

Qualitative dynamics and pattern formation of COVID-19 in the modified SIR model

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Abstract SIR (susceptible-infective-recovery) model is a widely investigated model to explain the time evolution of infectious diseases. Outbreak of infectious diseases is affected by diffusion of infected, which is true especially in COVID-19 outbreak. Therefore, it is imperative to construct a diffusion network in the model for spatial consideration; However, the inclusion of a diffusion network is seldom considered for the studies. In this work, we first modified the SIR model for COVID-19 and then performed its stability and bifurcation analysis in qualitative research. Based on our analysis, we propose some of the advice to mitigate the spread of COVID-19. Then, a random diffusion network is constructed, which shows its vital role in the Turing instability and bifurcation. We noticed that the stability of network-organized SIR could be determined by the maximum of eigenvalues of the network matrix. The maximum of eigenvalues of the network matrix is proportional to network

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connection rate and infection rate of the network. Therefore, these two rates play a critical role in Turing instability. We perform the numerical simulations to verify the analytical results. We try to explain the spread mechanism of infectious diseases and provide some feasible strategies based on our analysis of these two models. Also, the reduced system method for a network-organized system is proposed, which is a novel approach to investigate the complex system.

Keywords Diffusion network · Bifurcation · Pattern formation · SIR model

1 Introduction

COVID-19 was first reported at the end of 2019, and within short span of time, it becomes a pandemic[1]. The early transmission dynamics and its assessment were investigated through data driven models[2,3]. However, in such a short interval, the available data is often incomplete or insufficient. Therefore, we need mathematical models to analyze spread and prevention of epidemic and pandemic diseases. Mathematical models have been enhancing our understanding of epidemic qualitatively and quantitatively[4,5,6,7,8,9]. SIR model or its modified versions are often considered to investigate the transmission dynamics of infectious diseases[4]. For example, a delay SIR model with finite incubation times was proposed and shown that incubation times play an important role in the stability of epidemic diseases[5]. The extension of SIR model was investigated, which presents a way to address more realistic behaviours of infectious diseases[6]. Further, a pulse vaccination strategy in a SIR epidemic model has been investigated and it is found that it is more effective than no vaccination and continuous vaccination[7]. Generalized delayed SIR model with vaccination and treatment have shown the success of strategy depended on initial size of the susceptible population[8]. Also, a multi-group SIR epidemic model with stochastic perturbation was discussed and illustrates how the infectious disease prevailed[9]. Global dynamics of SIR model with switched transmission rate was investigated to explain a switching prevention strategy[10]. Recently, a deterministic SIR model was constructed and it is found that passive surveillance is the most effective way of detecting the presence of African swine fever in wild boar[11]. SIR model was also employed to fit the epidemic curve of COVID-19 outbreak[12]. Although, the diffusion network was treated as an important factor of infectious diseases[13,14], the role of the diffusion network in the distribution of the infected was seldom investigated, especially in the SIR model with random network. And Jin, et al showed the importance of random network in the SIR model[15,16]. Therefore, the spatial consideration has been included in this study.

Turing pattern was first proposed in the year 1952, which could show the distribution of different species, and the distribution in space and time is called pattern formation[17]. It was found that cellular networks also played a vital role in the Turing instability, and some extensive networks were proposed as well[18]. Pattern formation was used to understand how a single

and homogeneous egg cell achieves the intricate complexity of the adult[19]. Pattern formation could be explained and predicted through a linear Turing analysis in discrete cell lattices[20]. Recently, some theoretical research methods of network-organized Turing pattern about the un-directed network was provided, which laid the basis of the study of network-organized pattern dynamics[21]. The theory of pattern formation on directed networks was presented[22]. The subsequent research showed key features of Turing systems are determined purely by network topology. Specifically, the network topology affects the stability of the network-organized system[23]. The effect of network topology on the stability of the network-organized system was investigated [24]. However, the pattern dynamics of the SIR model with the diffusion network was never investigated.

Several measures were proposed and discussed, such as the isolation of infected, vaccine, wearing mask, and herd immunity to mitigate the spread of COVID-19,[25]. We still do not know which measures are more proper and which are not. However, we are still in the learning phase about the nature of this spread based on the regions. The delay in our learning and the nature of the spread of COVID-19 limit our knowledge about effective measures. It is well known that human relations in society can be treated as a large network, the epidemic can spread along the network when the COVID-19 occurs. In general, the spreading along the network could induce the outbreak of COVID-19, However, diffusion does not work in the stability of the reaction-diffusion SIR system. Namely, traditional diffusion no longer explains the spread of infectious diseases in SIR model. Therefore, some dynamics mechanisms and biological mechanisms should be proposed to investigate the diffusive behavior of the epidemic. In order to understand the dynamical & biological mechanism of the outbreak of COVID-19 and to explore the effective measures, we will investigate the dynamics behavior of a modified SIR model and the effect of the random network on the distribution of COVID-19. We first propose a modified SIR model based on the characteristic of COVID-19, analyze the reason why the outbreak of COVID-19 occurs, and show the importance of the strategies. Furthermore, we construct a network-organized SIR model, show the connection probability and the infection rate of network play an essential role in Turing instability. In addition, we find the maximum of eigenvalues of the network matrix, which is proportional to the connection probability and the infection rate of the network, affects the stability of the Turing system directly. Turing instability is important due to spatial consideration, which is specifically very important in the case of COVID-19 outbreak.

2 Model of the modified SIR system

A general SIR model[5] can be expressed as

$$\begin{aligned}\frac{dS}{dt} &= \alpha - \beta S - \gamma SI, \\ \frac{dI}{dt} &= \gamma SI - \beta I - \delta I, \\ \frac{dR}{dt} &= \delta I - \beta R.\end{aligned}$$

where S (susceptible), I (infectious), R (removed or recovery) individuals, α is the birth rate of vulnerable groups, β is the death rate, γ is the infection rate, δ is the recovery rate.

However, we all know that the death rate of COVID-19 for S, I, R is different[26]. Meanwhile, the infection rate of diagnosed with COVID-19 is also different from the participant infection rate (i.e., that could infect others), because most of the infected can be isolated. Nevertheless, the incubation of the COVID-19 is difficult to check, and it is infective when it is in the incubation, so the amount of the participant infection is different from the final diagnosis. Due to the characteristic of COVID-19, a modified SIR model can be written as

$$\begin{aligned} \frac{dS}{dt} &= \alpha - \beta_1 S - \gamma_1 SI, \\ \frac{dI}{dt} &= \gamma_2 SI - \beta_2 I - \delta I, \\ \frac{dR}{dt} &= \delta I - \beta_3 R. \end{aligned} \quad (1)$$

where α is the birth rate of vulnerable groups. β_1 is the death rate of the vulnerable groups, β_2 is the death rate of the infected groups, β_3 is the death rate of the recovery groups, in general, $\beta_3 \leq \beta_1 \leq \beta_2$ holds. γ_1 is the infection rate (including both isolated and un-isolated infected individuals), γ_2 is the infection rate of the participant infection(including the incubation and the un-isolated infected individuals), δ is the recovery rate.

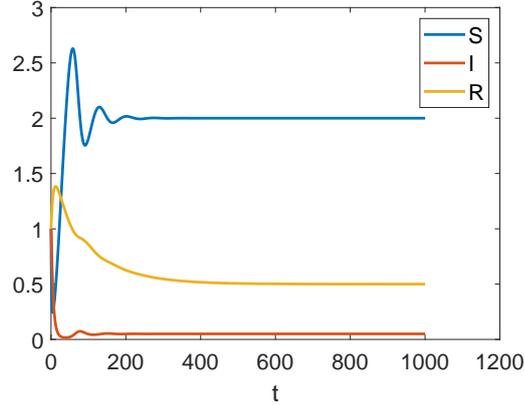


Fig. 1 The equilibrium point E_2 of the system(1) is stable when the default of parameters is $\alpha = 0.1, \beta_1 = 0.01, \beta_2 = 0.1, \beta_3 = 0.01, \gamma_1 = 0.8, \gamma_2 = 0.1, \delta = 0.1$ and initial value $(1, 1, 1)$.

There are two equilibrium points $E_1 = (S^*, I^*, R^*) = (\frac{\alpha}{\beta_1}, 0, 0)$, $E_2 = (S^*, I^*, R^*) = (\frac{\beta_2 + \delta}{\gamma_2}, \frac{\alpha \gamma_2 - \beta_1 \delta - \beta_1 \beta_2}{\gamma_1 (\beta_2 + \delta)}, \frac{\delta (\alpha \gamma_2 - \beta_1 \delta - \beta_1 \beta_2)}{\gamma_1 \beta_3 (\delta + \beta_2)})$ in the system(1). In general, the equilibrium point E_1 means the COVID-19 disappears, and the equilibrium point E_2 means the outbreak of the COVID-19. Then, we analyze the

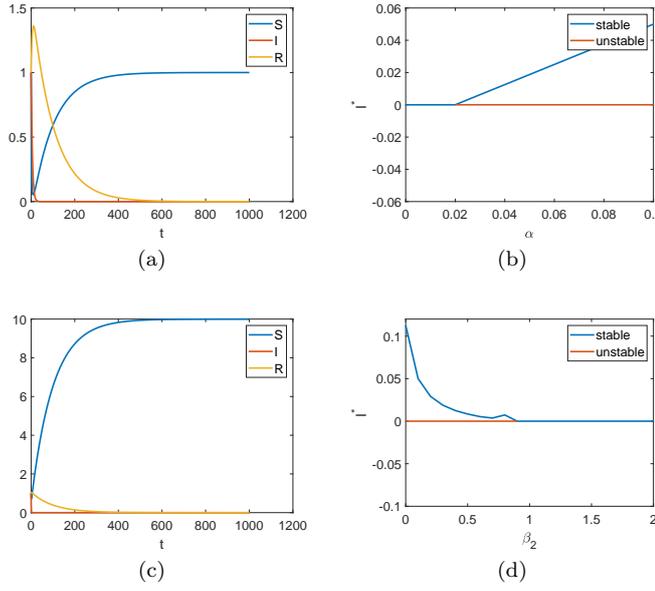


Fig. 2 The stability and bifurcation of the system(1).(a)The equilibrium point E_1 is stable when $\alpha = 0.01$.(b)The bifurcation about α .(c)The equilibrium point E_1 is stable when $\beta_2 = 1.5$.(d)The bifurcation about β_2

stability of E_1, E_2 in the following, and the corresponding Jacobian matrix is

$$B = \begin{pmatrix} -\beta_1 - \gamma_1 I^* & -\gamma_1 S^* & 0 \\ \gamma_2 I^* & \gamma_2 S^* - \delta - \beta_2 & 0 \\ 0 & \delta & -\beta_3 \end{pmatrix}.$$

And then the corresponding characteristic equation can be obtained

$$A = |B - \lambda E| = (\lambda + \beta_3)[(\lambda + \beta_1 + \gamma_1 I^*)(\lambda - \gamma_2 S^* + \delta + \beta_2) + \gamma_1 \gamma_2 S^* I^*] = 0.$$

For the equilibrium E_1 , the corresponding characteristic equation is

$$A = (\lambda + \beta_3)(\lambda + \beta_1)(\lambda - \gamma_2 S^* + \delta + \beta_2) = 0$$

and $\lambda_1 = -\beta_1, \lambda_2 = -\beta_3, \lambda_3 = \gamma_2 \frac{\alpha}{\beta_1} - \delta - \beta_2$

For the equilibrium E_2 , the corresponding characteristic equation is

$$A = (\lambda + \beta_3)[\lambda^2 + (\beta_1 + \gamma_1 I^*)\lambda + \gamma_1 \gamma_2 S^* I^*] = 0.$$

and $\lambda_1 = -\beta_3, \lambda_{2,3} = \frac{-(\beta_1 + \gamma_1 I^*) \pm \sqrt{(\beta_1 + \gamma_1 I^*)^2 - 4\gamma_1 \gamma_2 S^* I^*}}{2}$.

There are two states for epidemic in this paper, the outbreak of COVID-19 and eventually disappear with time. The outbreak of COVID-19 occurs when the system does not stay at the equilibrium E_1 , namely the pattern formation is heterogeneous. COVID-19 will disappear when the system stays at the

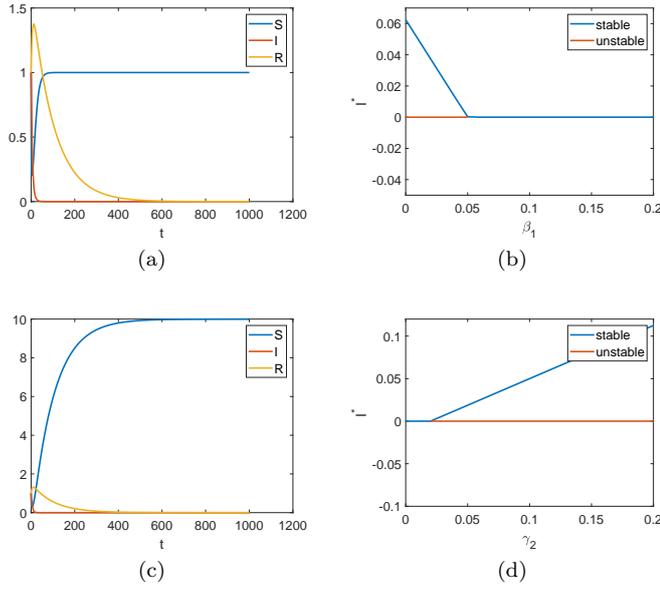


Fig. 3 The stability and bifurcation of the system(1).(a)The equilibrium point E_1 is stable when $\beta_1 = 0.1$.(b)The bifurcation about β_1 .(c)The equilibrium point E_1 is stable when $\gamma_2 = 0.01$.(d)The bifurcation about γ_2

equilibrium E_1 , and the corresponding pattern formation show its homogeneous distribution. Namely, E_1 is stable when $\alpha\gamma_2 \leq \beta_1\delta + \beta_1\beta_2$, E_2 is stable when $\alpha\gamma_2 \geq \beta_1\delta + \beta_1\beta_2$. It is well known that the infectious diseases can be effectively controlled when the equilibrium E_1 of system (1) is stable and the infectious diseases will spread far and wide when the equilibrium E_2 is stable. In general, it is necessary to take some measures to prevent the outbreak of the infectious diseases when the outbreak of COVID-19 occurs.

In this paper, a qualitative approach was applied to explain the outbreak of COVID-19. Some measures were provided based on the theoretical analysis and turned out to be effective for the prevention of COVID-19. The parameter values are set to be consistent with qualitative situation, in general, the death rate of infectious groups is higher than the death rate of vulnerable groups.

The default parameters are $\alpha = 0.1$, $\beta_1 = 0.01$, $\beta_2 = 0.1$, $\beta_3 = 0.01$, $\gamma_1 = 0.8$, $\gamma_2 = 0.1$, $\delta = 0.1$ and initial value $(1, 1, 1)$, and the equilibrium E_2 is stable [Fig.1], which means there is outbreak of COVID-19. As we all know that the birth rate α of vulnerable groups will become lower when the resistance of most people for the infectious diseases is strong (or the vaccine strategy is efficient). Meanwhile, the equilibrium E_1 of the system (1) will tend to be stable for lower birth rate $\alpha = 0.01$, and the infectious diseases disappear quickly[Fig.2(a)]. Therefore, the birth rate α of vulnerable groups depends on the degree of immunity for COVID-19. Namely, the herd immunity plays a vital role in the prevention of infection diseases, because in this case most of the people

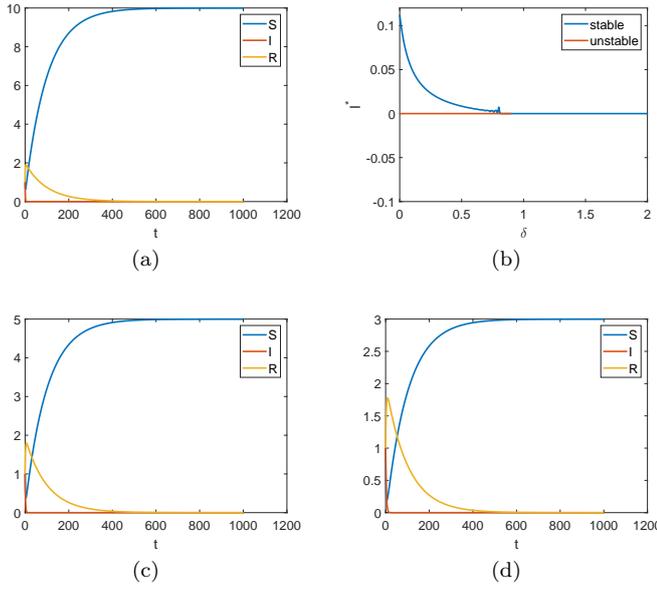


Fig. 4 The stability and bifurcation of the system(1).(a)The equilibrium point E_1 is stable when $\delta = 1$.(b)The bifurcation about δ .(c)The equilibrium point E_1 is stable when $\alpha = 0.05$, $\delta = 0.5$.(d)The equilibrium point E_1 is stable when $\alpha = 0.03$, $\gamma_2 = 0.05$, $\delta = 0.3$.

are immune to the disease. However, if the larger population is vulnerable for COVID-19, which also means the herd immunity does not work in this case and α is larger. [Figure 2(b) shows the stability of the equilibrium about α]. Above all, herd immunity strategy depends on the ratio of vulnerable groups [2(b)]. Also, some measures can be taken, for example, the isolation measure can be adopted; The vulnerable groups should decrease the aggregation during the epidemic. Although the total number of vulnerable people can not be reduced, it turns out that the susceptible rate can be decreased. Therefore, it should be an appeal to the government to optimize the health care system, which could provide security for more people. Also, it is a crucial preparation for dealing with the epidemic in the future.

We are more concerned about the death rate β_2 of the infected people. Although the higher death rate can induce the disappearance of the infectious diseases [Fig. 2(c,d)], it is an unacceptable way to prevent the infectious diseases. If we increase the death rate β_2 from 0.01 to 0.1, we can stop the outbreak of disease, however this is an unacceptable strategy [Fig. 3(a,b)]. The isolation is an important and most acceptable measure to prevent the further spread of infectious diseases, which cuts off the transmission of infectious diseases and reduces the source. γ_1 includes the isolated infected individuals and the unisolated infected individuals, γ_2 includes the incubation and the unisolated infected individuals. And γ_1 is greater than γ_2 when the incubation is few; γ_1 is less than γ_2 when the incubation is relatively much more. And

the unisolated infected individuals no longer have any infections, namely the unisolated infected individuals is not considered in the infection rates. And what's actually involved is γ_2 (the incubation and the unisolated infected individuals) due to the isolation. Also, the theoretical results show γ_1 does not affect the stability of the equilibrium E_1 . It means if we decrease infection rate of the participant infection (i.e. γ_2) from 0.1 to 0.01, the outbreak disappears, which we can see in the [3(c)]. Moreover, the outbreak of infectious diseases may occur when the amount of the incubation is large, because the incubation of COVID-19 could infect others. Therefore testing incubation people in a large scale and the isolation of the infected are an important strategy for the prevention of infection, especially some places where the number of infected grows exponentially[3(d)].

For the recovery rate δ of the infected, the lower recovery rate is not good for the prevention of infection[1]. The 10 fold increase of recovery rate (δ) (i.e. from 0.1 to 1.0) could make the infectious diseases disappear quickly[4(a)]. We notice that the recovery rate is a vital factor to prevent the outbreak[4(b)]. The higher recovery rate (including self-cure) could also lead to the generation of herd immunity, which in turn enhances immune system of the recovered to the COVID-19. The above analysis gives us very important insights for the dynamics of the outbreak of the COVID-19 and based on this analysis, we can suggest some of the strategy. For example, reducing the birth rate of the vulnerable and improving the rate of recovery could make the system stable[4(c)], these two combined measures may control the spread of the COVID-19. So the comprehensive strategy effectively stop the spread, which a single measure can not achieve efficiently[4(d)]. It's also explained that the effectiveness of the vaccine and isolation strategy. Although the high efficacy drugs are difficult to obtain for this current epidemic, it is a crucial preparation for dealing with the epidemic in the future.

3 Model of the network-organized SIR system

It is well known that the outbreak of infectious diseases is affected by the diffusion of the infected, especially valid in spread of the COVID-19. Considering this important factor, the system (1) can be rewritten as

$$\begin{aligned} \frac{\partial S}{\partial t} &= \alpha - \beta_1 S - \gamma_1 S I, \\ \frac{\partial I}{\partial t} &= \gamma_2 S I - \beta_2 I - \delta I + d\nabla^2 I, \\ \frac{\partial R}{\partial t} &= \delta I - \beta_3 R. \end{aligned} \quad (2)$$

and the corresponding Jacobian matrix is

$$B = \begin{pmatrix} -\beta_1 - \gamma_1 I^* & -\gamma_1 S^* & 0 \\ \gamma_2 I^* & \gamma_2 S^* - \delta - \beta_2 - dk^2 & 0 \\ 0 & \delta & -\beta_3 \end{pmatrix}.$$

In order to consider the Turing instability of system (2), we suppose the equilibrium E_1, E_2 of system (1) is stable. Then the characteristic equation for the

equilibrium E_1 is

$$A = (\lambda + \beta_3)(\lambda + \beta_1)(\lambda - \gamma_2 S^* + \delta + \beta_2 + dk^2) = 0$$

And $\lambda_{k1} = -\beta_1, \lambda_{k2} = -\beta_3, \lambda_{k3} = \gamma_2 \frac{\alpha}{\beta_1} - \delta - \beta_2 - dk^2$. Namely $\lambda_{k1} < 0$ for E_1 , the same is true for E_2 , where

$$\lambda_{k1} = -\beta_3, \lambda_{k2,k3} = \frac{-(dk^2 + \beta_1 + \gamma_1 I^*) \pm \sqrt{(dk^2 + \beta_1 + \gamma_1 I^*)^2 - 4(dk^2(\beta_1 + \gamma_1 I^*) + \gamma_1 \gamma_2 S^* I^*)}}{2}.$$

Because, Turing instability occurs only when a $\lambda_{ki} > 0 (i = 1, 2, 3)$ holds, it never happen in this system(2). Namely, diffusion can not induce the outbreak of COVID-19. In fact, traditional diffusion no longer explains the spread of infectious diseases in SIR model. Therefore, we try to transfer equation (2) into the network scheme. Given the probability p (the infectious connection probability means the number of the way of infection), the number of nodes N (the number of populations or regions). From the 1st people to the N th people, that everyone infects each other can be treated as a random process, which can construct a network by p and N . First, a random value r is chosen in every loop(for $i = 1 : N$ and $j = i + 1 : N$); Then if $r < p$, that means the infection exists between i th and j th, namely node i and j link each other; Finally, we repeat the above process till a symmetric random network is constructed.

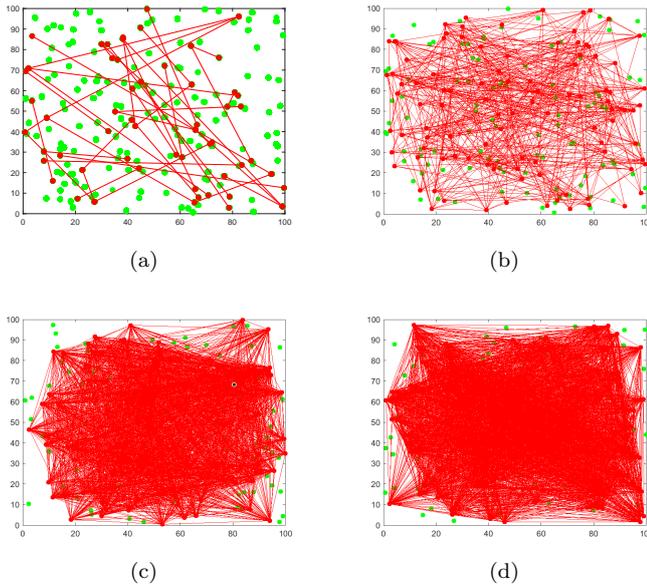


Fig. 5 The infection of network(Red represent the infected and green represent the immune). (a) The network is generated by $p = 0.01$.(b) The network is generated by $p = 0.1$.(c) The network is generated by $p = 0.5$.(d) The network is generated by $p = 0.7$.

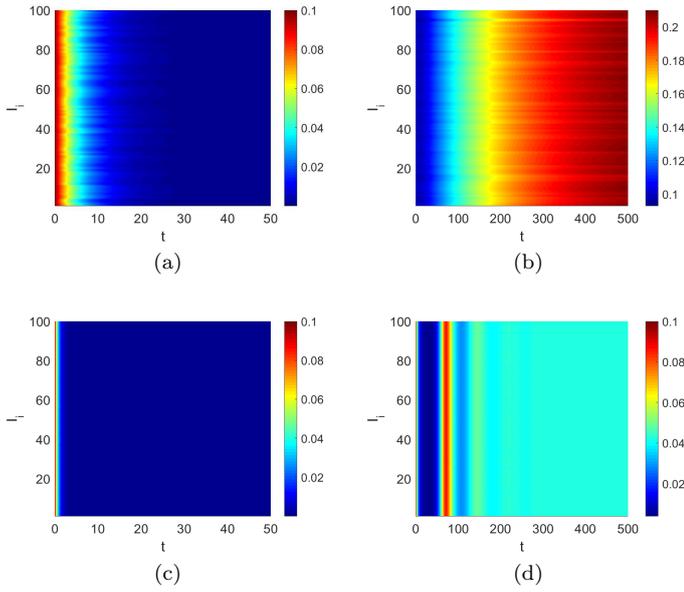


Fig. 6 The stability of the system(3) about p when $q = 0.02$. (a) The system is stable when $p = 0.01, \alpha = 0.01$.(b) Turing instability occurs when $p = 0.1, \alpha = 0.01$.(c) The system is stable when $p = 0.1, \beta_2 = 1.5$.(d) Turing instability occurs when $p = 0.7, \beta_2 = 1.5$

From the construction process, we know the COVID-19 infectious probability is low when p is small. The infected network is sparse[5(a)], which means the immunity of most people for the infectious diseases is strong. Herd immune can be considered to prevent the spread of COVID-19, because most people is immune for COVID-19.

On the other hand, the infectious probability of people will be high when p is not small, which means larger population do not have immunity[5(b,c,d)], and the outbreak of COVID-19 will occur quickly. In other words, the herd immunity does not work when the infectious probability is high. This also shows that behavior of infection is not only closely related to the dynamics of SIR model, but also the infection network. Here, we combine the dynamics of SIR model and the infection network to investigate the dynamical behavior of infection. For the above network, we can transfer it into a symmetric matrix A . First, a null matrix A is given; Then if node i and j connects in a loop, $A(i, j) = 1, A(j, i) = 1$ are assigned. Finally, we repeat the above process till a symmetric matrix A is obtained. Suppose r_i is the i th row sum(degree)of A , and $A = A - \mathbf{diag}(r_1, \dots, r_N)$ (the degree) is a Laplacian matrix(the elements of the main diagonal represents the spreading rate of network through diffusion and the other non-zero element is the infection rate by the receipt of diffusion). We treat matrix A as a diffusion matrix, meanwhile the number of the infected is invariable (all row sum are zero). Finally, assume $A = A + q\mathbf{diag}(r_1, \dots, r_N)$ and q is infection rate through network(we know that the amount of susceptibility,

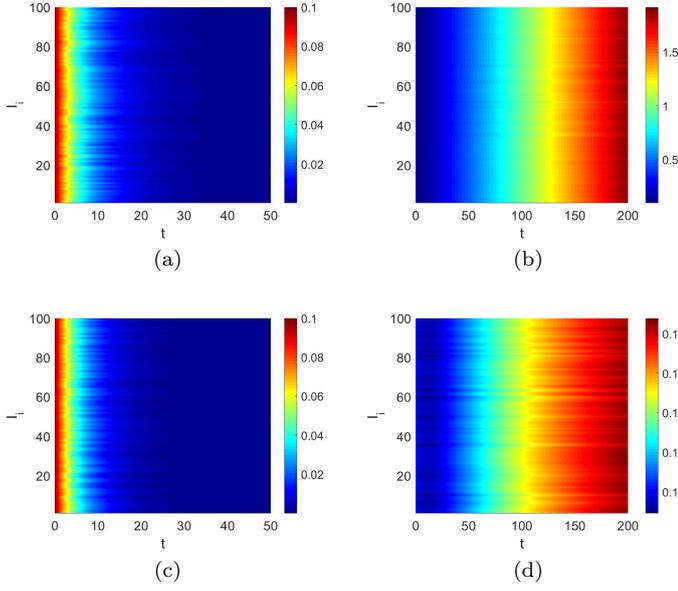


Fig. 7 The stability of the system(3) about p when $q = 0.02$.(a) The system is stable when $p = 0.01, \beta_1 = 0.1$.(b)Turing instability occurs when $p = 0.1, \beta_1 = 0.1$.(c) The system is stable when $p = 0.01, \gamma_2 = 0.01$.(d)Turing instability occurs when $p = 0.1, \gamma_2 = 0.01$

infection and recovery varies in the infectious process, we redefine the ratio of the spreading rate and the receipt in the diffusive matrix). Here the social distance p is different from the isolation in this paper, although the social distance may decrease the infection rate, it could infect some people, even it may not work for a highly contagious disease q . The isolation here means there is no contact.

In general, the number of the infected will increase by diffusion. After this consideration, the system(2)(suppose the infected is not isolated and move freely in 1st place) on network can be written as

$$\begin{aligned} \frac{dS_i}{dt} &= \alpha - \beta_1 S_i - \gamma S_i I_i, \\ \frac{dI_i}{dt} &= \gamma S_i I_1 - \beta_2 I_i - \delta I_i + d \sum_{j=1}^N A_{ij} I_j, \\ \frac{dR_i}{dt} &= \delta I_i - \beta_3 R_i. \end{aligned} \quad (3)$$

where $i = 1, \dots, N$, only the infected in the first city(or place) affect others.

And we obtain

$$B = \begin{pmatrix} -\beta_1 - \gamma_1 I^* & -\gamma_1 S^* & 0 \\ \gamma_2 I^* & \gamma_2 S^* - \delta - \beta_2 + d\Lambda_k & 0 \\ 0 & \delta & -\beta_3 \end{pmatrix}.$$

Then the corresponding characteristic equation is

$$A = |B - \lambda E| = (\lambda + \beta_3)[(\lambda + \beta_1 + \gamma_1 I^*)(\lambda - \gamma_2 S^* + \delta + \beta_2 - d\Lambda_k) + \gamma_1 \gamma_2 S^* I^*] = 0.$$

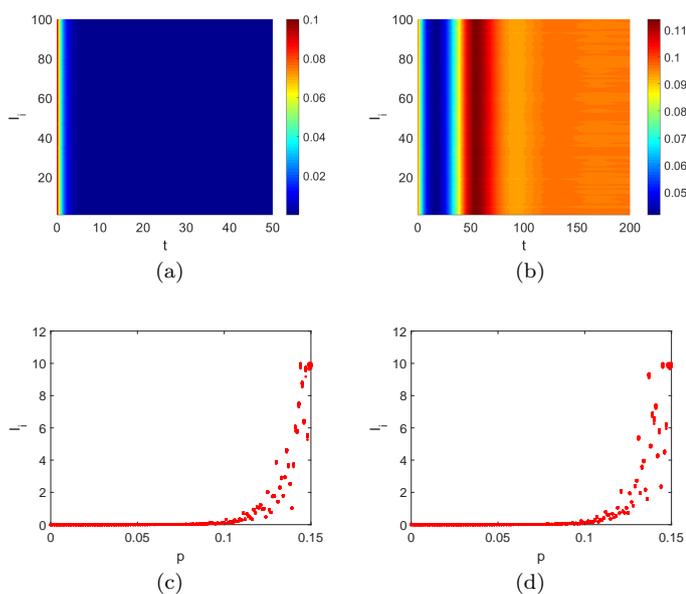


Fig. 8 The stability of the system(3) about p when $q = 0.02$.(a) The system is stable when $p = 0.1, \delta = 1$.(b) Turing instability occurs when $p = 0.5, \delta = 1$.(c) The bifurcation about p when $\gamma_2 = 0.01$.(d) The bifurcation about p when $\alpha = 0.01$.

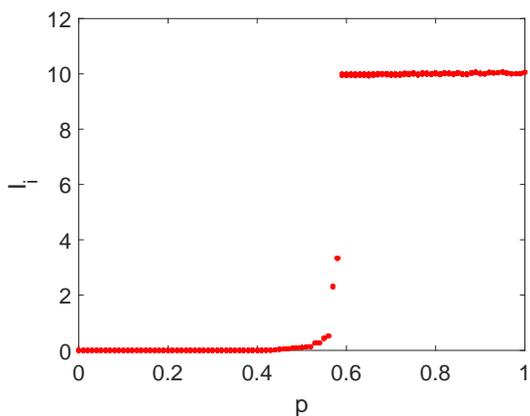


Fig. 9 The bifurcation about p when $\delta = 1$ and $q = 0.02$.

For the equilibrium E_1 , the corresponding characteristic equation is

$$A = (\lambda + \beta_3)(\lambda + \beta_1)(\lambda - \gamma_2 S^* + \delta + \beta_2 - d\Lambda_k) = 0$$

and $\lambda_{k1} = -\beta_1, \lambda_{k2} = -\beta_3, \lambda_{k3} = \gamma_2 \frac{\alpha}{\beta_1} - \delta - \beta_2 + d\Lambda_k$. So Turing instability occurs when $d\Lambda_k > \delta + \beta_2 - \gamma_2 \frac{\alpha}{\beta_1}$ holds.

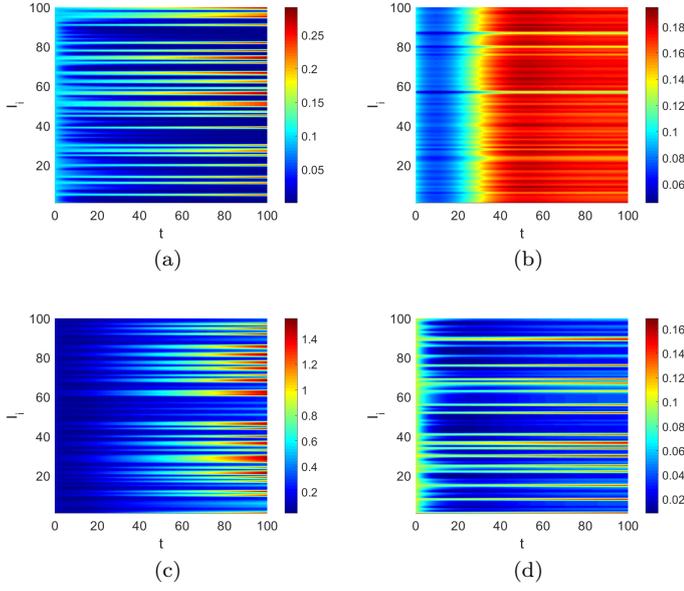


Fig. 10 The stability of the system(3) about q (a) Turing instability occurs when $p = 0.01, q = 0.1, \alpha = 0.01$.(b) Turing instability occurs when $p = 0.1, q = 0.15, \beta_2 = 1.5$.(c) Turing instability occurs when $p = 0.01, q = 0.1, \beta_1 = 0.1$.(d) Turing instability occurs when $p = 0.01, q = 0.1, \gamma_2 = 0.01$

For the equilibrium E_2 , the corresponding characteristic equation is

$$A = (\lambda + \beta_3)[(\lambda + \beta_1 + \gamma_1 I^*)(\lambda - d\Lambda_k) + \gamma_1 \gamma_2 S^* I^*] = 0.$$

and $\lambda_1 = -\beta_3, \lambda_{2,3} = \frac{-(-d\Lambda_k + \beta_1 + \gamma_1 I^*) \pm \sqrt{(-d\Lambda_k + \beta_1 + \gamma_1 I^*)^2 - 4(-d\Lambda_k(\beta_1 + \gamma_1 I^*) + \gamma_1 \gamma_2 S^* I^*)}}{2}$.
So Turing instability occurs when $d\Lambda_k > \beta_1 + \gamma_1 I^*$ or $\frac{\gamma_1 \gamma_2 S^* I^*}{\beta_1 + \gamma_1 I^*} < d\Lambda_k < \beta_1 + \gamma_1 I^*$ holds.

From the above analysis, we know the eigenvalues of network matrix play an important role in the stability of the network-organized system (3), especially the maximum eigenvalue. And Turing instability occurs when $d\Lambda_k > \delta + \beta_2 - \gamma_2 \frac{\alpha}{\beta_1} > 0$. So the stability of system (3) depends on

$$\begin{aligned} \frac{dS_i}{dt} &= \alpha - \beta_1 S_i - \gamma S_i I_i, \\ \frac{dI_i}{dt} &= \gamma S_i I_1 - \beta_2 I_i - \delta I_i + d\Lambda_i I_i, \\ \frac{dR_i}{dt} &= \delta I_i - \beta_3 R_i. \end{aligned} \quad (4)$$

Namely Turing instability occurs when $d\Lambda_{max} > \delta + \beta_2 - \gamma_2 \frac{\alpha}{\beta_1} > 0$ holds. And the stability of system (4) with $i = 1$ can be determined by the following system

$$\begin{aligned} \frac{dS}{dt} &= \alpha - \beta_1 S - \gamma SI, \\ \frac{dI}{dt} &= \gamma SI - \beta_2 I - \delta I + d\Lambda_{max} I, \\ \frac{dR}{dt} &= \delta I - \beta_3 R. \end{aligned} \quad (5)$$

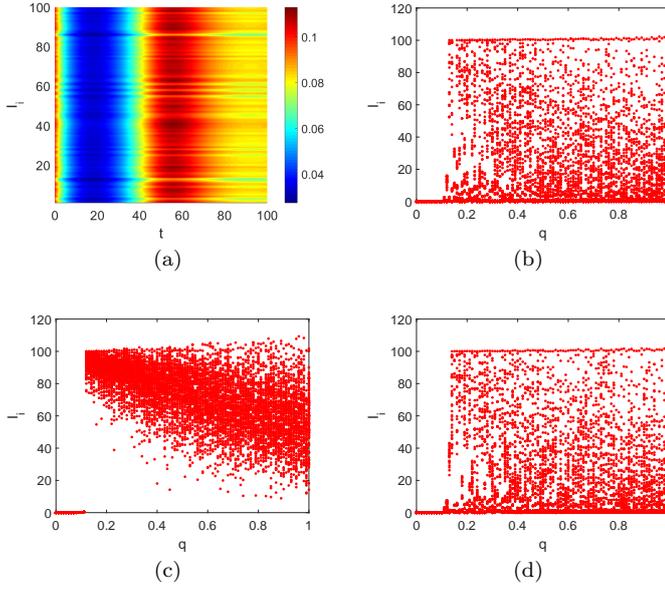


Fig. 11 The stability of the system(3) about q (a) Turing instability occurs when $p = 0.1, q = 0.1, \delta = 1$.(b)The bifurcation about q when $p = 0.01, \gamma_2 = 0.01$.(c) The bifurcation about q when $p = 0.1, \delta = 1$.(d)The bifurcation about q when $p = 0.01, \alpha = 0.01$

where Λ_{max} is the maximum of Λ_i , and it is the reduced system for a network-organized system.

Although the network matrix is generated randomly with p , the range of its eigenvalues can be estimated. Based on the Gershgorin circle theorem[28], A is a $N \times N$ matrix, $R_i = \sum_{i \neq j} |a_{ij}|$ and $D(a_{ii}, R_i) = \{\Lambda_i | |\Lambda_i - a_{ii}| \leq R_i\}$. Therefore every eigenvalue of A lies within at least one of the Gershgorin discs $D(a_{ii}, R_i)$. Here $R_i = r_i, a_{ii} = (q-1)r_i$, namely the eigenvalue $qr_i - 2r_i \leq \Lambda_i \leq qr_i \approx Npq$, and $\Lambda_{max} \approx Npq$. So the system (5) is approximate to the following system,

$$\begin{aligned} \frac{dS}{dt} &= \alpha - \beta_1 S - \gamma SI, \\ \frac{dI}{dt} &= \gamma SI - \beta_2 I - \delta I + dgNpqI, \\ \frac{dR}{dt} &= \delta I - \beta_3 R. \end{aligned} \quad (6)$$

where g is uncertain and the system is unstable when $dgNpq > \delta + \beta_2 - \gamma_2 \frac{\alpha}{\beta_1} > 0$, otherwise, it is stable.

Based on the numerical simulation of (1), Turing instability of system (3) is investigated, which means the outbreak of COVID-19. It is well known that the social network and trans-regional diffusion of COVID-19 can induce the increase of the infected, even global outbreak. In contrast, the COVID-19 will approach to zero when the social network and trans-regional diffusion is small [Fig.6-Fig.13]. In order to further prevent the spread of COVID-19, some measure should be taken. In our simulation, the system of (2) is stable when all

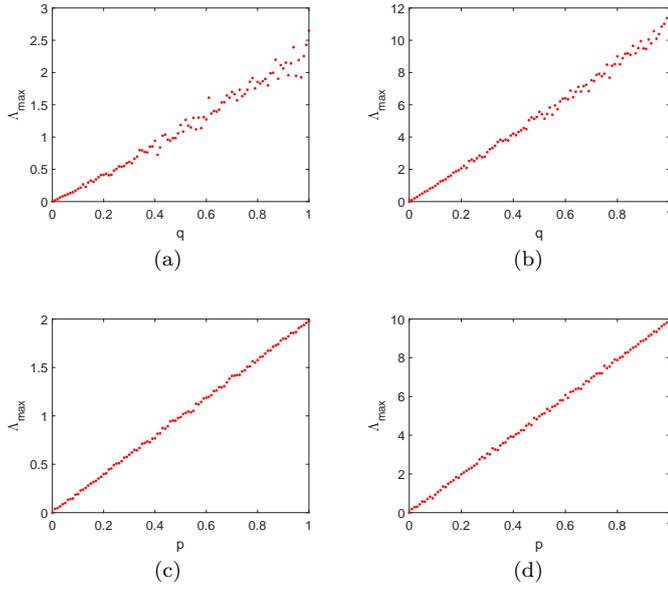


Fig. 12 The relationship of Λ_{max} and p, q . (a) The relationship of Λ_{max} and q when $p = 0.01$. (b) The relationship of Λ_{max} and q when $p = 0.1$. (c) The relationship of Λ_{max} and p when $q = 0.02$. (d) The bifurcation about p when $q = 0.1$.

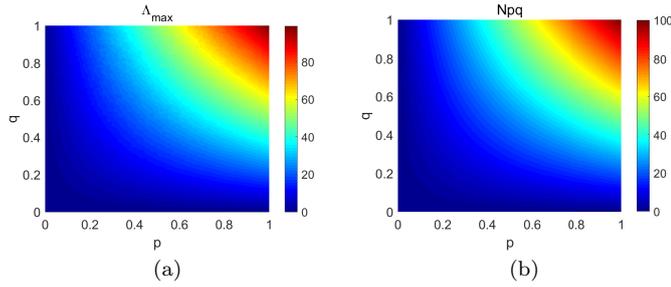


Fig. 13 The distribution of Λ_{max}, Npq about p and q . (a) The distribution of Λ_{max} about p and q . (b) The distribution of Npq about p and q .

I_i tend to zero and the system of (2) is unstable (Turing instability) when a I_i does not approach zero. For example, we can make the infected of COVID-19 gradually disappear by decreasing the susceptible rate ($\alpha = 0.01$) and the social contacts ($p = 0.01$) when the social contacts exist in the system [Fig. 6(a) shows the pattern formation is stable]. The excessive contacts ($p = 0.1$) of the infected lead to the more infected [Fig. 6(b) and it shows the occurrence of Turing instability]. Although the higher death rate of the infected could cut off the resource of the infection ($p = 0.1, \beta_2 = 1.5$), and make the infected approach to zero [Fig. 6(c)]. Surprisingly, even if death rate ($\beta_2 = 1.5$) is high,

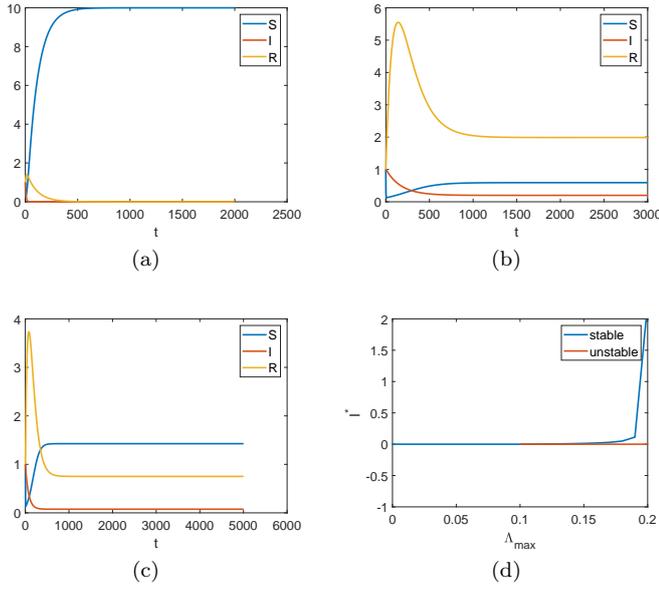


Fig. 14 The stability of the system(5). (a) The equilibrium point E_1 is stable when $\gamma_2 = 0.01, p = 0.01, q = 0.02$ and $\Lambda_{max} = 0.0362$. (b) The equilibrium point E_1 is unstable when $\gamma_2 = 0.01, p = 0.1, q = 0.02$ and $\Lambda_{max} = 0.2057$. (c) The equilibrium point E_1 is unstable when $\gamma_2 = 0.01, p = 0.01, q = 0.1$ and $\Lambda_{max} = 0.2179$. (d) The bifurcation about Λ_{max}

but through human exposures ($p = 0.7$) of COVID-19 spreads rapidly, which lead to more death eventually [Fig. 6(d)]. Meanwhile so is the death rate of the susceptible [Fig. 7(a,b)]. In ancient times, maybe increasing the death rate was a way to prevent the spread of infection diseases, because the medical constraints and the poor transportation. However, human civilisation is now advanced with better technological innovations. The isolation can effectively prevent the spread of disease [Fig. 7(c)]. However, just a few infected cause an infection in the whole community through social contact [Fig. 7(d)]. Therefore, the isolation and the decrease of social contacts is necessary to stop the spread of COVID-19. Moreover, increasing the recovery (self-cure) rate is the best way to prevent the spread of COVID-19 [Fig. 8(a)]. It is difficult to achieve due to the development of a vaccine. The other option is to live with COVID-19. If no symptoms or mild symptoms for most of the infected exists, or most of the infected can recover quickly by the treatment, eventually almost all the people live with COVID-19 [Fig. 8(b)]. So the herd immunity also depends on a strong health care system. In addition, Fig. 8(c,d) and Fig. 9 show the variation of the equilibrium about p . The equilibrium gradually becomes nonzero with the variation of p [Fig. 8(c,d)] when we stabilize the system (1) through the isolation (γ_2) and the susceptible rate α . Fig. 9 shows the strategy of increasing the recovery has a better resistance for the spread of COVID-19. By comparing [Fig. 8(c,d)] and [Fig. 9], we know the strategy of increasing the recovery of

the infected is better than the isolation and the decrease of susceptible rate. Because the increase of p means the isolation is not enough and the susceptible rate increases. Now we consider the infection rate q of network under the stable circumstances about p . As we all know that the system is stable when the infection rate $q = 0.02$ of network is small [Fig.6(a,c), Fig.7(a,c), Fig.8(a)]. Fig.10 and Fig.11(a) show the Turing instability induced by q , namely Turing instability occurs when the infection rate of network becomes larger [Fig.10, Fig.11(a)]. But the condition that the strategy of increasing the death rate stabilize the system (1) [Fig.10(b,c)] is worse than other ways [Fig.10(a,d), Fig.11(a)]. Namely the infection rate q of network also plays an important role in the system (3). Then we obtain the bifurcation of I_i about p, q [Fig.11(b,c,d) show the distribution of I_i about q]. From the numerical results, we know the isolation strategy [Fig.11(c)] is better at preventing the network diffusion of infection diseases. Although increasing the recovery rate [Fig.11(b)] and reducing the susceptible rate [Fig.11(d)] is efficient at the beginning, the infected will increase rapidly when the infection rate of network becomes large [Fig.11(b,d)]. In addition, it is found that p, q is proportional to Λ_{max} [Fig.12, Fig.13]. And it approaches the linearity relationship between Λ_{max} and q when $p = 0.01$ or $p = 0.1$ is fixed [Fig.12(a,b)], so is it when $q = 0.02$ or $q = 0.1$ is fixed [Fig.12(c,d)]. In other word, the magnitude of p, q affects the stability of system (5). the similar distribution [Fig.13] of Λ_{max} and Npq shows $\Lambda_{max} \approx Npq$ holds. And the distribution of Λ_{max} and Npq about p, q also verifies that the system (5) and the system (6) are approximately equal [Fig.13]. Also, the stability of system (5) is consistent with the stability of system (3) [Fig.14]. Such as Fig.7(c,d), Fig.11(d) correspond to Fig.14(a,b,c), respectively, which has the same stability about Λ_{max} and Npq between system (5) and system (3). Meanwhile the bifurcation about Λ_{max} accords with the condition of Turing instability of system (5) [Fig.14(d)]. In summary, although the infected of one city are not isolated, it also could induce the secondary outbreak through the network. Finally, the evolution of the infected cases of the COVID-19 epidemic through health commission of Hebei province in China from 2020.01.29 to 2020.1.9 is analyzed. From Fig.15, the infected cases increase at the beginning, and decrease when some measures are taken. However, secondary outbreak occurs because of the diffusion network (like the imported cases, the effect of other cities). Namely, the network diffusion plays a vital role in the prevention of COVID-19.

4 Conclusion

In order to explain the mechanism of the spread of COVID-19, and to predict the effective strategy, a qualitative approach was applied to explain the outbreak of COVID-19 due to the characteristic of COVID-19, and some measures were provided and turned out to be effective for the prevention of COVID-19. We first show the stability of the modified SIR model under different conditions and analyze the feasibility of some measures for COVID-19 in the actual situation. It is found that the isolation measure is an important ap-

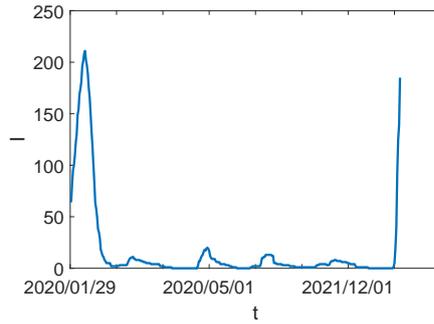


Fig. 15 The infected cases of the COVID-19 epidemic through health commission of Hebei province in China from 2020.01.29 to 2021.1.9

proach to reduce the infection rate: Reducing the birth rate of the vulnerable by isolation and improving the rate of recovery by good health care systems could make the system stable (COVID-19 disappears), and which shows that these are effective strategies. Then we show the effect of random network on the distribution of COVID-19 in a network-organized SIR model. And we find the connection probability and the infection rate of network play an essential role in Turing stability. We have seen that the infectious connection probability, which means the number of the way of infection, is important. Higher connection probability, which means lower social distancing, leads to Turing instability. Therefore, social distancing, along with higher recovery rates and higher isolation, are effective measures. Also, we find the maximum of eigenvalues of the network matrix, which is proportional to the connection probability and the infection rate of the network, affects the stability of the Turing system directly. Above all, some measures and methods proposed may play a vital role in the outbreak of COVID-19 and Turing instability based on the theoretical analysis. In summary, although the infected of one city are not isolated, it also could induce the secondary outbreak through the network. Also, the reduced system method for a network-organized system is proposed, which is a novel approach to investigate the complex system. Finally, It will be better to carry out analysis based on the literature or estimated from real data. However, the SIR model is considered on network, which make the estimation and collection of real data more difficult. We only show the effect of the diffusion network on the COVID-19 epidemic through the evolution of the infected cases of the COVID-19 epidemic in Hebei, China. Because the theoretical analysis of system holds for any parameters, the modified SIR model is valid for COVID-19, and the validity of the model used in this paper, as well as other data-based model, is predicated on having accurate data. Therefore we may do some research about real data in the next study.

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Conflict of interest

The authors declare no conflict of interest.

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