Heteroscedastic Process: A M-quantile Approach

Patrick Ferreira Patrocinio (patrick.ferreira-patrocinio@centralesupelec.fr)
CentraleSupelec/UFES https://orcid.org/0000-0002-8541-2447
Valderio A. Reisen
Federal University of Espirito Santo https://orcid.org/0000-0002-8313-7648
Pascal Bondon
Paris-Saclay Universite https://orcid.org/0000-0002-5158-7337
Edson Z. Monte
Federal University of Espirito Santo https://orcid.org/0000-0002-6878-5428
Ian M. Danilevicz
CentraleSupelec/UFMG https://orcid.org/0000-0003-4541-0524

Research Article

Keywords: GARCH, M-quantile, Robust, Outliers

Posted Date: July 12th, 2022

DOI: https://doi.org/10.21203/rs.3.rs-1848078/v1

Heteroscedastic Processes: A M-Quantile Approach

Patrick F. Patrocinio [1,3*], Valderio A. Reisen [1,2,3†], Pascal Bondon [3†], Edson Z. Monte [1†] and Ian M. Danilevicz [2,3†]

[1*]PPGEco and Department of Economics, Federal University of Espírito Santo, Av. Fernando Ferrari, Vitória, 29075-910, Espírito Santo, Brazil.

[2]DEST and Department of Statistics, Federal University of Minas Gerais, Av. Pres. Antônio Carlos, Belo Horizonte, 31270-901, Minas Gerais, Brazil.

[3]Laboratoire des Signaux et Systèmes, Université Paris-Saclay, CNRS, CentraleSupélec, Joliot Curie, Gif-sur-Yvette 91190, France.

*Corresponding author(s). E-mail(s): patrick.ferreira-patrocinio@centralesupelec.fr; Contributing authors: valderio.reisen@ufes.br; pascal.bondon@centralesupelec.fr; edsonzambon@yahoo.com.br; ian.meneghel-danilevicz@centralesupelec.fr;

†These authors contributed equally to this work.

Abstract

As is well known, outliers are quite common observations in different application areas and these types of data can cause large biases in the estimates of the mean, variance, correlation and, consequently, in the parameter estimates. Thus, robust estimation methods are needed to obtain reliable statistical models. There are empirical evidences that the financial time series and the distributions of returns are not well approximated by Gaussian models, which is an assumption generally considered to model these data. Therefore, both quantile and M-regression methods have been suggested to estimate GARCH model. In this paper, these two methodologies are combined to obtain a robust estimator for conditional volatility. Empirical evidence indicates that the proposed method seems to be more resistant to additive outliers than the M- and Quantile regressor estimators. Some technical issues are addressed, and an application illustrates the usefulness of the method in a real data set.

Keywords: GARCH, ***M***-quantile, Robust, Outliers

1 Introduction

Much attention has been paid to the study of methods in the context of processes that contain atypical observations (outliers) among other particularities. In the case of the effect of outliers in estimating time series models, several authors have shown that the presence of outliers can dramatically deteriorate the estimates of a time series model.

The effects of outliers with respect to model identification, estimation and forecasting time series data depend on the type of outlier. There are four types of outliers proposed in the literature of univariate time series: additive outliers, innovation outliers, level shifts and temporal changes (see, for example, Chen and Liu (1993), Tularam and Saeed (2016), Reisen et al. (2019)).

Additive outlier is quite common in practical problems and seems to be more dangerous than the other outlier types in time series. For example, in the standard structure of Box-Jenkis models, Ledolter (1989) showed that the range predictions in Integrated Autoregressive Moving Averages (ARIMA) models are considerably sensitive to additive outliers. Chang, Tiao, and Chen (1988) and Chen and Liu (1993) demonstrated that the estimated parameters of the ARMA model become biased when the data contains outliers. In the case of long-memory and periodic time series, see, for example, the recent papers by Reisen, Lévy-Leduc, and Taqqu (2017) and Sarnaglia, Reisen, Lévy-Leduc, and Bondon (2021).

The effect of additive outliers on the estimation of heterocedastic models is clearly discussed in Franses and Ghijsels (1999), Mendes (2000), Carnero (2003), Carnero, Peña, and Ruiz (2005) and Carnero, Peña, and Ruiz (2012). These authors study the bias of the sample autocovariance and of the different estimation methods in ARCH and GARCH models. In general, they show that additive outliers can substantially distort the estimation of the parameters of the ARCH(q) and GARCH(p, q) models in the same way as in the standard linear time series models.

As is well discussed in the literature, the estimators derived from the M-regression method are robust alternative approaches to obtain estimates of the parameters in time series contaminated by outliers or generated by probability distributions with heavy tails, see for example Bai, Rao, and Wu (1992), Wu (2007) and Li (2008). In addition to M-regression, the applying of quantile regression method offer some advantages, which are: (i) not depend of the error distribution. (ii) provide useful information about the error distribution. (iii) can indicates the presence of asymmetry on the series, see for example Xiao and Koenker (2009), Lee and Noh (2013) and Zheng, Qianqian, Li, and Xiao (2016).

In this study we consider the M-quantile regression method, proposed by Breckling and Chambers (1988), to estimate the parameters of the GARCH(p,q) model. In this framework, the conditional distribution of the response variable is characterized in terms of different location parameters, the M-quantiles. Although these have a less intuitive interpretation than standard quantiles, M-quantile regression also offers a number of advantages. (i) it easily allows for robust estimation. (ii) it can trade robustness and efficiency in inference by selecting the tuning constant of the influence function. (iii) it offers computational stability due to the wide range of available continuous influence functions with respect to the more standard absolute value used in the quantile regression context.

Since outliers typically appear in microeconomics and financial series, robust estimation methods for estimating heteroscedastic time series models become an important research topic from both applied and theoretical points of view. A thorough search of the relevant literature on robust estimation in heteroscedastic processes indicates that there are not much articles devoted to this topic. According to the best of our knowledge, the most recent literature on the subject are Muler and Yohai (2008), Mukherjee (2008) and Iqbal (2013).

Therefore, to fill part of this gap in this theme, this paper proposes the use of a M-quantile regression to estimate the parameters of the GARCH(p,q) model, denoted in this paper as MQGARCH(p,q). A simulation study is carried out to show the performance of the proposed estimation method in the context of contaminated and non-contaminated heteroscedastic processes with additive outliers. For comparison purpose, the classical Quasi Maximun Likelihood (QML), M-regression and the Quantile methods are considered in the simulation. Daily returns data of two financial indices are used to illustrate the use of the robust method in real problems.

This paper is organized as follows: in Section 2 the GARCH(p,q) model and the M-estimation methods are presented. In Section 3 an empirical robustness study using the M-quantile method is carried out. In Section 4 a real application is reported. Finally, the conclusion of this study is provided in Section 5.

2 The GARCH model and the estimation methods

Let Θ be a compact subset of $(0,\infty)^{p+1} \times (0,1)^q$ where p and q are the maximum nonzero lags in the GARCH(p,q) model. Let a process $\{X_t\}_{\{t\in\mathbb{Z}\}} := \{X_t\}$ with $\mathrm{E}(X_t^4) < \infty$. $\{X_t\}$ is defined as a GARCH(p,q) model, with orders p and q, if satisfies

$$\begin{cases} X_t = \sigma_t \varepsilon_t \\ \sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i X_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2 \end{cases}, \tag{1}$$

where $\omega > 0$, $\alpha_i \geq 0$, and $\beta_j \geq 0$ are constants, $\{\varepsilon_t, t \in \mathbb{Z}\} \sim IID(0,1)$ and ε_t is independent of $\{X_{t-h}, h \geq 1\}$ for all $t \in \mathbb{Z}$. Additionally, we assume that $\sum_{i=1}^{\max(p,q)}(\alpha_i + \beta_i) < 1$, where $\alpha_i = 0$ for $i > p$ and $\beta_j = 0$ for $j > q$. Under these conditions, $\{X_t\}$ becomes a unique strict stationary process (see, e.g., Fan and Yao (2002) and Francq and Zakoian (2019)). Note that if $q = 0$, $\{X_t\}$ is an ARCH(p) model.

Now, let $\boldsymbol{\theta} = (\omega, \alpha_1, ..., \alpha_p, \beta_1, ..., \beta_q)^{'} \in \Theta$. Theorem 1 in Berkes, Hovath, and Kokoszka (2003) establishes the following representation of σ_t^2

$$\sigma_t^2 = c_0 + \sum_{i=1}^{\infty} c_i X_{t-i}^2, \tag{2}$$

where $c_0 = \frac{\omega}{1-\sum_{j=1}^{q}\beta_j}$ and the coefficients $c_i's$ are determined by the equation $\sum_{i=1}^{\infty} c_i z^i = \frac{\sum_{i=1}^{p}\alpha_i \ z^i}{1-\sum_{j=1}^{q}\beta_j z^j}$, z ≤ 1, see, e.g, Section 4.2.2 in Fan and Yao (2002) . Under the model assumptions given above, $c_1, c_2, ...$ decay exponentially fast.

2.1 M-estimators for GARCH process

Using Equation 2, the variance function on Θ of the process in Equation 1 can be defined as

$$v_{X_t}(\boldsymbol{\theta}) = c_0(\boldsymbol{\theta}) + \sum_{j=1}^{\infty} c_j(\boldsymbol{\theta}) X_{t-j}^2 \ , \boldsymbol{\theta} \in \Theta \ , t \in \mathbb{Z}, \tag{3}$$

where the coefficients $\{c_j(\boldsymbol{\theta}), j \geq 0\}$ are given above see, also, Berkes et al. (2003) and Mukherjee (2008).

Remark 1 Under some conditions, Theorem 2.4 in Berkes et al. (2003) shows that $v_{X_t}(\boldsymbol{\theta}) = \sigma_t$, $t \in \mathbb{Z}$, is the unique almost sure representation.

As an example, consider the GARCH(2,2) model with $\boldsymbol{\theta} = (\omega, \alpha_1, \alpha_2, \beta_1, \beta_2)^{'}$. Thus, the coefficients $\{c_j(\boldsymbol{\theta}), j \geq 0\}$ are given as follows

$$c_0(\boldsymbol{\theta}) = \frac{\omega}{1 - \beta_1 - \beta_2} \ , \ c_1(\boldsymbol{\theta}) = \alpha_1 \ , \ c_2(\boldsymbol{\theta}) = \alpha_2 + \beta_1 \alpha_1$$

and

$$c_j(\boldsymbol{\theta}) = \beta_1 c_{j-1}(\boldsymbol{\theta}) + \beta_2 c_{j-2}(\boldsymbol{\theta}), j \geq 3.$$

Other examples are given in Liu and Mukherjee (2020).

Let $X_1, X_2, ..., X_n$ be a sample from the process $\{X_t\}$. Following Mukherjee (2008), the M-estimator of $\boldsymbol{\theta}$, say $\hat{\boldsymbol{\theta}}_{\boldsymbol{M}}$, is the solution of

$$\hat{\boldsymbol{\theta}}_{\boldsymbol{M}} = \underset{\boldsymbol{\theta}\in\Theta}{\operatorname{argmin}}\, \hat{\boldsymbol{M}}_n(\boldsymbol{\theta}), \tag{4}$$

where $\hat{\boldsymbol{M}}_n(\boldsymbol{\theta}) = \sum_{t=1}^{n} \hat{\boldsymbol{m}}_t(\boldsymbol{\theta})$,

$$\hat{\boldsymbol{m}}_t(\boldsymbol{\theta}) = \rho(X_t/\hat{v}_t^{1/2}(\boldsymbol{\theta})) + \frac{1}{2}\log(\hat{v}_t(\boldsymbol{\theta}))\ , \tag{5}$$

$$\hat{v}_t(\boldsymbol{\theta}) = c_0(\boldsymbol{\theta}) + \sum_{j=1}^{t-1} c_j X_{t-j}^2\ , \boldsymbol{\theta} \in \Theta\ , 2 \le t \le n, \tag{6}$$

and $\rho(x)$, $x \in \mathbb{R}$, is the loss-function. Equivalently, $\hat{\boldsymbol{\theta}}_{\boldsymbol{M}}$ can be computed using the derivatitive of $\hat{\boldsymbol{M}}_n(\boldsymbol{\theta})$, say $\hat{\boldsymbol{M}}_n^{'}(\boldsymbol{\theta})$, satisfying the equation

$$\hat{\boldsymbol{M}}_n^{'}(\boldsymbol{\theta}) = \sum_{i=1}^{n} \frac{\hat{v}_{X_t}^{'1/2}(\boldsymbol{\theta})}{2\hat{v}_{X_t}^{1/2}(\boldsymbol{\theta})}\left(1 - H\left(\frac{X_t}{\hat{v}_{X_t}^{1/2}(\boldsymbol{\theta})}\right)\right) = 0, \tag{7}$$

where $H(x) := x\psi(x)$ is the score function, and $\psi(x)$ is the derivative of $\rho(x)$.

The loss-function $\rho(x)$ has to satisfies the following assumptions: (I) $\rho(0) = 0$, (II) $\rho(x) = \rho(-x)$ (or $\psi(x) = -\psi(-x)$), (III) $0 \le x \le x^* \implies \rho(x) \le \rho(x^*)$, (IV) $\sup_x(\rho(x)) < \infty$, and (V) $\rho(x)$ has second derivative almost surely (a.s.).

Remark 2 Under some conditions and as $n \to \infty$, Mukherjee (2008) estates the following asymptotic property on $\hat{\boldsymbol{\theta}}_{\boldsymbol{M}}$:

- $n^{1/2}(\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_H) \to N[0, \sigma^2(H)\boldsymbol{G}^{-1}]$, such that $\boldsymbol{\theta}_H := (c_H\omega, c_H\alpha_1, \cdots, c_H\alpha_p, \beta_1, \cdots, \beta_q)$

where $H(x)$ is defined previously,

$$\sigma^2(H) := 4\ \mathsf{var}\left(H\left(\frac{\varepsilon_t}{c_H^{1/2}}\right)\right) / \operatorname{E}\left(\frac{\varepsilon_t}{c_H^{1/2}} H^{'}\left(\frac{\varepsilon_t}{c_H^{1/2}}\right)\right),$$

$$\boldsymbol{G} := \operatorname{E}\left(v_{X_1}^{'}(\boldsymbol{\theta}_H) v_{X_1}^{'T}(\boldsymbol{\theta}_H)/v_{X_1}^2(\boldsymbol{\theta}_H)\right)$$

and c_H is a positive real number satisfying $\operatorname{E}\left(H\left(\frac{\varepsilon_t}{c_H^{1/2}}\right)\right) = 1$, see, also, Muler and Yohai (2008) and Boudt and Croux (2010).

There are many candidate functions for $H(x)$ (or $\rho(x)$). Here, we will consider the two classical ones, Huber and QML, which are given as follows:

1. $H_H(x) = \begin{cases} x^2, & |x| \le k \\ k|x|, & \text{otherwise} \end{cases}$ and $\rho_H(x) = \begin{cases} \frac{1}{2}x^2, & |x| \le k \\ k|x| - \frac{1}{2}k^2, & \text{otherwise} \end{cases}$, for the Huber estimation method,

2. $H_{QML}(x) = x^2$ and $\rho_{QML}(x) = \frac{1}{2}x^2$, for the QML estimator method, where k is a tuning constant.

2.2 The M-quantile estimator

In the M-quantile regression estimation approach, we consider the loss function proposed by Breckling and Chambers (1988) with Huber loss function $\rho_H(x)$ defined previously. The M-quantile loss function is then given by

$$\rho_\tau(x) = \begin{cases} \rho_H\{(1-\tau)x\} & (x < 0) \\ \rho_H(\tau x) & \text{otherwise} \end{cases}, \tag{8}$$

where τ is the quantile satisfying $0 < \tau < 1$. As an alternative to obtain a M-quantile regression estimator, Breckling and Chambers (1988) also propose the influence function

$$\psi_\tau(x) = \begin{cases} (1-\tau)\psi(x) & (x < 0) \\ \tau\psi(x) & \text{otherwise} \end{cases}. \tag{9}$$

.

Remark 3 Although Equation 8 does not satisfy Assumption II, reasonable results are reported in Section 3. Additionally, as pointed out by Breckling and Chambers (1988), the use of Equation 8 instead 9 provides a better approximation to the sample τth quantile. Therefore, the choice of Equation 8 is justified.

Thus, the M-quantile estimator of $\boldsymbol{\theta}_\tau$, say $\hat{\boldsymbol{\theta}}_{\boldsymbol{M},\boldsymbol{\tau}}$, is the solution of

$$\hat{\boldsymbol{\theta}}_{\boldsymbol{M},\boldsymbol{\tau}} = \underset{\boldsymbol{\theta}_{\boldsymbol{\tau}} \in \Theta}{\operatorname{argmin}} \, \hat{\boldsymbol{M}}_{n,\tau}(\boldsymbol{\theta}_{\boldsymbol{\tau}}),\ 0 < \tau < 1, \tag{10}$$

where $\hat{\boldsymbol{M}}_{n,\tau}(\boldsymbol{\theta}_{\boldsymbol{\tau}}) = \sum_{t=1}^{n} \hat{\boldsymbol{m}}_{t,\tau}(\boldsymbol{\theta}_{\boldsymbol{\tau}})$,

$$\hat{\boldsymbol{m}}_{t,\tau}(\boldsymbol{\theta}_\tau) = \rho_\tau(X_t/\hat{v}_{t,\tau}^{1/2}(\boldsymbol{\theta}_{\boldsymbol{\tau}})) + \frac{1}{2}\log(\hat{v}_{t,\tau}(\boldsymbol{\theta}_\tau)) \tag{11}$$

and, for each τ, $\hat{v}_{t,\tau}(\cdot)$ is computed as in Equation 6.

Now, the robustness property of the M-quantile regression method is discussed. Firstly, suppose that ρ satisfies the following assumption:

(VI) $\rho(x)$ is convex and $\psi(-\infty) < 0 < \psi(\infty)$. Note that the Huber loss function satisfy the assumptions (I)-(VI), see for example Huber (1964, 1984)

Denote by $\hat{\epsilon}_A$ the additional breakdown point and $\hat{\epsilon}_{SR}$ the simplified replacement breakdown point.

For ρ satisfying (VI), any sample size n, such that $1 \leq t \leq n$, $\hat{\epsilon}_A$ and $\hat{\epsilon}_{SR}$ are given as follows

$$\hat{\epsilon}_A = \hat{\epsilon}_{SR} = \min\left\{\frac{-\psi(-\infty)}{\psi(\infty)-\psi(-\infty)}, \frac{\psi(\infty)}{\psi(\infty)-\psi(-\infty)}\right\}, \tag{12}$$

if ψ is bounded. On the other hand, $\hat{\epsilon}_A = 1/(n+1)$, $\hat{\epsilon}_{SR} = 1/n$ if ψ is unbounded. For more details see, for example, Zhang and Li (1998) and He, Jureckova, Konker, and Portnoy (1990). Note that the QML estimator has a breakdown point equal to zero, because his influence function is unbounded.

Example 1 Chose the Huber loss function in Equation 8 and suppose that ρ_τ satisfy (VI). Let $\psi^*_\tau = \frac{\partial \rho_\tau}{\partial x}$, so ψ^*_τ is given by

$$\psi^*_\tau(x) = \begin{cases} \psi_H\{(1-\tau)x\}(1-\tau) & (x<0) \\ \tau\psi_H(\tau x) & \text{otherwise} \end{cases}, \tag{13}$$

where ψ^*_τ is bounded and $\rho^{'}_H(x) = \psi_H(x)$ is defined as

$$\psi_H(x) = \begin{cases} x, & |x|<k \\ ksign(x), & \text{otherwise.} \end{cases}$$

Therefore, the asymptotic breakdown point of ρ_τ, denoted by $\hat{\epsilon}_{\rho_\tau}$, is given by

$$\hat{\epsilon}_{\rho_\tau} = \min\left\{\frac{(1-\tau)^2}{\tau^2+(1-\tau)^2}, \frac{\tau^2}{\tau^2+(1-\tau)^2}\right\}.$$

It is important to note that the breakdown point of ρ_τ not depend of tuning constant k (see Huber (1984) and Chao (1986)). Furthermore, we have $\hat{\epsilon}_{\rho_\tau} = 1/2 \iff \tau = 0.5$. Therefore, is expected that the M-quantile estimator at $\tau = 0.5$ be more robust than QML estimator.

3 Empirical Study

Let $\{X_t, t \in T\}$ be a sample from a process as defined in Equation 1 and let $\{Z_t, t \in T\}$ be a sample of the process defined by

$$Z_t = X_t + mI_t^{(T_1)}, \tag{14}$$

where the parameter m represents the magnitude of the outlier, and $I_t^{(T_1)}$ is a random variable with probability p of the occurrence of outliers, defined as a random variable with

$$\mathbb{P}(I_t = -1) = \mathbb{P}(I_t = 1) = p/2 \text{ and } \mathbb{P}(I_t = 0) = 1-p\ ,$$

where $\mathbb{E}[I_t] = 0$ and $Var(I_t) = p$. I_t is the product of $Bernoulli(p)$ and *Rademacher* random variables; the latter equals 1 or -1, both with probability $1/2$. X_t and I_t are independent random variables. Note that, if $m = 0.0$ $\{Z_t, t \in T\}$ is a time series with no outlier. Other methods for including outliers in time series can be found at Carnero et al. (2005) and Carnero et al. (2012)

Three sample sizes $\{100, 500, 1000\}$ generated according to a GARCH(1,1) model are examined in simulation over 100 repetitions. The contaminated data $\{Z_t, t \in \mathbb{Z}\}$, were generated from Equation 14 with $p = \{1\%, 0.2\%, 0.1\%\}$ for each sample size respectively with magnitudes $m = 0$ (no outliers) and 7. The following method estimators will be used to carry out a comparison study: QML, Huber and Quantile (proposed by Lee and Noh (2013)). We consider the following true parameter values: $\omega = 0.1$, $\alpha_1 = 0.1$, $\beta_1 = 0.8$ where

$$\begin{cases} X_t = \sigma_t \varepsilon_t \\ \sigma_t^2 = \omega + \alpha_1 X_{t-1} + \beta_1 \sigma_{t-1}^2 \end{cases}, \tag{15}$$

with $\{\varepsilon_t, t \in \mathbb{Z}\} \sim N(0,1)$. The tuning constant $k = 1.5$ was considered for the Huber and M-quantile estimators. Additionally, in each instance we estimate the set of values $\tau = \{0.05, 0.25, 0.45, 0.50\}$ for the Quantile and M-quantile estimator methods.

Remark 4 To formulate the quantile regression problem for GARCH models Lee and Noh (2013) proposed a reparametrization approach, addressed below, since under the Equation 1 the τth of $\{X_t\}$ conditional on the past observations up to time $t-1$ is not identifiable. So, the model is reformulated as

$$\begin{cases} X_t = \sqrt{h_t} u_t \\ h_t = 1 + \sum_{i=1}^{p} \gamma_i X_{t-1} + \sum_{j=1}^{q} \beta_j h_{t-j}, \end{cases}$$

where $h_t = \sigma_t^2/\omega$, $u_t = \sqrt{\omega}\varepsilon_t$ and $\gamma_i = \alpha_i/\omega$. Since ω is unknown, the parameter $\hat{\omega}$ is considered, where $\hat{\omega}$ is a consistent estimator of ω based on the QML method. See Lee and Noh (2013).

Tables 1, 2, 3 presents the empirical mean and MSE (Mean Squared Error) of the estimators considered in this study. The item (a) of the aforementioned tables shows the empirical Mean and MSE for the results of all estimators here considered for series generated without outliers. It can be seen that, in general, the classical QML and the M-quantile methods perform similarly, that is, under the scenario of a non-contaminated time series both estimation methods lead to comparable results, with estimates close to the real values of $\boldsymbol{\theta}$,

The item (b) of the tables below presents the empirical mean and MSE for the estimators of $\boldsymbol{\theta}$ considering the series with outliers ($m = 7$ and $p = \{1\%, 0.2\%, 0.1\%\}$). As can be perceived from the tables, the classical QML estimator is clearly affected by additive outliers, which is in line with results discussed in Carnero et al. (2012). On the other hand, the robust one, proposed in this study, keeps almost the same mean and MSE of the non-contaminated scenario. This simple empirical investigation leads to conclusions that the classical QML estimator is completely influenced by the outliers while, in general, the M-quantile is not. Therefore, the M-estimation method proposed in this paper can be an alternative estimator to deal with a heteroscedastic time series with possible additive outliers or not.

Table 1: Empirical Mean and MSE of GARCH(1,1) model estimated by QML, Huber, quantile and M-quantile methods, considering $n = 100$

(a) Whithout Outliers

	Mean			MSE		
	$\hat{\omega}$	$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{\omega}$	$\hat{\alpha}_1$	$\hat{\beta}_1$
QML Method						
	0.1826	0.0908	0.7264	0.0405	0.0114	0.0695
Huber Method						
	0.2465	0.1251	0.6009	0.0897	0.0252	0.1605
Quantile Method						
τ						
0.05	0.1749	0.1201	0.5758	0.0432	0.0312	0.1564
0.25	0.1772	0.1019	0.5925	0.0450	0.0214	0.1508
0.45	0.1981	0.1596	0.5886	0.0460	0.0446	0.1519
0.50	0.1892	0.0937	0.6216	0.0391	0.0220	0.1360
M-quantile Method						
τ						
0.05	0.1815	0.0923	0.6700	0.0594	0.0028	0.0957
0.25	0.2364	0.0845	0.6553	0.1031	0.0028	0.1086
0.45	0.2170	0.1270	0.6125	0.0723	0.0099	0.1129
0.50	0.2081	0.1320	0.6411	0.0444	0.0129	0.1086

(b) Whith Outliers

	Mean			MSE		
	$\hat{\omega}$	$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{\omega}$	$\hat{\alpha}_1$	$\hat{\beta}_1$
QML Method						
	0.2454	0.1364	0.6874	0.1001	0.0474	0.1130
Huber Method						
	0.3011	0.1264	0.5729	0.1222	0.0343	0.1727
Quantile Method						
τ						
0.05	0.2214	0.0901	0.6578	0.0857	0.0340	0.1569
0.25	0.2541	0.1254	0.6715	0.0999	0.0620	0.1565
0.45	0.2191	0.1397	0.6218	0.0918	0.0607	0.1576
0.50	0.2137	0.1490	0.6221	0.1091	0.0727	0.1444
M-quantile Method						
τ						
0.05	0.2149	0.0743	0.6667	0.0461	0.0060	0.1298
0.25	0.2255	0.0849	0.6925	0.0764	0.0072	0.1105
0.45	0.194	0.0896	0.6763	0.0587	0.0115	0.1139
0.50	0.2389	0.0855	0.6802	0.1182	0.0186	0.1169

Table 2: Empirical Mean and MSE of GARCH(1,1) model estimated by QML, Huber, quantile and M-quantile methods, considering $n = 500$

(a) Whithout Outliers

	Mean			MSE		
	$\hat{\omega}$	$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{\omega}$	$\hat{\alpha}_1$	$\hat{\beta}_1$
QML Method						
	0.1554	0.1029	0.7374	0.0233	0.0018	0.0336
Huber Method						
	0.1445	0.1172	0.7335	0.0212	0.0034	0.0399
Quantile Method						
τ						
0.05	0.1429	0.0965	0.6570	0.0277	0.0061	0.0840
0.25	0.1394	0.1038	0.6683	0.0254	0.0122	0.0879
0.45	0.1489	0.1035	0.6851	0.0218	0.0060	0.0786
0.50	0.1424	0.1074	0.6586	0.0225	0.0092	0.1052
M-quantile Method						
τ						
0.05	0.1352	0.0935	0.7697	0.0294	0.0045	0.0531
0.25	0.1057	0.0719	0.7408	0.0098	0.0022	0.0349
0.45	0.0906	0.0660	0.7223	0.0053	0.0014	0.0415
0.50	0.0870	0.0609	0.7051	0.0051	0.0013	0.0402

(b) Whith Outliers

	Mean			MSE		
	$\hat{\omega}$	$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{\omega}$	$\hat{\alpha}_1$	$\hat{\beta}_1$
QML Method						
	0.2152	0.1176	0.6947	0.0576	0.0098	0.0827
Huber Method						
	0.1538	0.1036	0.6786	0.0423	0.0062	0.0656
Quantile Method						
τ						
0.05	0.1922	0.1014	0.6141	0.0593	0.0065	0.1082
0.25	0.2036	0.0992	0.6620	0.0550	0.0157	0.1080
0.45	0.2184	0.1141	0.6800	0.0554	0.0072	0.0795
0.50	0.1908	0.1133	0.6092	0.0533	0.0108	0.1194
M-quantile Method						
τ						
0.05	0.1832	0.1036	0.7366	0.0314	0.0060	0.0705
0.25	0.1670	0.0661	0.6935	0.0201	0.0039	0.0577
0.45	0.1631	0.0653	0.7150	0.0190	0.0037	0.0468
0.50	0.1637	0.0715	0.7074	0.0209	0.0043	0.0448

Table 3: Empirical Mean and MSE of GARCH(1,1) model estimated by QML, Huber, quantile and M-quantile methods, considering $n = 1000$

(a) Whithout Outliers

	Mean			MSE		
	$\hat{\omega}$	$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{\omega}$	$\hat{\alpha}_1$	$\hat{\beta}_1$
QML Method						
	0.1206	0.1060	0.7741	0.0033	0.0010	0.0060
Huber Method						
	0.1396	0.1157	0.7395	0.0097	0.0018	0.0190
Quantile Method						
τ						
0.05	0.0985	0.0944	0.6911	0.0027	0.0040	0.0653
0.25	0.1040	0.0991	0.6582	0.0032	0.0029	0.0880
0.45	0.0982	0.0955	0.6769	0.0030	0.0033	0.0716
0.50	0.1187	0.1014	0.7266	0.0090	0.0032	0.0444
M-quantile Method						
τ						
0.05	0.1276	0.1107	0.7872	0.0097	0.0025	0.0200
0.25	0.0808	0.0760	0.8041	0.0047	0.0023	0.0169
0.45	0.0805	0.0836	0.7601	0.0026	0.0019	0.0156
0.50	0.0929	0.0862	0.7718	0.0031	0.0020	0.0109

(b) Whith Outliers

	Mean			MSE		
	$\hat{\omega}$	$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{\omega}$	$\hat{\alpha}_1$	$\hat{\beta}_1$
QML Method						
	0.1458	0.1030	0.7620	0.0131	0.0032	0.0164
Huber Method						
	0.1550	0.1146	0.7273	0.0150	0.0024	0.0272
Quantile Method						
τ						
0.05	0.1419	0.0866	0.6782	0.0128	0.0050	0.0690
0.25	0.1396	0.0954	0.6572	0.0120	0.0028	0.0760
0.45	0.1396	0.1123	0.6830	0.0136	0.0046	0.0807
0.50	0.1461	0.0973	0.7174	0.0132	0.0037	0.0494
M-quantile Method						
τ						
0.05	0.1329	0.1024	0.7558	0.0172	0.0039	0.0292
0.25	0.1095	0.0700	0.7341	0.0093	0.0029	0.0205
0.45	0.0936	0.0683	0.7524	0.0067	0.0025	0.0203
0.50	0.0930	0.0703	0.7374	0.0088	0.0023	0.0188

4 Real Application

In this section, we analyzed the daily returns of two financial indices, Continuous Assisted Quotation Index (CAC40) and the São Paulo Stock Exchange Index (Ibovespa). In each series, CAC40 and Ibovespa, between the period from January 3, 2011, to December 30, 2021 was generated 2811 and 2719 observations, respectively. Let p_t be the price of the financial index in time t and r_t the log return defined as $r_t = 100 \log \frac{p_t}{p_{t-1}}$. It is assumed that r_t follows a GARCH process according to the definition given in Equation 1, where $p = q = 1$, that is, a GARCH(1,1) model. While the French equity markets are mature and established over the sample period, the emerging market in Brazil experienced more considerable volatility and more dramatic jumps in prices.

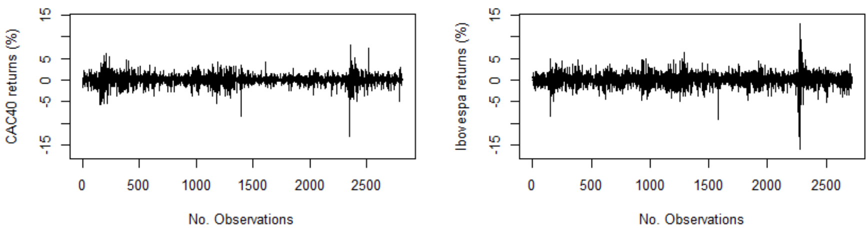

Fig. 1: Plot of the Daily Return Series of CAC40 and Ibovespa Indices.

As can be perceived there are possible outliers in both series, conform the figure 1, which is usual phenomenon in financial time series.

Table 4: Descriptive statistics of the daily log-returns of the financial indices CAC40 and Ibovespa

	CAC40	Ibovespa
Mean	0.0217	0.0149
Std. Deviation	1.2672	1.5988
Maximum	8.0561	13.0223
Minimum	-13.0983	-15.9930
Skewness	-0.7031	-0.8471
Kurtosis	8.5321	11.9224

Table 4 reports some summary statistics of the data. As has been documented extensively in the literature, both indices display negative skewness and excess kurtosis. The French CAC40 Index on average returned about 0.0217% per day, slightly above that of the Ibovespa Index. In addition, the Ibovespa Index came with a much higher risk than the CAC40. Each of both series was fitted as a GARCH(1,1) model using the QML, Huber, Quantile and

M-quantile estimators discussed in the previous section. These estimates are displayed in Tables 5, 6 and 7, respectively. The series contain outliers, thus important differences estimates are expected, as showed by empirical study.

Table 5: Fitted GARCH(1,1) models for the daily returns series

	CAC40			Ibovespa		
Estimates	$\hat{\omega}$	$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{\omega}$	$\hat{\alpha}_1$	$\hat{\beta}_1$
QML	0.0545	0.1357	0.8316	0.0993	0.0835	0.8717
Huber	0.0248	0.0890	0.8602	0.0619	0.0565	0.8906

Table 6: GARCH(1,1) models estimated by quantile regression for the daily returns series

τ	0.01	0.05	0.5	0.95	0.99
CAC40					
$\hat{\omega}$	-	-	-	-	-
$\hat{\alpha}_1$	0.0722	0.1058	0.1360	0.1435	0.1336
$\hat{\beta}_1$	0.7120	0.8236	0.7374	0.8307	0.8256
Ibovespa					
$\hat{\omega}$	-	-	-	-	-
$\hat{\alpha}_1$	0.1047	0.0626	0.0839	0.0883	0.0741
$\hat{\beta}_1$	0.8201	0.8213	0.8274	0.8653	0.9028

Table 7: GARCH(1,1) models estimated by M-quantile regression for the daily returns series

τ	0.01	0.05	0.5	0.95	0.99
CAC40					
$\hat{\alpha}_0$	0.0460	0.0437	0.0246	0.0127	0.0118
$\hat{\alpha}_1$	0.0762	0.0964	0.1182	0.1284	0.1174
$\hat{\beta}_1$	0.8256	0.8298	0.8402	0.8453	0.8645
Ibovespa					
$\hat{\alpha}_0$	0.0945	0.0924	0.0222	0.0227	0.0201
$\hat{\alpha}_1$	0.0620	0.0619	0.0594	0.0399	0.0386
$\hat{\beta}_1$	0.8617	0.8634	0.8902	0.9359	0.9391

It is important to mention that in a GARCH(1,1) model, the M-estimators should consistently estimate β_1, as can be observed for QML, Huber and M-quantile regression estimation, where $\hat{\beta}_1$ are quite close each other. This is not necessarily truth to $\hat{\omega}$ and $\hat{\alpha}_1$ parameters, because in a real case we can not calculate c_H satisfying $E\left(H\left(\frac{\varepsilon_t}{c_H{}^{1/2}}\right)\right) = 1$ for the Huber loss function. This is reflected in the Tables 5 and 7. This results are in line with Mukherjee (2008), Boudt and Croux (2010) and Liu and Mukherjee (2020)

From Table 6 and 7, it is seen that for the CAC40 series, the M-quantile and quantile regression estimates of $\hat{\alpha}_1$ for $\tau < 0.50$ tend to yield larger values than those for $\tau > 0.50$. On the other hand, from the same tables, it is seen that for the Ibovespa series, the M-quantile and quantile regression estimates of $\hat{\alpha}_1$ for $\tau < 0.50$ tend to yield smaller values than those for $\tau > 0.50$. This suggests the presence of asymmetry in both series, thus a method based on average effect can be not recommended.

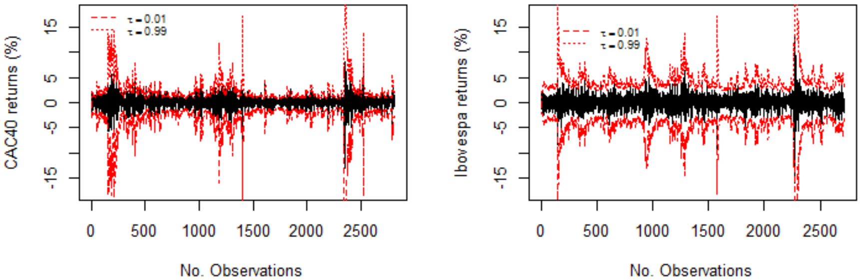

Fig. 2: Estimated conditional quantiles for CAC40 and Ibovespa returns.

Although we focus on the estimation of robust parameters for GARCH models, Figure 2 shows the value at risk (VaR) estimated for $\tau \in \{0.01, 0.99\}$, because the proposed estimation procedure also provides an alternative approach to VaR, since the evaluation of VaR is explicitly a conditional quantile estimation problem, see for example, Xiao and Koenker (2009) and Zheng et al. (2016).

Estimated GARCH volatilities, using the four methods considered in this study, are plotted in Figures 3 and 4. For the quantile and M-quantile methods $\tau = 0.5$ was considered. Clearly, QML tends to estimate larger volatilities compared to robust methods which may have important implications on real financial applications. For example, large volatility will provide large VaR, consequently increasing the uncertainty associated with returns, implying that the risk will seem larger. Furthermore, in particular, the M-quantile method, when compared with QML, does not seem to be significantly affected by high observations of $\{X_t\}$. In these senses, the volatilities are well-modelled by using the M-quantile estimation method.

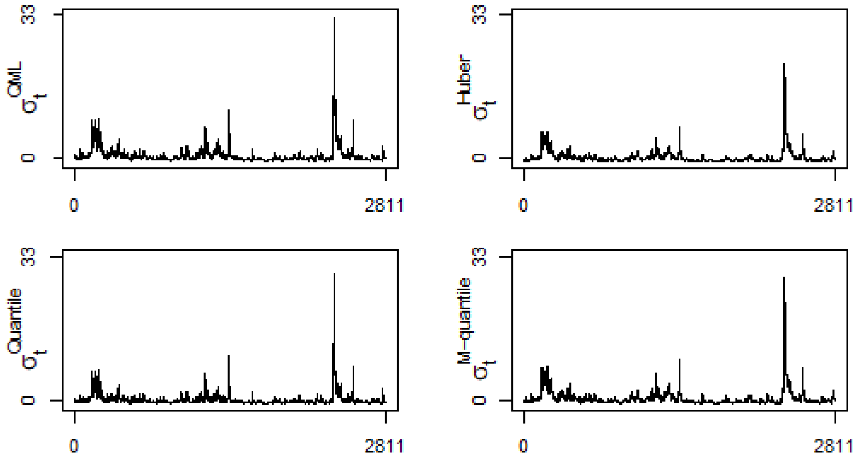

Fig. 3: Estimated volatility for the CAC40 index.

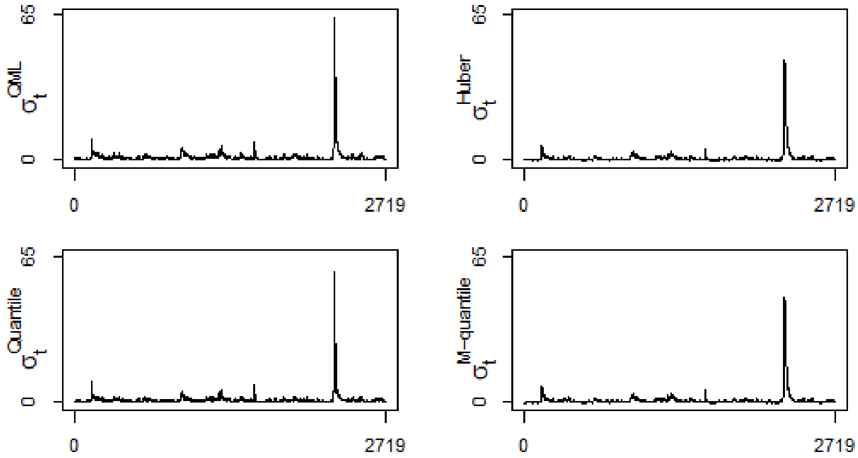

Fig. 4: Estimated volatility for the Ibovespa index.

5 Conclusion

In this paper, we propose estimating GARCH models using the M-quantile regression method, which is robust against aberrant observations (outliers). When there are no outliers a Monte Carlo study shows that the GARCH(1,1) model parameters estimated by M-quantile, are quite close to QML, which is a well-established and commonly method used to model heteroscedastic time series. On the other hand, was shown that the parameters of the GARCH(1,1) model estimated by QML suffer dramatic effects due to the contamination of the series, which can lead to spurious interpretations. Although other robust methods were considered, the simulation results showed that the M-quantile method is more reliable in terms of MSE. In addition, a real application using financial data set showed that the estimated volatility by M-quantile regression is not significantly affected by aberrant observations and seems to perform well in the series considered. Therefore, the use of the MQGARCH(1,1) model is strongly encouraged.

References

Bai, Z.D., Rao, C.R., Wu, Y. (1992). M-estimation of multivariate linear regression parameters under a convex discrepancy function. *Statistica Sinica*, *2*(1), 237–254.

Berkes, I., Hovath, L., Kokoszka, P. (2003). GARCH process: Structure and estimation. *Bernoulli*, *9*(2), 201–227.

Boudt, K., & Croux, C. (2010). Robust m-estimation of multivariate garch models. *Computational Statistics and Data Analysis*, *54*, 2459–2469.

Breckling, J., & Chambers, R. (1988). M-quantiles. *Biometrika*, *75*(4), 761–771.

Carnero, M.A. (2003). *Heterocedasticidad condicional, atípicos y cambios de nivel en series temporales financieras* (Unpublished doctoral dissertation). Universidad Carlos III de Madrid.

Carnero, M.A., Peña, D., Ruiz, E. (2005). Effects of outliers on the identification and estimation of GARCH models. *Journal of Time Series Analysis*, *28*, 471-497.

Carnero, M.A., Peña, D., Ruiz, E. (2012). Estimating GARCH volatility in the presence of outliers. *Econometrics Latters*, *114*, 86-90.

Chang, I., Tiao, G.C., Chen, C. (1988). Estimation of time series parameters in presence of outliers. *Technometrics*, *30*, 1936-204.

Chao, M. (1986). On m and p estimators that have breakdown point equal to 1/2. *Statistics and Probability Letter*, *4*, 127–131.

Chen, C., & Liu, L. (1993). Joint estimation of model parameters and outlier effects in time series. *Journal of the American Statistical Association*, *89*, 284-297.

Fan, J., & Yao, Q. (2002). *Nonlinear time series: Nonparametric and parametric methods*. Springer.

Francq, C., & Zakoian, J.-M. (2019). *GARCH models: Structure, statistical inference and financial applications*. Wiley.

Franses, P.H., & Ghijsels, H. (1999). Additive outliers, GARCH and forecasting volatility. *Internatonal Journal Forecasting*, *15*, 1–9.

He, X., Jureckova, J., Konker, R., Portnoy, S.L. (1990). Tail behavior of regression estimators and their breakdown points. *Econometrica*, *58*, 1995–1214.

Huber, P.J. (1964). Robust estimation of a location parameter. *Annals of Statistics*, *1*(53), 73–101.

Huber, P.J. (1984). Finite sample breakdown point of M- and P-estimators. *Annals of Statistics*, *12*, 119–126.

Iqbal, F. (2013). Robust estimation for the orthogonal GARCH model. *The Manchester school*, *81*, 904–924.

Ledolter, J. (1989). The effect of additive outliers on the forecast from ARIMA models. *International Journal of Forecasting*, *5*, 231-240.

Lee, S., & Noh, J. (2013). Quantile regression estimator for GARCH models. *Scandinavian Journal of Statistics*, *40*(1), 2–20.

Li, T.H. (2008). Laplace periodogram for time series analysis. *Journal of the American Statistical Association*, *103*, 757-768.

Liu, H., & Mukherjee, K. (2020, 01). M-estimation in garch models without higher order moments. *arXiv*.

Mendes, S.V.M. (2000). Assessing the bias of maximum likelihood estimates of contaminated GARCH models. *Journal of Statistical Computation and Simulation*, *67*, 359-376.

Mukherjee, K. (2008). *M*-estimation in GARCH models. *Econometric Theory*, *24*, 1530–1553.

Muler, N., & Yohai, V.J. (2008). Robust estimates for GARCH models. *Journal of Statistical Planning and Inference*, *138*, 2918–2940.

Reisen, V.A., Lévy-Leduc, C., Taqqu, M. (2017). An *M*-estimator for the long-memory parameter. *Journal of Statistical Planning and Inference*, *187*, 44–55.

Reisen, V.A., Sgrancio, A.M., Lévy-Leduc, C., Bondon, P., Monte, E.Z., Cotta, H.H.A., Ziegelmann, F. (2019). Robust factor modelling for high-dimensional time series: An application to air pollution data. *Applied Mathematics and Computation*, *346*, 842–852.

Sarnaglia, A.J.Q., Reisen, V.A., Lévy-Leduc, C., Bondon, P. (2021). *M*-regression spectral estimator for periodic ARMA models. an empirical investigation. *Stochastic Environmental Research and Risk Assessment volume*, *35*, 653–664.

Tularam, A., & Saeed, T. (2016, 01). Oil-price forecasting based on various univariate time-series models. *American Journal of Operations Research*, *06*, 226-235.

Wu, W.B. (2007). M-estimation of linear models with dependent errors. *The Annals of Statistics*, *35*(2), 495 – 521.

Xiao, Z., & Koenker, R. (2009). Conditional quantile estimation for generalized autoregressive conditional heteroscedasticity models. *Journal of the American Statistical Association*, *104*(488), 1696–1712.

Zhang, J., & Li, G. (1998). Breakdown properties of location M-estimators. *The Annals of Statistics*, *26*(3), 1170–1189.

Zheng, Y., Qianqian, Z., Li, G., Xiao, Z. (2016, 10). Hybrid quantile regression estimation for time series models with conditional heteroscedasticity. *SSRN Electronic Journal.*

Statements and Declarations

Funding. This work was supported by Espirito Santo State Research Foundation - FAPES (grant no. 012/2022-1) and Université Paris-Saclay - CentraleSupélec (grant no. 1941199416F01).

Acknowledge. Part of this paper was revised when Prof. Valdério Reisen was visiting Université Paris-Saclay - CentraleSupélec (from November 2021 to April 2022). This author is indebted to Université Paris-Saclay - CentraleSupélec for its financial support.

Non-financial interests. The authors have no competing interests to declare that are relevant to the content of this article.