Existence and stability of solutions for linear and nonlinear damping of q−fractional Duffing-Rayleigh problem

Mohamed Houas
FIMA, UDBKM, Khemis Miliana University
Mohammad Esmael Samei (mesamei@basu.ac.ir)
Bu-Ali Sina University

Research Article

Keywords: Fractional q-calculus, Uniqueness, Existence, Duffing-Rayleigh equation, Ulam- Hyers stability

Posted Date: July 15th, 2022

DOI: https://doi.org/10.21203/rs.3.rs-1843587/v1

Existence and stability of solutions for linear and nonlinear damping of q−fractional Duffing-Rayleigh problem

Mohamed Houas[1], **Mohammad Esmael Samei**[2,*]

[1] Laboratory, FIMA, UDBKM, Khemis Miliana University, Algeria

[2] Department of Mathematics, Faculty of Basic Science, Bu-Ali Sina University, Hamedan, Iran

Abstract

In this current paper, by using $q-$fractional calculus, we study the Duffing-Rayleigh type problem with sequential fractional $q-$derivative of the Caputo type. We investigate the existence and uniqueness of solutions by applying some classical fixed pointtheorems. Also we define and study the Ulam-Hyers and the Ulam-Hyers-Rassias stabilities of solutions for our problem. An example is presented to illustrate the main results.

(2010) Mathematics Subject Classifications: 26A33, 39B72, 34C45.
Keywords: Fractional $q-$calculus; Uniqueness; Existence; Duffing-Rayleigh equation; Ulam-Hyers stability.

1 Introduction

The study of nonlinear differential equations involving $q-$calculus has gained intensive interest in the last years, these types of equations have several applications in diverse fields and thus have evolved into multidisciplinary subjects, see for instance [7, 8, 16]. Also nonlinear differential equations with fractional $q-$calculus have been investigated by several scholars, see for example [9, 13, 20, 22, 23, 29] and the references cited therein. Recently, many scientific researchers, have studied the uniqueness, existence and Hyers-Ulam stability (H-U$\mathbb{S}$) of solutions for some differential equations with fractional quantum calculus, see the papers [10, 18, 25]. Considerable attention has been given to the study of the existence and uniqueness of solutions of sequential $q-$differential equations, see the research works [1–3, 15, 16].

So, in this current paper, we study the existence and U$\mathbb{S}$ of solutions for sequential $q-$fractional Duffing-Rayleigh problem ($q-\mathbb{F}$D-R$\mathbb{P}$). The Duffing-Rayleigh problem (D-R$\mathbb{P}$) has played a very important role and has many applications in applied sciences, for more details, see [6, 11, 14, 26]. The fractional version of the D-R$\mathbb{P}$ is given by

$$D^2z+\left(\lambda_1+\lambda_2\left(D^1z\right)^2\right)D^1z+\xi D^\eta s\left(\tau\right)+f\left(s\right)=\varphi\left(\tau\right),$$

* Corresponding author. *E-mail addresses:* m.houas.st@univ-dbkm.dz (M. Houas), mesamei@basu.ac.ir, mesamei@gmail.com (M.E. Samei).

where $\lambda_i, i = 1, 2$ denote coefficients of linear and nonlinear dampings, D^η is the fractional derivative of order η,

$$f(s) = w_0^2 s + ks^3,$$

is the unequivocally nonlinear function representing restoring force, with w_0 being the frequency of degenerate linear system and k the concentrated of nonlinearity; $\varphi(\tau)$ is Gaussian white noise with intensity $2D$ [19].

Nonlinear D-R oscillators involving the fractional calculus have recently been studied by several scholars [5, 12, 17, 21, 30]. Xiao *et al.* have used the collocation method to study the Rayleigh oscillator with a small fractional damping problem

$$D^2 s(\tau) - \epsilon\Big[1 - (D^\eta s(\tau))^2\Big] D^\eta s(\tau) + s(\tau) = 0, \qquad 0 < \epsilon \leq 1,$$

under conditions $s(0) = b_1, D^1 s(0) = b_2$, where $0 < \eta < 1$ and D^η is the Caputo fractional derivative [28]. Also Zhang *et al.* in [30], studied the fractional Caputo modified D-R oscillator given by $D^\eta s(\tau) = u$, $0 < \eta \leq 1$, and

$$D^\eta u(\tau) = c_1 s - c_2 s^3 + \epsilon\Big[\upsilon u(1 - |u|) + \psi \cos(\kappa\tau)\Big], \qquad \tau \in \Omega := [0, T],$$

where $\upsilon, c_i, i = 1, 2$ are respectively, nonlinear damping, linear and nonlinear restoring parameters, ψ and κ are respectively, amplitude and frequency of the external force and ϵ is a small nonnegative constant. Recently in [24], Selvam *et al.* discussed the H-U$\mathbb{S}$ of solutions for discrete forced $\mathbb{F}$D-R$\mathbb{E}$ described by

$$D^\eta s(\tau) + \gamma s(\tau + \eta) + \zeta (s(\tau + \eta))^3 + \varphi(\tau + \eta), \qquad \tau \in \Omega \cap \mathbb{N}_{2-\eta},$$

with $s(0) = b_1$, $D^1 s(0) = b_2$, where D^η $(1 < \eta \leq 2)$ is the Caputo derivative of order η, γ and ζ are used to control the stiffness, $\varphi : \mathbb{Q} \to \mathbb{R}$ is the driving force with $b_i \in \mathbb{R}^+, i = 1, 2$.

In this current paper, we discuss the existence, uniqueness and U$\mathbb{S}$ of solutions for sequential $q - \mathbb{F}$D-R$\mathbb{P}$, $(0 < q < 1)$ given by

$$\begin{cases} D_q^\eta \left(D_q^\mu \left(D_q^\vartheta + \theta\right)\right) s(\tau) = p(\tau) - \phi(\tau, s(\tau)) \\ \quad -\lambda\varphi\left(\tau, s(\tau), D_q^\alpha s(\tau)\right) \\ \quad -\delta\psi\left(\tau, s(\tau), D_q^\beta s(\tau)\right), & \tau \in \Omega := [0, 1], \\ s(0) = \Lambda_1, \\ \left(D_q^\vartheta + \theta\right) s(1) = \Lambda_2, \\ D_q^\mu \left(\left(D_q^\vartheta + \theta\right)\right) s(\omega) = \Lambda_3, & \Lambda_i \in \mathbb{R}, i = 1, 2, 3, \end{cases} \tag{1.1}$$

where θ, λ, $\delta \in \mathbb{R}^+$, $0 < \omega < 1$, $0 < \eta$, μ, $\vartheta < 1$, $\alpha < \vartheta$, $\beta < \vartheta$ and $D_q^\varkappa, \varkappa \in \{\eta, \mu, \vartheta, \alpha, \beta\}$ is the Caputo fractional $q-$derivative of order $\varkappa$, $p : \Omega \to \mathbb{R}, \phi : \Omega \times \mathbb{R} \to \mathbb{R}$ and $\varphi, \psi : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ are given continuous functions.

In Section 2, we recall some essential definition of fractional quantum calculus. Section 3 contains our main results in this work, while an example is presented to support the validity of our obtained results. stability results are extensively discussed in Section 4. An illustrative example with some needed algorithms for the problem are given in Section 5. Finally, in Section 6, conclusion are presented.

2 Preliminaries regarding $q-$fractional calculus

The operator $D_q^{\varkappa}$ is the fractional $q-$derivative of the Caputo type of order $\varkappa$ is given by

$$D_q^{\varkappa}\left[s\left(\tau\right)\right]=I_q^{[\varkappa]-\varkappa}\left[D_q^{[\varkappa]}\left[s\left(\tau\right)\right]\right],\qquad \varkappa>0,$$

and $D_q^0\left[s\left(\tau\right)\right]=s\left(\tau\right)$, where $[\varkappa]$ is the smallest integer greater than or equal to $\varkappa$. The fractional $q-$integral of the Riemann-Liouville type [4, 18], defined by

$$I_q^{\varkappa}\left[s\left(\tau\right)\right]=\frac{1}{\Gamma_q\left(\varkappa\right)}\int_0^{\tau}\left(\tau-qr\right)^{(\varkappa-1)}s\left(r\right)\,\mathrm{d}_qr,\qquad \kappa>0,$$

and $I_q^0\left[s\left(\tau\right)\right]=s\left(\tau\right)$, where the $q-$gamma function is defined by

$$\Gamma_q\left(\varkappa\right)=\left(1-q\right)^{(\varkappa-1)}\left(1-q\right)^{1-\kappa},\qquad 0<q<1,$$

and satisfies $\Gamma_q\left(\varkappa+1\right)=[\varkappa]_q\Gamma_q\left(\varkappa\right)$, such that $[\varkappa]_q=(1-q^{\varkappa})(1-q)^{-1}$,

$$\left(1-q\right)^{(n)}=\begin{cases}1, & n=0,\\ \left(1-q\right)^{(n)}=\prod_{l=0}^{n-1}\left(1-q^{l+1}\right), & n\in\mathbb{N}.\end{cases}$$

Algorithm 1 shows MATLAB function to obtain $q-$gamma function. By using Algorithm 2, we can calculate this type $q-$integral. We recall the the following lemmas [4, 18].

Lemma 2.1. *Let $\varkappa$, $\sigma\geq 0$ and s be a function defined in Ω. Then*

$$I_q^{\varkappa}\left[I_q^{\sigma}\left[s\left(\tau\right)\right]\right]=I_q^{\varkappa+\sigma}\left[s\left(\tau\right)\right]\ \&\ D_q^{\varkappa}I_q^{\varkappa}\left[s\left(\tau\right)\right]=s\left(\tau\right).$$

Lemma 2.2. *Let $\varkappa\in\mathbb{R}^+\backslash\mathbb{N}$. Then, the following equality is valid*

$$I_q^{\varkappa}\left[D_q^{\varkappa}\left[s\left(\tau\right)\right]\right]=s\left(\iota\right)-\sum_{m=0}^{n-1}\frac{\tau^m}{\Gamma_q\left(m+1\right)}D_q^m s\left(0\right),$$

such that n is the smallest integer greater than or equal to $\varkappa$.

Lemma 2.3. *For $\varkappa\in\mathbb{R}_+$ and $\sigma>-1$, we have*

$$I_q^{\varkappa}\left[\tau^{(\sigma)}\right]=\frac{\Gamma_q\left(\sigma+1\right)}{\Gamma_q\left(\varkappa+\sigma+1\right)}\tau^{(\varkappa+\sigma)}.$$

If $\sigma=0$, we can obtain $I_q^{\varkappa}\left[1\right]=\frac{1}{\Gamma_q(\varkappa+1)}\tau^{(\varkappa)}$.

Lemma 2.4. *[27] Assume that W is a completely continuous self operator on a Banach space F and the set*

$$\Big\{s \in F \,:\, s = \varrho W(s)\,,\ 0 < \varrho < 1\Big\},$$

is bounded. Then, there exists $s^{\circ} \in W$ such that $W(s^{\circ}) = s^{\circ}$.

In order to study the problem (1.1), we need the following space

$$E = \Big\{s \,:\, s,\ D_q^{\alpha} s \,\&\, D_q^{\beta} s \in C(\Omega, \mathbb{R})\Big\},$$

endowed with the norm

$$\|s\|_E = \|s\| + \left\|D_q^{\alpha} s\right\| + \left\|D_q^{\beta} s\right\| = \sup_{\tau\in\Omega} |s(\tau)| + \sup_{\tau\in\Omega} \left|D_q^{\alpha} s(\tau)\right| + \sup_{\tau\in\Omega} \left|D_q^{\beta} s(\tau)\right|.$$

Then it is well known that $(E, \|.\|_E)$ is a Banach space.

3 Existence of solutions for $q - \mathbb{F}$D-R$\mathbb{P}$

In this part, we demonstrate the existence and uniqueness of solution of problem (1.1).

Lemma 3.1. *Suppose that $l \in C(\Omega, \mathbb{R})$. Then the $q-$fractional problem*

$$\begin{cases} D_q^{\eta}\left(D_q^{\mu}\left(D_q^{\vartheta} + \theta\right)\right) s(\tau) = l(\tau)\,, \quad \tau \in \Omega, \\ s(0) = \Lambda_1, \\ \left(D_q^{\vartheta} + \theta\right) s(1) = \Lambda_2, \\ D_q^{\mu}\left(\left(D_q^{\vartheta} + \theta\right)\right) s(\omega) = \Lambda_3, \qquad \Lambda_i \in \mathbb{R},\ i = 1, 3, \end{cases} \tag{3.1}$$

where $0 < \eta,\ \mu,\ \vartheta \leq 1,\ \theta > 0,\ 0 < \omega < 1,$ admits the following solution

$$\begin{aligned} s(\tau) = & \int_0^{\tau} \frac{(\tau - q\xi)^{(\eta+\mu+\vartheta-1)}}{\Gamma_q(\eta+\mu+\vartheta)} l(\xi)\ \mathrm{d}_q\xi \\ & - \int_0^{\tau} \frac{\theta(\tau - q\xi)^{(\vartheta-1)}}{\Gamma_q(\vartheta)} l(\xi)\ \mathrm{d}_q\xi \\ & - \int_0^{\omega} \frac{\tau^{\mu+\vartheta}(\omega - q\xi)^{(\eta-1)}\, l(\xi)}{\Gamma_q(\mu+\vartheta+1)\,\Gamma_q(\eta)}\, \mathrm{d}_q\xi \\ & - \int_0^{1} \frac{\tau^{\vartheta}(1 - q\xi)^{(\eta+\mu-1)}}{\Gamma_q(\vartheta+1)\,\Gamma_q(\eta+\mu)} l(\xi)\ \mathrm{d}_q\xi \\ & + \int_0^{\omega} \frac{\tau^{\vartheta}(\omega - q\xi)^{(\eta-1)}}{\Gamma_q(\vartheta+1)\,\Gamma_q(\mu+1)\,\Gamma_q(\eta)} l(\xi)\ \mathrm{d}_q\xi \\ & + \frac{\Lambda_3}{\Gamma_q(\mu+\vartheta+1)}\tau^{\mu+\vartheta} + \frac{\Gamma_q(\mu+1)\,\Lambda_2 - \Lambda_3}{\Gamma_q(\vartheta+1)\,\Gamma_q(\mu+1)}\tau^{\vartheta} + \Lambda_1, \end{aligned} \tag{3.2}$$

Proof. Applying Lemma 2.2, we can write

$$D_q^{\mu}\left(D_q^{\vartheta}+\theta\right)s\left(\tau\right)=I_q^{\eta}\left[l\left(\tau\right)\right]+e_0,\qquad e_0\in\mathbb{R}. \tag{3.3}$$

Now, using the operator I_q^{μ}, we get

$$\left(D_q^{\vartheta}+\theta\right)s\left(\tau\right)=I_q^{\eta+\mu}\left[l\left(\tau\right)\right]+e_0I_q^{\mu}\left[1\right]+e_1,\qquad e_i\in\mathbb{R},i=0,1. \tag{3.4}$$

It follows that

$$s\left(\tau\right)=I_q^{\eta+\mu+\vartheta}\left[l\left(\tau\right)\right]-\theta I_q^{\vartheta}\left[s\left(\tau\right)\right]+e_0I_q^{\mu+\vartheta}\left[1\right]+e_1I_q^{\vartheta}\left[1\right]+e_2, \tag{3.5}$$

where $e_i\in\mathbb{R}$, $(i=0,1,2)$. By using the boundary conditions, we get

$$\begin{aligned}
e_0&=\Lambda_3-I_q^{\eta}\left[l\left(\omega\right)\right],\\
e_1&=\Lambda_2-I_q^{\mu+\vartheta}\left[l\left(1\right)\right]-\frac{\Lambda_3}{\Gamma_q\left(\mu+1\right)}+\frac{1}{\Gamma_q\left(\mu+1\right)}I_q^{\eta}\left[l\left(\omega\right)\right],\\
e_2&=\Lambda_1.
\end{aligned}$$

Hence, we obtain (3.2). □

Using Lemma 3.1, we define an operator $W:E\to E$ as follows

$$\begin{aligned}
&Ws\left(\tau\right) \qquad (3.6)\\
&=\int_0^{\tau}\frac{\left(\tau-q\xi\right)^{(\eta+\mu+\vartheta-1)}}{\Gamma_q\left(\eta+\mu+\vartheta\right)}\Big(p\left(\xi\right)-\phi_s^*\left(\xi\right)-\lambda\varphi_s^*\left(\xi\right)-\delta\psi_s^*\left(\xi\right)\Big)\,\mathrm{d}_q\xi\\
&-\int_0^{\tau}\frac{\theta\left(\tau-q\xi\right)^{(\vartheta-1)}}{\Gamma_q\left(\vartheta\right)}s\left(r\right)\,\mathrm{d}_q\xi\\
&-\int_0^{\omega}\frac{\tau^{\mu+\vartheta}\left(\omega-q\xi\right)^{(\eta-1)}}{\Gamma_q\left(\mu+\vartheta+1\right)\Gamma_q\left(\eta\right)}\left(p\left(\xi\right)-\phi_s^*\left(r\right)-\lambda\varphi_s^*\left(\xi\right)-\delta\psi_s^*\left(\xi\right)\right)\,\mathrm{d}_q\xi\\
&-\int_0^{1}\frac{\tau^{\vartheta}\left(1-q\xi\right)^{(\eta+\mu-1)}}{\Gamma_q\left(\vartheta+1\right)\Gamma_q\left(\eta+\mu\right)}\left(p\left(\xi\right)-\phi_s^*\left(\xi\right)-\lambda\varphi_s^*\left(\xi\right)-\delta\psi_s^*\left(r\right)\right)\,\mathrm{d}_q\xi\\
&+\int_0^{\omega}\frac{\tau^{\vartheta}\left(\omega-q\xi\right)^{(\eta-1)}}{\Gamma_q\left(\vartheta+1\right)\Gamma_q\left(\mu+1\right)\Gamma_q\left(\eta\right)}\left(p\left(\xi\right)-\phi_s^*\left(\xi\right)-\lambda\varphi_s^*\left(\xi\right)-\delta\psi_s^*\left(\xi\right)\right)\,\mathrm{d}_q\xi\\
&+\frac{\Lambda_3\tau^{\mu+\vartheta}}{\Gamma_q\left(\mu+\vartheta+1\right)}+\frac{\Gamma_q\left(\mu+1\right)\Lambda_2-\Lambda_3\tau^{\vartheta}}{\Gamma_q\left(\mu+1\right)\Gamma_q\left(\vartheta+1\right)}+\Lambda_1.
\end{aligned}$$

Also, we have

$$D_q^{\nu}\left(Ws\right)\left(\tau\right)=\int_0^{\tau}\frac{\left(\tau-q\xi\right)^{(-\nu)}}{\Gamma_q\left(1-\nu\right)}\left(D_qWs\right)\left(\xi\right)\,\mathrm{d}_q\xi,\qquad \nu=\alpha,\beta, \tag{3.7}$$

where

$$
\begin{aligned}
&D_q\left(Ws\right)(\tau)\\
&=\int_0^{\tau}\frac{(\tau-q\xi)^{(\eta+\mu+\vartheta-2)}}{\Gamma_q\left(\eta+\mu+\vartheta-1\right)}\left(p\left(\xi\right)-\phi_s^*\left(\xi\right)-\lambda\varphi_s^*\left(\xi\right)-\delta\psi_s^*\left(\xi\right)\right)\mathrm{d}_q\xi\\
&-\int_0^{\tau}\frac{\theta\left(\tau-q\xi\right)^{(\vartheta-2)}}{\Gamma_q\left(\vartheta-1\right)}s\left(\xi\right)\mathrm{d}_q\xi\\
&-\int_0^{\omega}\frac{\left(\omega-q\xi\right)^{(\eta-1)}\left[\mu+\vartheta\right]_q\tau^{\mu+\vartheta-1}}{\Gamma_q\left(\eta\right)\Gamma_q\left(\mu+\vartheta+1\right)}\left(p\left(\xi\right)-\phi_s^*\left(\xi\right)-\lambda\varphi_s^*\left(\xi\right)-\delta\psi_s^*\left(\xi\right)\right)\mathrm{d}_q\xi\\
&-\int_0^{1}\frac{\left[\vartheta\right]_q\tau^{\vartheta-1}\left(1-q\xi\right)^{(\eta+\mu-1)}}{\Gamma_q\left(\vartheta+1\right)\Gamma_q\left(\eta+\mu\right)}\left(p\left(\xi\right)-\phi_s^*\left(\xi\right)-\lambda\varphi_s^*\left(\xi\right)-\delta\psi_s^*\left(r\right)\right)\mathrm{d}_q\xi\\
&+\int_0^{\omega}\frac{\left[\vartheta\right]_q\tau^{\vartheta-1}\left(\omega-q\xi\right)^{(\eta-1)}}{\Gamma_q\left(\vartheta+1\right)\Gamma_q\left(\mu+1\right)\Gamma_q\left(\eta\right)}\left(p\left(\xi\right)-\phi_s^*\left(\xi\right)-\lambda\varphi_s^*\left(\xi\right)-\delta\psi_s^*\left(\xi\right)\right)\mathrm{d}_q\xi\\
&+\frac{\Lambda_3\left[\mu+\vartheta\right]_q\tau^{\mu+\vartheta-1}}{\Gamma_q\left(\mu+\vartheta+1\right)}+\frac{\left(\Gamma_q\left(\mu+1\right)\Lambda_2-\Lambda_3\right)\left[\vartheta\right]_q\tau^{\vartheta-1}}{\Gamma_q\left(\vartheta+1\right)\Gamma_q\left(\mu+1\right)}.
\end{aligned}
$$

Before stating and proving the main results, we impose the following hypotheses.

(H_1) : The functions φ and ψ are continuous over $\Omega\times\mathbb{R}^2$, ϕ is continuous over $\Omega\times\mathbb{R}$ and p is continuous over Ω.

(H_2) : There exists constant $a_j>0, j=1,2,3$ such that for all $\tau\in\Omega$ and $s_i,t_i\in\mathbb{R}$, $(i=1,2)$, we have

$$
\begin{aligned}
\left|\varphi\left(\tau,s_1,s_2\right)-\varphi\left(\tau,t_1,t_2\right)\right|&\leq a_1\left(\left|s_1-t_1\right|+\left|s_2-t_2\right|\right),\\
\left|\psi\left(\tau,s_1,s_2\right)-\psi\left(\tau,t_1,t_2\right)\right|&\leq a_2\left(\left|s_1-t_1\right|+\left|s_2-t_2\right|\right),
\end{aligned}
$$

and $\left|\phi\left(\tau,s_1\right)-\psi\left(\tau,t_1\right)\right|\leq a_3\left|s_1-t_1\right|$.

(H_3) : There exist a constants $\Delta_i>0, i=1,2,3,4$ such that for all $\tau\in\Omega$ and $s,t\in\mathbb{R}$, we have

$$
\left|\varphi\left(\tau,s,t\right)\right|\leq\Delta_1,\quad\left|\psi\left(\tau,s,t\right)\right|\leq\Delta_2,\quad\left|\phi\left(\tau,s\right)\right|\leq\Delta_3,
$$

and $\left|p\left(\tau\right)\right|\leq\Delta_4$.

Theorem 3.2. *Assume that (H_1) and (H_2) hold and that*

$$
\lambda a_1+\delta a_2+a_3<\left(1-\theta\Phi\right)\Theta^{-1},\tag{3.8}
$$

where

$$
\begin{aligned}
\Phi&:=\frac{1}{\Gamma_q\left(\vartheta+1\right)}+\frac{1}{\Gamma_q\left(2-\alpha\right)\Gamma_q\left(\vartheta\right)}+\frac{1}{\Gamma_q\left(2-\beta\right)\Gamma_q\left(\vartheta\right)},\\
\Theta&:=\Pi+\frac{\Sigma}{\Gamma_q\left(2-\alpha\right)}+\frac{\Sigma}{\Gamma_q\left(2-\beta\right)},
\end{aligned}\tag{3.9}
$$

and

$$\Pi := \frac{1}{\Gamma_q(\eta+\mu+\vartheta+1)} + \frac{\omega^{(\eta)}}{\Gamma_q(\mu+\vartheta+1)\Gamma_q(\eta+1)} + \frac{1}{\Gamma_q(\vartheta+1)\Gamma_q(\eta+\mu+1)} + \frac{\omega^{(\eta)}}{\Gamma_q(\vartheta+1)\Gamma_q(\mu+1)\Gamma_q(\eta+1)}, \tag{3.10}$$

$$\Sigma := \frac{1}{\Gamma_q(\eta+\mu+\vartheta)} + \frac{[\mu+\vartheta]_q\,\omega^{(\eta)}}{\Gamma_q(\mu+\vartheta+1)\Gamma_q(\eta+1)} + \frac{[\vartheta]_q}{\Gamma_q(\vartheta+1)\Gamma_q(\eta+\mu+1)} + \frac{[\vartheta]_q\,\omega^{(\eta)}}{\Gamma_q(\vartheta+1)\Gamma_q(\mu+1)\Gamma_q(\eta+1)}.$$

Then, the $q-\mathbb{F}D$-$R\mathbb{P}$ (1.1) has a unique solution.

Proof. Let us define

$$\rho \geq \frac{\Theta(\lambda+\delta+2)\nabla+\Theta^*}{1-[\Theta(\lambda a_1+\delta a_2+a_3)+\theta\Phi]}, \tag{3.11}$$

here

$$\Theta^* := \Pi^* + \frac{\Sigma^*}{\Gamma_q(2-\alpha)} + \frac{\Sigma^*}{\Gamma_q(2-\beta)}, \tag{3.12}$$

$$\Pi^* := \frac{|\Lambda_3|}{\Gamma_q(\mu+\vartheta+1)} + \frac{\Gamma_q(\mu+1)|\Lambda_2|+|\Lambda_3|}{\Gamma_q(\vartheta+1)\Gamma_q(\mu+1)} + |\Lambda_1|,$$

$$\Sigma^* := \frac{|\Lambda_3|[\mu+\vartheta]_q}{\Gamma_q(\mu+\vartheta+1)} + \frac{(\Gamma_q(\mu+1)|\Lambda_2|+|\Lambda_3|)[\vartheta]_q}{\Gamma_q(\vartheta+1)\Gamma_q(\mu+1)}.$$

and $\nabla = \max\{\nabla_i, i=1,2,3,4\}$, here ∇_i are given by

$$\nabla_1 = \sup_{\tau\in\Omega}|\varphi(\tau,0,0)|, \quad \nabla_2 = \sup_{\tau\in\Omega}|\psi(\tau,0,0)|, \quad \nabla_3 = \sup_{\tau\in\Omega}|\phi(\tau,0)|,$$

and $\nabla_4 = \sup_{\tau\in\Omega}|p(\tau)|$. Then we show that $WB_\rho \subset B_\rho$, where

$$B_\rho = \Big\{ s\in E \,:\, \|s\|_E \leq \rho \Big\}.$$

Thanks to (H_2), we get

$$\begin{aligned} |\varphi_t^{*,\alpha}(\tau)| &= \left|\varphi\left(\tau, s(\tau), D_q^\alpha s(\tau)\right)\right| \\ &\leq \left|\varphi\left(\tau, s(\tau), D_q^\alpha s(\tau)\right) - \varphi(\tau,0,0)\right| + |\varphi(\tau,0,0)| \\ &\leq a_1\left(\|s\| + \left\|D_q^\alpha s\right\|\right) + \nabla_1 \\ &\leq a_1\|s\|_E + \nabla_1 \\ &\leq a_1\rho + \nabla_1, \end{aligned} \tag{3.13}$$

$$\begin{aligned} \left|\psi_s^{*,\beta}(\tau)\right| &= \left|\psi\left(\tau, s(\tau), D_q^\beta s(\tau)\right)\right| \\ &\leq \left|\psi\left(\tau, s(\tau), D_q^\beta s(\tau)\right) - \psi(\tau,0,0)\right| + |\psi(\tau,0,0)| \\ &\leq a_2\left(\|s\| + \left\|D_q^\beta s\right\|\right) + \nabla_2 \\ &\leq a_2\|s\|_E + \nabla_2 \\ &\leq a_2\rho + \nabla_2, \end{aligned}$$

and

$$\begin{aligned}|\phi_s^*(\tau)| &= |\phi(\tau,s)| \\ &\leq |\phi(\tau,s)-\phi(\tau,0)|+|\phi(\tau,0)| \\ &\leq a_3\|s\|+\nabla_3 \\ &\leq a_3\|s\|_E+\nabla_3 \\ &\leq a_3\rho+\nabla_3.\end{aligned} \tag{3.14}$$

Then, thanks to (3.13) and (3.14), we can write

$$\begin{aligned}\|W(s)\| \leq &\Big[(\lambda a_1+\delta a_2+a_3)\rho+(\lambda+\delta+2)\nabla\Big] \\ &\times\Bigg(\frac{1}{\Gamma_q(\eta+\mu+\vartheta+1)}+\frac{\omega^{(\eta)}}{\Gamma_q(\eta+\mu+1)\Gamma_q(\eta+1)} \\ &+\frac{1}{\Gamma_q(\vartheta+1)\Gamma_q(\eta+\mu+1)} \\ &+\frac{\omega^{(\eta)}}{\Gamma_q(\vartheta+1)\Gamma_q(\mu+1)\Gamma_q(\eta+1)}\Bigg) \\ &+\frac{\theta}{\Gamma_q(\vartheta+1)}+\frac{|\Lambda_3|}{\Gamma_q(\mu+\vartheta+1)} \\ &+\frac{\Gamma_q(\mu+1)|\Lambda_2|+|\Lambda_3|}{\Gamma_q(\vartheta+1)\Gamma_q(\mu+1)}+|\Lambda_1| \\ =&\left[\Pi(\lambda a_1+\delta a_2+a_3)+\frac{\theta}{\Gamma_q(\vartheta+1)}\right]\rho \\ &+\Pi(\lambda+\delta+2)\nabla+\Pi^*.\end{aligned}$$

Also, one can observe that

$$\begin{aligned}\|D_qW(s)\| \leq &\Big[(\lambda a_1+\delta a_2+a_3)\rho+(\lambda+\delta+2)\nabla\Big] \\ &\times\Bigg(\frac{1}{\Gamma_q(\eta+\mu+\vartheta)}+\frac{[\mu+\vartheta]_q\,\omega^{(\eta)}}{\Gamma_q(\mu+\vartheta+1)\Gamma_q(\eta+1)} \\ &+\frac{[\vartheta]_q}{\Gamma_q(\vartheta+1)\Gamma_q(\eta+\mu+1)} \\ &+\frac{[\vartheta]_q\,\omega^{(\eta)}}{\Gamma_q(\vartheta+1)\Gamma_q(\mu+1)\Gamma_q(\eta+1)}\Bigg) \\ &+\frac{\theta\rho}{\Gamma_q(\vartheta)}+\frac{|\Lambda_3|[\mu+\vartheta]_q}{\Gamma_q(\mu+\vartheta+1)} \\ &+\frac{(\Gamma_q(\mu+1)|\Lambda_2|+|\Lambda_3|)[\vartheta]_q}{\Gamma_q(\vartheta+1)\Gamma_q(\mu+1)} \\ =&\left(\Sigma(\lambda a_1+\delta a_2+a_3)+\frac{\theta}{\Gamma_q(\vartheta)}\right)\rho \\ &+\Sigma(\lambda+\delta+2)\nabla+\Sigma^*.\end{aligned}$$

Using (3.7), we can write

$$\left\|D_q^{\alpha}W(s)\right\| \leq \frac{1}{\Gamma_q(2-\alpha)}\Bigg[\left(\Sigma(\lambda a_1+\delta a_2+a_3)+\frac{\theta}{\Gamma_q(\vartheta)}\right)\rho + \Sigma(\lambda+\delta+2)\nabla+\Sigma^*\Bigg],$$

and

$$\left\|D_q^{\beta}W(s)\right\| \leq \frac{1}{\Gamma_q(2-\beta)}\Bigg[\left(\Sigma(\lambda a_1+\delta a_2+a_3)+\frac{\theta}{\Gamma_q(\vartheta)}\right)\rho + \Sigma(\lambda+\delta+2)\nabla+\Sigma^*\Bigg].$$

In consequence, we get

$$\begin{aligned}
\|W(s)\|_W &= \|W(s)\| + \left\|D_q^{\beta}W(s)\right\| + \left\|D_q^{\beta}W(s)\right\| \\
&\leq \left[\left(\Pi+\frac{\Sigma}{\Gamma_q(2-\alpha)}+\frac{\Sigma}{\Gamma_q(2-\beta)}\right)(\lambda a_1+\delta a_2+a_3)+\theta\Phi\right]\rho \\
&\quad + \left(\Pi+\frac{\Sigma}{\Gamma_q(2-\alpha)}+\frac{\Sigma}{\Gamma_q(2-\beta)}\right)(\lambda+\delta+2)\nabla \\
&\quad + \left(\Pi^*+\frac{\Sigma^*}{\Gamma_q(2-\alpha)}+\frac{\Sigma^*}{\Gamma_q(2-\beta)}\right) \\
&= [\Theta(\lambda a_1+\delta a_2+a_3)+\theta\Phi]\rho+\Theta(\lambda+\delta+2)\nabla+\Theta^* \leq \rho,
\end{aligned}$$

which means that $WB_\rho \subset B_\rho$. For $s,t \in B_\rho$ and by (H_2), we have

$$\begin{aligned}
&|Ws(\tau)-Wt(\tau)| \\
&\quad = \int_0^{\tau}\frac{(\tau-q\xi)^{(\eta+\mu+\vartheta-1)}}{\Gamma_q(\eta+\mu+\vartheta)}\Big(|\phi_s^*(\xi)-\phi_t^*(\xi)| \\
&\qquad + \lambda|\varphi_s^*(\xi)-\varphi_s^*(\xi)|+|\delta\psi_s^*(\xi)-\delta\psi_t^*(\xi)|\,\mathrm{d}_q\xi\Big) \\
&\qquad + \int_0^{\tau}\frac{\theta(\tau-q\xi)^{(\vartheta-1)}}{\Gamma_q(\vartheta)}|s(\xi)-t(\xi)|\,\mathrm{d}_q\xi \\
&\qquad + \int_0^{\omega}\frac{\tau^{\mu+\vartheta}(\omega-q\xi)^{(\eta-1)}}{\Gamma_q(\mu+\vartheta+1)\Gamma_q(\eta)}\Big(|\phi_s^*(\xi)-\phi_t^*(\xi)| \\
&\qquad + \lambda|\varphi_s^*(\xi)-\varphi_s^*(\xi)|+|\delta\psi_s^*(\xi)-\delta\psi_t^*(\xi)|\Big)\,\mathrm{d}_q\xi \\
&\qquad + \int_0^{1}\frac{\tau^{\vartheta}(1-q\xi)^{(\eta+\mu-1)}}{\Gamma_q(\vartheta+1)\Gamma_q(\eta+\mu)}\Big(|\phi_s^*(\xi)-\phi_t^*(\xi)| \\
&\qquad + \lambda|\varphi_s^*(\xi)-\varphi_s^*(\xi)|+|\delta\psi_s^*(\xi)-\delta\psi_t^*(\xi)|\Big)\,\mathrm{d}_q\xi
\end{aligned}$$

$$+\int_0^{\omega}\frac{\tau^{\vartheta}\left(\omega-q\xi\right)^{(\eta-1)}}{\Gamma_q\left(\vartheta+1\right)\Gamma_q\left(\mu+1\right)\Gamma_q\left(\eta\right)}\Big(\left|\phi_s^*\left(\xi\right)-\phi_t^*\left(\xi\right)\right|$$
$$+\lambda\left|\varphi_s^*\left(\xi\right)-\varphi_s^*\left(\xi\right)\right|+\left|\delta\psi_s^*\left(\xi\right)-\delta\psi_t^*\left(\xi\right)\right|\Big)\,\mathrm{d}_q\xi.$$

So, we obtain

$$\left\|W\left(s\right)-W\left(t\right)\right\|\leq\left[\Pi\left(\lambda a_1+\delta a_2+a_3\right)+\frac{\theta}{\Gamma_q\left(\vartheta+1\right)}\right]\left\|s-t\right\|_E.$$

On the other hand, we have

$$\left\|D_qW\left(s\right)-D_qW\left(t\right)\right\|\leq\left[\Sigma\left(\lambda a_1+\delta a_2+a_3\right)+\frac{\theta}{\Gamma_q\left(\vartheta\right)}\right]\left\|s-t\right\|_E.$$

By (3.7), we get

$$\begin{aligned}&\left\|D_q^{\alpha}W\left(s\right)-D_q^{\alpha}W\left(t\right)\right\|\\&\qquad\leq\frac{1}{\Gamma_q\left(2-\alpha\right)}\left[\Sigma\left(\lambda a_1+\delta a_2+a_3\right)+\frac{\theta}{\Gamma_q\left(\vartheta\right)}\right]\left\|s-t\right\|_E,\end{aligned}$$

$$\begin{aligned}&\left\|D_q^{\beta}W\left(s\right)-D_q^{\beta}W\left(t\right)\right\|\\&\qquad\leq\frac{1}{\Gamma_q\left(2-\beta\right)}\left[\Sigma\left(\lambda a_1+\delta a_2+a_3\right)+\frac{\theta}{\Gamma_q\left(\vartheta\right)}\right]\left\|s-t\right\|_E.\end{aligned}$$

Consequently, we obtain

$$\left\|W\left(s\right)-W\left(t\right)\right\|_E\leq\left[\Theta\left(\lambda a_1+\delta a_2+a_3\right)+\theta\Phi\right]\left\|s-t\right\|_E.$$

By the Banach fixed point theorem, W has a fixed point which is a solution of (1.1). □

Now, we prove the existence of solutions for the $q-\mathbb{F}$D-R$\mathbb{P}$ (1.1) by applying lemma 2.4.

Theorem 3.3. *Assume that conditions* $(H_i)_{i=1,3}$ *hold. If* $\theta<\Phi^{-1}$, *where* Φ *is given by* (3.7), *then the* $q-\mathbb{F}$D-R$\mathbb{P}$ (1.1) *has at least one solution.*

Proof. The operator W is continuous in view of the continuity of functions p,ϕ,φ and ψ, Hypothesis (H_1). Now, we demonstrate that the operator W is completely continuous. Firstly, we prove that W maps bounded sets of B into bounded sets of B. Let us put

$$B_{\varepsilon}=\Big\{s\in E\,:\,\left\|s\right\|_E\leq\varepsilon,\varepsilon>0\Big\}.$$

Then for all $s \in B_\varepsilon$ and thanks to (H_3) and (3.7), we can write

$$\begin{aligned}\|W(s)\| \leq & \left(\frac{1}{\Gamma_q(\eta+\mu+\vartheta+1)}+\frac{\omega^{(\eta)}}{\Gamma_q(\mu+\vartheta+1)\Gamma_q(\eta+1)}\right.\\ & \left.+\frac{1}{\Gamma_q(\vartheta+1)\Gamma_q(\eta+\mu+1)}+\frac{\omega^{(\eta)}}{\Gamma_q(\vartheta+1)\Gamma_q(\mu+1)\Gamma_q(\eta+1)}\right)\\ & \times\left(\lambda\Delta_1+\delta\Delta_2+\sum_{i=3}^{4}\Delta_i\right)+\frac{\varepsilon\theta}{\Gamma_q(\vartheta+1)}+\frac{|\Lambda_3|}{\Gamma_q(\mu+\vartheta+1)}\\ & +\frac{\Gamma_q(\mu+1)|\Lambda_2|+|\Lambda_3|}{\Gamma_q(\vartheta+1)\Gamma_q(\mu+1)}+|\Lambda_1|\\ = & \,\Pi\left(\lambda\Delta_1+\delta\Delta_2+\sum_{i=3}^{4}\Delta_i\right)+\frac{\varepsilon\theta}{\Gamma_q(\vartheta+1)}+\Pi^*.\end{aligned}$$

On the other hand, for any $s \in B_\varepsilon$, we get

$$\|D_qW(s)\| \leq \Sigma\left(\lambda\Delta_1+\delta\Delta_2+\sum_{i=3}^{4}\Delta_i\right)+\frac{\varepsilon\theta}{\Gamma_q(\vartheta)}+\Sigma^*.$$

Thanks to (3.7), we have

$$\left\|D_q^\alpha W(s)\right\| \leq \frac{1}{\Gamma_q(2-\alpha)}\left[\Sigma\left(\lambda\Delta_1+\delta\Delta_2+\sum_{i=3}^{4}\Delta_i\right)+\frac{\varepsilon\theta}{\Gamma_q(\vartheta)}+\Sigma^*\right],$$

$$\left\|D_q^\beta W(s)\right\| \leq \frac{1}{\Gamma_q(2-\beta)}\left[\Sigma\left(\lambda\Delta_1+\delta\Delta_2+\sum_{i=3}^{4}\Delta_i\right)+\frac{\varepsilon\theta}{\Gamma_q(\vartheta)}+\Sigma^*\right].$$

From the above inequalities, it follows that the operator W is uniformly bounded. Next, we prove that W is equicontinuous. Let $s \in B_\varepsilon$ and $\tau_1, \tau_2 \in \Omega$, with $\tau_2 < \tau_1$. Then by (H_3), we have

$$\begin{aligned}&|Ws(\tau_1)-Ws(\tau_2)|\\ &\leq\left(\frac{\left[(\tau_1-\tau_2)^{\eta+\mu+\vartheta}+\left|\tau_1^{\eta+\mu+\vartheta}-\tau_2^{\eta+\mu+\vartheta}\right|\right]}{\Gamma_q(\eta+\mu+\vartheta+1)}\right.\\ &\quad+\frac{\omega^{(\eta)}\left|\tau_2^{\mu+\vartheta}-\tau_1^{\mu+\vartheta}\right|}{\Gamma_q(\mu+\vartheta+1)\Gamma_q(\eta+1)}+\frac{\left|\tau_2^{\vartheta}-\tau_1^{\vartheta}\right|}{\Gamma_q(\vartheta+1)\Gamma_q(\eta+\mu+1)}\\ &\quad\left.+\frac{\omega^{(\eta)}\left|\tau_1^{\vartheta}-\tau_2^{\vartheta}\right|}{\Gamma_q(\vartheta+1)\Gamma_q(\mu+1)\Gamma_q(\eta+1)}\right)\left(\lambda\Delta_1+\delta\Delta_2+\sum_{i=3}^{4}\Delta_i\right)\\ &\quad+\frac{\varepsilon\theta\left[(\tau_2-\tau_1)^{\vartheta}+\left|\tau_2^{\vartheta}-\tau_1^{\vartheta}\right|\right]}{\Gamma_q(\vartheta+1)}+\frac{|\Lambda_3|\left|\tau_1^{\mu+\vartheta}-\tau_2^{\mu+\vartheta}\right|}{\Gamma_q(\mu+\vartheta+1)}\\ &\quad+\frac{(\Gamma_q(\mu+1)|\Lambda_2|+|\Lambda_3|)\left|\tau_1^{\vartheta}-\tau_2^{\vartheta}\right|}{\Gamma_q(\vartheta+1)\Gamma_q(\mu+1)}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&|D_qWs(\tau_1) - D_qWs(\tau_2)| \\
&\leq \Bigg(\frac{\left[(\tau_1-\tau_2)^{\eta+\mu+\vartheta-1} + \left|\tau_1^{\eta+\mu+\vartheta-1} - \tau_2^{\eta+\mu+\vartheta-1}\right|\right]}{\Gamma_q(\eta+\mu+\vartheta)} \\
&\quad + \frac{\omega^{(\eta)}[\mu+\vartheta]_q \left|\tau_2^{\mu+\vartheta-1} - \tau_1^{\mu+\vartheta-1}\right|}{\Gamma_q(\mu+\vartheta+1)\Gamma_q(\eta+1)} + \frac{[\vartheta]_q \left|\tau_2^{\vartheta-1} - \tau_1^{\vartheta-1}\right|}{\Gamma_q(\vartheta+1)\Gamma_q(\eta+\mu+1)} \\
&\quad + \frac{\omega^{(\eta)}[\vartheta]_q \left|\tau_1^{\vartheta-1} - \tau_2^{\vartheta-1}\right|}{\Gamma_q(\vartheta+1)\Gamma_q(\mu+1)\Gamma_q(\eta+1)} \Bigg) \left(\lambda\Delta_1 + \delta\Delta_2 + \sum_{i=3}^{4}\Delta_i\right) \\
&\quad + \frac{\varepsilon\theta\left[(\tau_2-\tau_1)^{\vartheta-1} + \left|\tau_2^{\vartheta-1} - \tau_1^{\vartheta-1}\right|\right]}{\Gamma_q(\vartheta)} \\
&\quad + \frac{|\Lambda_3|[\mu+\vartheta]_q \left|\tau_1^{\mu+\vartheta-1} - \tau_2^{\mu+\vartheta-1}\right|}{\Gamma_q(\mu+\vartheta+1)} \\
&\quad + \frac{(\Gamma_q(\mu+1)|\Lambda_2| + |\Lambda_3|)[\vartheta]_q \left|\tau_1^{\vartheta-1} - \tau_2^{\vartheta-1}\right|}{\Gamma_q(\vartheta+1)\Gamma_q(\mu+1)}.
\end{aligned}$$

By (3.7), we have

$$\left|D_q^{\alpha}Ws(\tau_1) - D_q^{\alpha}Ws(\tau_2)\right| \leq \frac{1}{\Gamma_q(2-\alpha)}\left|D_qWs(\tau_1) - D_qWs(\tau_2)\right|,$$

$$\left|D_q^{\beta}Ws(\tau_1) - D_q^{\beta}Ws(\tau_2)\right| \leq \frac{1}{\Gamma_q(2-\beta)}\left|D_qWs(\tau_1) - D_qWs(\tau_2)\right|.$$

Thanks to the above inequalities, we can state that $\|Ws(\tau_1) - Ws(\tau_2)\|_E \to 0$ as $\tau_1 \to \tau_2$. By the Arzelà-Ascoli theorem, we conclude that W is a completely continuous operator. Finally, it will be proved that the set Ψ, given by

$$\Psi = \left\{ s \in E \,:\, s = \varsigma W(s),\ 0 < \varsigma < 1 \right\},$$

is bounded. Let $s \in \Psi$, then $s = \varsigma W(s)$ for some $0 < \varsigma < 1$. Thus for any $\tau \in \Omega$, we can write

$$s(\tau) = \varsigma W(s).$$

Then by (H_3), we get

$$\|s\| \leq \varsigma\Pi\left(\lambda\Delta_1 + \delta\Delta_2 + \sum_{i=3}^{4}\Delta_i\right) + \frac{\varsigma\theta\|s\|_E}{\Gamma_q(\vartheta+1)} + \varsigma\Pi^*.$$

On the other hand,

$$\|D_qs\| \leq \varsigma\Sigma\left(\lambda\Delta_1 + \delta\Delta_2 + \sum_{i=3}^{4}\Delta_i\right) + \frac{\varsigma\theta\|s\|_E}{\Gamma_q(\vartheta)} + \varsigma\Sigma^*.$$

Using (3.7), we can obtain

$$\left\|D_q^{\alpha}s\right\| \leq \frac{1}{\Gamma_q(2-\alpha)}\left[\varsigma\Sigma\left(\lambda\Delta_1+\delta\Delta_2+\sum_{i=3}^{4}\Delta_i\right)+\frac{\varsigma\theta\left\|s\right\|_E}{\Gamma_q(\vartheta)}+\varsigma\Sigma^*\right],$$

$$\left\|D_q^{\beta}s\right\| \leq \frac{1}{\Gamma_q(2-\beta)}\left[\varsigma\Sigma\left(\lambda\Delta_1+\delta\Delta_2+\sum_{i=3}^{4}\Delta_i\right)+\frac{\varsigma\theta\left\|s\right\|_E}{\Gamma_q(\vartheta)}+\varsigma\Sigma^*\right].$$

It follows from above inequalities, that

$$\left\|s\right\|_E \leq \frac{\Theta\left(\lambda\Delta_1+\delta\Delta_2+\sum_{i=3}^{4}\Delta_i\right)+\Pi^*+\Theta^*}{1-\theta\Phi} < +\infty,$$

where Θ, Θ^*, Π^* and Φ are given by (3.10) and (3.12). This shows that the set Ψ is bounded. Thanks to Lemma 2.4, we deduce that W has at least one fixed point, which is a solution of $q-\mathbb{F}$D-R$\mathbb{P}$ (1.1). □

4 Ulam-Hyers stability of q − $\mathbb{F}$D-R$\mathbb{P}$

In this part, the Ulam-stability type of the $q-\mathbb{F}$D-R$\mathbb{P}$ (1.1) will be discussed.

Definition 4.1. *The $q-\mathbb{F}D$-R$\mathbb{P}$ in* (1.1) *is stable*

- *in* H-U *sense if there exists a real number $D_{\varphi^*,\psi^*,\phi^*,p}>0$ such that for each $d>0$ and for each solution t of the inequality*

$$\left|D_q^{\eta}\left(D_q^{\mu}\left(D_q^{\vartheta}+\theta\right)\right)t(\tau)-\left[p(\tau)-\phi_t^*(\tau)-\lambda\varphi_t^{*,\alpha}(\tau)-\delta\psi_t^{*,\beta}(\tau)\right]\right| \leq d, \tag{4.1}$$

  $\forall\,\tau\in\Omega$, there exists a solution s of the $q-\mathbb{F}D$-R$\mathbb{P}$ (1.1) with

$$\left\|t-s\right\|_E \leq D_{\varphi^*,\psi^*,\phi^*,p}d,$$

- *in Ulam-Hyers-Rassias sense with respect to $m\in C(\Omega,\mathbb{R}_+)$ if there exists a real number $D_{\varphi^*,\psi^*,\phi^*,p}>0$ such that for each $d>0$ and for each solution t of the inequality*

$$\left|D_q^{\eta}\left(D_q^{\mu}\left(D_q^{\vartheta}+\theta\right)\right)t(\tau)-\left[p(\tau)-\phi_t^*(\tau)-\lambda\varphi_t^{*,\alpha}(\tau)-\delta\psi_t^{*,\beta}(\tau)\right]\right| \leq dm(\tau), \tag{4.2}$$

  $\forall;\tau\in\Omega$, there exists a solution s of the $q-\mathbb{F}D$-R$\mathbb{P}$ (1.1) with

$$\left\|t-s\right\|_E \leq D_{\varphi^*,\psi^*,\phi^*,p}dm(\tau).$$

Remark 4.2. *A function $t\in E$ is a solution of the inequality* (4.1) *iff there exists a function $h:\Omega\to\mathbb{R}$ (which depend on t) such that*

$$|h(\tau)| \leq d, \qquad \forall\,\tau\in\Omega,$$

and

$$D_q^{\eta}\left(D_q^{\mu}\left(D_q^{\vartheta}+\theta\right)\right)t(\tau)-\left[p(\tau)-\phi_t^*(\tau)-\lambda\varphi_t^{*,\alpha}(\tau)-\delta\psi_t^{*,\beta}(\tau)\right]+h(\tau), \qquad \tau\in\Omega.$$

Theorem 4.3. *If conditions* $(H_i)_{i=1,2}$ *and* (3.8) *are valid. Then, the* $q-\mathbb{F}D$*-R*$\mathbb{P}$ (1.1) *is stable in* H-U *sense.*

Proof. Let us denote by $s \in E$ the unique solution of the problem

$$\begin{cases} D_q^{\eta}\left(D_q^{\mu}\left(D_q^{\vartheta}+\theta\right)\right)s(\tau)=k(\tau)-\phi_s^*(\tau)-\lambda\varphi_s^{*,\alpha}(\tau)-\delta\psi_s^{*,\beta}(\tau), \\ \\ t(0)=s(0), \\ \left(D_q^{\vartheta}+\theta\right)t(1)=\left(D_q^{\vartheta}+\theta\right)s(1), \\ D_q^{\mu}\left(\left(D_q^{\vartheta}+\theta\right)\right)t(\omega)=D_q^{\mu}\left(\left(D_q^{\vartheta}+\theta\right)\right)s(\omega), \end{cases} \tag{4.3}$$

such that $t \in E$ is a solution of the inequality (4.1). Thanks to Remark 4.2, we can write

$$\begin{aligned} t(\tau)=&I_q^{\eta+\mu+\vartheta}\left[l_t(\tau)\right]-\theta I_q^{\vartheta}\left[t(\tau)\right]+e_0^{'}I_q^{\mu+\vartheta}[1] \\ &+e_1^{'}I_q^{\vartheta}[1]+e_2^{'}+I_q^{\eta+\mu+\vartheta}\left[h(\tau)\right], \qquad e_i^{'}\in\mathbb{R},\ (i=0,1,2), \end{aligned}$$

where

$$l_t(\tau)=p(\tau)-\phi_t^*(\tau)-\lambda\varphi_t^{*,\alpha}(\tau)-\delta\psi_t^{*,\beta}(\tau),$$

and $|h(\tau)|\leq d$, $\tau\in\Omega$. By Lemma 3.1 and Remark 4.2, we have

$$|t(\tau)-Wt(\tau)|=\left|I_q^{\eta+\mu+\vartheta}\left[h(\tau)\right]\right|\leq dI_q^{\eta+\mu+\vartheta}[1].$$

Using Lemma 2.3, we obtaint

$$\|t-W(t)\|_E\leq\frac{d}{\Gamma_q(\eta+\mu+\vartheta)}. \tag{4.4}$$

On the other hand, we have

$$\begin{aligned} |t(\tau)-s(\tau)|=&\Big|t(\tau)-I_q^{\eta+\mu+\vartheta}\left[l_s(\tau)\right]-\theta I_q^{\vartheta}\left[s(\tau)\right]+e_0I_q^{\mu+\vartheta}[1] \\ &+e_1I_q^{\vartheta}[1]+e_2\Big| \\ =&|t(\tau)-Wt(\tau)+Wt(\tau)-Ws(\tau)| \\ \leq&|t(\tau)-Wt(\tau)|+|Wt(\tau)-Ws(\tau)|. \end{aligned}$$

Thanks to (4.4) and (H_2), we can write

$$\|t-s\|_E\leq\frac{d}{\Gamma_q(\eta+\mu+\vartheta)}+\left[\Theta(\lambda a_1+\delta a_2+a_3)+\theta\Phi\right]\|s-t\|_E.$$

Then

$$\|t-s\|_E\leq\frac{d}{\Gamma_q(\eta+\mu+\vartheta)\left[1-(\Theta(\lambda a_1+\delta a_2+a_3)+\theta\Phi)\right]}, \tag{4.5}$$

if we put

$$D_{\varphi^*,\psi^*,\phi^*,p}:=\frac{1}{\Gamma_q(\eta+\mu+\vartheta)\left[1-(\Theta(\lambda a_1+\delta a_2+a_3)+\theta\Phi)\right]},$$

then

$$\|t-s\|_E\leq D_{\varphi^*,\psi^*,\phi^*,p}d.$$

Hence, the $q-\mathbb{F}$D-R$\mathbb{P}$ (1.1) is stable in H-U sense. □

Theorem 4.4. *If conditions* $(H_i)_{i=1,2}$ *and* (3.8) *are valid. Suppose there exist* $m \in C(\Omega, \mathbb{R}_+)$ *is nondecreasing and* $\xi_m > 0$ *such that*

$$I_q^{\eta+\mu+\vartheta}[m(\tau)] \leq \xi_m m(\tau). \tag{4.6}$$

Then the $q - \mathbb{F}D$*-R*$\mathbb{P}$ (1.1) *is stable in* H-U-R *sense.*

Proof. From Remark 4.2, we have

$$\begin{aligned} t(\tau) = {} & I_q^{\eta+\mu+\vartheta}[l_t(\tau)] - \theta I_q^{\vartheta}[t(\tau)] + e_0^{'} I_q^{\mu+\vartheta}[1] + e_1^{'} I_q^{\vartheta}[1] \\ & + e_2^{'} + I_q^{\eta+\mu+\vartheta}[h(\tau)], \end{aligned}$$

where $|h(\tau)| \leq dm(\tau), \tau \in \Omega$ and $t \in E$ is a solution of the inequality (4.2). Let us denote by $s \in E$ the unique solution of (4.3), then using Lemma 3.1, we get‘

$$\begin{aligned} |t(\tau) - Wt(\tau)| = {} & \left| I_q^{\eta+\mu+\vartheta}[h(\tau)] \right| \leq d I_q^{\eta+\mu+\vartheta}[m(\tau)] \\ & \leq d\xi_m m(\tau). \end{aligned} \tag{4.7}$$

From these relations, we have

$$\begin{aligned} |t(\tau) - s(\tau)| = {} & \Big| t(\tau) - I_q^{\eta+\mu+\vartheta}[l_s(\tau)] - \theta I_q^{\vartheta}[s(\tau)] \\ & + e_0 I_q^{\mu+\vartheta}[1] + e_1 I_q^{\vartheta}[1] + e_2 \Big| \\ = {} & |t(\tau) - Wt(\tau) + Wt(\tau) - Ws(\tau)| \\ \leq {} & |t(\tau) - Wt(\tau)| + |Wt(\tau) - Ws(\tau)|. \end{aligned}$$

Thanks to (H_2) and by (4.7), we get

$$\|t - s\|_E \leq d\xi_m m(\tau) + [\Theta(\lambda a_1 + \delta a_2 + a_3) + \theta\Phi] \|s - t\|_E.$$

Then, we have

$$\|t - s\|_E \leq \frac{\xi_m}{1 - [\Theta(\lambda a_1 + \delta a_2 + a_3) + \theta\Phi]} dm(\tau), \qquad \tau \in \Omega. \tag{4.8}$$

If we take

$$D_{\varphi^*,\psi^*,\phi^*,p} := \frac{\xi_m}{1 - [\Theta(\lambda a_1 + \delta a_2 + a_3) + \theta\Phi]},$$

we can obtain

$$\|t - s\|_E \leq D_{\varphi^*,\psi^*,\phi^*,p} dm(\tau), \qquad \tau \in \Omega.$$

Therefore, the $q - \mathbb{F}$D-R$\mathbb{P}$ (1.1) is stable in H-U-R sense. □

5 Example with algorithms

Consider the following $q-\mathbb{F}$D-R$\mathbb{P}$, $q\in\{0.2,0.5,0.87\}\subseteq(0,1)$,

$$\begin{cases} D_q^{\frac{2}{3}}\left(D_q^{\frac{\ln 2}{3}}\left(D_q^{\frac{\sqrt{e}}{2e}}+\frac{1}{13}\right)\right)s(\tau)=\frac{\sinh\tau+2}{1+e^{\tau}}-\frac{\sin(s(\tau))}{21(e^{t^2}+2)} \\ \quad -\frac{\cos(\sqrt{e})}{50\sqrt{\pi}}\left(\frac{\cosh(e^{\tau}+2)}{5}+\frac{|s(\tau)|}{43\left(\ln\left(\tau+\frac{\sqrt{5}}{5}\right)+2\right)\left(\left|\frac{1}{2}+s(\tau)\right|\right)}\right. \\ \quad \left.+\frac{\tan^{-1}\left(D_q^{\frac{1}{2}}s(\tau)\right)}{43\left(\ln\left(\tau+\frac{\sqrt{5}}{5}\right)+2\right)}\right) \\ \quad -\frac{\ln 2}{55e}\left(\frac{\ln(e^{\tau}+2)+1}{2\pi}+\frac{\arctan(s(\tau))}{37\left(e^{\tau+3}+\frac{\pi}{3}\right)}\right. \\ \quad \left.+\frac{\left|D_q^{\frac{3}{4}}s(\tau)\right|}{37\left(\frac{\pi}{3}+e^{\tau+3}\right)\left(\left|D_q^{\frac{3}{4}}s(\tau)\right|+1\right)}\right), \\ z(0)=\frac{\sqrt{5}}{12}, \\ \left(D_q^{\frac{\sqrt{e}}{2e}}+\frac{1}{13}\right)s(1)=\frac{\sqrt{\pi}}{5}, \\ D_q^{\frac{\ln 2}{3}}\left(\left(D_q^{\frac{\sqrt{e}}{2e}}+\frac{1}{13}\right)\right)s\left(\frac{6}{7}\right)=\frac{\ln 3}{5}, \end{cases} \tag{5.1}$$

and

$$\begin{aligned} &\left|D_q^{\frac{2}{3}}\left(D_q^{\frac{\ln 2}{3}}\left(D_q^{\frac{\sqrt{e}}{2e}}+\frac{1}{13}\right)\right)t(\tau)-\left(\frac{\sinh\tau+2}{1+e^{\tau}}-\phi_s^*(\tau)\right.\right. \\ &\qquad\left.\left.-\frac{\cos(\sqrt{e})}{50\sqrt{\pi}}\varphi_s^{*,\alpha}(\tau)-\frac{\ln 2}{55e}\psi_s^{*,\beta}(\tau)\right)\right| \\ &\qquad\leq d, \\ &\left|D_q^{\frac{2}{3}}\left(D_q^{\frac{\ln 2}{3}}\left(D_q^{\frac{\sqrt{e}}{2e}}+\frac{1}{13}\right)\right)t(\tau)-\left(\frac{\sinh\tau+2}{1+e^{\tau}}-\phi_s^*(\tau)\right.\right. \\ &\qquad\left.\left.-\frac{\cos(\sqrt{e})}{50\sqrt{\pi}}\varphi_s^{*,\alpha}(\tau)-\frac{\ln 2}{55e}\psi_s^{*,\beta}(\tau)\right)\right| \\ &\qquad\leq dm(\tau), \end{aligned}$$

where

$$\begin{aligned} \varphi_s^{*,\alpha}(\tau)=&\frac{\cosh(e^{\tau}+2)}{5}+\frac{|s(\tau)|}{43\left(\ln\left(\tau+\frac{\sqrt{5}}{5}\right)+2\right)\left(\left|\frac{1}{2}+s(\tau)\right|\right)} \\ &+\frac{\tan^{-1}\left(D_q^{\frac{1}{2}}s(\tau)\right)}{43\left(\ln\left(\tau+\frac{\sqrt{5}}{5}\right)+2\right)}, \end{aligned}$$

$$\psi_s^{*,\beta}(\tau) = \frac{\ln(e^{\tau}+2)+1}{2\pi} + \frac{\arctan(s(\tau))}{37\left(e^{\tau+3}+\frac{\pi}{3}\right)} + \frac{\left|D_q^{\frac{3}{4}} s(\tau)\right|}{37\left(\frac{\pi}{3}+e^{\tau+3}\right)\left(\left|D_q^{\frac{3}{4}} s(\tau)\right|+1\right)},$$

$$\phi_s^*(\tau) = \frac{\sin(s(\tau))}{21(e^{t^2}+2)}.$$

We have

$$\lambda = \frac{\cos(\sqrt{e})}{50\sqrt{\pi}}, \quad \delta = \frac{\ln 2}{55e}, \quad \theta = \frac{1}{13},$$

$$a_1 = \frac{1}{43\left(\ln\frac{\sqrt{5}}{5}+2\right)}, \quad a_2 = \frac{1}{37\left(e^3+\frac{\pi}{3}\right)}, \quad a_3 = \frac{1}{63},$$

$$[\vartheta]_q = \begin{cases} 0.690983, & q=0.2, \\ 0.585786, & q=0.5, \\ 0.517401, & q=0.87, \end{cases} \qquad [\mu+\vartheta]_q = \begin{cases} 0.864585, & q=0.2, \\ 0.795069, & q=0.5, \\ 0.744589, & q=0.87, \end{cases} \tag{5.2}$$

$$\lambda a_1 + \delta a_2 + a_3 = \frac{\sqrt{5}}{12} \times \frac{1}{43\left(\ln\frac{\sqrt{5}}{5}+2\right)} + \frac{\ln 2}{55e} \times \frac{1}{37\left(e^3+\frac{\pi}{3}\right)} + \frac{1}{63} \simeq 0.015905,$$

and by (3.9), (3.10), (3.12) and using the given data, we find that

Table 1: Numerical results of ϕ, Π, Σ, Θ and $(1-\theta\Phi)\Theta^{-1}$, of $q-\mathbb{F}$D-R$\mathbb{P}$ (5.1) whenever $q=0.2$.

n	Φ	Π	Σ	Θ	$(1-\theta\Phi)\Theta^{-1}$
			$\boldsymbol{q=0.2}$		
1	−0.3484	1.8556	1.2867	3.0646	0.3351
2	−0.2661	1.8449	1.2697	3.0220	0.3377
3	−0.2512	1.8428	1.2663	3.0137	0.3382
4	−0.2483	1.8424	1.2657	3.0120	0.3383
5	−0.2477	1.8423	1.2655	3.0117	0.3384
6	−0.2476	1.8423	1.2655	3.0116	0.3384
7	−0.2476	1.8423	1.2655	3.0116	0.3384

$$\Phi_i = \frac{1}{\Gamma_q(\vartheta+1)} + \frac{1}{\Gamma_q(2-\alpha)\Gamma_q(\vartheta)} + \frac{1}{\Gamma_q(2-\beta)\Gamma_q(\vartheta)}$$
$$= \frac{1}{\Gamma_q\left(\frac{\sqrt{e}}{2e}+1\right)} + \frac{1}{\Gamma_q\left(2-\frac{1}{2}\right)\Gamma_q\left(\frac{\sqrt{e}}{2e}\right)} + \frac{1}{\Gamma_q\left(2-\frac{3}{4}\right)\Gamma_q\left(\frac{\sqrt{e}}{2e}\right)}$$
$$\simeq \begin{cases} -0.248293, & q=0.2, \\ 0.1835834, & q=0.5, \\ 0.0837188, & q=0.87, \end{cases}$$

Table 2: Numerical results of ϕ, Π, Σ, Θ and $(1-\theta\Phi)\,\Theta^{-1}$, of $q-\mathbb{FD}$-R$\mathbb{P}$ (5.1) whenever $q=0.5$.

n	Φ	Π	Σ	Θ	$(1-\theta\Phi)\,\Theta^{-1}$
			$\boldsymbol{q=0.5}$		
1	0.0991	0.8800	0.5559	1.1934	0.8316
2	0.1593	0.8694	0.5147	1.1384	0.8677
3	0.1768	0.8658	0.4969	1.1166	0.8834
4	0.1836	0.8643	0.4886	1.1069	0.8907
5	0.1866	0.8636	0.4846	1.1023	0.8942
6	0.1880	0.8633	0.4826	1.1001	0.8959
7	0.1887	0.8632	0.4817	1.0990	0.8967
8	0.1890	0.8631	0.4812	1.0984	0.8972
9	0.1892	0.8631	0.4809	1.0981	0.8974
10	0.1892	0.8630	0.4808	1.0980	0.8975
11	0.1893	0.8630	0.4807	1.0979	0.8975
12	0.1893	0.8630	0.4807	1.0979	0.8976
13	0.1893	0.8630	0.4807	1.0979	0.8976
14	0.1893	0.8630	0.4807	1.0979	0.8976

$$
\begin{aligned}
\Pi_i &= \frac{1}{\Gamma_q(\eta+\mu+\vartheta+1)} + \frac{\omega^{(\eta)}}{\Gamma_q(\mu+\vartheta+1)\,\Gamma_q(\eta+1)} \\
&\quad + \frac{1}{\Gamma_q(\vartheta+1)\,\Gamma_q(\eta+\mu+1)} + \frac{\omega^{(\eta)}}{\Gamma_q(\vartheta+1)\,\Gamma_q(\mu+1)\,\Gamma_q(\eta+1)}, \\
&= \frac{1}{\Gamma_q\left(\frac{2}{3}+\frac{\ln 2}{3}+\frac{\sqrt{e}}{2e}+1\right)} + \frac{\left(\frac{6}{7}\right)^{\left(\frac{2}{3}\right)}}{\Gamma_q\left(\frac{\ln 2}{3}+\frac{\sqrt{e}}{2e}+1\right)\Gamma_q\left(\frac{2}{3}+1\right)} \\
&\quad + \frac{1}{\Gamma_q\left(\frac{\sqrt{e}}{2e}+1\right)\Gamma_q\left(\frac{2}{3}+\frac{\ln 2}{3}+1\right)} \\
&\quad + \frac{\left(\frac{6}{7}\right)^{\left(\frac{2}{3}\right)}}{\Gamma_q\left(\frac{\sqrt{e}}{2e}+1\right)\Gamma_q\left(\frac{\ln 2}{3}+1\right)\Gamma_q\left(\frac{2}{3}+1\right)}, \\
&\simeq \begin{cases} 1.842326, & q=0.2, \\ 0.863329, & q=0.5, \\ 0.147619, & q=0.87, \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\Sigma_i &= \frac{1}{\Gamma_q(\eta+\mu+\vartheta)} + \frac{[\mu+\vartheta]_q\,\omega^{(\eta)}}{\Gamma_q(\mu+\vartheta+1)\,\Gamma_q(\eta+1)} \\
&\quad + \frac{[\vartheta]_q}{\Gamma_q(\vartheta+1)\,\Gamma_q(\eta+\mu+1)} \\
&\quad + \frac{[\vartheta]_q\,\omega^{(\eta)}}{\Gamma_q(\vartheta+1)\,\Gamma_q(\mu+1)\,\Gamma_q(\eta+1)} \\
&= \frac{1}{\Gamma_q\left(\frac{2}{3}+\frac{\ln 2}{3}+\frac{\sqrt{e}}{2e}\right)} + \frac{\left[\frac{\ln 2}{3}+\frac{\sqrt{e}}{2e}\right]_q\left(\frac{6}{7}\right)^{\left(\frac{2}{3}\right)}}{\Gamma_q\left(\frac{\ln 2}{3}+\frac{\sqrt{e}}{2e}+1\right)\Gamma_q\left(\frac{2}{3}+1\right)}
\end{aligned}
$$

Table 3: Numerical results of ϕ, Π, Σ, Θ and $(1-\theta\Phi)\,\Theta^{-1}$, of $q-\mathbb{FD}$-R$\mathbb{P}$ (5.1) whenever $q=0.87$.

n	Φ	Π	Σ	Θ	$(1-\theta\Phi)\,\Theta^{-1}$
			$\boldsymbol{q=0.87}$		
1	0.0389	0.1336	0.1601	0.1733	5.7517
2	0.0761	0.1359	0.1331	0.1634	6.0843
3	0.0830	0.1391	0.1175	0.1605	6.1889
4	0.0837	0.1423	0.1073	0.1601	6.2043
5	0.0828	0.1451	0.1001	0.1607	6.1839
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
31	0.0709	0.1635	0.0725	0.1717	5.7924
32	0.0708	0.1635	0.0724	0.1717	5.7908
33	0.0708	0.1636	0.0723	0.1718	5.7894
34	0.0707	0.1636	0.0723	0.1718	5.7882
35	0.0707	0.1637	0.0723	0.1719	5.7871
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
55	0.0706	0.1639	0.0720	0.1721	5.7805
56	0.0706	0.1639	0.0720	0.1721	5.7804
57	0.0706	0.1639	0.0720	0.1721	5.7804
58	0.0706	0.1639	0.0720	0.1721	5.7803
59	0.0706	0.1639	0.0720	0.1721	5.7803
60	0.0706	0.1639	0.0720	0.1721	5.7803
61	0.0706	0.1639	0.0720	0.1721	5.7802
62	0.0706	0.1639	0.0720	0.1721	5.7802
63	0.0706	0.1639	0.0720	0.1721	5.7802
64	0.0706	0.1640	0.0720	0.1721	5.7802
65	0.0706	0.1640	0.0720	0.1721	5.7801
66	0.0706	0.1640	0.0720	0.1721	5.7801
67	0.0705	0.1640	0.0720	0.1721	5.7801

$$+\frac{\left[\frac{\sqrt{e}}{2e}\right]_q}{\Gamma_q\left(\frac{\sqrt{e}}{2e}+1\right)\Gamma_q\left(\frac{2}{3}+\frac{\ln 2}{3}+1\right)}$$
$$+\frac{\left[\frac{\sqrt{e}}{2e}\right]_q\left(\frac{6}{7}\right)^{\left(\frac{2}{3}\right)}}{\Gamma_q\left(\frac{\sqrt{e}}{2e}+1\right)\Gamma_q\left(\frac{\ln 2}{3}+1\right)\Gamma_q\left(\frac{2}{3}+1\right)}$$
$$\simeq\begin{cases}1.2655036, & q=0.2,\\ 0.4826411, & q=0.5,\\ 0.0947624, & q=0.87,\end{cases}$$

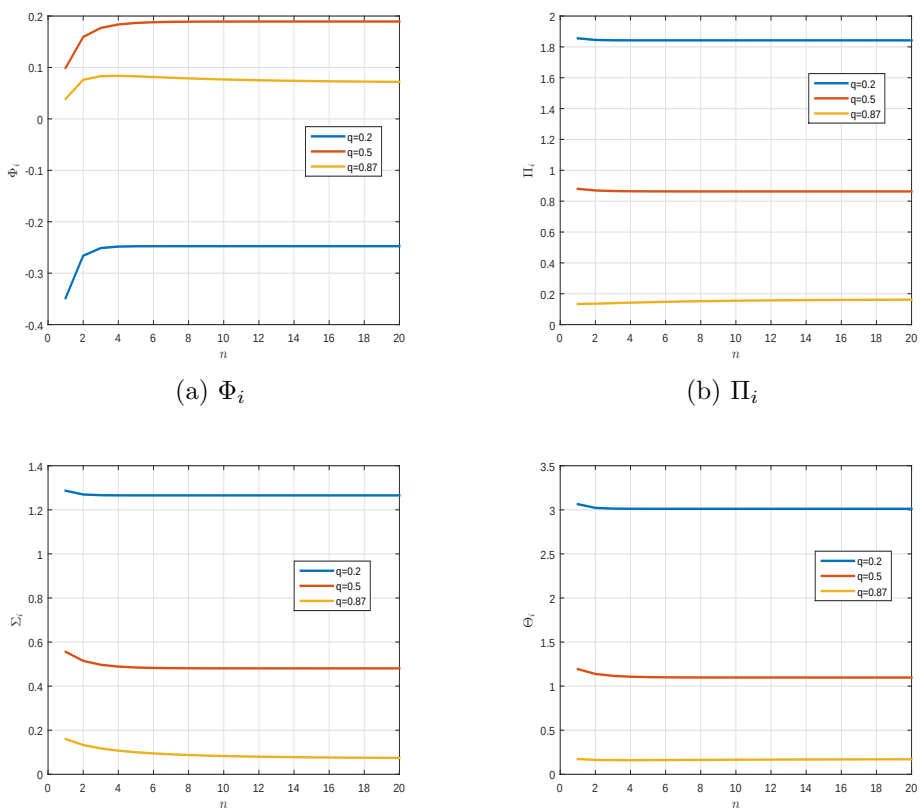
$$\Theta=\Pi+\frac{\Sigma}{\Gamma_q(2-\alpha)}+\frac{\Sigma}{\Gamma_q(2-\beta)}$$
$$=\Pi+\frac{\Sigma}{\Gamma_q\left(2-\frac{1}{2}\right)}+\frac{\Sigma}{\Gamma_q\left(2-\frac{3}{4}\right)}\simeq\begin{cases}2.095123, & q=0.2,\\ 0.889252, & q=0.5,\\ 0.152063, & q=0.87,\end{cases}$$

and

$$\lambda a_1+\delta a_2+a_3\simeq 0.015905$$

$$< \begin{cases} 0.486366, & q = 0.2, \\ 1.108352, & q = 0.5, \\ 6.058737, & q = 0.87, \end{cases}$$
$$\simeq (1 - \theta\Phi)\,\Theta^{-1}.$$

Thus Inequality (3.8) in Theorem 3.2 hold. Tables 1, 2 and 3 show the numerical results of ϕ, Π, Σ, Θ and $(1-\theta\Phi)\,\Theta^{-1}$, for $q = 0.2, 0.5, 0.87$. We can see graphical representation of ϕ_i, Π_i, Σ_i and Θ_i of $q-\mathbb{FD}$-R$\mathbb{P}$ (5.1) for three cases of $q = 0.2$, 0.5, 0.87 q in Figure 1. Also, Figure 2 show values of $(1-\theta\Phi_i)\,\Theta_i^{-1}$, for $q = 0.2$, 0.5, 0.87. In the other hand, once can see



(a) Φ_i (b) Π_i

(c) Σ_i (d) Θ_i

Figure 1: Graphical representation of ϕ_i, Π_i, Σ_i and Θ_i of $q-\mathbb{FD}$-R$\mathbb{P}$ (5.1) for $q = 0.2$, 0.5, 0.87.

that

$$\nabla_1 = \sup_{\tau\in\Omega} |\varphi(\tau,0,0)| = \sup_{\tau\in\Omega} \left| \frac{\sin(s(\tau))}{21(e^{\tau^2}+2)} \right| = 0,$$
$$\nabla_2 = \sup_{\tau\in\Omega} |\psi(\tau,0,0)| = \sup_{\tau\in\Omega} \left| \frac{\cosh(e^{\tau}+2)}{5} \right| = 11.198,$$
$$\nabla_3 = \sup_{\tau\in\Omega} |\phi(\tau,0)| = \sup_{\tau\in\Omega} \left| \frac{\ln(e^{\tau}+2)+1}{2\pi} \right| = 0.4060,$$
$$\nabla_4 = \sup_{\tau\in\Omega} |p(\tau)| = \sup_{\tau\in\Omega} \left| \frac{\sinh\tau+2}{1+e^{\tau}} \right| = 1.$$

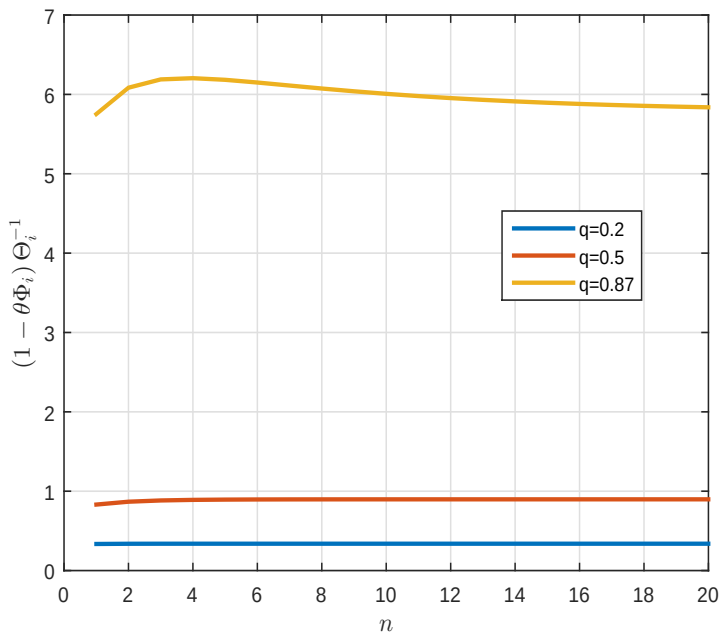


Figure 2: Graphical representation of $(1-\theta\Phi_i)\,\Theta_i^{-1}$, for $q = 0.2,\ 0.5,\ 0.87$.

Hence $\nabla = \max \nabla_i \simeq 11.198$. So, by Theorem 3.2, $q-\mathbb{FD}$-R$\mathbb{P}$ (5.1) has a unique solution on $[0,1]$ and by Theorem 4.3 is H-U$\mathbb{S}$ with

$$\|t-s\|_E \leq \begin{cases} 17.1739\,d, & q=0.2, \\ 8.1707\,d, & q=0.5, \\ 7.3773\,d, & q=0.87, \end{cases} \qquad d>0.$$

If we take $m(\tau) = \tau^{\frac{\sqrt{2}}{2}}$, then we obtain

$$I_q^{\frac{2}{3}+\frac{\ln 2}{3}+\frac{\sqrt{e}}{2e}}\left[m(\tau)\right] \leq 0.78719\tau^{\frac{\sqrt{2}}{2}} = \xi_m m(\tau).$$

Hence, the condition (4.6) is satisfied with $m(\tau) = \tau^{\frac{\sqrt{2}}{2}}$ and $\xi_m = 0.78719$. Tables 4 and 5 show H-U$\mathbb{S}$ and H-U-R$\mathbb{S}$ for $q = 0.2, 0.5, 0.87$. We can see graphical representation of these results of $q-\mathbb{FD}$-R$\mathbb{P}$ (5.1) for three cases of $q = 0.2,\ 0.5,\ 0.87$ q in Figures 3 and 4. It follows from Theorem 4.4, $q-\mathbb{FD}$-R$\mathbb{P}$ (5.1) is H-U-R$\mathbb{S}$ with

$$\|t-s\|_E \leq \begin{cases} 1.0297\,d\tau^{\frac{\sqrt{2}}{2}}, & q=0.2, \\ 1.0331\,d\tau^{\frac{\sqrt{2}}{2}}, & q=0.5, \\ 1.0083\,d\tau^{\frac{\sqrt{2}}{2}}, & q=0.87, \end{cases} \qquad d>0, \tau\in[0,1]\,.$$

Algorithm 3 shows the MATLAb lines to calculate values of all variables in Example 1.

6 Conclusion

The $q-\mathbb{FD}$-R$\mathbb{P}$ has been investigated in this work in details. The investigation of this particular equation provides us with a powerful tool in modeling most scientific phenomena without the

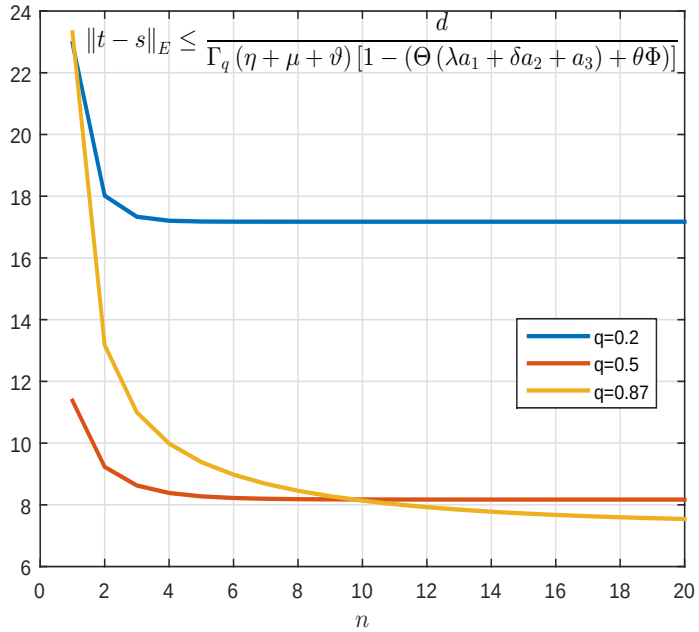


Figure 3: Graphical representation of H-U$\mathbb{S}$ for $q = 0.2$, 0.5, 0.87.

Table 4: Numerical results of H-U$\mathbb{S}$ inequality (4.5) for $q-\mathbb{F}$D-R$\mathbb{P}$ (5.1) for $q = 0.2$, 0.5, 0.87.

n	$q = 0.2$	$q = 0.5$	$q = 0.87$
1	22.9611	11.3750	23.3089
2	18.0161	9.2300	13.1796
3	17.3337	8.6264	10.9944
4	17.2056	8.3839	9.9860
5	17.1803	8.2741	9.3848
6	17.1752	8.2216	8.9780
7	17.1742	8.1960	8.6820
8	17.1740	8.1833	8.4565
⋮	⋮	⋮	⋮
14	17.1739	8.1709	7.7781
15	17.1739	8.1708	7.7214
16	17.1739	8.1707	7.6734
17	17.1739	8.1707	7.6325
18	17.1739	8.1707	7.5975
⋮	⋮	⋮	⋮
69	17.1739	8.1707	7.3774
70	17.1739	8.1707	7.3774
71	17.1739	8.1707	7.3774
72	17.1739	8.1707	7.3774
73	17.1739	8.1707	7.3773
74	17.1739	8.1707	7.3773
75	17.1739	8.1707	7.3773

need to remove most parameters which have an essential role in the physical interpretation of the studied phenomena. $q-\mathbb{F}$D-R$\mathbb{P}$ (1.1) has been studied under some B.Cs. An example has been provided to support our results' validity and applicability in fields of physics and engineering.

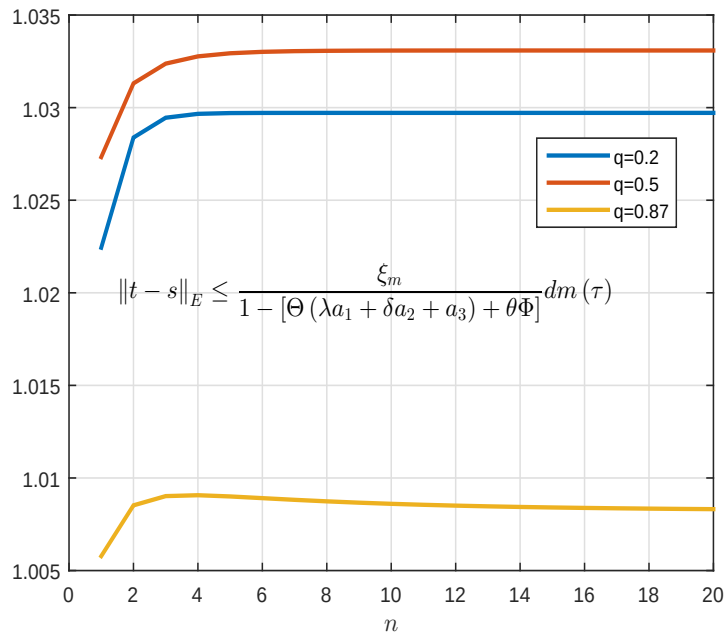


Figure 4: Graphical representation of H-U-R$\mathbb{S}$ for $q = 0.2$, 0.5, 0.87.

Table 5: Numerical results of H-U-R$\mathbb{S}$ inequality (4.5) for $q-\mathbb{F}$D-R$\mathbb{P}$ (5.1) for $q = 0.2$, 0.5, 0.87.

n	$q = 0.2$	$q = 0.5$	$q = 0.87$
1	1.0224	1.0273	1.0058
2	1.0284	1.0313	1.0085
3	1.0295	1.0324	1.0090
4	1.0297	1.0328	1.0091
5	1.0297	1.0329	1.0090
6	1.0297	1.0330	1.0089
7	1.0297	1.0331	1.0088
8	1.0297	1.0331	1.0087
⋮	⋮	⋮	⋮
16	1.0297	1.0331	1.0084
17	1.0297	1.0331	1.0084
18	1.0297	1.0331	1.0083
19	1.0297	1.0331	1.0083
20	1.0297	1.0331	1.0083

Declarations

Availability of Data and Materials

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Competing interests

The authors declare that they have no competing interests.

Funding

Not applicable.

Authors' contributions
MH: Actualization, methodology, formal analysis, validation, investigation, initial draft and was a major contributor in writing the manuscript. **MES**: Actualization, methodology, formal analysis, validation, investigation, software, simulation, initial draft and was a major contributor in writing the manuscript. All authors read and approved the final manuscript.

Acknowledgments
Not applicable.

References

[1] Abdeljawad, T., Samei, M.E.: Applying quantum calculus for the existence of solution of $q-$integro-differential equations with three criteria. Discrete & Continuous Dynamical Systems-Series S. **14**(10), 3351–3386 (2021)

[2] Agarwal, R.P., Ahmad, B., Alsaedi, A., Al-Hutami, H.: Existence theory for $q-$antiperiodic boundary value problems of sequential $q-$fractional integrodifferential equations. Abstr. Appl. Anal. **2014**, 1–12 (2014)

[3] Agarwal, R.P., Ahmad, B., Alsaedi, A., Al-Hutami, H.: Sequential fractional $q-$difference equations with nonlocal sub-strip boundary conditions. Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis **22**, 1–12 (2015)

[4] Annaby, M., Mansour, Z.: q–Fractional Calculus and Equations. Springer Heidelberg, Cambridge (2012). DOI 10.1007/978-3-642-30898-7

[5] Bendahou, A., Dahmani, Z.: Solvability of a sequential problem of duffing rayleigh type. Turkish J. Ineq. **5**(2), 21–32 (2021)

[6] Chatterjee, S., Dey, S.: Nonlinear dynamics of two harmonic oscillators coupled by rayleigh type self exciting force. Nonlinear Dynamics **72**(1), 113–128 (2013)

[7] Finkelstein, R., Marcus, E.: Transformation theory of the $q-$oscillator. J. Math. Phys. **36**(6), 2652–2672 (1995)

[8] Freund, P.G.O., Zabrodin, A.V.: The spectral problem for the $q-$knizhnik-zamolodchikov equation and continuous $q-$jacobi polynomials. Commun. Math. Phys. **173**(1), 17–42 (1995)

[9] Hajiseyedazizi, S.N., Samei, M.E., Alzabut, J., Chu, Y.: On multi-step methods for singular fractional $q$integro-differential equations. Open Mathematics **19**, 1378–1405 (2021). DOI 10.1515/math-2021-0093

[10] Jiang, M., Huang, R.: Existence and stability results for impulsive fractional $q-$difference equation. Journal of Applied Mathematics and Physics **8**(7), 1413–1423 (2020)

[11] Kaplan, B.Z., Horen, Y.: Switching-mode counterparts of the rayleigh and van-der-pol oscillators. International Journal of Circuit Theory & Applications **28**(1), 31–49 (2000)

[12] Kwuimy, C.K., Nbendjo, B.N.: Active control of horseshoes chaos in a driven rayleigh oscillator with fractional order deflection. Physics Letters A. **375**(39), 3442–3449 (2012)

[13] Liang, S., Samei, M.E.: New approach to solutions of a class of singular fractional $q-$differential problem via quantum calculus. Adv. Differ. Equ. **2020**, 14 (2020)

[14] Liu, C., Kuo, C., Chang, J.: Solving the optimal control problems of nonlinear duffing oscillators by using an iterative shape functions method. Computer Modeling in Engineering & Sciences **122**(1), 33–48 (2020)

[15] Mishra, S.K., Samei, M.E., Chakraborty, S.K., Ram, B.: On q-variant of dai–yuan conjugate gradient algorithm for unconstrained optimization problems. Nonlinear Dynamic **104**, 2471–2496 (2021). DOI 10.1007/s11071-021-06378-3

[16] Phuong, N.D., Etemad, S., Rezapour, S.: On two structures of the fractional $q-$sequential integro-differential boundary value problems. Math Meth Appl Sci. **45**(2), 618–639 (2022)

[17] Pirmohabbati, P., Sheikhani, A.H.R., Najafi, H.S., Ziabari, A.A.: Numerical solution of full fractional duffing equations with cubic-quintic-heptic nonlinearities. AIMS Mathematics **5**(2), 1621–1641 (2020)

[18] Rajkovic, P.M., Marinkovic, S.D., Stankovic, M.S.: On $q-$analogues of caputo derivative and mittag-leffer function. Fract. Calc. Appl. Anal. **10**, 359–373 (2007)

[19] Ran-Ran, Z., Wei, X., Gui-Dong, Y., Qun, H.: Response of a duffing rayleigh system with a fractional derivative under gaussian white noise excitation. Chinese Physics B. **24**(2), 20–24 (2015)

[20] Rezapour, S., Samei, M.E.: On the existence of solutions for a multi-singular pointwise defined fractional $q-$integro-differential equation. Bound. Value. Probl. **2020**, 38 (2020)

[21] Rostami, M., Haeri, M.: Undamped oscillations in fractional-order duffing oscillator. Signal processing **107**, 361–367 (2015)

[22] S., C.A.G., Roshid, H.O., Inc, M., Akinyemi, L., Rezazadeh, H.: On soliton solutions for perturbed fokaslenells equation. Optical and Quantum Electronics **54**, 370 (2022). DOI 10.1007/s11082-022-03796-4

[23] Samei, M.E., Ahmadi, A., Hajiseyedazizi, S.N., Mishra, S.K., Ram, B.: The existence of non-negative solutions for a nonlinear fractional q–differential problem via a different numerical approach. Journal of Inequalities and Applications **2021**, 75 (2021). DOI 10.1186/s13660-021-02612-z

[24] Selvam, A.G.M., Baleanu, D., Alzabut, J., Vignesh, D., Abbas, S.: On hyers-ulam mittag-leffler stability of discrete fractional duffing equation with application on inverted pendulum. Adv. Differ. Equ. **2020**, 456 (2020)

[25] Sheng, Y., Zhang, T.: Some results on the $q-$calculus and fractional $q-$differential equations. Mathematics **10**(1), 1–15 (2022)

[26] Siewe, M.S., Cao, H., Sanjuan, M.A.: Effect of nonlinear dissipation on the basin boundaries of a driven two-well rayleigh-duffing oscillator. Chaos, Solitons & Fractals **39**(3), 1092–1099 (2009)

[27] Smart, D.R.: Fixed point theorems. Cambridge University Press, Cambridge (1980)

[28] Xiao, M., Jiang, G., Cao, J.: Asymptotic solutions and circuit implementations of a rayleigh oscillator including cubic fractional damping terms, circuits, systems. Signal Processing **35**(6), 2041–2053 (2016)

[29] Zhai, C., Ren, J.: The unique solution for a fractional $q-$difference equation with three-point boundary conditions. Indagationes Mathematicae **29**(3), 948–961 (2018)

[30] Zhang, Y.L., Li, C.K.: Fractional modified duffing-rayleigh system and its synchronization. Nonlinear Dyn. **88**(4), 3023–3041 (2017)

Supplement

Algorithm 1: MATLAB function for calculation $q-$gamma function.

```
function p = qGamma(q,kappa,n)
s=1;
for k=0:n
    s=s*(1-q^(k+1))/(1-q^(x+k-1));
end;
p = s*(1-q)^(1-x);
end
```

Algorithm 2: MATLAB function for calculation the fractional $q-$integral of the Riemann-Liouville type.

```
function g=Iq_sigma(q,sigma,tau,n,fun)
p=0;
for k=0:n
    s=1;
    for i=0:n
        s=s*(1-q^(sigma+i-1))*(1-q^(k+i))...
        /((1-q^(i+1))*(1-q^(sigma+k+i-1)));
    end
    p=p+s*q^k*eval(subs(fun,tau*q^k));
end;
g=round(p*(tau^sigma)*(1-q)^sigma,6);
end
```

Algorithm 3: MATLAB lines for calculation all variables in Example 1.

```
clear;
format long;
syms v e;
q=[0.2 0.5 0.87];
[xq yq]=size(q);
k=120;
eta=2/3; mu=log(2)/3; vartheta=sqrt(exp(1))/(2*sqrt(exp(1)));
lambda=cos(sqrt(exp(1)))/(50*sqrt(pi));
Δ=log(2)/(55*exp(1)); theta =1/13;
alpha=1/2; beta=3/4;
a_1=1/(43*log(sqrt(5)/5)+2);
a_2=1/(37*((exp(1))^3+ pi/3));
a_3=1/63;
omega=6/7;
Lambda_1=sqrt(5)/12;
Lambda_2=sqrt(pi)/5;
Lambda_3=log(3)/5;
```

```

t_0 = 0; T = 1;
nabla=11.198;
column=1;
for s=1:yq
    for n=1:k
        paramsmatrix(n, column)=n;
        G1=qGamma(q(s),vartheta+1,n);
        G2=qGamma(q(s),vartheta,n);
        G3=qGamma(q(s),2-alpha,n);
        G4=qGamma(q(s),2-beta,n);
        paramsmatrix(n, column+1)=G1;
        paramsmatrix(n, column+2)=G2;
        paramsmatrix(n, column+3)=G3;
        paramsmatrix(n, column+4)=G4;
        Phi=1/G1+1/(G2*G3)+1/(G2*G4);
        paramsmatrix(n, column+5)=Phi;
        G5=qGamma(q(s),eta+mu+vartheta+1,n);
        G6=qGamma(q(s),mu+vartheta+1,n);
        G7=qGamma(q(s),eta+1,n);
        G8=qGamma(q(s),eta+mu+1,n);
        G9=qGamma(q(s),mu+1,n);
        paramsmatrix(n, column+6)=G5;
        paramsmatrix(n, column+7)=G6;
        paramsmatrix(n, column+8)=G7;
        paramsmatrix(n, column+9)=G8;
        paramsmatrix(n, column+10)=G9;
        Pi=1/G5+omega^eta/(G6*G7)+1/(G1*G8)+omega^eta/(G1*G9*G7);
        paramsmatrix(n, column+11)=Pi;
        H1=braketq(q(s),mu+vartheta);
        H2=braketq(q(s),vartheta);
        G10=qGamma(q(s),eta+mu+vartheta,n);
        paramsmatrix(n, column+12)=H1;
        paramsmatrix(n, column+13)=H2;
        paramsmatrix(n, column+14)=G10;
        Sigma=1/G10+H1*omega^eta/(G6*G7)+H2/(G1*G8)+H2*omega^eta/(G1*G9*G7);
        paramsmatrix(n, column+15)=Sigma;
        Theta=Pi+Sigma/G3 + Sigma/G4;
        paramsmatrix(n, column+16)=Theta;
        paramsmatrix(n,column+17)=lambda*a_1+Δ*a_2+a_3;
        paramsmatrix(n,column+18)=(1-theta*Phi)/Theta;
        paramsmatrix(n,column+19)=1/(G10*(Theta ...
            *(lambda*a_1+Δ*a_2+a_3) + theta*Phi));
        paramsmatrix(n,column+20)=1/(1- (Theta ...
            *(lambda*a_1+Δ*a_2+a_3) + theta*Phi));
    end;
    column=column +21;
end;
```