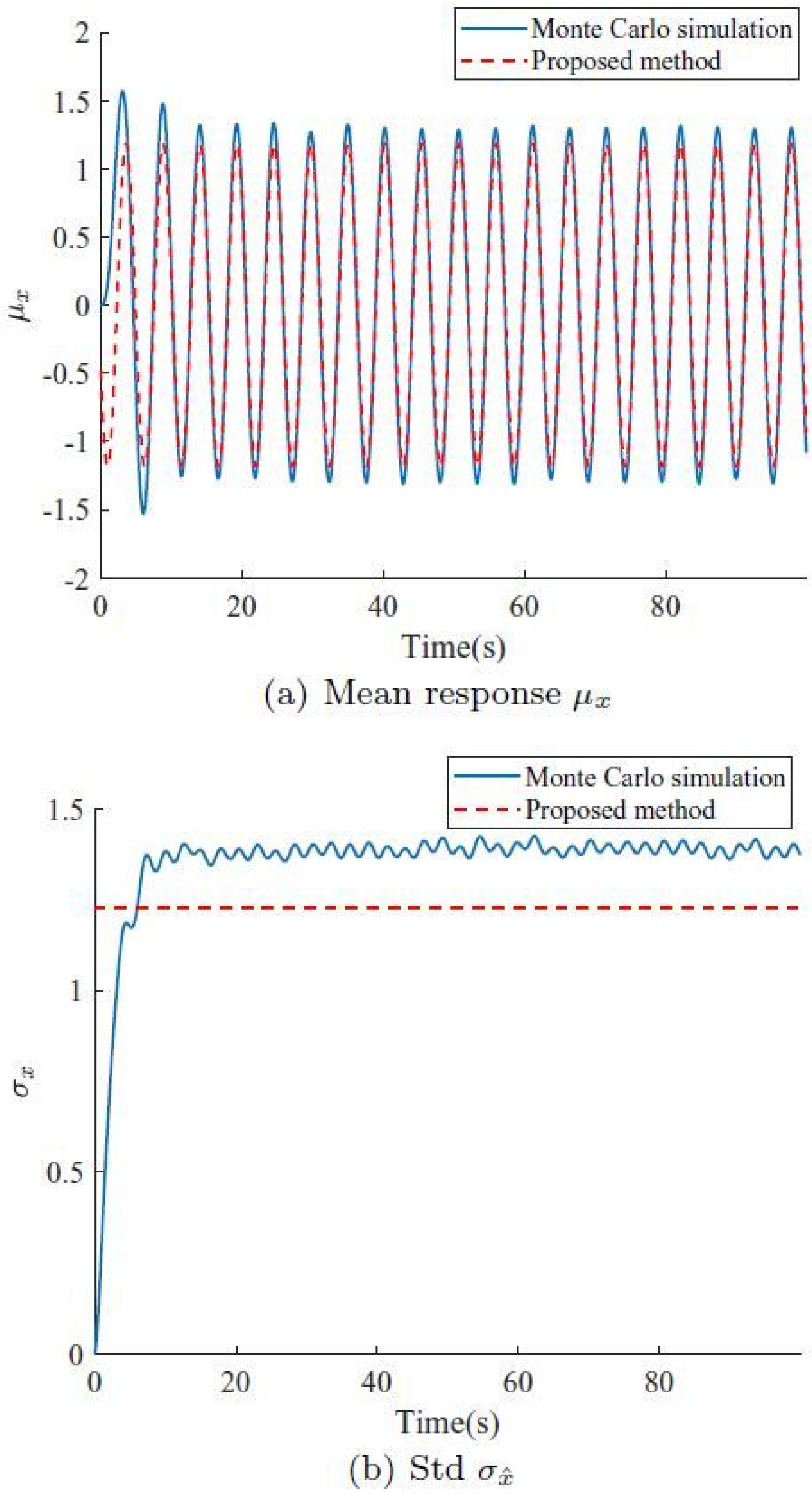
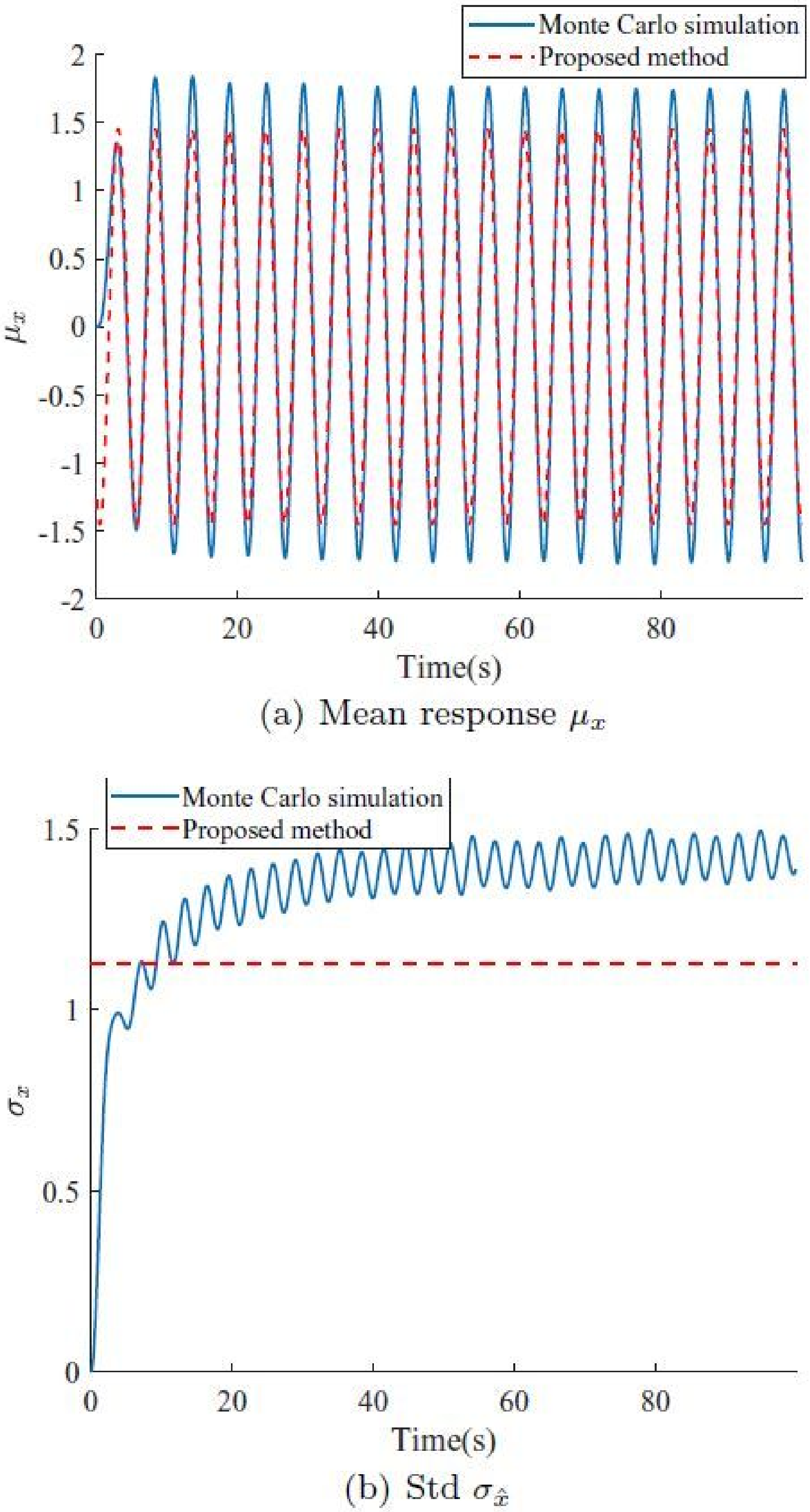


Figure 1

Displacement of the softening Bouc-Wen system sub- jected to combined excitation.

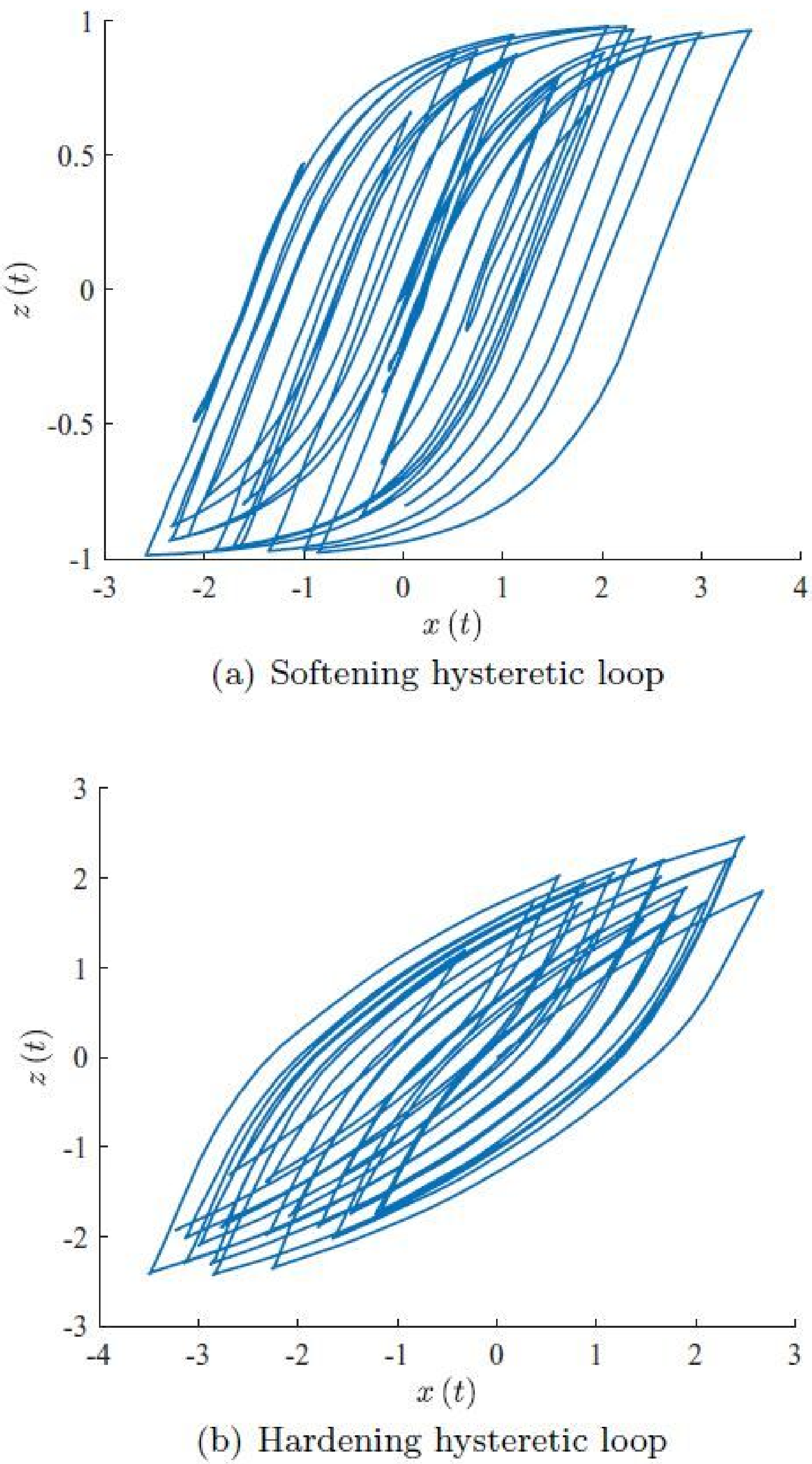


(a) Mean response μ_x

(b) Std $\sigma_{\hat{x}}$

Figure 2

Displacement of the hardening Bouc-Wen system sub- jected to combined excitation.



(a) Softening hysteretic loop

(b) Hardening hysteretic loop

Figure 3

Hysteretic loops of the Bouc-Wen dynamic system subjected to combined harmonic and stochastic excitation.

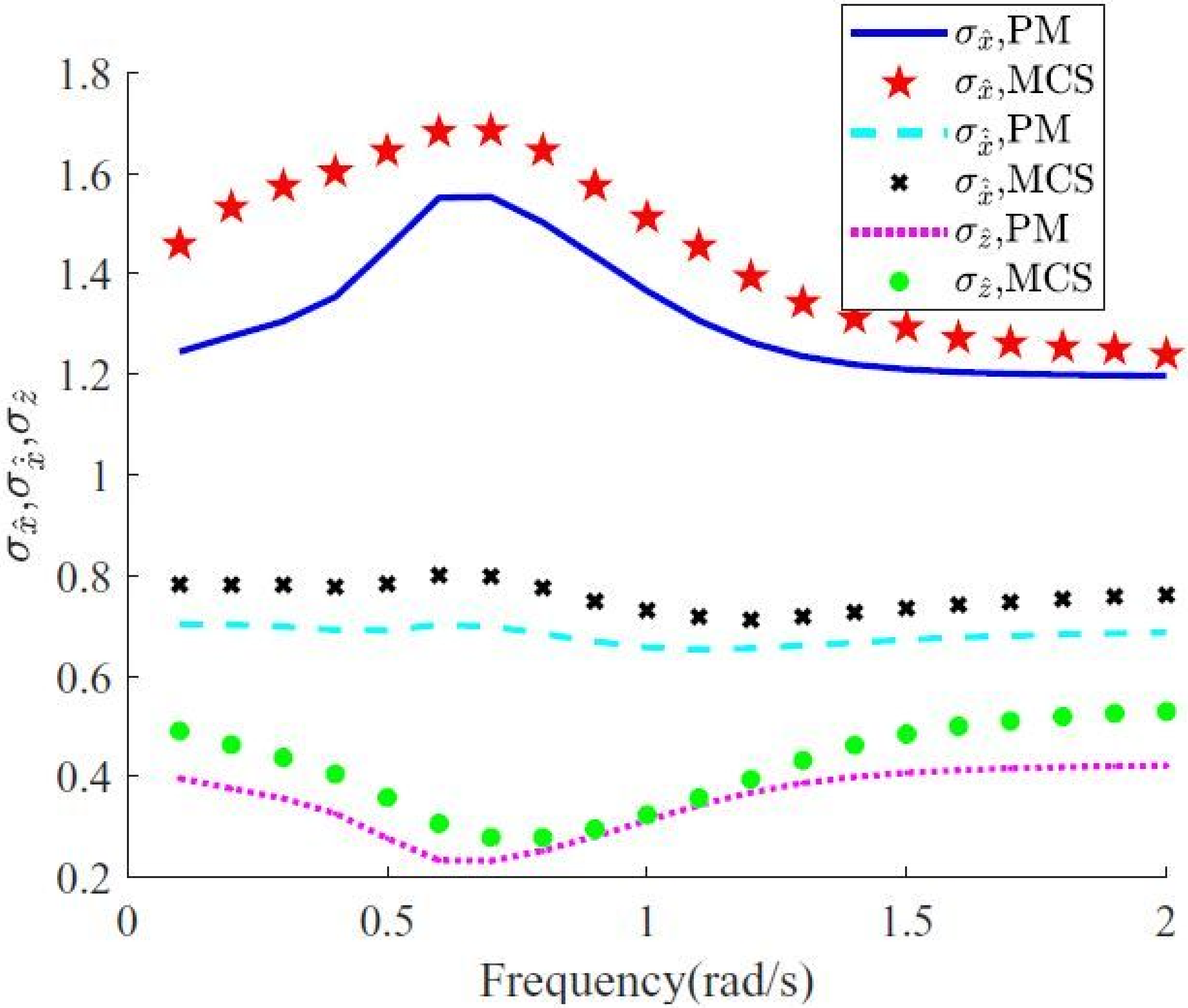


Figure 4

Standard deviation of the stochastic response compo- nent of a softening Bouc-Wen system subjected to combined stochastic excitation and harmonic excitation with di erent frequencies

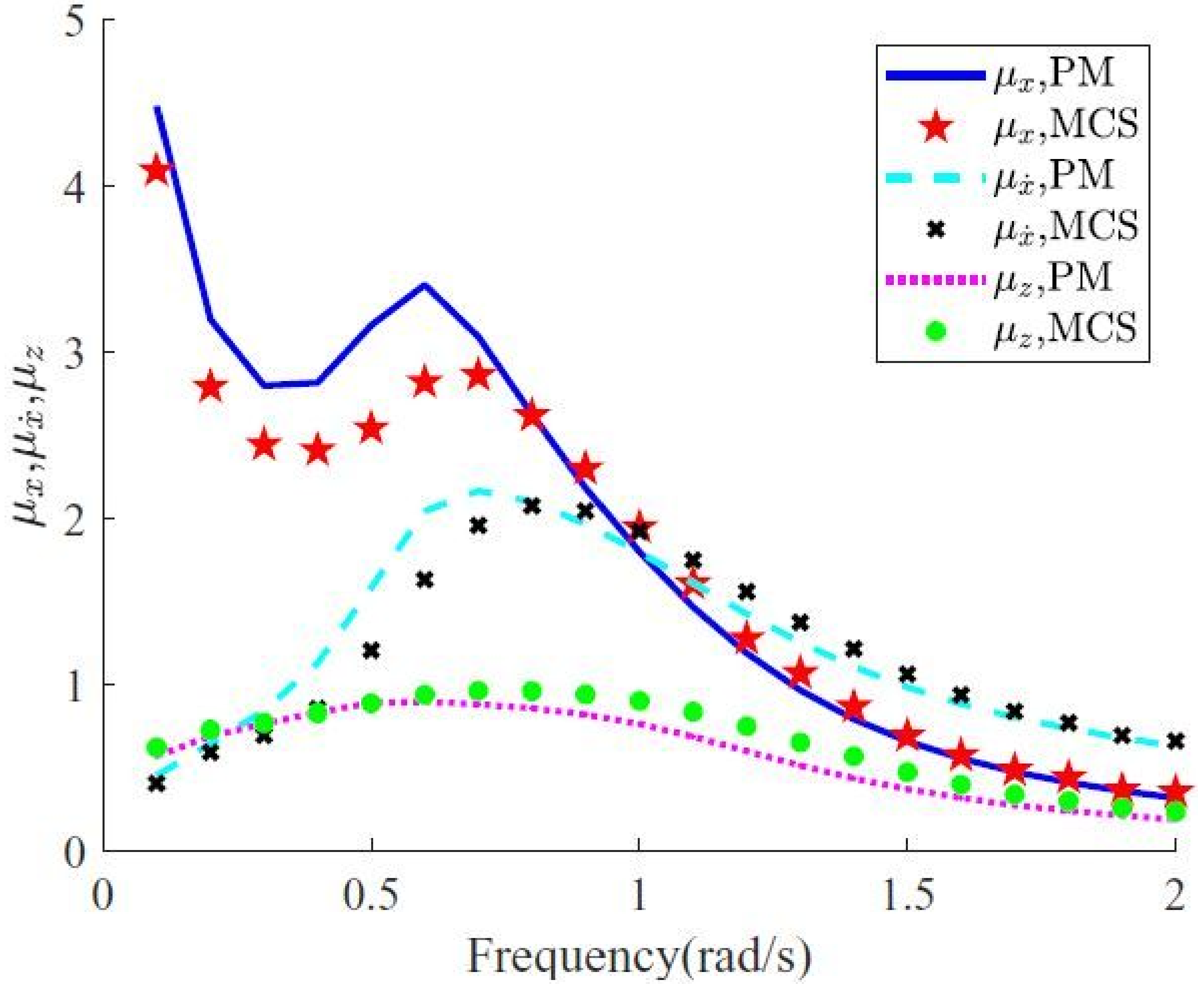


Figure 5

Amplitude of the deterministic response component of a softening Bouc-Wen system subjected to combined stochas- tic excitation and harmonic excitation with di erent frequen- cies

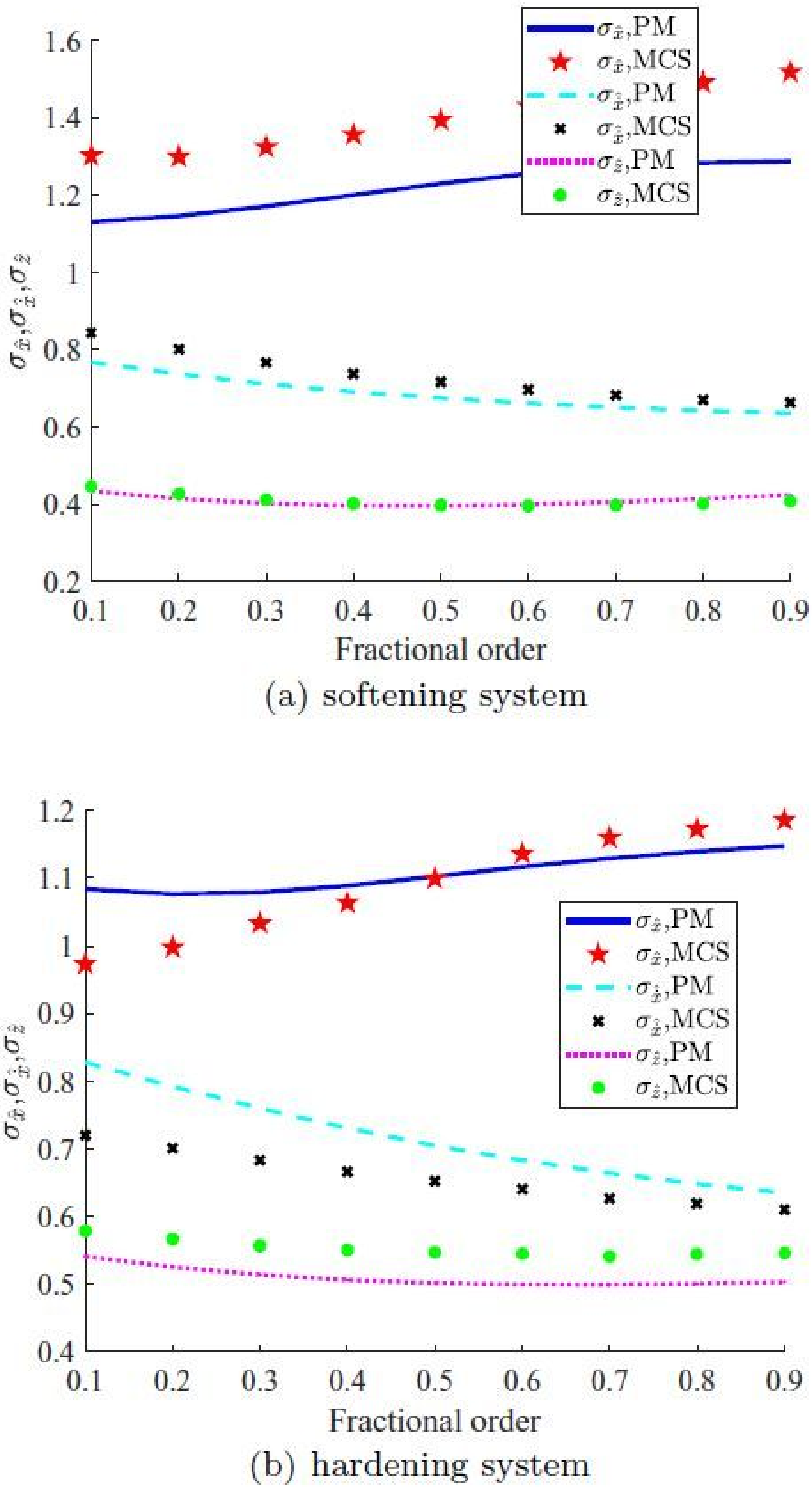
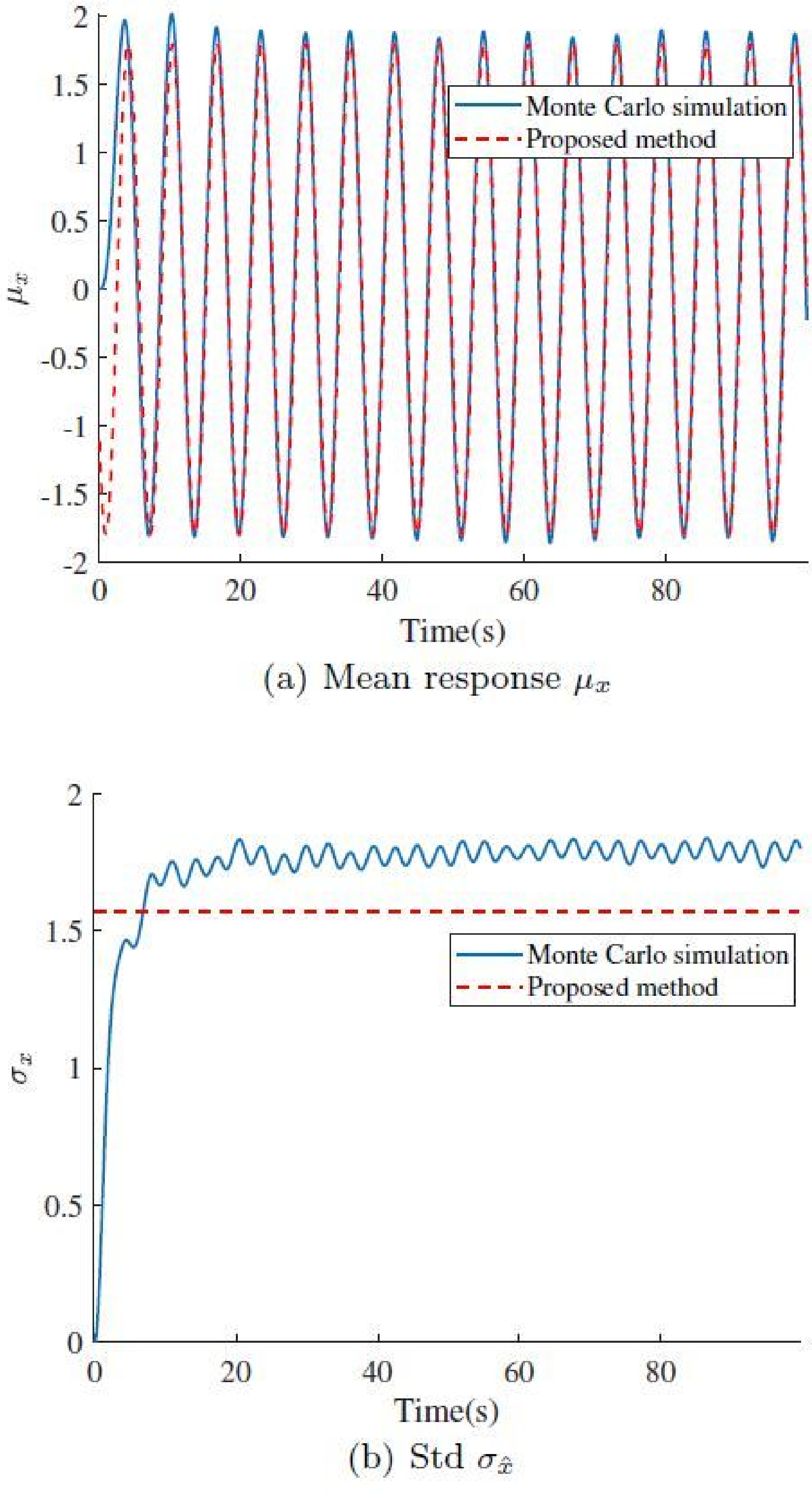


Figure 6

Standard deviation of the stochastic response compo- nent of softening and hardening Bouc-Wen systems subjected to combined stochastic excitation and harmonic excitation with di erent fractional orders

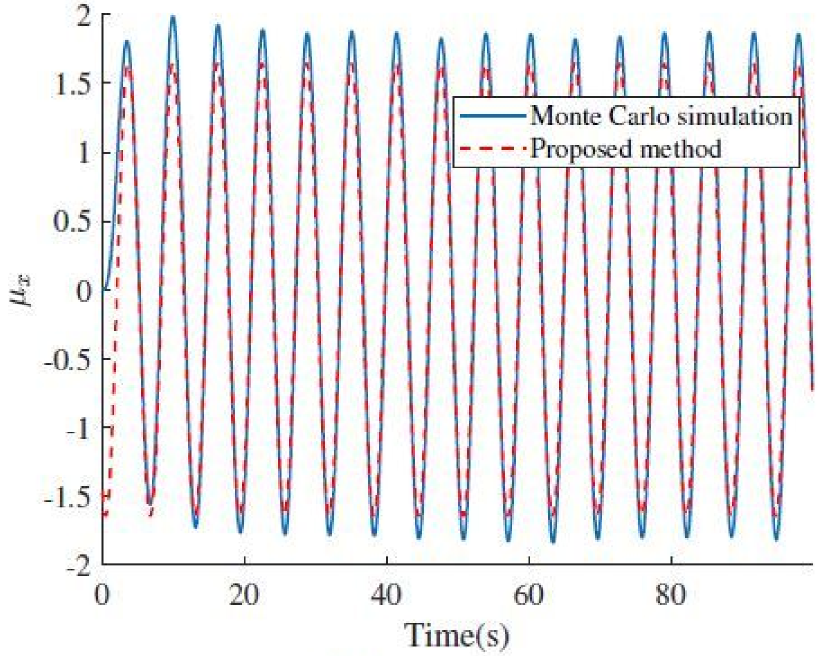


(a) Mean response μ_x

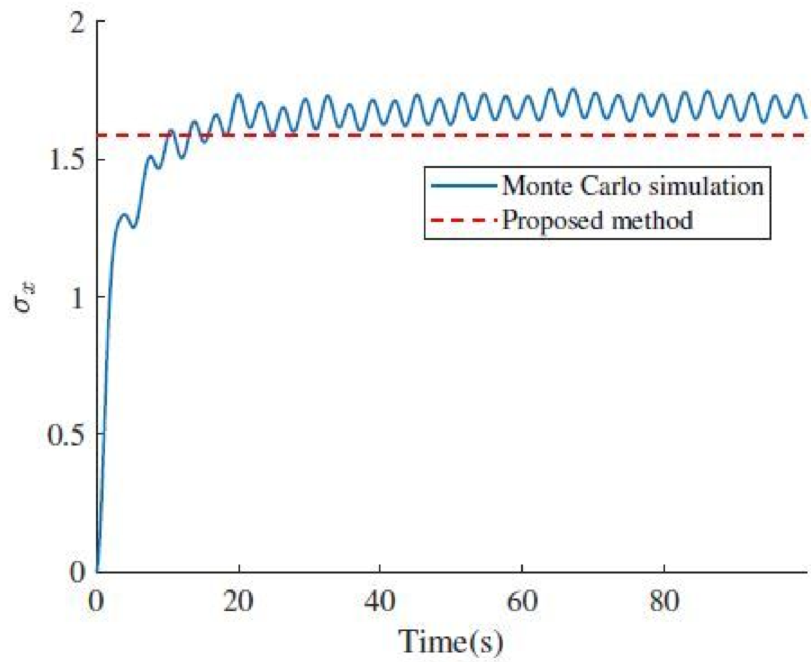
(b) Std $\sigma_{\hat{x}}$

Figure 7

Displacement of the softening Bouc-Wen system sub- jected to combined excitation.



(a) Mean response μ_x



(b) Std $\sigma_{\hat{x}}$

Figure 8

Displacement of the hardening Bouc-Wen system sub- jected to combined excitation.

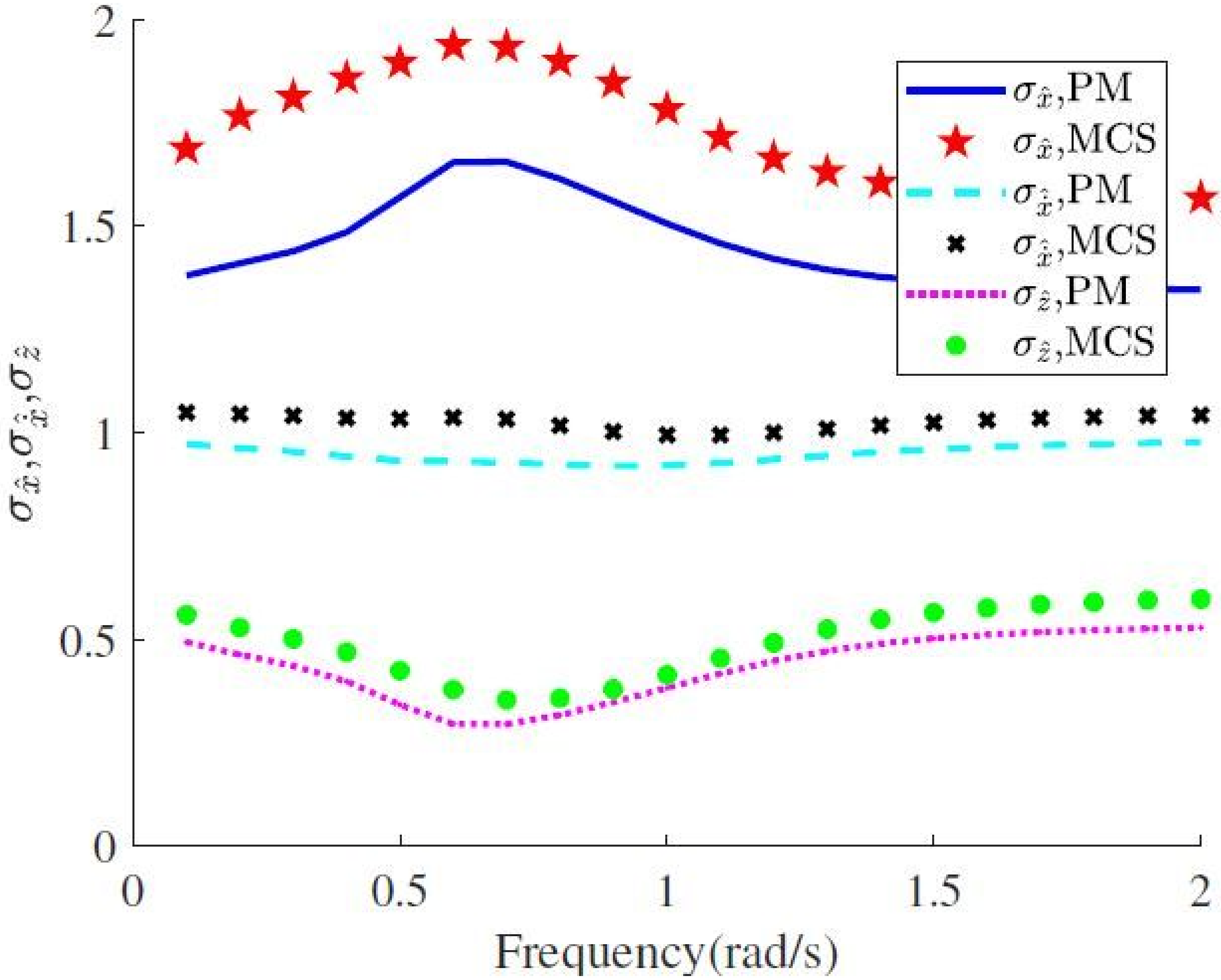


Figure 9

Standard deviation of the stochastic response compo- nent of a softening Bouc-Wen system subjected to combined stochastic excitation and harmonic excitation with di erent frequencies

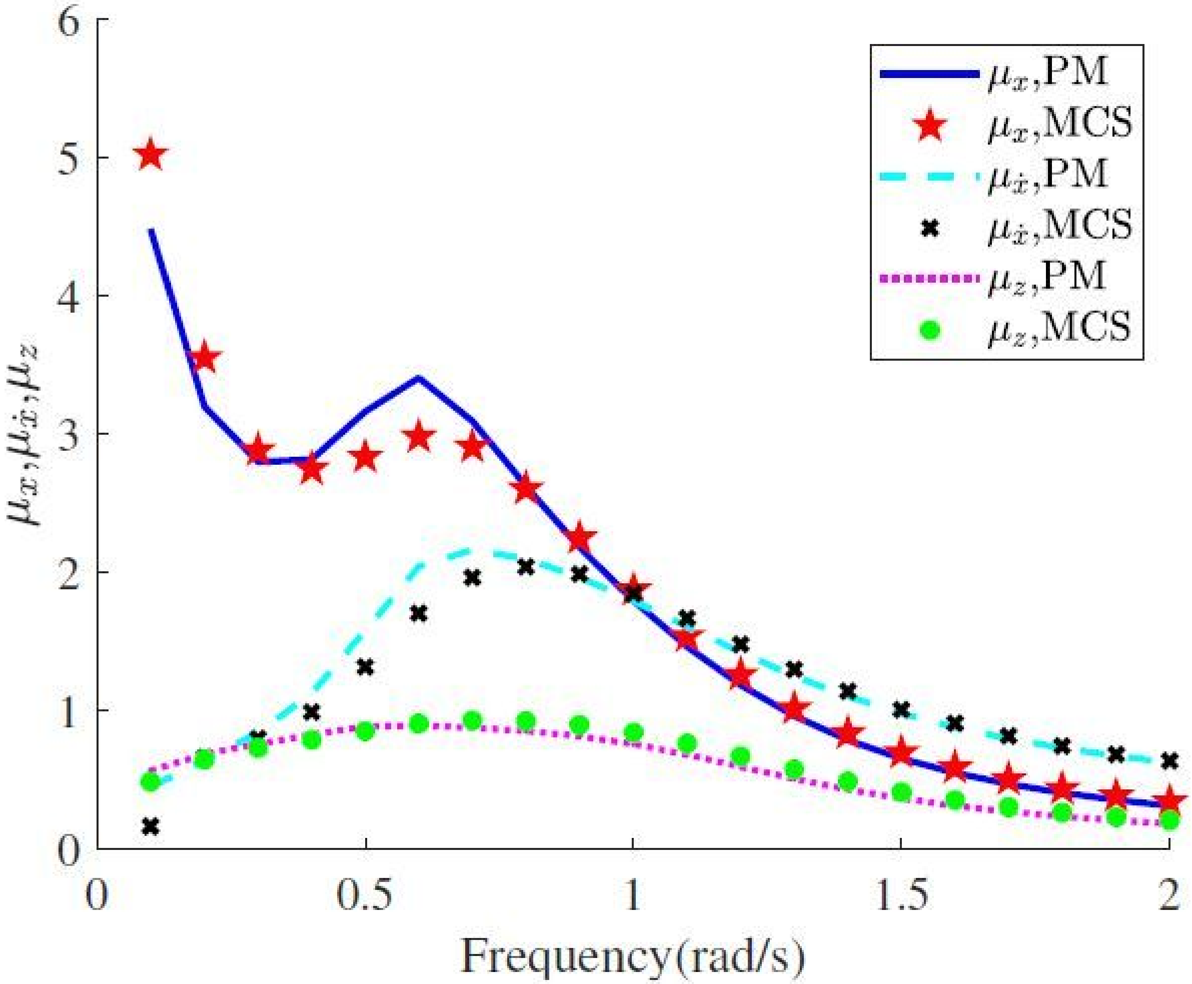


Figure 10

Amplitude of the deterministic response compo- nent of a softening Bouc-Wen system subjected to combined stochastic excitation and harmonic excitation with di erent frequencies

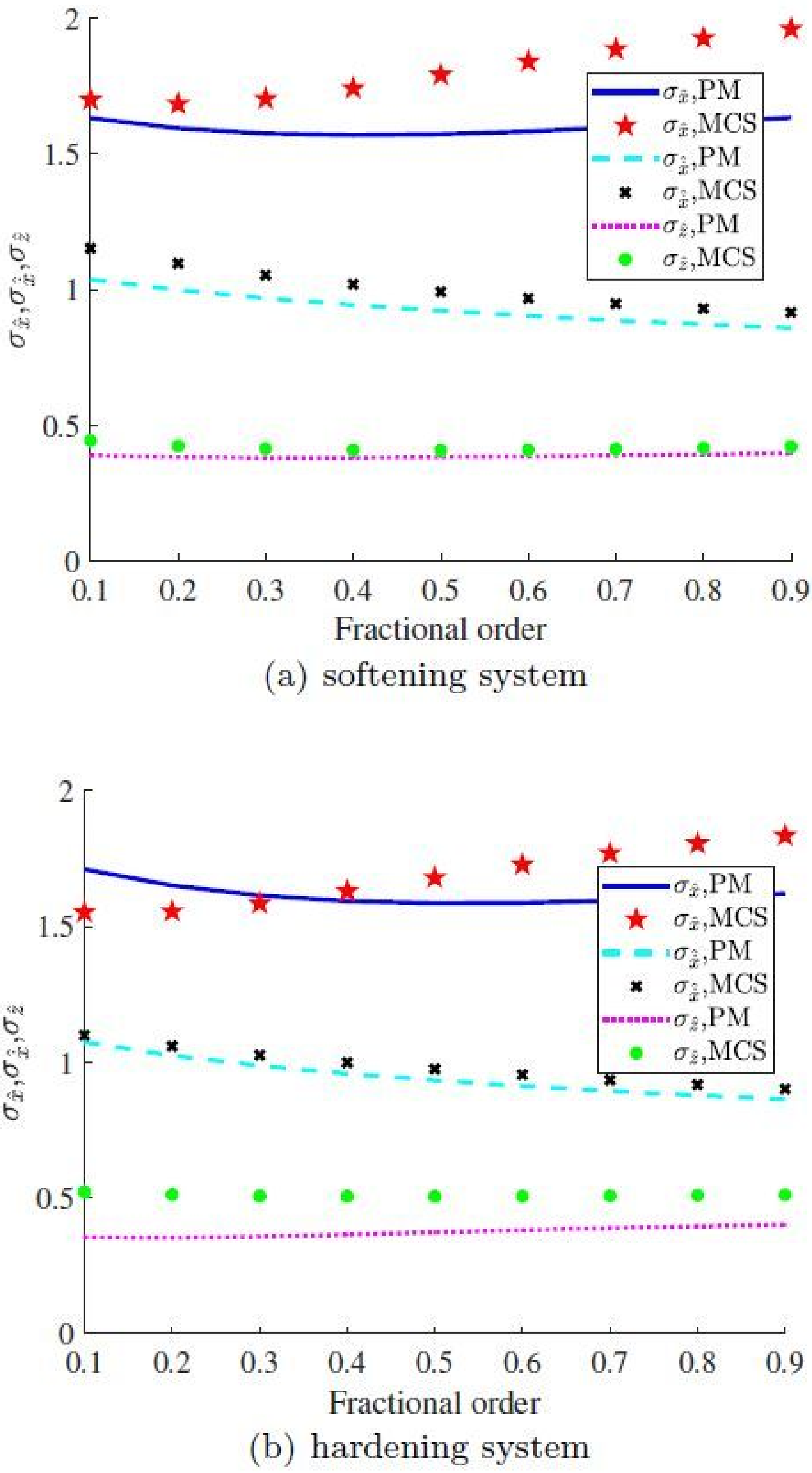


Figure 11

Standard deviation of the stochastic response compo- nent of softening and hardening Bouc-Wen systems subjected to combined stochastic excitation and harmonic excitation with di erent fractional orders