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Orbital b-Metric Spaces and Related Fixed Point Results on Advanced Nashine-Wardowski-Feng-Liu Type Contractions with Applications

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Abstract:

In this article we prove some novel fixed-point results for a pair of multivalued dominated mappings obeying a new generalized Nashine-Wardowski-Feng-Liu- type contraction for orbitally lower semi-continuous functions in complete orbital b-metric space. Furthermore, some new fixed-point theorems for dominated multivalued mappings are established in the scenario of ordered complete orbital b-metric space. Some examples are offered to demonstrate the validity of our new results' premise. To demonstrate the applicability of our findings, applications for a system of nonlinear Volterra type integral equations and fractional differential equations are shown. These results extend the theoretical results of Nashine et al. (Nonlinear Anal. Model. Control. 26(3)(2021), 522–533).

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Keywords: Fixed point; new generalized Nashine-Wardowski-Feng-Liu-type contraction; dominated multivalued mappings; integral equation; fractional differential equation; orbital b-metric space.

1. Introduction and Basic Preliminaries

Fixed point theory is a fascinating branch of mathematics that plays a critical and fundamental role in both applied and pure mathematics, including modern optimization, control theories, functional analysis, topology and geometry, economics, and modeling. Fixed point theory is a well-balanced mixture of sub algebra, topology, and configuration. In many fields, it is a critical investigation and detection tool. Fixed point theory is a fundamental and useful area of functional analysis. Fixed point theory is one of the most fascinating research topics in non-limited anatomy. Fixed point theory intends to develop not only nonlinear and functional analysis, but also economics, finance, computer science, and other subjects, in deciding problems for differential, integral, and random differential equations. As a result, the theory of fixed point arose as an analytic theory. Banach [9] developed the Banach contraction principle, which is the first well-known result in fixed point theory. It's a programmed for solving linear and nonlinear differential, integral, and functional equations in a variety of generalized spaces. There are many variations on the Banach contraction principle that involve contractive-type requirements that must be met on different distance spaces. Different multiplications of Banach's result in metric and b-metric spaces were studied by Bakhtin [10] and Czerwik [13-15]. Wardowski [35] proposed a new generalization of Banach's theorem known as F-contraction, as well as some new fixed-point results. After that, Sgroi et al. [32] established the existence of new theorems for set valued F-contraction and showed functional and integral equation applications. Nicolae [24] demonstrated fixed-point results for Feng-Liu type contractions that perform several functions. Padcharoen et al. [26] presented periodic theorems for a novel generalized -type F-contraction in modular-like-metric-spaces. Furthermore, Rasham et al. [28] used the first of Wardowski's three F-contraction conditions to prove fixed point theorems in complete $b-$like-metric-space using multivalued dominated F-contraction on the closed ball. Rasham et al. [31] went on to analyze various linked fuzzy dominated generalized contractive maps in modular-like metric spaces, and they demonstrated applications to examine the unique solution of integral and fractional differential equations. Nashine et al. [21] recently obtained fixed point theorems for set valued maps satisfying the Wardowski-Feng-Liu-type condition for orbitally lower semi continuous functions on orbital complete b-metric space, as well as illustrative examples to certify new results and a fractal integral equations application.

We prove some novel fixed-point theorems for a couple of multivalued dominated maps that fulfill Nashine-Wardowski-Feng-Liu-type contraction for orbitally lower semi-continuous functions in complete orbital b-metric proved by Nashine et al. [21]. In addition, a new existence theorem for a few multi-dominated maps meeting a new generalized

Nashine-Wardowski-Feng-Liu-type rational contractive condition for orbitally lower semi continuous functions in ordered full orbital b-metric space is presented. To validate our findings, we give examples and new definitions. Finally, we describe applications for nonlinear Volterra type integral and fractional differential equations to demonstrate the utility of our findings.

Definition 1.1 [22] A function $d_b: Y \times Y \rightarrow [0, \infty)$ satisfying the following axioms (for all $g, h, i \in Y$) is called a b-metric (d_b-metric):

i. If $d_b(g,h) = 0$, iff $g = h$;

ii. $d_b(g,h) = d_b(h,g)$;

iii. $d_b(g,h) \leq b[d_b(g,i) + d_b(i,h)]$.

The pair (Y, d_b) is called a b-metric space (abbreviated as BMS).

Example 1.2 [22] Let $Y = R^+ \cup \{0\}$. Define $d_b(g,h) = |g-h|^2$, $\forall \quad g,h \in Y$. Then (Y, d_b) is a BMS with constant $b = 2$.

Definition 1.3 [28] Let Q be a non-empty subset of Y and $g \in Y$. Then, $p_0 \in Q$ is said to be a best approximation in Q if

$$d_b(g,Q) = d_b(g,p_0), \text{where } d_b(g,Q) = \inf_{p \in Q} d_b(g,p).$$

Here $P(Y)$ denotes the set of all closed compact subsets of Y.

Definition 1.4 [28] A function $H_{d_b}: P(Y) \times P(Y) \rightarrow R^+$ defined by

$$H_{d_b}(R,E) = \max\left\{\sup_{x \in R} d_b(x,E), \sup_{m \in E} d_b(R,m)\right\}$$

undergoes all the axioms of a b-metric and known as Pompiue Hausdorff b-metric on $P(Y)$.

Definition 1.5 [35] An F-contraction is a mapping $T: \mathcal{M} \rightarrow \mathcal{M}$ satisfying the following:

there exists $\tau > 0$ such that (for every $\mu, y \in \mathcal{M}$)

$$d(T(\mu), T(y)) > 0 \Rightarrow \tau + F\big(d(T(\mu), T(y))\big) \leq F\big(d(\mu, y)\big).$$

Here the function $F: \mathbb{R}^+ \rightarrow \mathbb{R}$ satisfies the following conditions:

$(F1)$ F is strictly-increasing function;

$(F2)$ $\lim_{j \to +\infty} \rho_j = 0$ if and only if $\lim_{j \to +\infty} F(\rho_j) = -\infty$, for every positive sequence $\{\rho_j\}_{j=1}^{\infty}$;

$(F3)$ for each $g \in (0,1)$, $\lim_{j \to \infty} \rho_j^{g} F(\rho_j) = 0$;

$(F4)$ there exists $\tau > 0$ such that for every positive sequence $\{\rho_j\}$,

$\tau + F(b\rho_j) \leq F(\rho_{j-1})$ for all $j \in N$, then $\tau + F(b^j\rho_j) \leq (b^{j-1}\rho_{j-1})$ for all $j \in N$.

Definition 1.6 [21] Let $b \geq 1$ be a non-negative number and ξF_b^* represent the family of functions $F: \mathbb{R}^+ \to \mathbb{R}$ satisfying the conditions $(F1) - (F4)$ and

$(F5)$ $F(inf\ B) = inf\ F(B)$ for all $B \subset (0, \infty)$ with $inf\ B > 0$.

It is clear that ξF_b^* is a non-empty containing $F(g) = ln\ g$ or $F(g) = g + ln\ g$.

A function $f: \mathcal{H} \to \mathbb{R}$ is called a lower semi-continuous if for every sequence of positive numbers $\{\omega_j\}$ in $\mathcal{H}$ with $\lim_{j\to+\infty} \omega_j = \omega \in \mathcal{H}$, $f(\omega) \leq \lim_{j\to+\infty} \inf f(\omega_j)$.

Definition 1.7 [29] Let a non-empty set U where $K \subseteq U$ and $\alpha: U \times U \to [0, +\infty)$. A mapping $W : U \to P(U)$ satisfying

$$\alpha_*(Wg, Wh) = \inf\{\alpha(t,z): t\epsilon Wg, z\epsilon Wh\} \geq 1, \text{ whenever } \alpha(t,z) \geq 1, for\ all \quad t,z \in U$$

is called an α_* −admissible.

A mapping $W: U \longrightarrow P(U)$ satisfying $\alpha_*(a, \mathrm{Wa}) = \inf\{\alpha(\mathrm{a}, \mathrm{h}): \mathrm{h} \in \mathrm{Wa}\} \geq 1$ is said to be an α_* −dominated on E.

Example 1.8 [29] Define $\gamma: R \times R \to [0, \infty)$ and $G, R: R \to P(R)$ by

$$\gamma(o,p) = \begin{Bmatrix} 1 & if o > p \\ \frac{1}{4} & if o \leq p \end{Bmatrix},$$

and $Gs = [-4 + s, -3 + s]$ and $Rm = [-2 + m, -1 + m]$ respectively. Then G and R are γ_* −dominated, but they are not γ_* −admissible.

2. Main results

Let (Y, d_b) be an orbitally complete BMS and $\varkappa_0 \in Y$, let S and T be two multi-maps from Y to $P(Y)$. Let $\varkappa_1 \in S(\varkappa_o)$, then $d_b(\varkappa_o, S(\varkappa_o)) = d_b(\varkappa_0, \varkappa_1)$. Let $\varkappa_2 \in T(\varkappa_1)$ be such that $d_b(\varkappa_1, T(\varkappa_1)) = d_b(\varkappa_1, \varkappa_2)$. Continuing this way, we attain a sequence $\{TS(\varkappa_j)\}$ in Y, where $\varkappa_{2j+1} \in S(\varkappa_{2j})$, $\varkappa_{2j+2} \in T(\varkappa_{2j+1})$, $j \in \mathbb{N} \cup \{0\}$. Also, $d_b(\varkappa_{2j}, S(\varkappa_{2j})) = d_b(\varkappa_{2j}, \varkappa_{2j+1})$, $d_b(\varkappa_{2j+1}, T(\varkappa_{2j+1})) = d_b(\varkappa_{2j+1}, \varkappa_{2j+2})$, then $\{TS(\varkappa_j)\}$ be a sequence in Y generated by $\varkappa_0$. If, $S = T$, then we say $\{YS(\varkappa_j)\}$ instead of $\{TS(\varkappa_j)\}$.

Definition 2.1 Let S and T be two multi-maps from Y to $P(Y)$, $F \in \xi F_b^*$ and $\eta: (0, \infty) \to (0, \infty)$. For $y \in Y$ with $\max\{d_b(\varkappa, S(\varkappa)), d_b(y, T(y))\} > 0$, define a set $F_\eta^y \subseteq Y$ as:

$$F_\eta^y = \left\{ y \in S(\varkappa), z \in T(y): \quad F\ (d_b(y,z)) \leq F\ \left(b\left[max\left\{d_b(\varkappa, S(\varkappa)), d_b(y, T(y)), \frac{d_b(\varkappa, S(\varkappa)).d_b(y, T(y))}{1 + d_b(\varkappa, y)}\right\}\right]\right) + \eta\ (d_b(y,z)) \right\}.$$

Let $S, T: Y \to Y$, and for any $y_0 \in Y$, $O(y_0) = \{y_0, S(y_0), T(y_1), \ldots\ldots\}$ denotes the orbit of y_0. A mapping $f: Y \to \mathbb{R}$ is said (S,T)-orbitally lower semi continuous if $f(y) < \lim_{n\to\infty} \inf f(y_0)$ for all sequences $\{ST(y_n)\} \subset O(y_0)$ with $\lim_{n\to\infty} \{ST(y_n)\} = y \in Y$.

Definition 2.2 Let $S, T: Y \to P(Y)$ be a couple of multi-maps on (Y, d_b). An orbit for a pair (S,T) is a point $y_0 \in Y$ denoted by $O(y_0)$ and a sequence defined as $\{y_n: y_n \in ST(y_{n-1})\}$.

Definition 2.3 Let $S, T: Y \to P(y)$ be a couple of multi-maps on (Y, d_b). If Cauchy sequence $\{y_n: y_n \in ST(y_{n-1})\}$ converges in b-metric Y, then Y is said to be (S,T)-orbitally complete.

It is noted that an orbitally complete BMS may not be complete. Now we begin our main theorem.

Theorem 2.4 Let (Y, d_b) be an orbitally complete BMS. Let $y_0 \in Y, \alpha: Y \times Y \to [0,\infty)$ and $S, T: Y \to P(Y)$ be two α_*-dominated multi-maps and $F \in \xi F_b^*$. Assume the following properties hold:

i. The mappings $z \mapsto \max\{d_b(\varkappa, S(\varkappa)), d_b(y, T(y))\}$ are orbitally lower semi continuous;

ii. There exists functions $\tau, \eta: (0,\infty) \to (0,\infty)$ such that for all $t \geq 0$;

$$\tau(t) > \eta(t), \qquad \lim_{s\to t^+} inf\, \tau(t) > \lim_{s\to t^+} inf\, \eta(t);$$

iii. For all $\varkappa, y \in \{ST(\varkappa_j)\}$ with $\alpha(x,y) \geq 1$and $\max\{d_b(\varkappa, S(\varkappa)), d_b(y, T(y))\} > 0$, there exists $y \in F_\eta^\varkappa$ satisfying;

$$\tau(d_b(\varkappa, y)) + F\left(b\left[\max\left\{d_b(\varkappa, S(\varkappa)), d_b(y, T(y)), \frac{d_b(\varkappa, S(\varkappa)).\, d_b(y, T(y))}{1 + d_b(\varkappa, y)}\right\}\right]\right)$$
$$\leq F(d_b(\varkappa, y)). \qquad (2.1)$$

If (2.1) holds then S and T have a common fixed point q in Y.

Proof Suppose S and T have no any fixed point then for each $\varkappa, y \in Y$ we have $\max\{d_b(\varkappa, S(\varkappa)), d_b(y, T(y))\} > 0$. Since $S(\varkappa), T(y) \in P(Y)$ for every $\varkappa, y \in Y$ and $F \in \xi F_b^*$, it is simple to prove F_η^x is a non-empty set for all $\varkappa, y \in Y$(proof will follow in the first line of [19]). As $S, T: Y \to P(Y)$ are two $\alpha_* -$dominated multi-maps on $\{TS(x_j)\}$, so by definition, we have $(\alpha_*\ (\varkappa_{2j}, S(\varkappa_{2j})) \geq 1$ and $(\alpha_*(\varkappa_{2i+1}, T(\varkappa_{2j+1})) \geq 1$ for all $j \in \mathbb{N}$. As $\alpha_*(\varkappa_{2j}, S(\varkappa_{2j})) \geq 1$, this implies that $\inf\{\alpha(x_{2j}, b): b \in S(\varkappa_{2j})\} \geq 1$ and therefore, $\alpha(x_{2j}, x_{2j+1}) \geq 1$. If $\varkappa_0 \in Y$ is any initial point then $\varkappa_{2j}, \varkappa_{2j+1} \in F_\eta^{\varkappa_0}$ and using (2.1), we have

$$\tau\left(d_b(\varkappa_{2j}, \varkappa_{2j+1})\right) + F\left(b\left[\max\left\{d_b(\varkappa_{2j}, S(\varkappa_{2j})), d_b(\varkappa_{2j+1}, T(\varkappa_{2j+1})), \frac{d_b(\varkappa_{2j}, S(\varkappa_{2j})). d_b(\varkappa_{2j+1}, T(\varkappa_{2j+1}))}{1 + d_b(\varkappa_{2j}, \varkappa_{2j+1})}\right\}\right]\right) \leq F\left(d_b(\varkappa_{2j}, \varkappa_{2j+1})\right)$$

$$\tau\left(d_b(\varkappa_{2j}, \varkappa_{2j+1})\right) + F\left(b\left[\max\left\{d_b(\varkappa_{2j}, \varkappa_{2j+1}), d_b(\varkappa_{2j+1}, \varkappa_{2j+2}), \frac{d_b(\varkappa_{2j}, \varkappa_{2j+1}). d_b(\varkappa_{2j+1}, \varkappa_{2j+2})}{1 + d_b(\varkappa_{2j}, \varkappa_{2j+1})}\right\}\right]\right) \leq F\left(d_b(\varkappa_{2j}, \varkappa_{2j+1})\right),$$

$$\tau\left(d_b(\varkappa_{2j}, \varkappa_{2j+1})\right) + F\left(b\left[\max\{d_b(\varkappa_{2j}, \varkappa_{2j+1}), d_b(\varkappa_{2j+1}, \varkappa_{2j+2}), d_b(\varkappa_{2j+1}, \varkappa_{2j+2})\}\right]\right) \leq F\left(d_b(\varkappa_{2j}, \varkappa_{2j+1})\right).$$

This implies that,

$$\tau\left(d_b(\varkappa_{2j}, \varkappa_{2j+1})\right) + F\left(b\left[\max\{d_b(\varkappa_{2j}, \varkappa_{2j+1}), d_b(\varkappa_{2j+1}, \varkappa_{2j+2})\}\right]\right) \leq F\left(d_b(\varkappa_{2j}, \varkappa_{2j+1})\right). \tag{2.2}$$

If $\max\{d_b(\varkappa_{2j}, \varkappa_{2j+1}), d_b(\varkappa_{2j+1}, \varkappa_{2j+2})\} = d_b(\varkappa_{2j}, \varkappa_{2j+1})$, then from (2.2), we have

$$\tau\left(d_b(\varkappa_{2j}, \varkappa_{2j+1})\right) + F\left(b\left[d_b(\varkappa_{2j}, \varkappa_{2j+1})\right]\right) \leq F\left(d_b(\varkappa_{2j}, \varkappa_{2j+1})\right),$$

which is wrong due to $(F1)$, that is, F is strictly increasing. Therefore

$$\max\{d_b(\varkappa_{2j}, \varkappa_{2j+1}), d_b(\varkappa_{2j+1}, \varkappa_{2j+2})\} = d_b(\varkappa_{2j+1}, \varkappa_{2j+2}),$$

for each $j \in \mathbb{N} \cap \{0\}$, we have

$$\tau\left(d_b(\varkappa_{2j}, \varkappa_{2j+1})\right) + F\left(b\left[d_b(\varkappa_{2j+1}, \varkappa_{2j+2})\right]\right) \leq F\left(d_b(\varkappa_{2j}, \varkappa_{2j+1})\right), \tag{2.3}$$

for all $j \in \mathbb{N} \cap \{0\}$, from (2.3) and applying $(F4)$ we get,

$$\tau\left(d_b(\varkappa_{2j}, \varkappa_{2j+1})\right) + F\left(b^{2j+1}\left[d_b(\varkappa_{2j+1}, \varkappa_{2j+2})\right]\right) \leq F\left(b^{2j} d_b(\varkappa_{2j}, \varkappa_{2j+1})\right). \tag{2.4}$$

Since, $\varkappa_{j+1} \in F_\eta^y$ then by the definition of F_η^y we have

$$F\left(d_b(\varkappa_{2j}, \varkappa_{2j+1})\right) \leq F\left(d_b(\varkappa_{2j}, T\varkappa_{2j})\right) + \eta d_b(\varkappa_{2j}, \varkappa_{2j+1}),$$

which implies that

$$F\left(b^{2j}d_b\left(\varkappa_{2j},\varkappa_{2j+1}\right)\right)$$
$$\leq F\left(b^{2j}d_b\left(\varkappa_{2j},T\varkappa_{2j}\right)\right)$$
$$+\eta d_b\left(\varkappa_{2j},\varkappa_{2j+1}\right). \qquad (2.5)$$

From (2.4) and (2.5), we have

$$F\left(\ b^{2j+1}\left[d_b\left(\varkappa_{2j+1},\varkappa_{2j+2}\right)\right]\right)$$
$$\leq \ F\left(b^{2j}d_b\left(\varkappa_{2j},\varkappa_{2j+1}\right)\right)+\eta d_b\left(\varkappa_{2j},\varkappa_{2j+1}\right)-\tau\left(d_b\left(\varkappa_{2j},\varkappa_{2j+1}\right)\right). \qquad (2.6)$$

Similarly for each $j\in\mathbb{N}\cap\{0\}$ we have,

$$F\left(\ b^{2j}\left[d_b\left(\varkappa_{2j},\varkappa_{2j+1}\right)\right]\right)$$
$$\leq F\left(b^{2j-1}d_b\left(\varkappa_{2j-1},\varkappa_{2j}\right)\right)+\eta d_b\left(\varkappa_{2j-1},\varkappa_{2j}\right)$$
$$-\tau\left(d_b\left(\varkappa_{2j-1},\varkappa_{2j}\right)\right). \qquad (2.7)$$

Using (2.7) in (2.5) we have,

$$F\left(\ b^{2j+1}\left[d_b\left(\varkappa_{2j+1},\varkappa_{2j+2}\right)\right]\right)$$
$$\leq F\left(b^{2j-1}d_b\left(\varkappa_{2j-1},\varkappa_{2j}\right)\right)+\eta d_b\left(\varkappa_{2j},\varkappa_{2j+1}\right)+\eta d_b\left(\varkappa_{2j-1},\varkappa_{2j}\right)-\tau\left(d_b\left(\varkappa_{2j-1},\varkappa_{2j}\right)\right)$$
$$-\tau\left(d_b\left(\varkappa_{2j},\varkappa_{2j+1}\right)\right).$$

Now put $2j+1=g$ and also $d_b\left(\varkappa_{2j},\varkappa_{2j+1}\right)=\partial_g$ for all $\in\mathbb{N}\cap\{0\}$, $\partial_g>0$ and from (2.8) $\left\{\partial_g\right\}$ is decreasing. Therefore, there exists $\varepsilon>0$ such that $\lim_{g\to\infty}\partial_g=\varepsilon$. As $\varepsilon>0$, let $\gamma(t)=\lim inf_{t\to s^+}\tau(t)-\lim inf_{t\to s^+}\eta(t)\geq 0$. Then, using (2.6), we get

$$F\left(b^g\partial_g\right)\leq F\left(b^{g-1}\partial_g\right)-\gamma\left(\partial_g\right),$$
$$\leq F\left(b^{g-1}\partial_{g-1}\right)-\gamma\left(\partial_g\right)-\gamma\left(\partial_{g-1}\right),$$

… … … ..

$$\leq F\left(\partial_0\right)-\gamma\left(\partial_g\right)-\gamma\left(\partial_{g-1}\right)-\cdots-\gamma\left(\partial_0\right). \qquad (2.9)$$

Let l_g be the greatest number in $\{0,1,\ 2,3,\cdots,g-1\}$ such that,

$$\gamma\left(\partial_{l_g}\right)=\min\left\{\gamma\left(\partial_0\right),\gamma\left(\partial_1\right),\gamma\left(\partial_2\right),\cdots,\gamma\left(\partial_g\right)\right\}.$$

For every $g\in\mathbb{N}$, so the sequence $\left\{\partial_g\right\}$ is a non-decreasing. Now from equation (2.9) we get,

$$F\left(b^g\partial_g\right)\leq \ F\left(\partial_0\right)-g\gamma\left(\partial_{l_g}\right).$$

Similarly, from (2.6) we can obtain

$$F\left(b^{g+1}d_b\left(\varkappa_{g+1},T(\varkappa_{g+1})\right)\right)$$

$$\leq F\left(b^g d_b(\varkappa_0,\varkappa_1)\right)$$

$$-g\gamma\left(\partial_{l_g}\right). \tag{2.10}$$

Now for the sequence $\left\{\gamma\left(\partial_{l_g}\right)\right\}$ we have two cases.

Case I: For each, $g\in\mathbb{N}$, there exists $e>g$ such that $\gamma\left(\partial_{l_e}\right)>\gamma\left(\partial_{l_g}\right)$. Then we obtain a subsequence $\left\{\gamma\left(\partial_{l_{g_k}}\right)\right\}$ of $\left\{\partial_{l_g}\right\}$ with $\gamma\left(\partial_{l_{g_k}}\right)>\gamma\left(\partial_{l_{g_{k+1}}}\right)$ for all k. Since, $\partial_{l_{g_k}}\to\delta^+$, we deduce that

$$\lim_{t\to\delta^+}\inf\gamma\left(\partial_{l_{g_k}}\right)>0.$$

Hence,

$$F\left(b^{g_k}\partial_{g_k}\right)\leq F(\partial_0)-g^k\gamma\left(\partial_{l_{g_k}}\right),$$

For every k. Consequently, $\lim_{k\to\infty}F\left(b^{g_k}\partial_{g_k}\right)=-\infty$ and by $(F2)$ $\lim_{k\to+\infty}b^{g_k}\partial_{g_k}=0$, which is not true that $\lim_{k\to+\infty}\partial_{g_k}>0$ for $b>1$.

Case II: As $g_0\in\mathbb{N}$ so that $\gamma\left(\partial_{l_{g_0}}\right)>\gamma\left(\partial_{l_e}\right)$ for every $e>g_0$. Then

$$F(\partial_e)\leq F(\partial_0)-e\gamma\left(\partial_{l_e}\right),\qquad \text{for all } e>g_0.$$

Hence, $\lim_{e\to+\infty}F(\partial_e)=-\infty$ and by $(F2)$ $\lim_{e\to+\infty}\partial_e=0$, which contradicts the fact that $\lim_{e\to+\infty}\partial_e>0$. Thus, $\lim_{e\to+\infty}\partial_e=0$. From $(F3)$ there exists $k\in(0,1)$ such that $\lim_{g\to+\infty}\left(b^g\partial_g\right)^kF\left(b^g\partial_g\right)=-\infty$ and from inequality (2.10) the following holds for all $g\in\mathbb{N}$:

$$\left(b^g\partial_g\right)^kF\left(b^g\partial_g\right)-\left(b^g\partial_g\right)^kF(\partial_0)\leq\left(b^g\partial_g\right)^k\left(F(\partial_0)-g\gamma\left(\partial_{l_g}\right)\right)-\left(b^g\partial_g\right)^kF(\partial_0)$$

$$=-g\left(b^g\partial_g\right)^k\gamma\left(\partial_{l_g}\right)\leq 0. \tag{2.11}$$

Passing to limit as $g\to+\infty$ in (2.11), we obtain

$$\lim_{g\to+\infty}g\left(b^g\partial_g\right)^k\gamma\left(\partial_{l_g}\right)=0.$$

Since $\zeta=\lim_{g\to+\infty}\gamma\left(\partial_{l_g}\right)>0$ there exists $g_0\in\mathbb{N}$, such that $\gamma\left(\partial_{l_g}\right)>\frac{\zeta}{2}$ for all $g\neq g_0$. Thus,

$$g\left(b^g\partial_g\right)^k\frac{\zeta}{2}$$

$< g\left(b^g\partial_g\right)^k\gamma\left(\partial_{l_g}\right),$

for each $g > \ g_0$. Letting $g \to +\infty$ in (2.12), we have

$$0 \leq \lim_{g\to+\infty} \ g\left(b^g\partial_g\right)^k\frac{\zeta}{2} < \lim_{g\to+\infty} g\left(b^g\partial_g\right)^k\gamma\left(\partial_{l_g}\right) = 0.$$

That is,

$$\lim_{g\to+\infty} \ g\left(b^g\partial_g\right)^k$$

$= 0,$

from (2.13) there exists $g_1 \in \mathbb{N}$ such that $g\left(b^g\partial_g\right)^k \leq 1$ for all $g \ > \ g_1$

$$\left(b^g\partial_g\right)^k \leq \frac{1}{g},$$

$$\partial_g \leq \frac{1}{b^g g^{1/k}}, \qquad \text{for all } g > g_{1.}$$

Now, taking limit $g \to \ +\infty$ then series $\sum_{g=1}^{\infty} b^g\partial_g$ becomes convergent, and the sequence $\{\varkappa_g\}$ is a Cauchy in Y and it is obvious that Y is a complete orbitally BMS, there exists $q \in \ \sigma(\varkappa_0)$ so that $\varkappa_g \to q$ when $g \to \ +\infty$. By (2.10) and $(F2)$, we find $\lim_{g\to+\infty} d_b\left(\varkappa_g, T(\varkappa_g)\right) = 0$. Since $q \to d_b(\varkappa, T(\varkappa))$ is orbitally lower semi-continuous and $\alpha\left(\varkappa_g, T\left(\varkappa_g\right) > 1\right.$, we have

$$0 \leq d_b(q, T(q)) \leq \lim_{g\to+\infty} \inf d_b\left(\varkappa_g, T(\varkappa_g)\right) \leq \lim_{j\to+\infty} \inf d_b\left(\varkappa_g, \varkappa_{g+1}\right),$$

$$\leq \lim_{g\to+\infty} b\left[d_b\left(\varkappa_g, \varkappa\right) + d_b\left(\varkappa, \varkappa_{g+1}\right)\right] = 0.$$

Hence, $q \in T(q)$.

As the series $\sum_{g=1}^{\infty} b^g\partial_g$ is convergent, and the sequence $\{y_g\}$ is Cauchy in Y and Y is a complete orbitally BMS, there exists $q \in \ \sigma(\varkappa_0)$ such that $y_g \to q$ as $g \to \ +\infty$. Using (2.10) and $(F2)$, we get $\lim_{g\to+\infty} d_b\left(y_g, S(y_g)\right) = 0$. Since $q \to d_b(y, S(y))$ is orbitally lower semi-continuous and also $\alpha\left(y_g, S(y_g)\right) > 1$, we have

$$0 \leq d_b\left(q, S(q)\right) \leq \lim_{g\to+\infty} \inf d_b\left(y_g, S(y_g)\right) \leq \lim_{j\to+\infty} \inf d_b\left(y_g, y_{g+1}\right),$$

$$\leq \lim_{g\to+\infty} b\left[d_b\left(y_g, y\right) + d_b\left(y, y_{g+1}\right)\right] = 0.$$

Therefore, $q \in S(q)$. Hence, S has a fixed point. So, T and S both have common fixed point in Y.

Recall that $u \preccurlyeq A$ means there is $b \in A$ such that $u \preccurlyeq b$. The function $S: Y \to P(Y)$ is multi $\preccurlyeq$-dominated on A, if $u \preccurlyeq Su$ for any $u \in Y$.
We prove upcoming theorem for $\preccurlyeq$-dominated multi maps on $\{TS(c_j)\}$ in a complete orbitally B-M-S.

Theorem 2.6 Let $(Y, \preccurlyeq, d_b)$ be a complete orbitally BMS. Let $y_0 \in Y, \alpha: \ Y \times Y \to [0, \infty)$ and $S, T: Y \to P(Y)$ be two α_*-dominated multi-maps and $F \in \xi F_b^*$. Assume that following properties hold:

i. The mappings $z \mapsto \max\{d_b(\varkappa, S(\varkappa)), d_b(y, T(y))\}$ are orbitally lower semi continuous;

ii. There exists functions $\tau, \eta: (0, \infty) \to (0, \infty)$ such that for all $t \geq 0$;

$$\tau(t) > \eta(t), \qquad \lim_{s \to t^+} inf\, \tau(t) > \lim_{s \to t^+} inf\, \eta(t);$$

iii. For all $\varkappa, y \in \{ST(f_j)\}$ with either $x \preccurlyeq y$ or $y \preccurlyeq x$ and max $\{d_b(\varkappa, S(\varkappa)), d_b(y, T(y))\} > 0$, also $\{ST(f_j)\} \to f^*$, there exists $y \in F_{\eta, \preccurlyeq}^{\varkappa}$ satisfying;

$$\tau(d_b(\varkappa, y)) + F\left(b\left[\max\left\{d_b(\varkappa, S(\varkappa)), d_b(y, T(y)), \frac{d_b(\varkappa, S(\varkappa)).\, d_b(y, T(y))}{1 + d_b(\varkappa, y)}\right\}\right]\right)$$
$$\leq F(d_b(\varkappa, y)). \qquad (2.14)$$

Also if, (2.14) holds for f^*, $f^* \preccurlyeq f_j$ or $f_j \preccurlyeq f^*$ where $j = \{0,1,2,3, \dots\}$, then both S and T have a common fixed point f^* in Y.

Proof Let $\alpha: Y \times Y \to [0, +\infty)$ be a function defined by $\alpha(f, q) = 1$ for each $f \in Y$ with $f \preccurlyeq q$, and $\alpha(f, q) = 0$ for each comparable elements $f, q \in Y$. As S and T are two multi dominated maps on Y, so $f \preccurlyeq S(\varkappa)$ and $f \preccurlyeq T(y)$ for every $f \in Y$. This implies that $f \preccurlyeq u$ for every $u \in S(\varkappa)$ and $f \preccurlyeq u$ for each $f \in T(y)$. So, $\alpha(f, u) = 1$ for every $u \in S(\varkappa)$ and $\alpha(f, u) = 1$ for each $f \in T(y)$. This implies that $\inf\{\alpha(f, q): q \in S(\varkappa)\} = 1$, and $\inf\{\alpha(f, q): q \in T(y)\} = 1$.
Hence, $\alpha_*(f, S(\varkappa)) = 1$, $\alpha_*(f, T(y)) = 1$ for each $f \in Y$. So, $S, T: Y \to P(Y)$ are α_*-dominated multi mappings on Y. Moreover, (2.14) exists can be written as

$$\tau(d_b(\varkappa, y)) + F\left(b\left[\max\left\{d_b(\varkappa, S(\varkappa)), d_b(y, T(y)), \frac{d_b(\varkappa, S(\varkappa)).\, d_b(y, T(y))}{1 + d_b(\varkappa, y)},\right\}\right]\right) \leq F(d_b(\varkappa, y)),$$

for each f, q in $\{TS(f_j)\}$, with either $\alpha(f, q) \geq 1$ or $\alpha(q, f) \geq 1$. Then, by Theorem 2.4, the sequence $\{ TS(f_j)\}$ is convergent in Y as $\{TS(f_j)\} \to f^* \in Y$. Now, $f_j, f^* \in Y$ and either $f_j \preccurlyeq$

f^*, or $f^* \preccurlyeq f_j$ implies that the either $\alpha(f_j, f^*)$, or $\alpha(f^*, f_j) \geq 1$. Hence, all the conditions of Theorem 2.6 are satisfied. So, from Theorem 2.6, both S and T have a multi fixed point f^* in Y and $d_b(f^*, f^*) = 0$.

Example 2.7 Let $Y = [0, \infty)$ and consider a relation $\preccurlyeq$ on Y by defining $x \preccurlyeq y$ if and only if x divides y. Then it is easy to verify that $(Y, \preccurlyeq)$ is a partially ordered set. Now we define $d_b: Y \times Y \to \mathbb{R}$ by $d_b(p,q) =$

$$\begin{cases} 0 & if \quad x = y; \\ 3\left(\frac{1}{n} + \frac{1}{m}\right) & if \quad x = n, y = m \quad and \quad n \neq m; \\ \frac{1}{n} & if \quad x = n, y = 0, or \quad x = 0, y = n; \end{cases}$$

for each $p, q \in Y$. Then the function d_b becomes an ordered b-metric on Y with $b = 3$. Also, we define mappings $S, T: Y \to P(Y)$ by

$$S(p) = \{p, 3p, 5p\} \qquad and \qquad T(q) = \{2q, 4q, 6q\},$$

and $\alpha: Y \times Y \to R$ defined as

$$\alpha(p,q) = \begin{cases} 1 & if \quad p > q \\ \frac{1}{2} & otherwise. \end{cases}$$

Now, for all $p, q \in \{TS(x_j)\}$ with either $\alpha(p,q) \geq 1$ or $\alpha(q,p) \geq 1$. Define a function $F: \mathbb{R}^+ \to \mathbb{R}$ by $F(p) = \ln(p)$ for all $p \in \mathbb{R}^+$ and and $\tau(t) = 1/8, \eta(t) = 1/10$ for each $t \in (0, \infty)$. Then, clearly $F \in \xi F_b^*$ and $\tau(t) > \eta(t)$, $\lim_{s \to t^+} inf\, \tau(t) > \lim_{s \to t^+} inf\, \eta(t)$. As $p, q \in Y$, then

$$S(p) = \begin{cases} 0 & if \quad p = 0 \\ \frac{14}{3p} & if \quad p \neq 0 \end{cases} \qquad and \qquad T(q) = \begin{cases} 0 & if \quad q = 0 \\ \frac{13}{3q} & if \quad q \neq 0. \end{cases}$$

Therefore, $q \to max\{d_b(p, S(p)), d_b(q, T(q))\}$ is orbitally lower semicontinuous. For $q \in Y$, we have $q = 4p \in F^{\varkappa}_{\eta,\preccurlyeq}$, and for this q, we have

$$\tau(d_b(p,q)) + F\left(b\left[\max\left\{d_b(p, S(p)), d_b(q, T(q)), \frac{d_b(p, S(p)).\, d_b(q, T(q))}{1 + d_b(p,q)}\right\}\right]\right)$$

$$= \frac{1}{8} + ln\left[\!\left[3max\left\{\frac{14}{12p}, \frac{13}{12q}, \frac{7 \times 13}{9pq}\right\}\right]\!\right]$$

$$= \frac{1}{8} + ln\left(\frac{39}{12q}\right)$$

$$\leq ln\left(\frac{4}{3}\cdot\frac{39}{12q}\right) \leq ln\left(\frac{39}{12q}\right),$$

$$=F(d_b(p,q)).$$

Hence, all the requirements of our Theorem 2.6 hold for $F(p) = \ln(p)$ where $p > 0$.

3. Application to Integral Equation.

Using distinct generalized contractions in different settings of generalized metric spaces, a significant number of writers showed necessary and sufficient conditions for different types of linear and nonlinear (Volterra and Fredholm) type integrals in fixed point theory. Rasham et al., [30] presented the presence of new fixed point results for two families of multivalued mappings, and their main result was utilized to examine necessary conditions for the solution of nonlinear integral equations. For more recent fixed point results incorporating integral inclusions applications, click here (see [4, 22, 27, 32]).

Let $\mathcal{X} = (\mathrm{C}[0,1], \mathcal{R}_+)$ represent the set of continuous functions on $[0,1]$. Take an integral equation

$$\varkappa(t) = \frac{1}{2}\int_0^t \mathcal{K}\big(\varkappa, t, \varkappa(t)\big)dt\,;$$

where $\mathcal{K}: \mathrm{C}[0,1] \times \mathrm{C}[0,1] \times \mathcal{X} \to \mathcal{R}$ is defined and continuous for all $\varkappa, t \in \mathrm{C}[0,1]$. Our aim is to prove the solution of equation (3.1) for our main Theorem4.2.

Theorem 3.1 Let $\mathcal{X} = (\mathrm{C}[0,1], \mathcal{R})$ and let $S, T: \mathcal{X} \to \mathcal{X}$ defined as

$\mathrm{S}(\varkappa)(t) = \frac{1}{2}\int_0^t \mathcal{K}\big(\varkappa, t, \varkappa(t)\big)dt,\ and\ \ T(y)(t) = \frac{1}{2}\int_0^t \mathcal{K}\big(\varkappa, t, y(t)\big)dt\,, where\ \ \varkappa, y, t \in \mathrm{C}[0,1]$.

Also $\mathcal{K}: \mathrm{C}[0,1] \times \mathrm{C}[0,1] \times \mathcal{X} \to \mathcal{R}$ is defined as continuous on $\mathcal{R}$ for all $\varkappa, t \in \mathrm{C}[0,1]$ and following properties are hold:

i. There exists a continuous function $\mu: \mathcal{X} \to \mathcal{R}_+$ satisfying
$\max\{\left|\mathcal{K}\big(\varkappa, t, \varkappa(t)\big) - \mathcal{K}\big(\varkappa, t, S(\varkappa)(t)\big)\right|, \left|\mathcal{K}\big(\varkappa, t, y(t)\big) - \mathcal{K}\big(\varkappa, t, T(y)(t)\big), \right| \leq$
$\mu(t)|\varkappa(t) - y(t)|,$

ii. $\left(\int_0^t \mu(t)dt\right)^2 \leq e^{-\tau} = \tau\max\{d_b(\varkappa, S(\varkappa),), d_b\big(\varkappa, T(y)\big) > 0.$

iii. Also,
$D_b(\varkappa, y) = \max\left\{|\varkappa - S\varkappa|^2, |y - Ty|^2, \frac{|\varkappa - S\varkappa|^2.|y - Ty|^2}{1+|\varkappa - y|^2}\right\} = |y - Ty|^2.$

Then, (3.1) has a unique solution.

Proof We give an assurance here that both the multi-maps S and T hold all the necessary requirements of our main Theorem 2.4, for single valued mappings. Let $\varkappa \in \mathcal{X} = (\mathrm{C}[0,1], \mathcal{R}_+)$, then we have

$$|\mathcal{y}(t) - T(\mathcal{y})(t)|^2 \leq \frac{1}{4}\left(\int_0^t \left|\mathcal{K}\big(\varkappa, t, \varkappa(t)\big) - \mathcal{K}(\varkappa, t, T(\mathcal{y})(t))\right| dt\right)^2$$

$$\leq \frac{1}{4}\left(\int_0^t \mu(t)|\varkappa(t) - \mathcal{y}(t)|^2 dt\right)^2$$

$$\leq \frac{1}{4} d_b(\varkappa, \mathcal{y})\left(\int_0^t \mu(t) dt\right)^2$$

$$|\mathcal{y}(t) - T(\mathcal{y})(t)|^2 \leq \frac{1}{4} d_b(\varkappa, \mathcal{y}) e^{-\tau}$$

$$e^{\tau}|\mathcal{y}(t) - T(\mathcal{y})(t)|^2 \leq \frac{1}{4} d_b(\varkappa, \mathcal{y})$$

$$4e^{\tau}|\mathcal{y}(t) - T(\mathcal{y})(t)|^2 \leq d_b(\varkappa, \mathcal{y}).$$

Take $b = 2$ and ln on both sides

$$\ln e^{\tau} + \ln(b|\mathcal{y}(t) - T(\mathcal{y})(t)|^2) \leq \ln\big(d_b(\varkappa, \mathcal{y})\big).$$

Hence from Theorem 2.4, define the function F by $F(\varkappa) = \ln \varkappa$,

$$\tau d_b(\varkappa, \mathcal{y}) + \ln\left|b d_b\big(\mathcal{y}, T(\mathcal{y})\big)\right| \leq \ln\big(d_b(\varkappa, \mathcal{y})\big)$$

$$\tau d_b(\varkappa, \mathcal{y}) + F\left(b d_b\big(\mathcal{y}, T(\mathcal{y})\big)\right) \leq \mathrm{F}\big(d_b(\varkappa, \mathcal{y})\big).$$

So, all the conditions of our main Theorem 2.4 for single valued mappings are satisfied. Hence, the integral equation (3.1) has a unique common solution.

4. Application to Fractional Differential Equation

Many relevant aspects of fractional differentials were introduced and shown by Lacroix (1819). Later, a wide number of academics proved a variety of important fixed-point theorems in many types of metric spaces using various generalized contractions, as well as applications to fractional differential equations (see [21,23,29]). A variety of new models connected to the Caputo–Fabrizio derivative (CFD) have recently been constructed and demonstrated, see [18,34]. In this section, we will show that one of these types of models exists in b-metric spaces.

Let $I = [0,1]$ and $\mathrm{C}(I, \mathcal{R})$ be the space of continuous function. Define the metric $d_b: \mathrm{C}(I, \mathcal{R}) \times \mathrm{C}(I, \mathcal{R}) \to [0. \infty)$ by $d_b(u, v) = \|u - v\|_\infty^2 = \max_{l \in [0,L]} |u(l) - v(l)|^2$, for each $u, v \in \mathrm{C}(I, \mathcal{R})$.

Then $(\mathrm{C}(I, \mathcal{R}), d_b)$ is a complete orbital BMS. Let $\mathcal{K}_1: I \times \mathcal{R} \to \mathcal{R}$ be a mapping such that

$\mathcal{K}_1(l,g) \geq 0$ for all $l \in I$ and $g \geq 0$. We will investigate the following CFDF:

$$^{\mathrm{C}}D^v g(l) = \mathcal{K}_1\big(l, g(l)\big); \quad g \in \mathrm{C}(I,\mathcal{R}) \tag{4.1}$$

with boundary conditions $g(0) = 0, Ig(1) = g'(0)$.

Here, $^{\mathrm{C}}D^v$ represents the CFD of order v given as

$$^{\mathrm{C}}D^v g(l) = \frac{1}{\gamma(p-v)}\int_0^l \big((l-e)^{p-v-1} g(e)\big)de,$$

where $p-1 < v < p$ and $p = [n]+1$, and $I^v g$ is defined by

$$I^v g(l) = \frac{1}{\gamma(v)}\int_0^l \big((l-e)^{v-1} g(e)\big)de\,, \text{with } \ v > 0.$$

Then the equation (4.1) can be modified to

$$g(l) = \frac{1}{\gamma(v)}\int_0^l (l-e)^{v-1}\mathcal{K}_1\big(e, g(e)\big)de + \frac{2l}{\gamma(v)}\int_0^L\int_0^e (e-z)^{v-1}\mathcal{K}_1\big(z, g(z)\big)dzde.$$

The mapping $\mathcal{K}_1 : I \times \mathcal{R} \to \mathcal{R}$ satisfies the following conditions:

a) There exists $\tau > 0$, such that,

$$|\mathcal{K}_1(l,g) - \mathcal{K}_1(l,e)|^2 \leq \frac{e^{-\tau}\gamma(v+1)}{4}|g-e|^2,$$

for all $g, e > 0$ with $\tau = \tau d_b(e,g) > 0$.

b) There exists $q \in \mathrm{C}(I,\mathcal{R})$ so that for any $l \in I$,

$$q(l) = \frac{1}{\gamma(v)}\int_0^l (l-e)^{v-1}\mathcal{K}_1\big(e, u_0(e)\big)de + \frac{2l}{\gamma(v)}\int_0^L\int_0^e (e-z)^{v-1}\mathcal{K}_1\big(z, u_0(z)\big)dzde.$$

Theorem 4.1 The equation (4.1) admits a solution in $\mathrm{C}(I,\mathcal{R})$ if the conditions (a)-(b) are satisfied.

Proof Let $\mathcal{X} = \{u \in \mathrm{C}(I,\mathcal{R}) : u(l) \geq 0 \text{ for all } l \in I\}$ and define the mapping $\mathcal{R} : \mathcal{X} \to \mathcal{X}$ in accordance with the above-mentioned notations by

$$\mathcal{R}(q)(l) = \frac{1}{\gamma(v)}\int_0^l (l-e)^{v-1}\mathcal{K}_1\big(e, q(e)\big)de + \frac{2l}{\gamma(v)}\int_0^L\int_0^e (e-z)^{v-1}\mathcal{K}_1\big(z, q(z)\big)dzde.$$

We will check the conditions of Theorem 2.4. For this purpose, we have

$$|(q)(l)-\mathcal{R}(q)(l)|^2=\left|\begin{array}{c}\frac{1}{\gamma(v)}\int_0^l(l-e)^{v-1}\mathcal{K}_1\big(e,q(e)\big)\vartheta e\\ -\frac{1}{\gamma(v)}\int_0^l(l-e)^{v-1}\mathcal{K}_1\big(e,u(e)\big)\vartheta e\\ +\frac{2l}{\gamma(v)}\int_0^L\int_0^z(z-w)^{v-1}\mathcal{K}_1\big(w,q(w)\big)\vartheta w\vartheta z\\ -\frac{2l}{\gamma(v)}\int_0^L\int_0^z(z-w)^{v-1}\mathcal{K}_1\big(w,u(w)\big)\vartheta w\vartheta z\end{array}\right|^2,$$

which implies that

$$\begin{aligned}
&|(q)(l)-\mathcal{R}(q)(l)|^2\\
&\le\left(\begin{array}{c}\left|\int_0^l\left(\frac{1}{\gamma(v)}(l-e)^{v-1}\mathcal{K}_1\big(e,q(e)\big)-\frac{1}{\gamma(v)}(l-e)^{v-1}\mathcal{K}_1\big(e,u(e)\big)\right)\vartheta e\right|\\ +\left|\int_0^L\int_0^z\left(\frac{2}{\gamma(v)}(z-w)^{v-1}\mathcal{K}_1\big(w,q(w)\big)-\frac{2}{\gamma(v)}(z-w)^{v-1}\mathcal{K}_1\big(w,u(w)\big)\right)\vartheta w\vartheta z\right|\end{array}\right)^2\\
&\le\frac{1}{\gamma(v)}\frac{e^{-\tau}\gamma(v+1)}{4}\left(\int_0^l(l-e)^{v-1}\left(q(e)-u(e)\right)\vartheta e\right)^2\\
&+\frac{2}{\gamma(v)}\frac{e^{-\tau}\gamma(v+1)}{4}\left(\int_0^L\int_0^z(z-w)^{v-1}\big(q(w)-u(w)\big)\vartheta w\,\vartheta z\right)^2\\
&\le\frac{1}{\gamma(v)}\frac{e^{-\tau}\gamma(v+1)}{4}.|q-u|^2.\left(\int_0^l(l-e)^{v-1}\,\vartheta e\right)^2\\
&+\frac{2}{\gamma(v)}\frac{e^{-\tau}\gamma(v).\gamma(v+1)}{4(v).\gamma(v+1)}.|q-u|^2.\left(\int_0^L\int_0^z(z-w)^{v-1}\vartheta w\,\vartheta z\right)^2\\
&\le\left(\frac{e^{-\tau}\gamma(v).\gamma(v+1)}{4\gamma(v).\gamma(v+1)}\right).|q-u|^2+2e^{-\tau}B(v+1,1)\frac{\gamma(v).\gamma(v+1)}{4\gamma(v)\gamma.(v+1)}.|q-u|^2\\
&\le\frac{e^{-\tau}}{4}|q-u|^2+\frac{e^{-\tau}}{2}|q-u|^2<\frac{e^{-\tau}}{4}|q-u|^2,
\end{aligned}$$

where B is the beta function. The final inequality is written as:

$$4d_b\left((q)(l),\mathcal{R}\big(u(l)\big)\right)\le d_b\big((q),\mathcal{R}(u)\big)$$

$$\le e^{-\tau}d_b(q,u). \tag{4.2}$$

Define $F\big(q(l)\big)=\ln\big(q(l)\big)$ for all $q,u\in C(I,\mathcal{R}^+)$, then the equation (4.2) can be written as

$$\tau\ \ d_b(q,u)+\ln 4+\ln\left(d_b\big((q),\mathcal{R}(u)\big)\right)\le\ln\big(\ \ d_b(q,u)\big),$$

$$\tau\ \ d_b(q,u)+\ln\left(4d_b\big((q),\mathcal{R}(u)\big)\right)\le\ln\big(\ \ d_b(q,u)\big).$$

Consider $b=4$ we get,

$$\tau \ \ d_b(q,u) + \ln\left(bd_b\big((q),\mathcal{R}(u)\big)\right) \leq \ln\big(\ \ d_b(q,u)\big).$$

All the conditions of Theorem 2.4 for single valued mapping are verified and the mappings S and T admit a fixed point. Hence, equation (4.1) has a unique solution.

5. Conclusion

In this paper, we prove some new fixed-point theorems for coupled dominated multivalued mappings in complete orbital b-metric space that execute a novel extended Nashine-Wardawski-Feng-Liu-type-contraction for orbitally lower semi continuous functions. Furthermore, the presence of some new fixed- point outcomes for a pair of dominated multivalued mappings are established in the setting of ordered complete orbital b-metric space. A few instances are provided to support our new findings. To highlight the originality of our findings, applications for a system of nonlinear integral equations and fractional differential equations are offered. In addition, we improve and generalize the findings of Nashine et al. [21] and Rasham et al. [27-29], as well as many others [1,2,3,6,7,8,17,22,24,35]. We can broaden our horizons to include, fuzzy mappings, L-fuzzy mappings, intuitionistic fuzzy mappings, and bipolar fuzzy mappings.

DECLARATIONS

Availability of data and materials
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Authors contributions
T. R and M. S. S together discussed the problem and prepared this research article. M. N analyzed all multivalued hybrid pair results for dominated mappings and made necessary improvements in applications section. A. M; write the original draft of this paper, C. P writing—review and editing, T. R.; project administration, C. P.; funding acquisition. All authors have read and approved final version of the manuscript.

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