

Solving a Class of Equilibrium Problems With Equilibrium Constraints*

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Solving a class of equilibrium problems with equilibrium constraints ^{*}

Pei-Yu Li[†] and Ke-Wei Ding[‡]

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Abstract. An equilibrium problem with equilibrium constraints (EPEC) can be looked on as a generalized Nash equilibrium problem (GNEP) and the mathematical programs with equilibrium constraints (MPEC) whose constraints contain a parametric variational inequality or complementarity system. In this paper, we particularly consider a class of EPEC and solve its normalized stationary points where the multipliers of the leaders on the shared constraints are proportionable. We reformulate this kind of EPEC to a standard MPEC. In addition, we demonstrate the proposed approach on an EPEC model in similar products market.

Key words. Mathematical programs with equilibrium constraints, equilibrium problem with equilibrium constraints, normalized stationarity

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1 Introduction

An equilibrium problem with equilibrium constraints (EPEC) is a member of a new class of mathematical programs that often arise in engineering and economics applications. More generally, an EPEC is a mathematical program to find equilibria that simultaneously solve several mathematical programs with equilibrium constraints (MPEC), each of which is parameterized by decision variables of other MPEC. In particular, we assume the EPEC consists of K MPEC, and for each $\nu = 1, \dots, K$, the ν -th MPEC has the following form with

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independent decision variables $x^\nu \in \mathbb{R}^{n_\nu}$ and shared decision variables $y \in \mathbb{R}^{n_0}$:

$$\begin{aligned} \min \quad & f^\nu(x^\nu, y, \bar{x}^{-\nu}) \\ \text{s.t.} \quad & g^\nu(x^\nu, y, \bar{x}^{-\nu}) \leq 0, \quad h^\nu(x^\nu, y, \bar{x}^{-\nu}) = 0 \\ & 0 \leq G(x^\nu, y, \bar{x}^{-\nu}) \perp H(x^\nu, y, \bar{x}^{-\nu}) \geq 0. \end{aligned} \quad (1.1)$$

where $f^\nu : \mathbb{R}^n \rightarrow \mathbb{R}$, $g^\nu : \mathbb{R}^n \rightarrow \mathbb{R}^{p_\nu}$, $h^\nu : \mathbb{R}^n \rightarrow \mathbb{R}^{q_\nu}$, $G : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $H : \mathbb{R}^n \rightarrow \mathbb{R}^m$ are twice continuously differentiable functions in both $x = (x^\nu)_{\nu=1}^K$ and y , with $n = \sum_{\nu=0}^K n_\nu$. The notation $\bar{x}^{-\nu}$ means that $x^{-\nu} = (x^1, \dots, x^{\nu-1}, x^{\nu+1}, \dots, x^K) \in \mathbb{R}^{n-n_\nu-n_0}$ is not a variable but a fixed vector. This implies that we can view (1.1), denoted by MPEC ($\bar{x}^{-\nu}$), as being parameterized by $\bar{x}^{-\nu}$. Given $\bar{x}^{-\nu}$, we assume the solution set of the ν -th MPEC is nonempty and denote it by SOL (MPEC ($\bar{x}^{-\nu}$)).

The EPEC, associated with K MPEC defined as above, is to find a Nash equilibria $(x^*, y^*) \in \mathbb{R}^n$ such that

$$(x^{*,\nu}, y^*) \in \text{SOL}(\text{MPEC}(x^{*,-\nu})), \quad \nu = 1, \dots, K. \quad (1.2)$$

The EPEC has recently been studied by some researchers and used to model several problems in applications. Several EPEC models have been developed to study the strategic behavior of generating firms in deregulated electricity markets [1, 3, 10, 18]. A sequential nonlinear complementarity problem approach for solving EPEC is proposed in [21]. This approach is related to the relaxation technique for solving MPECs that relaxes the complementarity conditions and drives the relaxation parameter to zero [20]. Based on the strong stationarity conditions of each leader in a multi-leader-follower game, Leyffer and Munson [15] derived a family of nonlinear complementarity problem, nonlinear programming problem and MPEC formulations of the multi-leader-follower game. They also reformulated price-consistent multi-leader-follower game to a standard MPEC by imposing an additional restriction. See also various applications in economics such as Ehrenmann [2], Ehrenmann and Neuho [3], Hu [12], Murphy and Smeers [17], Su [22], Yao et al. [23]; and the algorithm investigation Su [21]. Guo and Lin [7, 8] reformulated various stationarities for EPECs as constrained equations and proposed a globally and superlinearly convergent algorithm to solve these constrained equations. Li [16] considered a class of EPECs which is completely separable. In [13, 14], Kulkarni et al. reformulated some EPEC models as MPECs by using potential games and shared constraints.

An equilibrium problem usually has many solutions and the uniqueness of an equilibrium point is expected only under very restrictive assumptions. If there are many equilibrium points, we could try to find all equilibrium points or as many equilibrium points as possible.

An alternative approach is to single out an equilibrium point that has some special property. The normalized equilibrium is such an equilibrium point that the Lagrange multipliers (shadow prices) associated with the shared constraints are equal among all players up to constant factors, and its uniqueness is guaranteed under appropriate conditions [19].

Note that EPEC is highly nonconvex and hence we study its stationary point. In this paper, we study normalized stationary points of a class of EPEC where the multipliers of the players on the shared constraints are proportionable. In economic terms, it means that the relative values of shadow prices associated with the common resources are identical for all players at any normalized stationary points. We reformulate this kind of EPEC to a standard MPEC by imposing some additional conditions and solve it by applying standard MPEC method, which generalizes the work of Leyffer and Munson [15] in which the multipliers are identical for all players. The EPEC model is different from the ones considered in [13, 14] in which the decision variables (on the lower level) vary among different leaders decision problems, while it is assumed in our model that the leaders have a common knowledge on the solution of the lower level equilibrium problem. Moreover, from the modeling perspective, our research focuses on the leadership role which is different from the framework in [13, 14]. Separability assumption of the EPEC model is also different from the ones studied in [7, 8] and [16].

This paper is organized as follows. In the next section we briefly introduce several stationarity condition for EPEC and show how equilibrium points can be computed reliably for EPEC by solving nonlinear optimization problems. Section 3 present normalized stationary points of a class of equilibrium problem with equilibrium constraints and reformulate to stationary points of an associated MPEC. In the section 4, we consider the EPEC model for similar products market in the same city and demonstrate the proposed approach on the model. In addition, some concluding remarks are given in Section 5.

The following notations will be used in the paper. We denote the transposed Jacobian matrix of a differentiable function $h : \mathbb{R}^s \rightarrow \mathbb{R}^t$ at a given point x by $\nabla h(x) \in \mathbb{R}^{s \times t}$. For a real-valued function $h(x, y)$ with the variable $x \in \mathbb{R}^s$ and $y \in \mathbb{R}^t$, the partial gradients with respect to x and y are denoted by $\nabla_x h(x, y) \in \mathbb{R}^s$ and $\nabla_y h(x, y) \in \mathbb{R}^t$, respectively.

2 Stationarity conditions and formulation

In this section, we first review the concepts of the weakly (W-) stationarity, Clarke (C-) stationarity, Mordukhovich (M-) stationarity and strongly (S-) stationarity for MPEC

(1.1)(see. e.g., [4, 20, 24]).

Definition 2.1 A vector (x^ν, y) is called *W-stationary point* of the MPEC (1.1), if there exist a vector multipliers $(\lambda^\nu, \mu^\nu, \psi^\nu, \sigma^\nu)$ such that $(x^\nu, y, \lambda^\nu, \mu^\nu, \psi^\nu, \sigma^\nu)$ satisfies the following conditions:

$$\begin{aligned}
& \nabla_{x^\nu} f^\nu(x^\nu, y, \bar{x}^{-\nu}) + \nabla_{x^\nu} g^\nu(x^\nu, y, \bar{x}^{-\nu})\lambda^\nu + \nabla_{x^\nu} h^\nu(x^\nu, y, \bar{x}^{-\nu})\mu^\nu \\
& \quad - \nabla_{x^\nu} G(x^\nu, y, \bar{x}^{-\nu})\psi^\nu - \nabla_{x^\nu} H(x^\nu, y, \bar{x}^{-\nu})\sigma^\nu = 0, \\
& \nabla_y f^\nu(x^\nu, y, \bar{x}^{-\nu}) + \nabla_y g^\nu(x^\nu, y, \bar{x}^{-\nu})\lambda^\nu + \nabla_y h^\nu(x^\nu, y, \bar{x}^{-\nu})\mu^\nu \\
& \quad - \nabla_y G(x^\nu, y, \bar{x}^{-\nu})\psi^\nu - \nabla_y H(x^\nu, y, \bar{x}^{-\nu})\sigma^\nu = 0, \\
& \quad 0 \leq -g^\nu(x^\nu, y, \bar{x}^{-\nu}) \perp \lambda^\nu \geq 0, \\
& \quad \quad \quad h^\nu(x^\nu, y, \bar{x}^{-\nu}) = 0, \\
& \quad 0 \leq G(x^\nu, y, \bar{x}^{-\nu}) \perp H(x^\nu, y, \bar{x}^{-\nu}) \geq 0, \\
& \text{if } G_i(x^\nu, y, \bar{x}^{-\nu}) > 0, \quad \text{then } \psi_i^\nu = 0, \quad i = 1, \dots, m, \\
& \text{if } H_i(x^\nu, y, \bar{x}^{-\nu}) > 0, \quad \text{then } \sigma_i^\nu = 0, \quad i = 1, \dots, m.
\end{aligned} \tag{2.1}$$

In addition, the vector (x^ν, y) is called

- (a) a C-stationary point if $\psi_i^\nu \sigma_i^\nu \geq 0$ when $G_i(x^\nu, y, \bar{x}^{-\nu}) = H_i(x^\nu, y, \bar{x}^{-\nu}) = 0$.
- (b) a M-stationary point if either $\min(\psi_i^\nu, \sigma_i^\nu) > 0$ or $\psi_i^\nu \sigma_i^\nu = 0$ when $G_i(x^\nu, y, \bar{x}^{-\nu}) = H_i(x^\nu, y, \bar{x}^{-\nu}) = 0$.
- (c) a S-stationary point if $\psi_i^\nu \geq 0, \sigma_i^\nu \geq 0$ when $G_i(x^\nu, y, \bar{x}^{-\nu}) = H_i(x^\nu, y, \bar{x}^{-\nu}) = 0$.

In [6, 15], it is shown that MPEC can be solved reliably and efficiently by replacing the complementarity constraint with $Ys \leq 0$, where Y is the diagonal matrix with y along its diagonal. Fletche and Leyffer [5] have shown that the strong stationarity of MPEC is equivalent to the KKT conditions of a NLP. Since we only consider finite-dimensional optimization problems, based the stationarity for MPEC (1.1), we define the stationarity for EPEC (1.2)(see. e.g., [11, 21]).

Definition 2.2 We say that a vector (x^*, y^*) is a *C-stationary(M-stationary, S-stationary) point* of the EPEC (1.2) if for each $\nu = 1, \dots, K$, $(x^{\nu,*}, y^*)$ is a *C-stationary(M-stationary, S-stationary) point* for the MPEC $(x^*, -\nu)$ (1.1).

We have known from [21] that if (x^*, y^*) is an (possibly local) equilibrium point of EPEC (1.2) and every MPEC of EPEC (1.2) satisfies an MPEC-LICQ, then there exist multipliers $(\lambda^*, \mu^*, \psi^*, \sigma^*)$ such that (x^*, y^*) is a S-stationary point of the EPEC (1.2). In the following two propositions, we will consider the relationship of C-stationary (M-stationary) point and (possibly local) equilibrium point in EPEC (1.2).

Proposition 2.1 *Let (x^*, y^*) be an (possibly local) equilibrium point of EPEC (1.2), if for each $\nu = 1, \dots, K$, the MPEC-MFCQ holds at $(x^{\nu,*}, y^*)$ for MPEC($x^{\nu,*}$)(1.1), then there exist multipliers $(\lambda^*, \mu^*, \psi^*, \sigma^*)$ such that (x^*, y^*) is a C-stationary point of the EPEC (1.2).*

Proof. Since (x^*, y^*) be an (possibly local) equilibrium point of the EPEC (1.2), it follows that for each $\nu = 1, \dots, K$, the point $(x^{\nu,*}, y^*)$ is a (local) minimizer of the MPEC($x^{\nu,*}$)(1.1). By applying the Theorem 2 of [20] for each $\nu = 1, \dots, K$, we can show that there exists multipliers $(\lambda^\nu, \mu^\nu, \psi^\nu, \sigma^\nu)$ such that $(x^{\nu,*}, y^*)$ is a C-stationary point of the MPEC($x^{\nu,*}$)(1.1) for each $\nu = 1, \dots, K$. We can easy obtain the conclusion from the definition of C-stationary point of the EPEC (1.2). ■

Proposition 2.2 *Let (x^*, y^*) be an (possibly local) equilibrium point of EPEC (1.2), if for each $\nu = 1, \dots, K$, the NNAMCQ holds at $(x^{\nu,*}, y^*)$ for MPEC($x^{\nu,*}$)(1.1), then there exist multipliers $(\lambda^*, \mu^*, \psi^*, \sigma^*)$ such that (x^*, y^*) is a M-stationary point of the EPEC (1.2).*

Proof. We can easy obtain the conclusion by applying the Corollary 2.1 of [24]. Since its proof is similar to Proposition 2.1, we omit the proof here. ■

3 Normalized stationary point of a class of equilibrium problem with equilibrium constraints

In this section, we will consider a special local equilibrium points of the EPEC and call it normalized stationary points of the EPEC.

Normalized equilibrium, first introduced by Rosen [19], is a special GNEP. To reduce the number of variables and constraints, which may make the problem more tractable, Leyffer and Munson [15] make a price-consistency assumption. This technique restricts the solutions considered to those for which the multipliers (prices) on the shared constraints are the same. In economic terms, this means that the relative values of shadow prices associated with the common resources are identical for all players at any normalized equilibrium. We need the following separability assumption for objective functions.

Definition 3.1 *We say that the EPEC (1.2) is relatively separable if the general constraints consist of a set of constraints independent of other decision variables and a set of constraints common across all players, that is,*

$$g^\nu(x, y) = \begin{bmatrix} \bar{g}^\nu(x^\nu) \\ \tilde{g}(x, y) \end{bmatrix}, h^\nu(x, y) = \begin{bmatrix} \bar{h}^\nu(x^\nu) \\ \tilde{h}(x, y) \end{bmatrix},$$

and the objective function consists of a separable term and a term relatively common across all players, that is,

$$f^\nu(x, y) = \bar{f}^\nu(x^\nu) + \beta^\nu \tilde{f}(x, y), \quad \nu = 1, \dots, K,$$

where $\beta^\nu > 0$.

Assume that the EPEC (1.2) is relatively separable, we can rewrite (1.1) as following for each $\nu = 1, \dots, K$,

$$\begin{aligned} \min_{x^\nu} \quad & \bar{f}^\nu(x^\nu) + \beta^\nu \tilde{f}(x, y) \\ \text{s.t.} \quad & \bar{g}^\nu(x^\nu) \leq 0, \\ & \tilde{g}(x^\nu, y, x^{-\nu}) \leq 0, \\ & \bar{h}^\nu(x^\nu) = 0, \\ & \tilde{h}(x^\nu, y, x^{-\nu}) = 0, \\ & 0 \leq G(x^\nu, y, x^{-\nu}) \perp H(x^\nu, y, x^{-\nu}) \geq 0. \end{aligned} \tag{3.1}$$

Definition 3.2 We say that a vector (x^*, y^*) is a normalized C -(M -, S -) stationary point of the EPEC (3.1), if (x^*, y^*) is a C -(M -, S -) stationary point of the EPEC (3.1) and the Lagrange multipliers associated with the shared constraints satisfying:

$$\frac{\tilde{\lambda}^\nu}{\tilde{\lambda}^0} = \frac{\tilde{\mu}^\nu}{\tilde{\mu}^0} = \frac{\psi^\nu}{\psi^0} = \frac{\sigma^\nu}{\sigma^0} = \beta^\nu, \quad \nu = 1, \dots, K, \tag{3.2}$$

where $\beta^\nu > 0$.

The following theorem relates normalized stationary point of relatively separable EPEC to a standard MPEC.

Theorem 3.1 Assume that the EPEC (1.2) is relatively separable. Then the C -(M -, S -) stationary point conditions of solving normalized equilibrium of the EPEC (1.2) are equivalent to the C -(M -, S -) stationary point conditions of the following MPEC:

$$\begin{aligned} \min_{x, y} \quad & \sum_{\nu=1}^K \frac{1}{\beta^\nu} \bar{f}^\nu(x^\nu) + \tilde{f}(x, y) \\ \text{s.t.} \quad & \bar{g}^\nu(x^\nu) \leq 0, \quad \nu = 1, \dots, K \\ & \tilde{g}(x, y) \leq 0 \\ & \bar{h}^\nu(x^\nu) = 0, \quad \nu = 1, \dots, K \\ & \tilde{h}(x, y) = 0 \\ & 0 \leq G(x, y) \perp H(x, y) \geq 0. \end{aligned} \tag{3.3}$$

Proof. Assume that the EPEC (1.2) is relatively separable, if there exists a vector multipliers $(\tilde{\lambda}^\nu, \tilde{\mu}^\nu, \rho^\nu, \tau^\nu, \psi^\nu, \sigma^\nu)$,

such that $(x^\nu, y, \tilde{\lambda}^\nu, \tilde{\mu}^\nu, \rho^\nu, \tau^\nu, \psi^\nu, \sigma^\nu)$ satisfies the following: $\nu = 1, \dots, K$,

$$\begin{aligned}
\nabla_{x^\nu} \bar{f}^\nu(x^{*,\nu}) + \nabla_{x^\nu} \beta^\nu \tilde{f}(x^*, y^*) + \nabla_{x^\nu} \bar{g}^\nu(x^{*,\nu}) \rho^\nu + \nabla_{x^\nu} \tilde{g}(x^*, y^*) \tilde{\lambda}^\nu + \nabla_{x^\nu} \bar{h}^\nu(x^{*,\nu}) \tau^\nu + \nabla_{x^\nu} \tilde{h}(x^*, y^*) \tilde{\mu}^\nu \\
- \nabla_{x^\nu} G(x^*, y^*) \psi^\nu - \nabla_{x^\nu} H(x^*, y^*) \sigma^\nu = 0, \\
\nabla_y \beta^\nu \tilde{f}(x^*, y^*) + \nabla_y \tilde{g}(x^*, y^*) \tilde{\lambda}^\nu + \nabla_y \tilde{h}(x^*, y^*) \tilde{\mu}^\nu - \nabla_y G(x^*, y^*) \psi^\nu - \nabla_y H(x^*, y^*) \sigma^\nu = 0, \\
0 \leq -\bar{g}^\nu(x^{*,\nu}) \perp \rho^\nu \geq 0, \\
0 \leq -\tilde{g}(x^*, y^*) \perp \tilde{\lambda}^\nu \geq 0, \\
\bar{h}^\nu(x^{*,\nu}) = 0, \tilde{h}(x^*, y^*) = 0, \\
0 \leq G(x^*, y^*) \perp H(x^*, y^*) \geq 0, \\
\text{if } G_i(x^*, y^*) > 0, \text{ then } \psi_i^\nu = 0, \quad i = 1, \dots, m, \\
\text{if } H_i(x^*, y^*) > 0, \text{ then } \sigma_i^\nu = 0, \quad i = 1, \dots, m.
\end{aligned} \tag{3.4}$$

In addition, when $G_i(x^*, y^*) = H_i(x^*, y^*) = 0$, we can call the vector (x^*, y^*) is, $\nu = 1, \dots, K$,

- (a) a C-stationary point if $\psi_i^\nu \sigma_i^\nu \geq 0$.
- (b) a M-stationary point if either $\min(\psi_i^\nu, \sigma_i^\nu) > 0$ or $\psi_i^\nu \sigma_i^\nu = 0$.
- (c) a S-stationary point if $\psi_i^\nu \geq 0, \sigma_i^\nu \geq 0$.

Since $\beta^\nu > 0, \nu = 1, \dots, K$ and

$$\frac{\tilde{\lambda}^\nu}{\tilde{\lambda}^0} = \frac{\tilde{\mu}^\nu}{\tilde{\mu}^0} = \frac{\psi^\nu}{\psi^0} = \frac{\sigma^\nu}{\sigma^0} = \beta^\nu, \quad \nu = 1, \dots, K, \tag{3.5}$$

we can reformulate (3.4) as follows:

$$\begin{aligned}
\nabla_{x^\nu} \frac{1}{\beta^\nu} \bar{f}^\nu(x^{*,\nu}) + \nabla_{x^\nu} \tilde{f}(x^*, y^*) + \nabla_{x^\nu} \bar{g}^\nu(x^{*,\nu}) \frac{\rho^\nu}{\beta^\nu} + \nabla_{x^\nu} \tilde{g}(x^*, y^*) \tilde{\lambda}^0 + \nabla_{x^\nu} \bar{h}^\nu(x^{*,\nu}) \frac{\tau^\nu}{\beta^\nu} + \nabla_{x^\nu} \tilde{h}(x^*, y^*) \tilde{\mu}^0 = 0 \\
- \nabla_{x^\nu} G(x^*, y^*) \psi^0 - \nabla_{x^\nu} H(x^*, y^*) \sigma^0 = 0 \\
\nu = 1, \dots, K \\
\nabla_y \tilde{f}(x^*, y^*) + \nabla_y \tilde{g}(x^*, y^*) \tilde{\lambda}^0 + \nabla_y \tilde{h}(x^*, y^*) \tilde{\mu}^0 - \nabla_y G(x^*, y^*) \psi^0 - \nabla_y H(x^*, y^*) \sigma^0 = 0, \\
0 \leq -\bar{g}^\nu(x^{*,\nu}) \perp \frac{\rho^\nu}{\beta^\nu} \geq 0 \\
\nu = 1, \dots, K \\
0 \leq -\tilde{g}(x^*, y^*) \perp \tilde{\lambda}^0 \geq 0, \\
\bar{h}^\nu(x^{*,\nu}) = 0, \\
\nu = 1, \dots, K \\
\tilde{h}(x^*, y^*) = 0, \\
0 \leq G(x^*, y^*) \perp H(x^*, y^*) \geq 0, \\
\text{if } G_i(x^*, y^*) > 0, \text{ then } \psi_i^0 = 0, \quad i = 1, \dots, m, \\
\text{if } H_i(x^*, y^*) > 0, \text{ then } \sigma_i^0 = 0, \quad i = 1, \dots, m.
\end{aligned} \tag{3.6}$$

Obviously, if there exists a vector multipliers $(\tilde{\lambda}^0, \tilde{\mu}^0, \psi^0, \sigma^0, \rho^1, \dots, \rho^K, \tau^1, \dots, \tau^K)$ such that the vector $(x^*, y^*, \tilde{\lambda}^0, \tilde{\mu}^0, \psi^0, \sigma^0, \rho^1, \dots, \rho^K, \tau^1, \dots, \tau^K)$ satisfies the (3.6), so the vector (x^*, y^*) is a W-stationary point of the MPEC (3.3).

Additionally, we have the following conclusion by $\beta^\nu > 0, \nu = 1, \dots, K$ and (3.5):

$$(a) \quad \psi^\nu \sigma^\nu \geq 0 \iff \psi^0 \sigma^0 \geq 0.$$

$$(b) \quad \text{either } \min(\psi^\nu, \sigma^\nu) > 0 \text{ or } \psi^\nu \sigma^\nu = 0 \iff \text{either } \min(\psi^0, \sigma^0) > 0 \text{ or } \psi^0 \sigma^0 = 0.$$

$$(c) \quad \psi^\nu \geq 0, \sigma^\nu \geq 0 \iff \psi^0 \geq 0, \sigma^0 \geq 0.$$

Note that the definition of C-(M,S-) stationary point of EPEC, we can easy get the conclusion. ■

Obviously, solving normalized stationary points of relatively separable EPEC is a generalization of the price-consistent multi-leader-follower games in [15]. By introducing the relatively separable assumption and finding normalized stationary points, we produce a model that may be easier to solve than the original standard EPEC. The following examples of Nash game shows the results for solving the normalized stationary points:

Example 1 (Leyffer's example) This problem is taken from [15].

$$\begin{aligned} \min_{x_1} \quad & x_1^2 + ax_1x_2 \\ \text{s.t} \quad & x_1 + x_2 = c, \\ \min_{x_2} \quad & x_2^2 + bx_1x_2 \\ \text{s.t} \quad & x_1 + x_2 = c, \end{aligned} \tag{3.7}$$

where a, b, and c are parameters. The NCP formulation of solving normalized stationary points for this problem is the system of equations

$$\begin{aligned} 2x_1 + ax_2 - a\lambda_0 &= 0 \\ 2x_2 + bx_1 - b\lambda_0 &= 0 \\ x_1 + x_2 &= c, \end{aligned}$$

where $(a\lambda_0, b\lambda_0)$ is Lagrange multiplier vectors for shared constraint. This problem have normalized equilibrium $(\frac{ac(b-2)}{2(ab-a-b)}, \frac{bc(a-2)}{2(ab-a-b)})$, which $\lambda_0 = \frac{abc-4c}{2(ab-a-b)}$. It is equivalent to computing a first-order critical point for the single optimization problem:

$$\begin{aligned} \min_{x_1, x_2} \quad & \frac{1}{a}x_1^2 + \frac{1}{b}x_2^2 + x_1x_2 \\ \text{s.t} \quad & x_1 + x_2 = c. \end{aligned}$$

Example 2 (Harker's example) This problem is taken from [9]. There are two players and they

solve the following problems:

$$\begin{aligned}
& \min_{x_1} && x_1^2 + \frac{8}{3}x_1x_2 - 34x_1 \\
& s.t && 0 \leq x_1 \leq 10 \\
& && x_1 + x_2 \leq 15, \\
& \min_{x_2} && x_2^2 + \frac{5}{4}x_1x_2 - 24.25x_2 \\
& s.t && 0 \leq x_2 \leq 10 \\
& && x_1 + x_2 \leq 15.
\end{aligned} \tag{3.8}$$

This is a GNEP with one shared constraint and the solution set is given by

$$SOL^{GNEP} = \{(5, 9)\} \cup \{(t, 15 - t) | 9 \leq t \leq 10\}.$$

If $(\frac{8}{3}\lambda_0, \frac{5}{4}\lambda_0)$, $\lambda_0 \geq 0$ is Lagrange multiplier vectors for shared constraint. This problem has normalized equilibrium $(5, 9)$ and $(\frac{47}{7}, \frac{58}{7})$, which $\lambda_0 = 0$ and $\lambda_0 = \frac{4}{7}$. It is equivalent to computing a first-order critical point for the single optimization problem:

$$\begin{aligned}
& \min_{x_1, x_2} && \frac{3}{8}x_1^2 - \frac{51}{4}x_1 + \frac{4}{5}x_2^2 - \frac{97}{5}x_2 + x_1x_2 \\
& s.t && 0 \leq x_1 \leq 10 \\
& && 0 \leq x_2 \leq 10 \\
& && x_1 + x_2 \leq 15.
\end{aligned}$$

4 Applications

In this section, we first discuss normalized stationary points of relatively separable EPEC arising from competition of manufacturer for similar products in the same city.

4.1 Model in similar products market

Since similar products have some same function but not exactly the same, similar products have different prices. We consider an oligopoly consisting of $K + F$ manufacturers that produce similar products noncooperatively before the market demand is realized. The first K manufacturers (herein leaders) have no capacity installed and thus have to decide now what their future output will be before the demand function is realized. The remaining F manufacturers (followers) have sufficient capacity installed and thus do not have to make a decision today, but instead they can wait to observe the quantities supplied by the K leaders as well as the realized demand function before making a decision on their supply quantities.

The market demand are characterized by inverse demand functions $p_\nu(x, y)$, $\nu = 1, \dots, K + F$, where $p_\nu(x, y)$ is the market price of the product made by the manufacturer ν , $x = (x_i)_{i=1}^K$,

x_i is the supply quantity of the leader i , and $y = (y_j)_{j=1}^F$, y_j is the supply quantity of the follower j .

Before market demand is realized, leader i chooses his quantity x_i . The leader's profit can be formulated as

$$R_i(x_i, X_{-i}, Y^*) = x_i p_i(x_i, X_{-i}, Y^*) - C_i(x_i), \quad (4.1)$$

where X_{-i} denotes the total bids by the other leaders, $Y^* = (y_j^*)_{j=1}^K$, where y_j^* is the strategies of the j follower. $x_i p_i(x_i, X_{-i}, Y^*)$ means the total revenue for leader i , and $C_i(x_i)$ denotes the cost function of leader i . The j th leader's decision problem is to choose the supply quantity x_i that maximizes its profit; that is,

$$\max_{x_i \in \mathcal{X}_i} R_i(x_i, X_{-i}, Y^*) = x_i p_i(x_i, X_{-i}, Y^*) - C_i(x_i), \quad (4.2)$$

where $\mathcal{X}_i := \{x_i \in [0, +\infty) \mid g_i(x_i) \leq 0, g(x, y) \leq 0\}$ is nonempty and bounded convex set, for each $i = 1, \dots, K$.

The j th follower chooses its supply quantity after observing the aggregate leaders supply X . Thus, the total revenue of the j th follower is $y_j p_j(X^*, y_j, Y_{-j})$, and its total cost is $C_j(y_j)$. Consequently, the j th follower profit is

$$R_j(X^*, y_j, Y_{-j}) = y_j p_j(X^*, y_j, Y_{-j}) - C_j(y_j), \quad (4.3)$$

where Y_{-j} denotes the total bids by the other followers, $X^* = (x_i^*)_{i=1}^K$, where x_i^* is the strategies of the leader i . The j th follower decision problem is

$$\max_{y_j \geq 0} R_j(X^*, y_j, Y_{-j}) = y_j p_j(X^*, y_j, Y_{-j}) - C_j(y_j). \quad (4.4)$$

Note that the existence of the multi-leader-follower games can be obtained under the following assumptions: $\forall \nu = 1, 2, \dots, K + F$,

- (A1) $p_\nu(\cdot)$ is twice continuously differentiable and decreasing,
- (A2) There holds $p_{\nu q}'(q) + q p_{\nu q}''(q) \leq 0$ for any $q \geq 0$,
- (A3) The cost functions $C_\nu(q_\nu)$, are twice continuously differentiable and their first and second derivatives are nonnegative for all $q_\nu \geq 0$.

Under Assumptions (A1)–(A3), one can easily show that $R^\nu(x, y)$ is concave, which guarantees the existence of multi-leader-follower games of the model. Additionally, we suppose that the multipliers of the leaders on the shared constraints are proportionable; that is,

$$\frac{\tilde{\lambda}_\nu}{\tilde{\lambda}_0} = \frac{\tilde{\mu}_\nu}{\tilde{\mu}_0} = \frac{\psi_\nu}{\psi_0} = \frac{\sigma_\nu}{\sigma_0} = \beta_\nu, \quad \nu = 1, \dots, K, \quad (4.5)$$

where $\beta_\nu > 0$.

If (4.4) is convex and satisfies a constraint qualification for each follower, then the condition that each follower chooses an optimal strategy is equivalent to the following KKT conditions:

$$0 \leq y_j \perp -\nabla_{y_j} R_j(X, y_j, Y_{-j}) \geq 0, \quad j = 1, \dots, F.$$

The aim of each leader i , $i = 1, \dots, K$, is to choose a strategy x_i that solves the following MPEC:

$$\begin{aligned} \min_{x_i \in \mathcal{X}_i} & -R_i(x_i, X_{-i}, Y) \\ \text{s.t.} & g_i(x_i) \leq 0, g(x, y) \leq 0, \\ & 0 \leq y_j \perp -\nabla_{y_j} R_j(X, y_j, Y_{-j}) \geq 0, \quad j = 1, \dots, F. \end{aligned}$$

4.2 Preliminary numerical results

Consider the above normalized stationary points of relatively separable EPEC with two leaders and one follower and set the data as follows:

- The inverse demand functions are given by

$$\begin{aligned} \text{Leader1} \quad P_1(x_1, x_2, y) & := a_1 - b_1(x_1 + x_2 + \frac{y}{x_1}), \\ \text{Leader2} \quad P_2(x_1, x_2, y) & := a_2 - b_2(x_1 + x_2 + \frac{y}{x_2}), \\ \text{follower} \quad P_3(x_1, x_2, y) & := a_3 - b_3(x_1 + x_2 + y). \end{aligned}$$

When $x_1 \in (0, \sqrt{y})$ is bigger, P_1 is bigger. On the other hand, when $x_1 \in (\sqrt{y}, +\infty)$ is bigger, P_1 is smaller. Actually, the supply quantities of leaders are far more than the follower's and they are in the same order of magnitude. Consequently, x_1 is in $(\sqrt{y}, +\infty)$ in general. when there is not enough demand for its product, the price goes down. While the product is in short supply relative to the demand, the price will be bid up. The effective level of various variables was different.

- The cost functions are given by

$$\begin{aligned} \text{Leader1} \quad C_1(x_1) & := c_1 x_1, \\ \text{Leader2} \quad C_2(x_2) & := c_2 x_2, \\ \text{follower} \quad C_3(y) & := c_3 y. \end{aligned}$$

- The constraint functions of the leaders are given by

$$\begin{aligned} g_1(x_1) &:= x_1 - d_1, \\ g_2(x_2) &:= x_2 - d_2, \\ g(x_1, x_2, y) &:= x_1 + x_2 + y - d_3. \end{aligned}$$

where $a_i \geq 0, b_i \geq 0, c_i \geq 0, i = 1, 2, 3; x_1 \geq 0, x_2 \geq 0, y \geq 0$.

Then the problem can be written as follows:

$$\begin{array}{ll} \min_{x_1, y} & b_1 x_1^2 - (a_1 - c_1)x_1 + b_1 x_1 x_2 + b_1 y \\ \text{s.t} & x_1 - d_1 \leq 0, \\ & x_1 + x_2 + y - d_3 \leq 0, \\ & 0 \leq y \perp 2b_3 y - (a_3 - c_3) \geq 0. \end{array} \quad \begin{array}{ll} \min_{x_2, y} & b_2 x_2^2 - (a_2 - c_2)x_2 + b_2 x_1 x_2 + b_2 y \\ \text{s.t} & x_2 - d_2 \leq 0, \\ & x_1 + x_2 + y - d_3 \leq 0, \\ & 0 \leq y \perp 2b_3 y - (a_3 - c_3) \geq 0. \end{array} \quad (4.6)$$

By Theorem 3.1, the normalized C-(M-,S-) stationary point of the EPEC (4.6) is the standard C-(M-,S-) stationary point of the following MPEC:

$$\begin{array}{ll} \min_{x, y} & x_1^2 - \frac{a_1 - c_1}{b_1} x_1 + x_2^2 - \frac{a_2 - c_2}{b_2} x_2 + x_1 x_2 \\ \text{s.t} & x_1 - d_1 \leq 0, \\ & x_2 - d_2 \leq 0, \\ & x_1 + x_2 + y - d_3 \leq 0, \\ & 0 \leq y \perp 2b_3 y - (a_3 - c_3) \geq 0, \end{array} \quad (4.7)$$

Conversely, if (x^*, y^*) is the C-(M-,S-) stationary point of (4.7) then (x^*, y^*) is the normalized C-(M-,S-) stationary point of the EPEC (4.6).

We set the parameters by

$$a_1 = 68, b_1 = 4, a_2 = 78, b_2 = 5, a_3 = 60, b_3 = 3, c_1 = 22, c_2 = 26, c_3 = 28.$$

We formulate the solving normalized stationary points of relatively separable EPEC to a standard MPEC and solve the MPEC in Matlab R2010a. The computational results are

$$(x_1, x_2, y) = (4.3667, 3.2667, 1.5167).$$

The results reveal that the proposed methods were able to solve normalized stationary points of relatively separable EPEC successfully.

Although the example is basic, it is thought to be competent to show the difficulties in this field of research and efficiency of our method.

5 Conclusions

We introduce relatively separable EPECs and solve its normalized stationary points, that result in a standard MPEC. In this approach, the special equilibrium problem with equilibrium constraints is solved by a single optimization problem, unlike traditional approach that solve a sequence of related optimization problems. Additionally, we provide numerical results and application demonstrating that our new approach.

6 Declarations

We would like to submit the enclosed manuscript entitled “Solving a class of equilibrium problems with equilibrium constraints”, which we wish to be considered for publication in “Soft Computing”. No conflict of interest exists in the submission of this manuscript, and manuscript is approved by author for publication. We would like to declare that the work described was original research that has not been published previously, and not under consideration for publication elsewhere, in whole or in part. The conception and design of the study owe to Pei-yu li, while the main contributions of Ke-wei Ding are analysis and interpretation of data.

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