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Essential extensions of filters in residuated lattices

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Abstract

We define the essential extension of a filter in the residuated lattice A associated to an ideal of $L(A)$ and investigate its related properties. We prove the residuated lattice A is a Boolean algebra, $G(RL)$ -algebra or MV -algebra if and only if the essential extension of $\{1\}$ associated to $A \setminus P$ is a Boolean filter, G -filter or MV -filter (for all $P \in SpecA$), respectively. Also, some properties of lattice of essential extensions are studied.

Key words and phrases: Residuated lattice, Essential extension of filters, Boolean filter, G -filter, MV -filter, Boolean algebra, $G(RL)$ -algebra, MV -algebra, MTL -algebra.

1 Introduction

Various logical algebras have been proposed as the semantical systems of non-classical logic systems, for example, residuated lattices, MV -algebras, BL -algebras, Gödel algebras, MTL -algebras, lattice implication algebras, etc. Among these logical algebras, residuated lattices are very basic and important algebraic structures because the other logical algebras are all particular cases of residuated lattices.

For the general development of residuated lattice, the filter theory plays an important role. Let F be a filter of residuated lattice A , I an ideal of $L(A)$, $x \in F$ and $a \in I$. Then $x \vee a \in F$. Generally, there exist elements of A , for instance z , such that $z \vee a \in F$, for some $a \in I$. In section 3, the set of these members is called the essential extension of F associated to I . Some of its useful properties are studied. Zhu and Xu [9] showed residuated lattice A

is a Boolean algebra, $G(RL)$ -algebra or MV -algebra if and only if every filter of residuated lattice A is a Boolean filter, G -filter or MV -filter, respectively. In this paper, we prove the residuated lattice A is a Boolean algebra, $G(RL)$ -algebra or MV -algebra if and only if the essential extension of $\{1\}$ associated to $A \setminus P$ is a Boolean filter, G -filter or MV -filter, respectively (for all prime filter P of residuated lattice A), respectively. At last, we consider the lattice of essential extension of F_0 associated to ideals of $L(A)$ and the lattice of essential extensions in A associated to I_0 , where F_0 is a filter of residuated lattice A and I_0 is an ideal of $L(A)$.

2 Preliminaries

A residuated lattice is an algebra $A = (A, \wedge, \vee, *, \rightarrow, 0, 1)$ with four binary operations $\wedge, \vee, *, \rightarrow$ and two constants $0, 1$ such that:

(Rl1) $(A, \wedge, \vee, 0, 1)$ is a bounded lattice,

(Rl2) $(A, *, 1)$ is a commutative monoid,

(Rl3) $*$ and $\rightarrow$ form an adjoint pair, i.e.,

$x * y \leq z$ if and only if $x \leq y \rightarrow z$, for all $x, y, z \in A$ [6].

On any residuated lattice A let us define a unary operation negation $-$ by $x^- = x \rightarrow 0$ for any $x \in A$.

A residuated lattice A is called :

(1) a $G(RL)$ -algebra if $x^2 = x$, for all $x \in A$, where $x^2 = x * x$.

(2) an MV -algebra if $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$, for all $x, y, z \in A$.

(3) a bounded commutative RL -monoid if $x \wedge y = x * (x \rightarrow y)$, for all $x, y \in A$ (the divisibility property).

(4) an MTL -algebra if $(x \rightarrow y) \vee (y \rightarrow x) = 1$, for all $x, y \in A$ (the pre-linearity equation).

(5) a BL -algebra if it satisfies the divisibility property and the pre-linearity equation.

Lemma 2.1.[4, 6] In any residuated lattice A , the following relations hold for all $x, y, z \in A$:

- (1) $1 \rightarrow x = x$, $x \rightarrow x = 1$,
- (2) $x \leq y$ if and only if $x \rightarrow y = 1$,
- (3) $x * y \leq x \wedge y$,
- (4) $x \leq y \rightarrow x$,
- (5) $x \rightarrow (y \rightarrow z) = (x * y) \rightarrow z = y \rightarrow (x \rightarrow z)$,
- (6) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$,
- (7) $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$,
- (8) If $x \leq y$, then $y \rightarrow z \leq x \rightarrow z$ and $z \rightarrow x \leq z \rightarrow y$,
- (9) $x \vee y \leq ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x)$,
- (10) $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$,

- (11) $x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y)$,
- (12) If $x \leq y$ then $x * z \leq y * z$,
- (13) $(x * y)^- = y \rightarrow x^-$,
- (14) $x * (y \vee z) = (x * y) \vee (x * z)$,
- (15) $x \vee (y * z) \geq (x \vee y) * (x \vee z)$,
- (16) $x^m \vee y^n \geq (x \vee y)^{mn}$, where $x^m = x * x * \dots * x$, (m times).

A nonempty subset F of residuated lattice A is called a filter of A if :

- (1) $x * y \in F$, for all $x, y \in F$,
- (2) $x \leq y$ and $x \in F$ imply $y \in F$.

A nonempty subset D of residuated lattice A is called a deductive system of A if :

- (1) $1 \in D$,
- (2) If $x \in D$ and $x \rightarrow y \in D$, then $y \in D$.

Proposition 2.2. [6] A nonempty subset F of residuated lattice A is a deductive system if and only if it is a filter.

A filter H of residuated lattice A is said to be maximal if it is a proper filter of A and is not a proper subset of any proper filter of A ; and a filter F is prime if, for all $x, y \in A$, $x \vee y \in F$ implies $x \in F$ or $y \in F$. Every maximal filter is prime filter.[6]

If X is a subset of residuated lattice A , denote by $\langle X \rangle$ the filter of A generated by X . For $X = \emptyset$, we have $\langle \emptyset \rangle = \{1\}$. If $X \neq \emptyset$, then $\langle X \rangle = \{a \in A \mid a \geq x_1 * \dots * x_n, \text{ for some } x_1, \dots, x_n \in X\}$. For $F, G \in \text{Fil}(A)$, we put $F \vee G = \langle F \cup G \rangle$, where $\text{Fil}(A) = \{F \mid F \text{ is a filter of } A\}$ [6].

Proposition 2.3. [6] If A is a nontrivial residuated lattice, then any proper filter of A can be extended to a prime filter.

Because each bounded commutative *Rl-monoid* is a residuated lattice, we can extend the definitions and theorems that follow similar axioms from bounded commutative *Rl-monoid* to residuated lattice. For example, because the proof of Theorems 3.1 and 3.5 [5] do not depend on the divisibility property, we can extend these from bounded commutative *Rl-monoid* to residuated lattice.

Let F be a filter of residuated lattice A and $B \subseteq A$. We define the extended filter of F associated to B as follows:

$$E_F(B) = \{x \in A \mid x \vee b \in F, \forall b \in B\}.$$

Theorem 2.4. [5] If F is a filter of residuated lattice A and $B \subseteq A$ then

- (1) $E_F(B)$ is a filter of A containing F ,
- (2) $B \subseteq E_F(E_F(B))$.

Definition 2.5. [9] A nonempty subset F of residuated lattice A is called:

- (a) a Boolean filter if it is a filter of A that satisfies the condition: $x \vee x^- \in F$, for all $x \in A$.
- (b) a G -filter if it is a filter of A that satisfies the condition: $x^2 \rightarrow y \in F$ implies $x \rightarrow y \in F$, for all $x, y \in A$.
- (c) an MV -filter if it is a filter of A that satisfies the condition: $((x \rightarrow y) \rightarrow y) \rightarrow ((y \rightarrow x) \rightarrow x) \in F$, for all $x, y \in A$.

Theorem 2.6. [9] Let F be a filter of residuated lattice A . Then F is a G -filter if and only if $x \rightarrow x^2 \in F$, for all $x \in A$.

Theorem 2.7. [9] Let A be a residuated lattice. Then the following conditions are equivalent:

- (1) A is a Boolean algebra,
- (2) Every filter of A is a Boolean filter of A ,
- (3) $\{1\}$ is a Boolean filter of A .

Theorem 2.8. [9] In any residuated lattice A , the following conditions are equivalent:

- (1) A is a $G(RL)$ -algebra,
- (2) Every filter of A is a G -filter of A ,
- (3) $\{1\}$ is a G -filter of A .

Theorem 2.9. [9] Let A be a residuated lattice. Then the following conditions are equivalent:

- (1) A is an MV -algebra,

(2) Every filter of A is an MV -filter of A ,

(3) $\{1\}$ is an MV -filter of A .

3 Essential extensions

Let F be a filter of residuated lattice A and I an ideal of $L(A)$. We define the essential extension of F associated to I as follows:

$$\overline{E_F(I)} = \{x \in A \mid x \vee a \in F, \exists a \in I\}.$$

It is clear that $E_F(I) \subseteq \overline{E_F(I)}$.

Theorem 3.1. Let F be a filter of residuated lattice A and I an ideal of $L(A)$. Then $\overline{E_F(I)}$ is a filter of A and $F \subseteq \overline{E_F(I)}$.

Proof. It is easy to see that $a \leq b$ and $a \in \overline{E_F(I)}$ imply $b \in \overline{E_F(I)}$. Clearly, $1 \in \overline{E_F(I)}$. Now let $a, b \in \overline{E_F(I)}$. Then $a \vee x \in F$ and $b \vee y \in F$, for some $x, y \in I$. By Lemma 2.1, we have $(a * b) \vee (x \vee y) \geq (a \vee x \vee y) * (b \vee x \vee y)$. Hence $(a * b) \vee (x \vee y) \in F$. We note that $x \vee y \in I$, because $x, y \in I$. Then $a * b \in \overline{E_F(I)}$. Therefore $\overline{E_F(I)}$ is a filter of A . Let $a \in F$. Using $a \leq a \vee x$, for $x \in I$ we have $a \vee x \in F$, for $x \in I$. Hence $F \subseteq \overline{E_F(I)}$. $\square$

Example 3.2. Let $A = \{0, n, a, b, c, d, 1\}$, with $0 < n < a < b, c < d < 1$, but b and c are uncomparable. Then A becomes a residuated lattice relative to the operations:

*	0	n	a	b	c	d	1
0	0	0	0	0	0	0	0
n	0	0	0	n	0	n	n
a	0	0	a	a	a	a	a
b	0	n	a	b	a	b	b
c	0	0	a	a	c	c	c
d	0	n	a	b	c	d	d
1	0	n	a	b	c	d	1

$\rightarrow$	0	n	a	b	c	d	1
0	1	1	1	1	1	1	1
n	c	1	1	1	1	1	1
a	n	n	1	1	1	1	1
b	0	n	c	1	c	1	1
c	n	n	b	b	1	1	1
d	0	n	a	b	c	1	1
1	0	n	a	b	c	d	1

We note that A is not a bounded commutative *Rl-monoid* (and so is not a *BL*-algebra) because $c \wedge n = n \neq 0 = c * (c \rightarrow n)$. Since $(b \rightarrow c) \vee (c \rightarrow b) = d \neq 1$, then A is not an *MTL*-algebra. A is not a *G(RL)*-algebra, since $n^2 = 0 \neq n$. Also we have $(b \rightarrow 0) \rightarrow 0 = 1 \neq b = (0 \rightarrow b) \rightarrow b$, hence A is not an *MV*-algebra.

It is easy to see that $F = \{1, d\}$ is a filter of A , $I = \{0, n, a, b\}$ is an ideal of $L(A)$ and $F \subset \{1, d, c\} = \overline{E_F(I)}$.

We denote $Id(L(A)) = \{I \mid I \text{ is an ideal of } L(A)\}$.

Lemma 3.3. If $F, G \in Fil(A)$ and $I, J \in Id(L(A))$ then

$$(1) \ I \subseteq J \text{ implies } \overline{E_F(I)} \subseteq \overline{E_F(J)},$$

$$(2) \ F \subseteq G \text{ implies } \overline{E_F(I)} \subseteq \overline{E_G(I)},$$

$$(3) \ \overline{E_F(I)} \cap \overline{E_G(I)} = \overline{E_{F \cap G}(I)},$$

$$(4) \ \overline{E_F(I)} \cap \overline{E_F(J)} = \overline{E_F(I \cap J)},$$

$$(5) \ \overline{E_F(I)} = \overline{E_{\overline{E_F(I)}}(I)}.$$

Proof. Parts (1) and (2) are clear.

(3) Using part (2) and $F \cap G \subseteq F, G$, we have $\overline{E_{F \cap G}(I)} \subseteq \overline{E_F(I)} \cap \overline{E_G(I)}$. Let $a \in \overline{E_F(I)} \cap \overline{E_G(I)}$. Then $a \vee x \in F$ and $a \vee y \in G$, for some $x, y \in I$. Since F and G are filters and I is an ideal we get $a \vee (x \vee y) \in F \cap G$ and $x \vee y \in I$. This means $a \in \overline{E_{F \cap G}(I)}$.

(4) By $I \cap J \subseteq I, J$ and part (1), $\overline{E_F(I \cap J)} \subseteq \overline{E_F(I)} \cap \overline{E_F(J)}$. Now consider $a \in \overline{E_F(I)} \cap \overline{E_F(J)}$. Then $a \vee x, a \vee y \in F$, for some $x \in I$ and $y \in J$. Since $x * y \in I \cap J$ and $a \vee (x * y) \geq (a \vee x) * (a \vee y)$ we get $a \in \overline{E_F(I \cap J)}$.

(5) Using Theorem 3.1, $\overline{E_F(I)} \subseteq \overline{E_{\overline{E_F(I)}}(I)}$. Now let $a \in \overline{E_{\overline{E_F(I)}}(I)}$. Then $a \vee x \in \overline{E_F(I)}$, for some $x \in I$. Hence $a \vee x \vee y \in F$, for some $x \in I$ and $y \in I$. Therefore $a \in \overline{E_F(I)}$. $\square$

Remark 3.4. It is easy to see that if P is a prime filter of residuated lattice A then $A \setminus P$ is an ideal of $L(A)$.

Remark 3.5. The ideal generated by a subset H of lattice L will be denoted by $[H]$ and $(H) = \{i \mid i \leq h_0 \vee h_1 \vee \dots \vee h_n, \text{ for some } h_0, h_1, \dots, h_n \in H\}$ (see [1,3]).

Theorem 3.6. If F is a filter of residuated lattice A and I a proper ideal of $L(A)$ then

$$\overline{E_F(I)} = \bigcap \{ \overline{E_F(A \setminus P)} \mid P \in \text{Spec}A, I \subseteq A \setminus P \}.$$

Proof. Let $P \in \text{Spec}A$ be such that $I \subseteq A \setminus P$. Using Lemma 3.3, $\overline{E_F(I)} \subseteq \overline{E_F(A \setminus P)}$. Hence $\overline{E_F(I)} \subseteq \bigcap \{ \overline{E_F(A \setminus P)} \mid P \in \text{Spec}A, I \subseteq A \setminus P \}$.

Now let $a \notin \overline{E_F(I)}$, we show that there exists $Q \in \text{Spec}A$ such that $I \subseteq A \setminus Q$ and $a \notin \overline{E_F(A \setminus Q)}$. Consider $\Lambda = \{J \mid J \in \text{Id}(L(A)), I \subseteq J \text{ and } a \notin \overline{E_F(J)}\}$. Since $I \in \Lambda$, $\Lambda \neq \emptyset$. Let $\mathcal{C}$ be a chain of members of Λ and $H = \bigcup_{J \in \mathcal{C}} J$. It is clear that H is an ideal, $I \subseteq H$. Also $\overline{E_F(H)} = \bigcup_{J \in \mathcal{C}} \overline{E_F(J)}$, because $J \subseteq H$ implies $\overline{E_F(J)} \subseteq \overline{E_F(H)}$ and so $\bigcup_{J \in \mathcal{C}} \overline{E_F(J)} \subseteq \overline{E_F(H)}$. If $a \in \overline{E_F(H)}$ then $a \vee x \in F$, for some $x \in H$. Hence for some $J \in \mathcal{C}$, $x \in J$ and $a \vee x \in F$. This means $a \in \overline{E_F(J)}$, for some $J \in \mathcal{C}$. Then $\overline{E_F(H)} \subseteq \bigcup_{J \in \mathcal{C}} \overline{E_F(J)}$. Hence $\overline{E_F(H)} = \bigcup_{J \in \mathcal{C}} \overline{E_F(J)}$. Since $a \notin \overline{E_F(J)}$, for all $J \in \mathcal{C}$, we get $a \notin \overline{E_F(H)}$. Therefore $\overline{E_F(H)}$ is an upper bound of the chain $\mathcal{C}$. Using Zorn's Lemma, Λ has a maximal element J_0 . Then $a \notin \overline{E_F(J_0)}$ and so $A \neq \overline{E_F(J_0)}$. This means $J_0 \neq A$. Consider $Q = A \setminus J_0$, we show that Q is a prime filter of A . An easy check shows that $1 \in Q$ and $x \leq y$ and $x \in Q$ imply $y \in Q$. Now let $x, y \in Q$, we show that $x * y \in Q$. $x, y \in Q$ imply $x, y \notin J_0$ and so $J_0 \subset (J_0 \cup \{x\})$ and $J_0 \subset (J_0 \cup \{y\})$. Since J_0 is maximal element of Λ , $(J_0 \cup \{x\}) \notin \Lambda$ and $(J_0 \cup \{y\}) \notin \Lambda$. Also $I \subseteq J_0 \subseteq (J_0 \cup \{x\}), (J_0 \cup \{y\})$, hence

$$a \in \overline{E_F((J_0 \cup \{x\}))} \cap \overline{E_F((J_0 \cup \{y\}))} = \overline{E_F((J_0 \cup \{x\}) \cap (J_0 \cup \{y\}))}.$$

Then there exists $b \in (J_0 \cup \{x\}) \cap (J_0 \cup \{y\})$ such that $a \vee b \in F$. This means $b \leq z \vee x$ and $b \leq t \vee y$, for some $z, t \in J_0$. Let $s = z \vee t$. Hence $s \in J_0$, $b \leq z \vee x \leq s \vee x$ and $b \leq t \vee y \leq s \vee y$. Then $a \vee b \leq a \vee s \vee x$

and $a \vee b \leq a \vee s \vee y$. Thus $a \vee s \vee x, a \vee s \vee y \in F$. By Lemma 2.1, $a \vee s \vee (x * y) \geq (a \vee s \vee x) * (a \vee s \vee y)$. Hence $a \vee s \vee (x * y) \in F$. If $x * y \notin Q$ then $x * y \in J_0$ and since $s \in J_0$ we have $s \vee (x * y) \in J_0$. Thus $a \in \overline{E_F(J_0)}$, this is a contradiction. Hence $x * y \in Q$. Therefore Q is a filter of A . Now let $c \vee d \in Q = A \setminus J_0$, thus $c \notin J_0$ or $d \notin J_0$. This means $c \in Q$ or $d \in Q$. Hence Q is a prime filter. We have $I \subseteq J_0 = A \setminus Q$. Also $a \notin \overline{E_F(J_0)} = \overline{E_F(A \setminus Q)}$. Thus there exists $Q \in \text{Spec}A$ such that $I \subseteq A \setminus Q$ and $a \notin \overline{E_F(A \setminus Q)}$. Therefore $\bigcap \{ \overline{E_F(A \setminus P)} \mid P \in \text{Spec}A, I \subseteq A \setminus P \} \subseteq \overline{E_F(I)}$. $\square$

As immediate consequences we get the following corollaries:

Corollary 3.7. If F is a filter of residuated lattice A then $F = \bigcap_{P \in \text{Spec}A} \overline{E_F(A \setminus P)}$.

By the above Corollary, Definition 2.5 and Theorem 2.6, we get:

Corollary 3.8. F is a Boolean filter, G -filter or MV - filter of residuated lattice A if and only if $\overline{E_F(A \setminus P)}$ is a Boolean filter, G -filter or MV - filter of residuated lattice A , for all $P \in \text{Spec}A$.

Zhu and Xu in [9] proved every filter of residuated lattice A is a Boolean filter, G -filter or MV - filter if and only if A is a Boolean algebra, $G(RL)$ -algebra or MV -algebra, respectively (see Theorems 2.7, 2.8, 2.9). In the following, using Corollary 3.8 and Theorems 2.7, 2.8, 2.9 we get:

Corollary 3.9. Let A be a residuated lattice. Then the following conditions are equivalent:

- (1) A is a Boolean algebra,
- (2) Every filter of A is a Boolean filter of A ,
- (3) $\{1\}$ is a Boolean filter of A ,
- (4) $\overline{E_{\{1\}}(A \setminus P)}$ is a Boolean filter of A , for all $P \in \text{Spec}A$.

Corollary 3.10. In any residuated lattice A , the following conditions are equivalent:

- (1) A is a $G(RL)$ -algebra,
- (2) Every filter of A is a G -filter of A ,

- (3) $\{1\}$ is a G -filter of A ,
- (4) $\overline{E_{\{1\}}(A \setminus P)}$ is a G -filter of A , for all $P \in \text{Spec}A$.

Corollary 3.11. Let A be a residuated lattice. Then the following conditions are equivalent:

- (1) A is an MV -algebra,
- (2) Every filter of A is an MV -filter of A ,
- (3) $\{1\}$ is an MV -filter of A ,
- (4) $\overline{E_{\{1\}}(A \setminus P)}$ is an MV -filter of A , for all $P \in \text{Spec}A$.

Corollary 3.12. If F is a filter of residuated lattice A and $a \in A$ then

$$\begin{aligned}
E_F(a) &= \bigcap \{ \overline{E_F(A \setminus P)} \mid P \in \text{Spec}A, a \notin P \} \\
&= \bigcap \{ \overline{E_F(A \setminus P)} \mid P \in \text{Spec}A, E_F(E_F(a)) \not\subseteq P \} \\
&= \bigcap \{ \overline{E_F(A \setminus P)} \mid P \in \text{Spec}A, E_F(a) \subseteq \overline{E_F(A \setminus P)} \}.
\end{aligned}$$

Proof. Let F be a filter of A and $a \in A$. At first we show that $E_F(a) = \overline{E_F([a])}$. Let $z \in \overline{E_F([a])}$, then $z \vee x \in F$, for some $x \in [a]$. Hence $z \vee x \in F$ and $x \leq a$. Thus $z \vee a \in F$, this means $z \in E_F(a)$. Therefore $\overline{E_F([a])} \subseteq E_F(a)$. Now let $z \in E_F(a)$ and so $z \vee a \in F$ and $a \in [a]$. Then $z \in \overline{E_F([a])}$. Hence $E_F(a) \subseteq \overline{E_F([a])}$.

Using Theorem 3.6, we have

$E_F(a) = \overline{E_F([a])} = \bigcap \{ \overline{E_F(A \setminus P)} \mid P \in \text{Spec}A, [a] \subseteq A \setminus P \}$. It is obvious that $a \notin P$ if and only if $[a] \subseteq A \setminus P$. Let $E_F(E_F(a)) \subseteq P$. By Theorem 2.4, $a \in E_F(E_F(a))$ and so $a \in P$. Hence $a \notin P$ implies $E_F(E_F(a)) \not\subseteq P$.

Now let $E_F(E_F(a)) \not\subseteq P$. Then there exists $x \in E_F(E_F(a))$ such that $x \notin P$.

Let $y \in E_F(a)$, hence $x \vee y \in F$ and since $x \in A \setminus P$ we get $y \in \overline{E_F(A \setminus P)}$. Therefore $E_F(E_F(a)) \not\subseteq P$ implies $E_F(a) \subseteq \overline{E_F(A \setminus P)}$. Hence

$$\begin{aligned}
E_F(a) &= \overline{E_F([a])} = \bigcap \{ \overline{E_F(A \setminus P)} \mid P \in \text{Spec}A, [a] \subseteq A \setminus P \} \\
&= \bigcap \{ \overline{E_F(A \setminus P)} \mid P \in \text{Spec}A, a \notin P \}
\end{aligned}$$

$$\begin{aligned}
&\supseteq \bigcap \{ \overline{E_F(A \setminus P)} \mid P \in \text{Spec} A, E_F(E_F(a)) \not\subseteq P \} \\
&\supseteq \bigcap \{ \overline{E_F(A \setminus P)} \mid P \in \text{Spec} A, E_F(a) \subseteq \overline{E_F(A \setminus P)} \} \\
&\supseteq E_F(a). \square
\end{aligned}$$

Definition 3.13. A filter F of residuated lattice A is called strong stable relative to an ideal I of $L(A)$ if $F = \overline{E_F(I)}$.

We recall that a filter F is called stable relative to a subset B of A if $F = E_F(B)$ [5].

Using Theorems 2.4, 3.1 and Lemma 3.3 we get the following corollaries:

Corollary 3.14. Every strong stable filter relative to an ideal I of $L(A)$ is a stable filter relative to I .

Corollary 3.15. If I and J are ideals of $L(A)$ and F a strong stable filter of residuated lattice A relative to J such that $I \subseteq J$ then F is a strong stable filter of A relative to I .

The converse of Corollary 3.14 is not true:

Example 3.16. In Example 3.2, $F = \{1, d\}$ and $I = \{0, n, a, b\}$ are filter and ideal of A and $L(A)$, respectively. It is easy to see that, $\overline{E_F(I)} = \{1, d, c\}$ and $E_F(I) = F$.

Proposition 3.17. In any residuated lattice A , every prime filter P is a strong stable filter relative to $A \setminus P$.

Proof. Using Theorem 3.1, $P \subseteq \overline{E_P(A \setminus P)}$
 $= \{x \in A \mid x \vee y \in P, \exists y \in A \setminus P\}$
 $= P. \quad (\text{because } P \in \text{Spec} A) \square$

Remark 3.18. In any residuated lattice A , every maximal filter M is a strong stable filter relative to $A \setminus M$ (because every maximal filter is a prime filter in any residuated lattice).

Remark 3.19. (Extension property) Let F be a filter of BL -algebra A containing a prime filter of A . Then F is a prime filter of BL -algebra A [7].

We note that F is a prime filter in MTL -algebra A if and only if $x \rightarrow y \in F$ or $y \rightarrow x \in F$, for all $x, y \in A$ [8]. Therefore we get Extension property holds in MTL -algebras. Hence using the above Proposition:

Corollary 3.20. If F is a filter of BL -algebra or MTL -algebra A such that F is containing a prime filter then there exists $P \in SpecA$ such that F is a strong stable filter relative to $A \setminus P$.

In the following we see that in any residuated lattice, Corollary 3.20 is not true:

Example 3.21. (a) In Example 3.2, it is easy to see that $P_1 = \{1\}$, $P_2 = \{1, d, b\}$, $P_3 = \{1, d, c\}$ and $P_4 = \{1, a, b, c, d\}$ are prime filters of A , $SpecA = \{P_1, P_2, P_3, P_4\}$ and $F = \{1, d\}$ is a filter of A . $\{1\} \subset F = \{1, d\}$ (We note that F is not a prime filter because $b \vee c = d \in F$, $b \notin F$ and $c \notin F$.) but $\overline{E_F(A \setminus P_1)} = A \neq F$, $\overline{E_F(A \setminus P_2)} = P_2 \neq F$, $\overline{E_F(A \setminus P_3)} = P_3 \neq F$ and $\overline{E_F(A \setminus P_4)} = P_1 \neq F$.

(b) Let $A = \{0, a, b, c, 1\}$. Define $*, \rightarrow, \wedge, \vee$ as follows:

$*$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

Then $(A, *, \rightarrow, \wedge, \vee, 0, 1)$ is a bounded commutative *Rl-monoid* which is not an MTL -algebra (since $(a \rightarrow b) \vee (b \rightarrow a) = b \vee a = c \neq 1$).

It is easy to see that $P_1 = \{1\}$, $P_2 = \{1, a, c\}$ and $P_3 = \{1, b, c\}$ are prime filters of A , $SpecA = \{P_1, P_2, P_3\}$ and $F = \{1, c\}$ is a filter of A . $\{1\} \subset F = \{1, c\}$ (We

note that F is not a prime filter because $a \vee b = c \in F$, $a \notin F$ and $b \notin F$.) but $\overline{E_F(A \setminus P_1)} = A \neq F$, $\overline{E_F(A \setminus P_2)} = P_2 \neq F$ and $\overline{E_F(A \setminus P_3)} = P_3 \neq F$.

Theorem 3.22. If F and F_0 are filters of residuated lattice A such that $F_0 \subseteq F$ then $F = \bigcap_{p \in \text{Spec} A} F \vee \overline{E_{F_0}(A \setminus P)}$.

Proof. Let $a \in \bigcap_{p \in \text{Spec} A} F \vee \overline{E_{F_0}(A \setminus P)}$. Then for all $p \in \text{Spec} A$ there exist $f_P \in F$ and $x_P \in \overline{E_{F_0}(A \setminus P)}$ such that $a \geq f_P * x_P$. Hence there exists $y_P \in A \setminus P$ such that $x_P \vee y_P \in F_0$.

We show that $\bigvee_{p \in \text{Spec} A} \langle y_P \rangle = A$. If $\bigvee_{p \in \text{Spec} A} \langle y_P \rangle \subsetneq A$ then there exists $P_0 \in \text{Spec} A$ such that $\bigvee_{p \in \text{Spec} A} \langle y_P \rangle \subseteq P_0$ (by Proposition 2.3) and so $\langle y_{P_0} \rangle \subseteq P_0$. This is a contradiction, because $y_{P_0} \notin P_0$. Then $0 \in \bigvee_{p \in \text{Spec} A} \langle y_P \rangle$. This means there exist $P_1, \dots, P_n \in \text{Spec} A$ such that $a_{P_1} * a_{P_2} * \dots * a_{P_n} = 0$, where $a_{P_1} \in \langle y_{P_1} \rangle, \dots, a_{P_n} \in \langle y_{P_n} \rangle$. Hence $a_{P_i} \geq y_{P_i}^{m_i}$, for some m_i , where $1 \leq i \leq n$. Let $m = \max\{m_1, \dots, m_n\}$. Then we get $y_{P_1}^m * \dots * y_{P_n}^m = 0$. We suppose $f = f_{P_1} * \dots * f_{P_n}$ and using Lemma 2.1,

$$\begin{aligned}
a &\geq (f_{P_1} * x_{P_1}) \vee (f_{P_2} * x_{P_2}) \vee \dots \vee (f_{P_n} * x_{P_n}) \\
&\geq (f_{P_1} * x_{P_1}^m) \vee (f_{P_2} * x_{P_2}^m) \vee \dots \vee (f_{P_n} * x_{P_n}^m) \vee 0 \\
&= (f_{P_1} * x_{P_1}^m) \vee (f_{P_2} * x_{P_2}^m) \vee \dots \vee (f_{P_n} * x_{P_n}^m) \vee (y_{P_1}^m * \dots * y_{P_n}^m) \\
&\geq (y_{P_1}^m \vee ((f_{P_1} * x_{P_1}^m) \vee (f_{P_2} * x_{P_2}^m) \vee \dots \vee (f_{P_n} * x_{P_n}^m))) * \dots * (y_{P_n}^m \vee ((f_{P_1} * x_{P_1}^m) \vee (f_{P_2} * x_{P_2}^m) \vee \dots \vee (f_{P_n} * x_{P_n}^m))) \\
&\geq (y_{P_1}^m \vee ((f * x_{P_1}^m) \vee (f * x_{P_2}^m) \vee \dots \vee (f * x_{P_n}^m))) * \dots * (y_{P_n}^m \vee ((f * x_{P_1}^m) \vee (f * x_{P_2}^m) \vee \dots \vee (f * x_{P_n}^m))) \\
&= (y_{P_1}^m \vee (f * (x_{P_1}^m \vee x_{P_2}^m \vee \dots \vee x_{P_n}^m))) * \dots * (y_{P_n}^m \vee (f * (x_{P_1}^m \vee x_{P_2}^m \vee \dots \vee x_{P_n}^m))) \\
&\geq (y_{P_1}^m \vee f) * (y_{P_1}^m \vee x_{P_1}^m \vee x_{P_2}^m \vee \dots \vee x_{P_n}^m) * \dots * (y_{P_n}^m \vee f) * (y_{P_n}^m \vee x_{P_1}^m \vee x_{P_2}^m \vee \dots \vee x_{P_n}^m) \\
&= (y_{P_1}^m \vee f) * \dots * (y_{P_n}^m \vee f) * (y_{P_1}^m \vee x_{P_1}^m \vee x_{P_2}^m \vee \dots \vee x_{P_n}^m) * \dots * (y_{P_n}^m \vee x_{P_1}^m \vee x_{P_2}^m \vee \dots \vee x_{P_n}^m) \\
&\geq (y_{P_1}^m \vee f) * \dots * (y_{P_n}^m \vee f) * ((y_{P_1} \vee x_{P_1})^{m^2} \vee x_{P_2}^m \vee \dots \vee x_{P_n}^m) * \dots * ((y_{P_n} \vee x_{P_n})^{m^2} \vee x_{P_1}^m \vee \dots \vee x_{P_{n-1}}^m).
\end{aligned}$$

We note that $y_{P_i} \vee x_{P_i} \in F_0 \subseteq F$ and $y_{P_i}^m \vee f \in F$, for all i and so $a \in F$. Hence $\bigcap_{p \in \text{Spec} A} F \vee \overline{E_{F_0}(A \setminus P)} \subseteq F$. It is clear that $F \subseteq \bigcap_{p \in \text{Spec} A} F \vee \overline{E_{F_0}(A \setminus P)}$. $\square$

Remark 3.23. We recall that a lattice is Brouwerian if it satisfies the identity $a \wedge (\vee b_i) = \vee(a \wedge b_i)$ (whenever the arbitrary unions exists)[3].

The lattice $(\text{Fil}(A), \subseteq)$ is a Brouwerian lattice (and so is a distributive lattice). For a family $\{F_i\}_{i \in I}$ of filters, $\wedge F_i = \cap F_i$ and $\vee F_i = \langle \cup F_i \rangle$. [2]

Theorem 3.24. If I is a proper ideal of $L(A)$ then

$(\overline{E(I)}, \wedge, \vee_{F_i}, \overline{E_{\{1\}}(I)}, \overline{E_A(I)})$ is a bounded complete Brouwerian lattice, where $\overline{E(I)} = \{\overline{E_F(I)} \mid F \in \text{Fil}(A)\}$, $\overline{E_F(I)} \wedge \overline{E_G(I)} = \overline{E_{F \cap G}(I)}$, $\overline{E_F(I)} \vee_{F_i} \overline{E_G(I)} = \overline{E_{F \vee G}(I)}$.

Proof. Using Lemma 3.3, $\overline{E_F(I)} \cap \overline{E_G(I)} = \overline{E_{F \cap G}(I)}$ and so $\overline{E_F(I)} \wedge \overline{E_G(I)} = \overline{E_{F \cap G}(I)}$. By $F, G \subseteq F \vee G$ and Lemma 3.3 we get $\overline{E_{F \vee G}(I)}$ is an upper bound of $\overline{E_F(I)}$ and $\overline{E_G(I)}$. Now let $\overline{E_F(I)}, \overline{E_G(I)} \subseteq \overline{E_H(I)}$ and $a \in \overline{E_{F \vee G}(I)}$. Hence there exist $f \in F, g \in G$ and $x \in I$ such that $a \vee x \geq f * g$. By Lemma 2.1, $a \vee x = (a \vee x) \vee (f * g) \geq (a \vee x \vee f) * (a \vee x \vee g) \geq (a \vee f) * (a \vee g)$. We note that $a \vee f, a \vee g \in \overline{E_H(I)}$, because $f \in F$ and $f \leq a \vee x \vee f$ imply $a \vee x \vee f \in F$ and so $a \vee f \in \overline{E_F(I)} \subseteq \overline{E_H(I)}$. Similarly, $a \vee g \in \overline{E_H(I)}$. Therefore $a \vee x \in \overline{E_H(I)}$. Then $a \vee x \vee y \in H$, for some $y \in I$. This means $a \in \overline{E_H(I)}$. Hence $\overline{E_{F \vee G}(I)}$ is the least upper bound of $\overline{E_F(I)}$ and $\overline{E_G(I)}$.

It is clear that $\overline{E_{\{1\}}(I)} \subseteq \overline{E_F(I)} \subseteq \overline{E_A(I)}$, for all $F \in \text{Fil}(A)$. Therefore $(\overline{E(I)}, \wedge, \vee_{F_i}, \overline{E_{\{1\}}(I)}, \overline{E_A(I)})$ is a bounded lattice. Now consider $\{\overline{E_{F_i}(I)}\}_{i \in I}$, we have $\wedge \overline{E_{F_i}(I)} = \overline{E_{\cap F_i}(I)}$. Hence $(\overline{E(I)}, \wedge, \vee_{F_i}, \overline{E_{\{1\}}(I)}, \overline{E_A(I)})$ is a bounded complete lattice.

We note that $\vee_{F_i} \overline{E_{H_i}(I)} = \overline{E_{\vee H_i}(I)}$, where $\{\overline{E_{H_i}(I)}\}_{i \in \Lambda} \subseteq \overline{E(I)}$, because $H_i \subseteq \vee H_i$ implies $\overline{E_{H_i}(I)} \subseteq \overline{E_{\vee H_i}(I)}$ and so $\overline{E_{\vee H_i}(I)}$ is an upper bound of $\overline{E_{H_i}(I)}$. Also if $\overline{E_{H_i}(I)} \subseteq \overline{E_K(I)}$, for all i and $a \in \overline{E_{\vee H_i}(I)}$ then there exist $x_{i_1} \in H_{i_1}, \dots, x_{i_n} \in H_{i_n}$ and $x \in I$ such that $a \vee x \geq x_{i_1} * \dots * x_{i_n}$, where $i_1, \dots, i_n \in \Lambda$. Using Lemma 2.1,

$$\begin{aligned} a \vee x &= (a \vee x) \vee (x_{i_1} * \dots * x_{i_n}) \geq (a \vee x_{i_1} \vee x) * \dots * (a \vee x_{i_n} \vee x) \\ &\geq (a \vee x_{i_1}) * \dots * (a \vee x_{i_n}). \end{aligned}$$

Using $x_{i_j} \in H_{i_j}$ and Theorem 3.1, we get $a \vee x_{i_j} \in \overline{E_{H_{i_j}}(I)} \subseteq \overline{E_K(I)}$ and so $a \vee x \in \overline{E_K(I)}$. This means there exists $y \in I$ such that $a \vee x \vee y \in K$. Then $a \in \overline{E_K(I)}$. Hence $\overline{E_{\vee H_i}(I)} \subseteq \overline{E_K(I)}$. Therefore $\overline{E_{\vee H_i}(I)}$ is the least upper bound of $\overline{E_{H_i}(I)}$, for all i .

At last, by the above Remark we have

$$\begin{aligned} \overline{E_F(I)} \wedge (\vee_{F_i} \overline{E_{H_i}(I)}) &= \overline{E_F(I)} \cap \overline{E_{\vee H_i}(I)} \\ &= \overline{E_{F \cap (\vee H_i)}(I)} \\ &= \overline{E_{\vee(F \cap H_i)}(I)} \end{aligned}$$

$$\begin{aligned}
&= \vee_{F_i} \overline{E_{F \cap H_i}(I)} \\
&= \vee_{F_i} (\overline{E_F(I)} \cap \overline{E_{H_i}(I)})
\end{aligned}$$

Therefore $(\overline{E(I)}, \wedge, \vee_{F_i}, \overline{E_{\{1\}}(I)}, \overline{E_A(I)})$ is a bounded complete Brouwerian lattice. $\square$

Example 3.25. In Example 3.2, an easy check shows that

$Fil A = \{F_1, F_2, F_3, F_4, F_5, F_6\}$, where $F_1 = \{1\}$, $F_2 = \{1, d\}$, $F_3 = \{1, d, b\}$, $F_4 = \{1, d, c\}$, $F_5 = \{1, a, b, c, d\}$, $F_6 = A$ and $I = \{0, n, a\}$ is an ideal of $L(A)$. Then $\overline{E_{F_1}(I)} = F_1$, $\overline{E_{F_2}(I)} = F_2$, $\overline{E_{F_3}(I)} = F_3$, $\overline{E_{F_4}(I)} = F_4$, $\overline{E_{F_5}(I)} = A$ and $\overline{E_{F_6}(I)} = A$. $\overline{E_{F_3}(I)} \vee \overline{E_{F_4}(I)} = F_3 \vee F_4 = F_5 \neq \overline{E_{F_i}(I)}$, for all $1 \leq i \leq 6$. Hence we get :

Corollary 3.26. Let I be a proper ideal of $L(A)$. In general, $(E(I), \subseteq)$ is not a sublattice of $(Fil(A), \subseteq)$.

Remark 3.27. Let X be a non-empty subset of residuated lattice A . We denote $X^\perp = \{a \in A \mid a \vee x = 1, \forall x \in X\}$. It is easy to see that $X^\perp$ is a filter of A .

Corollary 3.28. If I is a proper ideal of $L(A)$ then

$(\overline{E(I)}, \wedge, \vee_{F_i}, \overline{E_{\{1\}}(I)}, \overline{E_A(I)})$ is a Boolean algebra if and only if for all $F \in Fil(A)$ there exist $x \in F$ and $y \in F^\perp$ such that $x * y \in I$, where $(\overline{E_F(I)})' = \overline{E_{F^\perp}(I)}$.

Proof. Let for all $F \in Fil(A)$, there exist $x \in F$ and $y \in F^\perp$ such that $x * y \in I$. Then $0 \vee (x * y) = x * y \in F \vee F^\perp$ and so $0 \in \overline{E_{F \vee F^\perp}(I)}$. This means $\overline{E_F(I)} \vee_{F_i} (\overline{E_F(I)})' = \overline{E_F(I)} \vee_{F_i} \overline{E_{F^\perp}(I)} = \overline{E_{F \vee F^\perp}(I)} = A = \overline{E_A(I)}$. Also we have $\overline{E_F(I)} \wedge (\overline{E_F(I)})' = \overline{E_F(I)} \wedge \overline{E_{F^\perp}(I)} = \overline{E_{F \cap F^\perp}(I)} = \overline{E_{\{1\}}(I)}$. Also $(\overline{E(I)}, \wedge, \vee_{F_i}, \overline{E_{\{1\}}(I)}, \overline{E_A(I)})$ is a distributive lattice. Obviously, $\overline{E_F(I)} \wedge \overline{E_{\{1\}}(I)} = \overline{E_{\{1\}}(I)}$ and $\overline{E_F(I)} \vee_{F_i} \overline{E_A(I)} = \overline{E_A(I)}$, for all F . Hence $(\overline{E(I)}, \wedge, \vee_{F_i}, \overline{E_{\{1\}}(I)}, \overline{E_A(I)})$ is a Boolean algebra.

Conversely, let $(\overline{E(I)}, \wedge, \vee_{F_i}, \overline{E_{\{1\}}(I)}, \overline{E_A(I)})$ be a Boolean algebra. Then $A = \overline{E_F(I)} \vee_{F_i} (\overline{E_F(I)})' = \overline{E_F(I)} \vee_{F_i} \overline{E_{F^\perp}(I)} = \overline{E_{F \vee F^\perp}(I)}$. Then there exist $i \in I$ such that $0 \vee i \in F \vee F^\perp$. This means there exist $x \in F$ and $y \in F^\perp$ such that $i \geq x * y$. Therefore there exist $x \in F$ and $y \in F^\perp$ such that $x * y \in I$. $\square$

Remark 3.29. [3] Let L be a complete lattice and let a be an element of L . Then a is called compact if $a \leq \vee X$ for some $X \subseteq L$ implies that $a \leq \vee X_1$, for some finite $X_1 \subseteq X$. A complete lattice is called algebraic if every elements

is the join of compact elements (in the literature, algebraic lattices are also called compactly generated lattices).

Theorem 3.30. For all residuated lattice A , $(\overline{E(I)}, \subseteq)$ is an algebraic lattice, where I is an ideal in $L(A)$.

Proof. Let $I \in Id(L(A))$. At first we show that $\overline{E_{<a>}(I)}$ is a compact element in $\overline{E(I)}$, for all $a \in A$. Let $\{E_{F_\lambda}(I)\}_{\lambda \in \Lambda}$ be an arbitrary family of $\overline{E(I)}$ such that $\overline{E_{<a>}(I)} \subseteq \bigvee_{F_i} \{E_{F_\lambda}(I) \mid \lambda \in \Lambda\}$. Using the proof of Theorem 3.24, $\overline{E_{<a>}(I)} \subseteq \overline{E_{\bigvee_{\lambda \in \Lambda} F_\lambda}(I)}$. Then there exist $x_{\lambda_1} \in F_{\lambda_1}, \dots, x_{\lambda_n} \in F_{\lambda_n}$ and $x \in I$ such that $a \vee x \geq x_{\lambda_1} * \dots * x_{\lambda_n}$, where $\lambda_1, \dots, \lambda_n \in \Lambda$. This means $a \in \overline{E_{\bigvee_{i=1}^n F_{\lambda_i}}(I)}$ and so $<a> \subseteq \overline{E_{\bigvee_{i=1}^n F_{\lambda_i}}(I)}$. By Lemma 3.3, we get $\overline{E_{<a>}(I)} \subseteq \overline{E_{\bigvee_{i=1}^n F_{\lambda_i}}(I)} = \overline{E_{\bigvee_{i=1}^n F_{\lambda_i}}(I)} = \overline{E_{F_{\lambda_1}}(I)} \vee_{F_i} \overline{E_{F_{\lambda_2}}(I)} \vee_{F_i} \dots \vee_{F_i} \overline{E_{F_{\lambda_n}}(I)}$. Hence $\overline{E_{<a>}(I)}$ is a compact element in $\overline{E(I)}$, for all $a \in A$. Also for all $F \in Fil(A)$ we have $\overline{E_F(I)} = \overline{E_{\bigvee_{a \in F} <a>}(I)} = \bigvee_{F_i} \{\overline{E_{<a>}(I)} \mid a \in F\}$. Therefore $(\overline{E(I)}, \subseteq)$ is an algebraic lattice. $\square$

Remark 3.31. The ideals of distributive lattice L form a complete Brouwerian lattice[1,3].

Theorem 3.32. If F is a proper filter of residuated lattice A then $(\overline{E_F}, \wedge, \vee_{Id}, \overline{E_F(\{0\})}, \overline{E_F(A)})$ is a bounded complete Brouwerian lattice, where $\overline{E_F} = \{\overline{E_F(I)} \mid I \in Id(L(A))\}$, $\overline{E_F(I)} \wedge \overline{E_F(J)} = \overline{E_F(I \cap J)}$, $\overline{E_F(I)} \vee_{Id} \overline{E_F(J)} = \overline{E_F(I \vee J)}$.

Proof. Using Lemma 3.3, $\overline{E_F(I)} \cap \overline{E_F(J)} = \overline{E_F(I \cap J)}$. Hence we get, $\overline{E_F(I)} \wedge \overline{E_F(J)} = \overline{E_F(I \cap J)}$.

By Lemma 3.3 and $I, J \subseteq I \vee J$, we have $\overline{E_F(I)}, \overline{E_F(J)} \subseteq \overline{E_F(I \vee J)}$. Hence $\overline{E_F(I \vee J)}$ is an upper bound of $\overline{E_F(I)}, \overline{E_F(J)}$.

Now let $\overline{E_F(I)}, \overline{E_F(J)} \subseteq \overline{E_F(K)}$, where $K \in Id(L(A))$ and $a \in \overline{E_F(I \vee J)}$. Then $a \vee z \in F$, for some $z \in I \vee J$. Hence we get $a \vee i \vee j \in F$, for some $i \in I$ and $j \in J$. This means $a \vee i \in \overline{E_F(J)} \subseteq \overline{E_F(K)}$. Hence $a \vee i \vee k \in F$, for some $k \in K$ and $i \in I$. Then $a \vee k \in \overline{E_F(I)} \subseteq \overline{E_F(K)}$ and so $a \vee k \vee k_1 \in F$, for some $k, k_1 \in K$. Therefore $a \in \overline{E_F(K)}$. Hence $\overline{E_F(I \vee J)}$ is the least upper bound of $\overline{E_F(I)}$ and $\overline{E_F(J)}$. This means $\overline{E_F(I)} \vee_{Id} \overline{E_F(J)} = \overline{E_F(I \vee J)}$. It is clear that $\overline{E_F(\{0\})} \subseteq \overline{E_F(I)} \subseteq \overline{E_F(A)}$, for all $I \in Id(L(A))$. Hence $(\overline{E_F}, \wedge, \vee_{Id}, \overline{E_F(\{0\})}, \overline{E_F(A)})$ is a bounded lattice.

Now consider $\{\overline{E_F(I_i)}\}_{i \in \Lambda}$, since $\bigwedge \overline{E_F(I_i)} = \overline{E_F(\bigcap I_i)}$ we have

$(\overline{E_F}, \wedge, \vee_{Id}, \overline{E_F(\{0\})}, \overline{E_F(A)})$ is a bounded complete lattice.

Consider $\{I_i\}_{i \in \Lambda} \subseteq Id(L(A))$, we show that $\bigvee_{Id} \overline{E_F(I_i)} = \overline{E_F(\bigvee I_i)}$, where $\overline{E_F(I_i)} \in \overline{E_F}$. Using $I_i \subseteq \bigvee I_i$ and Lemma 3.3, we get $\overline{E_F(I_i)} \subseteq \overline{E_F(\bigvee I_i)}$. This

means $\overline{E_F(\vee I_i)}$ is an upper bound of $\overline{E_F(I_i)}$, for all i . Now let $\overline{E_F(I_i)} \subseteq \overline{E_F(K)}$, for all i and $a \in \overline{E_F(\vee I_i)}$. Then $a \vee x \in F$, for some $x \in \vee I_i$. This implies that there are $x_{j_1} \in I_{j_1}, \dots, x_{j_n} \in I_{j_n}$ such that $x \leq x_{j_1} \vee x_{j_2} \vee \dots \vee x_{j_n}$, where $j_1, \dots, j_n \in \Lambda$. Hence $a \vee x_{j_1} \vee x_{j_2} \vee \dots \vee x_{j_n} \in F$ and so $a \vee x_{j_1} \vee x_{j_2} \vee \dots \vee x_{j_{n-1}} \in \overline{E_F(I_{j_n})} \subseteq \overline{E_F(K)}$. Then there exists $k_1 \in K$ such that $a \vee x_{j_1} \vee x_{j_2} \vee \dots \vee x_{j_{n-1}} \vee k_1 \in F$ and so $a \vee x_{j_1} \vee x_{j_2} \vee \dots \vee x_{j_{n-2}} \vee k_1 \in \overline{E_F(I_{j_{n-1}})} \subseteq \overline{E_F(K)}$. This follows that there is $k_2 \in K$ such that $a \vee x_{j_1} \vee x_{j_2} \vee \dots \vee x_{j_{n-2}} \vee k_1 \vee k_2 \in F$. Continuing this process we get there are $k_1, \dots, k_n \in K$ such that $a \vee k_1 \vee \dots \vee k_n \in F$. Hence $a \in \overline{E_F(K)}$. Therefore $\overline{E_F(\vee I_i)}$ is the least upper bound of $\overline{E_F(I_i)}$, for all i . At last we show that $\overline{E_F(I)} \wedge (\vee_{Id} \overline{E_F(J_i)}) = \vee_{Id}(\overline{E_F(I)} \wedge \overline{E_F(J_i)})$. Using the above Remark, we have

$$\begin{aligned}
\overline{E_F(I)} \wedge (\vee_{Id} \overline{E_F(J_i)}) &= \overline{E_F(I)} \cap \overline{E_F(\vee J_i)} \\
&= \overline{E_F(I \cap (\vee J_i))} \\
&= \overline{E_F(\vee(I \cap J_i))} \\
&= \vee_{Id} \overline{E_F(I \cap J_i)} \\
&= \vee_{Id}(\overline{E_F(I)} \wedge \overline{E_F(J_i)}).
\end{aligned}$$

Therefore $(\overline{E_F}, \wedge, \vee_{Id}, \overline{E_F(\{0\}}), \overline{E_F(A)})$ is a bounded complete Brouwerian lattice. $\square$

In the following example we see that generally $(\overline{E_F}, \subseteq)$ is not a sublattice of $(Fil(A), \subseteq)$:

Example 3.33. In Example 3.2, it is easy to see that $Id(L(A)) = \{I_1, I_2, I_3, I_4, I_5, I_6, I_7\}$, where $I_1 = \{0\}$, $I_2 = \{0, n\}$, $I_3 = \{0, n, a\}$, $I_4 = \{0, n, a, b\}$, $I_5 = \{0, n, a, c\}$, $I_6 = \{0, n, a, b, c, d\}$, $I_7 = A$ and $Fil A = \{F_1, F_2, F_3, F_4, F_5, F_6\}$, where $F_1 = \{1\}$, $F_2 = \{1, d\}$, $F_3 = \{1, d, b\}$, $F_4 = \{1, d, c\}$, $F_5 = \{1, a, b, c, d\}$, $F_6 = A$. Then $\overline{E_{F_2}(I_1)} = \overline{E_{F_2}(I_2)} = \overline{E_{F_2}(I_3)} = F_2$, $\overline{E_{F_2}(I_4)} = F_4$, $\overline{E_{F_2}(I_5)} = F_3$, $\overline{E_{F_2}(I_6)} = \overline{E_{F_2}(I_7)} = F_6$ and $\overline{E_{F_2}(I_4)} \vee \overline{E_{F_2}(I_5)} = F_5 \neq \overline{E_{F_2}(I_i)}$, for all $1 \leq i \leq 7$.

Remark 3.34. Let A be an residuated lattice and I an ideal of $L(A)$. An easy check shows that $I^\perp$ is an ideal of $L(A)$ and $I \cap I^\perp = \{0\}$, where $I^\perp = \{x \in A \mid x \wedge y = 0, \forall y \in I\}$.

Corollary 3.35. If F is a proper filter of residuated lattice A then $(\overline{E_F}, \wedge, \vee_{Id}, ", \overline{E_F(\{0\})}, \overline{E_F(A)})$ is a Boolean algebra if and only if for all $I \in Id(L(A))$ there exist $x \in I$ and $y \in I^\perp$ such that $x \vee y \in F$, where $(\overline{E_F(I)})'' = \overline{E_F(I^\perp)}$.

Proof. Let for all $I \in Id(L(A))$ there exist $x \in I$ and $y \in I^\perp$ such that $x \vee y \in F$. This means $(x \vee y) \vee 0 \in F$. Since $x \vee y \in I \vee I^\perp$, we get $0 \in \overline{E_F(I \vee I^\perp)}$. Hence $\overline{E_F(I)} \vee_{Id} (\overline{E_F(I)})'' = \overline{E_F(I)} \vee_{Id} \overline{E_F(I^\perp)} = \overline{E_F(I \vee I^\perp)} = A = \overline{E_F(A)}$. Using Lemma 3.3, we have $\overline{E_F(I)} \wedge (\overline{E_F(I)})'' = \overline{E_F(I)} \wedge \overline{E_F(I^\perp)} = \overline{E_F(I \cap I^\perp)} = \overline{E_F(\{0\})}$. Also $(\overline{E_F}, \wedge, \vee_{Id}, \overline{E_F(\{0\})}, \overline{E_F(A)})$ is a distributive lattice. It is clear that $\overline{E_F(I)} \wedge \overline{E_F(\{0\})} = \overline{E_F(\{0\})}$ and $\overline{E_F(I)} \vee_{Id} \overline{E_F(A)} = \overline{E_F(A)}$, for all $I \in Id(LA)$. Therefore $(\overline{E_F}, \wedge, \vee_{Id}, ", \overline{E_F(\{0\})}, \overline{E_F(A)})$ is a Boolean algebra.

Conversely, let $(\overline{E_F}, \wedge, \vee_{Id}, ", \overline{E_F(\{0\})}, \overline{E_F(A)})$ be a Boolean algebra and $I \in Id(L(A))$. Then $\overline{E_F(I \vee I^\perp)} = \overline{E_F(I)} \vee_{Id} \overline{E_F(I^\perp)} = A$. Hence there exists $a \in I \vee I^\perp$ such that $0 \vee a \in F$. This means there exist $x \in I$, $y \in I^\perp$ and $a \in F$ such that $a \leq x \vee y$. Therefore there exist $x \in I$ and $y \in I^\perp$ such that $x \vee y \in F$. $\square$

Conclusion

In this paper, we introduced the notion of essential extension of a filter associative to an ideal in residuated lattices and got some of its properties. For instance, in a theorem, we proved an essential extension of a filter associative to an ideal I is the intersection of essential extensions of the same filter associative to $A \setminus P$, for all prime filter P such that $I \subseteq A \setminus P$. Using this theorem, it is proved that residuated lattice A is a Boolean algebra, $G(RL)$ -algebra or MV -algebra if and only if the essential extension of $\{1\}$ associated to $A \setminus P$ is a Boolean filter, G -filter or MV -filter (for all $P \in Spec A$), respectively.

We showed the lattice of essential extension of F_0 associated to ideals of $L(A)$ and the lattice of essential extensions in A associated to I_0 are bounded complete Brouwerian lattice, where F_0 is a filter of residuated lattice A and I_0 is an ideal of $L(A)$. Also the lattice of essential extensions in A associated to I_0 is an algebraic lattice.

Compliance with ethical standards

Conflict of interest The author declares that he has no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by the author.

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