On the Lacunary Statistical Convergence of Triple Sequences in Fuzzy Metric Spaces

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ON THE LACUNARY STATISTICAL CONVERGENCE OF TRIPLE SEQUENCES IN FUZZY METRIC SPACES

MEHMET GÜRDAL AND EKREM SAVAŞ

ABSTRACT. In this research paper, we analyze the lacunary statistical convergence and lacunary statistical Cauchy concepts of triple sequence in fuzzy metric space. We also introduce the concept of triple lacunary statistical completeness and prove some basic properties.

1. Introduction

In recent years, the fuzzy theory has emerged as the most active area of research in many branches of mathematics and engineering. In 1965, Zadeh [34] first introduced the fuzzy set theory and since then a large number of research papers have appeared by using the concept of fuzzy set (numbers) and fuzzification of many classical theories has also been made. One of the main problems in the theory of fuzzy topological spaces is to obtain an appropriate and consistent notion of a fuzzy metric space. Many authors have defined several concepts of fuzzy metric space in different ways [2, 8, 17, 18]. In [8, 9], George and Veeramani introduced and studied the concept of fuzzy metric space (in short FMS) with the help of continuous t -norms. Recently, several convergences in fuzzy metric spaces, as p -convergence, s -convergence, st -convergence and std -convergence, were studied by Gregori et al [12, 13, 14, 15].

Freedman and Sember introduced the concept of lower asymptotic density and defined the concept of convergence in density in [5]. Fast [4] officially introduced the definition of statistical convergence. Šalát [26] gave some basic properties of statistical convergence. The study of statistical convergence in triple sequence has been initiated by Şahiner et al. [25]. Fridy and Orhan defined the concept of lacunary statistical convergence [7]. Various applications of this concept can be found in [3, 6, 10, 16, 20, 21, 22, 23, 24, 27, 28, 29, 30, 32, 33].

In this research paper, we analyze the lacunary statistical convergence and lacunary statistical Cauchy concepts of triple sequence in fuzzy metric space. We also

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introduce the concept of lacunary statistical completeness which would provide a more general framework to study the completeness of fuzzy metric spaces.

2. PRELIMINARIES

We recall the following basic concepts from [1, 8, 11, 18, 31]. They will be needed in the course of the paper. By $\mathbb{N}$ and $\mathbb{R}$, we mean the set of all natural and real numbers, respectively.

Definition 1. A binary operation $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous t -norm if $([0, 1], *)$ is a topological monoid with unit 1 such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ ($a, b, c, d \in [0, 1]$).

Example 1. $a * b = \min\{a, b\}$ and $a * b = a.b$ are two common examples of continuous t -norms.

Definition 2. The 3-tuple $(X, M, *)$ is said to be a fuzzy metric space if X is a nonempty set, $*$ is a continuous t -norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions for all $x, y, z \in X$ and $s, t > 0$:

- (i) $M(x, y, t) > 0$;
- (ii) $M(x, y, t) = 1$ if and only if $x = y$;
- (iii) $M(x, y, t) = M(y, x, t)$;
- (iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
- (v) $M(x, y, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous.

If $(X, M, *)$ is a fuzzy metric space, then we will call $(M, *)$, or simply M , a fuzzy metric on X .

Let (X, d) be a metric space. Denote by $a.b$ the usual multiplication for all $a, b \in [0, 1]$, and let M_d be the fuzzy set defined on $X^2 \times (0, \infty)$ by

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}.$$

Then $(M_d, \cdot)$ is a fuzzy metric on X called standard fuzzy metric space [8].

Definition 3. Let $(X, M, *)$ be a fuzzy metric space. We define open ball $B_M(x, r, t)$ with centre $x \in X$ and radius r , $0 < r < 1$, $t > 0$ as

$$B_M(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$$

Let $(X, M, *)$ be a fuzzy metric space. Define

$$\tau_M = \{A \subset X : x \in A \text{ iff there exists } t > 0, r \in (0, 1) \text{ such that } B_M(x, r, t) \subset A\}.$$

Then τ_M is a topology on X . George and Veeramani [8] proved that $\{B_M(x, r, t) : x \in X, t > 0, r \in (0, 1)\}$ forms a base of a topology τ_M in X .

The next characterization of convergent sequences was gave in [8] .

Proposition 1. *Let $(X, M, *)$ be a fuzzy metric space and τ_M be the topology induced by the fuzzy metric. Then for a sequence $\{x_n\}$ in X , x_n converges to x in X if and only if $M(x_n, x, t)$ tends to 1 as n tends to ∞ , for $t > 0$.*

It is easy to see that $\{x_n\}$ converges to x_0 which is equivalent to the following :

For each $r \in (0, 1)$ and each $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_0, t) > 1 - r$ for all $n > n_0$.

Definition 4. *A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is said to be a Cauchy sequence if for each ε , $0 < \varepsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq n_0$.*

Definition 5. *A fuzzy metric space is said to be complete if every Cauchy sequence is convergent.*

3. TRIPLE LACUNARY STATISTICAL CONVERGENCE IN FMS

Li et al. [19] introduced the concepts of statistical convergence in fuzzy metric spaces. In this section, we present the notion of lacunary statistically convergent and lacunary statistically Cauchy of triple sequences in fuzzy metric spaces.

The idea is based on the notion of natural density of subsets of $\mathbb{N}$, the set of all positive integers which is defined as follows: The natural density of a subset A of $\mathbb{N}$ denoted as $\delta(A)$ is defined by $\delta(A) = \lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : k \in A\}|$. Here, for a subset K of a set X , $|K|$ will denote the cardinality of K . It is straightforward to show that $\delta(A) \in [0, 1]$ and $\delta(\mathbb{N} - A) = 1 - \delta(A)$ if $\delta(A)$ exists. A is said to be statistically dense provided that $\delta(A) = 1$.

We now recall that the concept of statistical convergence for triple sequences was presented by Şahiner, Gürdal and Düden [25] as follows:

A function $x : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}$ (or $\mathbb{C}$) is called a real (complex) triple sequence. A triple sequence (x_{jkl}) is said to be convergent to L in Pringsheim's sense if for every $\varepsilon > 0$, there exists $n_0(\varepsilon) \in \mathbb{N}$ such that $|x_{jkl} - L| < \varepsilon$ whenever $j, k, l \geq n_0$.

Definition 6. *A subset K of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ is said to have natural density $\delta_3(K)$ if*

$$\delta_3(K) = P - \lim_{n, k, l \rightarrow \infty} \frac{|K_{nkl}|}{nkl}$$

exists, where the vertical bars denote the number of (n, k, l) in K such that $p \leq n$, $q \leq k$, $r \leq l$. Then, a real triple sequence $x = (x_{pqr})$ is said to be statistically convergent to L in Pringsheim's sense if for every $\varepsilon > 0$,

$$\delta_3(\{(n, k, l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : p \leq n, q \leq k, r \leq l, |x_{pqr} - L| \geq \varepsilon\}) = 0.$$

Furthermore, a new type of sequence called triple lacunary sequence was introduced in Esi and Savaş [3]. The triple sequence $\theta_3 = \theta_{r,s,t} = \{(j_r, k_s, l_t)\}$ is called triple lacunary sequence if there exist three increasing sequences of integers such that

$$j_0 = 0, \quad h_r = j_r - j_{r-1} \rightarrow \infty \text{ as } r \rightarrow \infty,$$

$$k_0 = 0, \quad h_s = k_s - k_{s-1} \rightarrow \infty \text{ as } s \rightarrow \infty,$$

and

$$l_0 = 0, \quad h_t = l_t - l_{t-1} \rightarrow \infty \text{ as } t \rightarrow \infty.$$

Let $k_{r,s,t} = j_r k_s l_t$, $h_{r,s,t} = h_r h_s h_t$ and $\theta_{r,s,t}$ is determined by

$$I_{r,s,t} = \{(j, k, l) : j_{r-1} < j \leq j_r, k_{s-1} < k \leq k_s \text{ and } l_{t-1} < l \leq l_t\},$$

$$q_r = \frac{j_r}{j_{r-1}}, q_s = \frac{k_s}{k_{s-1}}, q_t = \frac{l_t}{l_{t-1}} \text{ and } q_{r,s,t} = q_r q_s q_t.$$

Let $D \subset \mathbb{N} \times \mathbb{N} \times \mathbb{N}$. The number

$$\delta_{\theta_3}(D) = \lim_{r,s,t} \frac{1}{h_{r,s,t}} |\{(j, k, l) \in I_{r,s,t} : (j, k, l) \in D\}|$$

is said to be the $\theta_{r,s,t}$ -density of D , provided the limit exists.

We start this section with the following definition.

Definition 7. Let $(X, M, *)$ be a fuzzy metric space. A triple sequence $\{x_{jkl}\}$ in X is said to be convergent to $L \in X$ if, for each $r \in (0, 1)$ and each $t > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$M(x_{jkl}, L, t) > 1 - r \text{ whenever } j > n_0, k > n_0, l > n_0.$$

Definition 8. Let $(X, M, *)$ be a fuzzy metric space. A triple sequence $\{x_{jkl}\}$ in X is said to be statistically convergent to $L \in X$ if

$$\delta(\{(j, k, l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : M(x_{jkl}, L, t) > 1 - r\}) = 1$$

for all $r \in (0, 1)$ and $t > 0$.

In this case we write $x_{jkl} \rightarrow L (S(\text{sts}_3))$. In such a case we say that $\{x_{jkl}\}$ is sts_3 -convergent to L (or, $\{x_{jkl}\}$ sts_3 -converges to L). It is obvious that

$$\delta(\{(j, k, l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : M(x_{jkl}, L, t) > 1 - r\}) = 1 \Leftrightarrow \lim_{r,s,t \rightarrow \infty} \frac{|\{j \leq r, k \leq s, l \leq t : M(x_{jkl}, L, \varepsilon) > 1 - r\}|}{rst} = 1.$$

Example 2. Let $X = [0, 3]$. Denote $a * b = ab$ and for all $a, b \in [0, 1]$. For all $x, y \in \mathbb{R}$ and every $t > 0$, consider

$$M(x, y, t) = \frac{t}{t + |x - y|}.$$

In this case observe that $(X, M, *)$ is a fuzzy metric space. Now define a sequence $x = \{x_{jkl}\}$ whose terms are given by

$$x_{jkl} = \begin{cases} jkl, & j, k, l = \frac{r(r+3)}{2}, r \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases}$$

It is immediate to see that $\{x_{jkl}\}$ is not convergent to 0, but $\{x_{jkl}\}$ is sts_3 -convergent to 0.

Definition 9. Let $(X, M, *)$ be a fuzzy metric space and $\theta_3 = \theta_{r,s,t}$ be a lacunary triple sequence. A triple sequence $\{x_{jkl}\}$ is said to be lacunary statistically convergent to $L \in X$, written as $\text{sts}_{\theta_3}\text{-}\lim x_{jkl} = L$, if, for all $r \in (0, 1)$ and $t > 0$,

$$\lim_{r,s,t} \frac{1}{h_{r,s,t}} |\{(j, k, l) \in I_{r,s,t} : M(x_{jkl}, L, t) > 1 - r\}| = 1$$

or

$$\delta_{\theta}(\{(j, k, l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : M(x_{jkl}, L, t) > 1 - r\}) = 1.$$

In this case we write $x_{jkl} \rightarrow L(S_{\theta_3}(\text{sts}))$, where L is said to be $\text{sts}_{\theta_3}\text{-}\lim$.

Theorem 1. Let $(X, M, *)$ be a fuzzy metric space and $\theta_3 = \theta_{r,s,t}$ be a lacunary triple sequence. If a triple sequence $\{x_{jkl}\}$ in X is lacunary statistically convergent in X , then $\text{sts}_{\theta_3}\text{-}\lim$ is unique.

Proof. Suppose that $x_{jkl} \rightarrow L_1(S_{\theta_3}(\text{sts}))$ and $x_{jkl} \rightarrow L_2(S_{\theta_3}(\text{sts}))$. Let L_1 and L_2 be two distinct points in X and $t > 0$. Then $0 < M(L_1, L_2, t) < 1$. Put $1 - \varepsilon \in (M(L_1, L_2, t), 1)$. For each $1 - s \in (1 - \varepsilon, 1)$, there exists $1 - s$ such that $(1 - s) * (1 - s) \geq 1 - \varepsilon$. Let

$$B_{L_1} = \left\{ y \in X : M\left(L_1, y, \frac{t}{2}\right) > 1 - s \right\}$$

and

$$B_{L_2} = \left\{ y \in X : M\left(L_2, y, \frac{t}{2}\right) > 1 - s \right\}.$$

We claim that $B_{L_1} \cap B_{L_2} = \emptyset$. In fact, if there exists $z \in B_{L_1} \cap B_{L_2}$, then we have

$$\begin{aligned} M(L_1, L_2, t) &\geq M\left(L_1, z, \frac{t}{2}\right) * M\left(z, L_2, \frac{t}{2}\right) \\ &\geq (1 - s) * (1 - s) \geq 1 - \varepsilon \\ &> M(L_1, L_2, t) \end{aligned}$$

which is a contradiction. Since

$$\left\{ y \in X : M \left(L_2, y, \frac{t}{2} \right) > 1 - s \right\} \subset \left\{ x \in X : M \left(x, L_1, \frac{t}{2} \right) \leq 1 - s \right\},$$

it follows that

$$\begin{aligned} & \left\{ (j, k, l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : M \left(x_{jkl}, L_2, \frac{t}{2} \right) > 1 - s \right\} \\ (1) \quad & \subseteq \left\{ (j, k, l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : M \left(x_{jkl}, L_1, \frac{t}{2} \right) \leq 1 - s \right\}. \end{aligned}$$

Then we may write

$$\begin{aligned} & \lim_{r,s,t} \frac{1}{h_{r,s,t}} \left| \left\{ (j, k, l) \in I_{r,s,t} : M \left(x_{jkl}, L_1, \frac{t}{2} \right) \leq 1 - s \right\} \right| \\ & = 1 - \lim_{r,s,t} \frac{1}{h_{r,s,t}} \left| \left\{ (j, k, l) \in I_{r,s,t} : M \left(x_{jkl}, L_1, \frac{t}{2} \right) > 1 - s \right\} \right| \\ & = 1 - 1 = 0. \end{aligned}$$

By (1), we get

$$\begin{aligned} 1 & = \lim_{r,s,t} \frac{1}{h_{r,s,t}} \left| \left\{ (j, k, l) \in I_{r,s,t} : M \left(x_{jkl}, L_2, \frac{t}{2} \right) > 1 - s \right\} \right| \\ & \leq \lim_{r,s,t} \frac{1}{h_{r,s,t}} \left| \left\{ (j, k, l) \in I_{r,s,t} : M \left(x_{jkl}, L_1, \frac{t}{2} \right) \leq 1 - s \right\} \right| = 0. \end{aligned}$$

which is a contradiction. Therefore, we conclude that sts_{θ_3} -lim is unique. This completes the proof of the theorem. $\square$

Theorem 2. Let $(X, M, *)$ be a fuzzy metric space and $\theta_3 = \theta_{r,s,t}$ be a lacunary triple sequence. If triple sequence $\{x_{jkl}\}$ in X is convergent to L , then $\{x_{jkl}\}$ is lacunary statistical convergent to L . But converse need not be true.

Proof. Suppose that $\{x_{jkl}\}$ is convergent to L . Let $r \in (0, 1)$ and $t > 0$. Then there exists $n_0 \in \mathbb{N}$ such that $M(x_{jkl}, L, t) > 1 - r$ for all $j > n_0, k > n_0, l > n_0$. Hence

$$|\{(j, k, l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : M(x_{jkl}, L, \varepsilon) > 1 - r\}| \geq (j - j_0)(k - k_0)(l - l_0).$$

Therefore

$$\lim_{r,s,t} \frac{1}{h_{r,s,t}} |\{(j, k, l) \in I_{r,s,t} : M(x_{jkl}, L, t) > 1 - r\}| = 1.$$

that is $x_{jkl} \rightarrow L(S_{\theta_3}(\text{sts}))$. We complete the proof. $\square$

The following example shows that the converse of Theorem 2 is not valid.

Example 3. Let $(\mathbb{R}, |\cdot|)$ denote the space of real numbers with the usual metric, and let $a * b = ab$ and for all $a, b \in [0, 1]$. For all $x, y \in \mathbb{R}$ and every $t > 0$, consider

$$M(x, y, t) = \frac{t}{t + |x - y|}.$$

In this case observe that $(\mathbb{R}, M, *)$ is a fuzzy metric space (see [8]). Now define a sequence $x = \{x_{jkl}\}$ whose terms are given by

$$x_{jkl} = \begin{cases} jkl, & \begin{matrix} j_r - [\sqrt{h_r}] + 1 \leq j \leq j_r, \\ k_s - [\sqrt{h_s}] + 1 \leq s \leq k_s, \\ l_t - [\sqrt{h_t}] + 1 \leq l \leq l_t, \quad r, s, t \in \mathbb{N} \end{matrix} \\ 0, & \text{otherwise.} \end{cases}$$

It is immediate to see that $\{x_{jkl}\}$ is not convergent to 0. We claim that $\{x_{jkl}\}$ is sts_{θ_3} -convergent to 0. In fact, let $r \in (0, 1)$ and $t > 0$. Put

$$A = \{(j, k, l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : M(x_{jkl}, x, t) \leq 1 - r\}.$$

Then

$$\begin{aligned} A &= \{(j, k, l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : M(x_{jkl}, x, t) \leq 1 - r\} \\ &= \left\{ (j, k, l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \begin{matrix} j_r - [\sqrt{h_r}] + 1 \leq j \leq j_r, \\ k_s - [\sqrt{h_s}] + 1 \leq k \leq k_s, \quad l_t - [\sqrt{h_t}] + 1 \leq l \leq l_t, \quad r, s, t \in \mathbb{N} \end{matrix} \right\} \\ &= K \end{aligned}$$

and so, we get

$$\frac{1}{h_{r,s,t}} |A| \leq \frac{1}{h_{r,s,t}} |K| \leq \frac{\sqrt{h_r} \sqrt{h_s} \sqrt{h_t}}{h_{r,s,t}},$$

which implies that

$$\lim_{h_{r,s,t}} \frac{1}{h_{r,s,t}} |\{(j, k, l) \in I_{r,s,t} : M(x_{jkl}, x, t) \leq 1 - r\}| = 0$$

Consequently,

$$\lim_{h_{r,s,t}} \frac{1}{h_{r,s,t}} |\{(j, k, l) \in I_{r,s,t} : M(x_{jkl}, 0, t) > 1 - r\}| = 1$$

for all $r \in (0, 1)$ and $t > 0$.

Theorem 3. Let $x = \{x_{jkl}\}$ be a triple sequence in a fuzzy metric space $(X, M, *)$. Then, for any lacunary triple sequence θ_3 , $\text{sts}_{\theta_3}\text{-}\lim x = L$ if and only if there exists a subset $K = \{(j, k, l) : j, k, l = 1, 2, \dots\} \subset \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ such that $\delta_{\theta_3}(K) = 1$ and $\{x_{jkl}\}_{(j,k,l) \in K}$ converges to L .

Proof. Suppose that $\text{sts}_{\theta_3}\text{-}\lim x = L$. Let

$$K_s = \left\{ (j, k, l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : M(x_{jkl}, L, t) > 1 - \frac{1}{s} \right\}.$$

Then, for $t > 0$ and each $s = 1, 2, \dots$, we have that

$$(2) \quad \delta_{\theta_3}(K_s) = 1$$

and

$$(3) \quad K_{s+1} \subseteq K_s.$$

For $(j, k, l) \in K_s$, we need to show that $\{x_{jkl}\}$ converges to L . Now, suppose that $\{x_{jkl}\}$ is not convergent to L . Therefore there is $\mu > 0$ such that

$$\{(j, k, l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : M(x_{jkl}, L, t) \leq 1 - \mu\}$$

for infinitely many terms. Let

$$K_\mu = \{(j, k, l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : M(x_{jkl}, L, t) \leq 1 - \mu\}, \quad \mu > \frac{1}{s}.$$

Then $\delta_{\theta_3}(K_\mu) = 0$ and by (2), $K_s \subset K_\mu$. Hence $\delta_{\theta_3}(K_s) = 0$ which contradicts (3). So $\{x_{jkl}\}$ converges to L .

Conversely, suppose that there exists a subset $K = \{(j, k, l) : j, k, l = 1, 2, \dots\} \subset \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ such that $\delta_{\theta_3}(K) = 1$ and $\{x_{jkl}\}$ converges to L , i.e. there exists $k_0 \in \mathbb{N}$ such that all $r \in (0, 1)$ and $t > 0$,

$$M(x_{jkl}, L, t) > 1 - r, \quad \forall j, k, l \geq k_0$$

Now

$$\begin{aligned} K_M(r, t) &= \{(j, k, l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : M(x_{jkl}, L, t) \leq 1 - r\} \\ &\subseteq \mathbb{N} \times \mathbb{N} \times \mathbb{N} - \{(j_{k_0+1}, k_{k_0+1}, l_{k_0+1}), (j_{k_0+2}, k_{k_0+2}, l_{k_0+2}), \dots\}. \end{aligned}$$

This implies that $\delta_{\theta_3}(K_M(r, t)) \leq 1 - 1 = 0$. Therefore

$$\lim_{r, s, t} \frac{1}{h_{r, s, t}} |\{(j, k, l) \in I_{r, s, t} : M(x_{jkl}, L, t) > 1 - r\}| = 1.$$

Hence $\text{sts}_{\theta_3}\text{-}\lim x_{jkl} = L$. This completes the proof of the theorem. $\square$

4. TRIPLE LACUNARY STATISTICALLY COMPLETE FUZZY METRIC SPACES

In [19], Li et al. has defined the concept of statistically Cauchy sequences in fuzzy metric space. In this section we define lacunary statistically Cauchy triple sequences with respect to an fuzzy metric space. And we consider some relations between lacunary statistical completeness and the ordinary completeness in fuzzy metric space. Now, we define the Cauchy and lacunary statistical convergence of triple sequences in a fuzzy metric space.

Definition 10. Let $(X, M, *)$ be a fuzzy metric space. A triple sequence $\{x_{jkl}\}$ in X is called Cauchy if for each $\alpha \in (0, 1)$ and each $t > 0$, there exists $N \in \mathbb{N}$ such that

$$M(x_{jkl}, x_{pqr}, t) > 1 - \alpha$$

whenever $j \geq p \geq N$, $k \geq q \geq N$, $l \geq r \geq N$.

Definition 11. Let $(X, M, *)$ be a fuzzy metric space and $\theta_3 = \theta_{r,s,t}$ be a lacunary triple sequence. A triple sequence $\{x_{jkl}\}$ in X is said to be lacunary statistically Cauchy sequence, if, for all $\alpha \in (0, 1)$ and $t > 0$, there exists $M, M', M'' \in \mathbb{N}$ such that for all $j, p \geq M''$, $k, q \geq M'$, $l, r \geq M$,

$$\delta_{\theta_3}(\{(j, k, l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : M(x_{jkl}, x_{pqr}, t) > 1 - \alpha\}) = 1.$$

In such a case we say that $\{x_{jkl}\}$ is sts_{θ_3} -Cauchy.

Theorem 4. Let $\{x_{jkl}\}$ be a triple sequence in a fuzzy metric space $(X, M, *)$ and $\theta_3 = \theta_{r,s,t}$ be a lacunary triple sequence. If $\{x_{jkl}\}$ is a lacunary statistically convergent, then $\{x_{jkl}\}$ is a lacunary statistically Cauchy.

Proof. Let $\{x_{jkl}\}$ be lacunary statistically convergent to L , i.e., $x_{jkl} \rightarrow L (S_{\theta_3}(\text{sts}))$. Let $\alpha \in (0, 1)$ and $t > 0$. Then there exists $\alpha_1 \in (0, \alpha)$ such that $(1 - \alpha_1) * (1 - \alpha_1) > 1 - \alpha$. According to Theorem 3, there exists a subset $K = \{(j, k, l) : j, k, l = 1, 2, \dots\} \subset \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ such that $\delta_{\theta_3}(K) = 1$ and $\{x_{jkl}\}$ converges to L . Thus there exists $n_0 \in \mathbb{N}$ such that

$$M(x_{jkl}, L, \frac{t}{2}) > 1 - \alpha_1 \text{ for all } j > n_0, k > n_0, l > n_0.$$

Let $(p, q, r) \in \{(j, k, l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : M(x_{jkl}, L, \frac{t}{2}) > 1 - \alpha_1\}$. Then

$$\begin{aligned} M(x_{pqr}, x_{jkl}, t) &\geq M(x_{pqr}, L, t/2) * M(x_{jkl}, L, t/2) \\ &\geq (1 - \alpha_1) * (1 - \alpha_1) > 1 - \alpha. \end{aligned}$$

It follows that

$$\begin{aligned} (4) \quad &\left\{ (j, k, l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : M\left(x_{jkl}, L, \frac{t}{2}\right) > 1 - \alpha_1 \right\} \\ &\subseteq \{(j, k, l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : M(x_{jkl}, x_{pqr}, t) > 1 - \alpha_1\}. \end{aligned}$$

Since $\delta_{\theta_3}(\{(j, k, l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : M(x_{jkl}, L, \frac{t}{2}) > 1 - \alpha_1\}) = 1$ and (4), we get that

$$\delta_{\theta_3}(\{(j, k, l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : M(x_{jkl}, x_{pqr}, t) > 1 - \alpha_1\}) = 1.$$

So $\{x_{jkl}\}$ is sts_{θ_3} -Cauchy. This completes the proof of the theorem. $\square$

The following example shows that the converse of the above theorem need not be true.

Example 4. Let $X = \{x_{jkl} : (j, k, l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}\}$, where $x_{jkl} = 1 - \frac{1}{(j+1)(k+1)(l+1)}$ ($j, k, l \in \mathbb{N}$) and $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$, and let M be a fuzzy set on $X^2 \times (0, \infty)$ defined as follows:

$$M(x, y, t) = \begin{cases} 1, & x = y \\ \min\{x, y\}, & x \neq y, \end{cases}$$

for all $x, y \in X$ and $t > 0$. Then $(X, M, *)$ is a fuzzy metric space and triple sequence $\{x_{jkl}\}$ in $(X, M, *)$ is sts_{θ_3} -Cauchy, but it is not sts_{θ_3} -convergent.

Let $\alpha \in (0, 1)$ and $t > 0$. Then there exists $p, q, r \in \mathbb{N}$ such that $\frac{1}{(p+1)(q+1)(r+1)} < \alpha$. Hence

$$M(x_{jkl}, x_{pqr}, t) = x_{pqr} = 1 - \frac{1}{(p+1)(q+1)(r+1)} > 1 - \alpha$$

for all $j > p, k > q, l > r$. Thus $\delta_{\theta}(\{(j, k, l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : M(x_{jkl}, x_{pqr}, t) > 1 - \alpha\}) = 1$ which means that $\{x_{jkl}\}$ is sts_{θ_3} -Cauchy. Let $L \in X$. Then there exists $p, q, r \in \mathbb{N}$ such that $L = x_{pqr} = 1 - \frac{1}{(p+1)(q+1)(r+1)}$. Now, fix $t_0 = \alpha_0 = \frac{1}{3(p+1)(q+1)(r+1)}$. Then

$$M(x_{jkl}, L, t_0) = M(x_{jkl}, x_{pqr}, t_0) = x_{pqr} = 1 - \frac{1}{(p+1)(q+1)(r+1)} \leq 1 - \alpha_0$$

for all $j > p, k > q, l > r$. Hence

$$\delta_{\theta}(\{(j, k, l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : M(x_{jkl}, L, t_0) \leq 1 - \alpha_0\}) = 1,$$

which implies that $\delta_{\theta}(\{(j, k, l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : M(x_{jkl}, L, t_0) > 1 - \alpha_0\}) = 0$. So $\{x_{jkl}\}$ is not sts_{θ_3} -convergent.

Corollary 1. If a triple sequence in a fuzzy metric space is Cauchy, then it is sts_{θ_3} -Cauchy. However, the converse is false.

Now using a similar technique in the proof of Theorem 3, one can get the following result at once.

Theorem 5. Let $(X, M, *)$ be a fuzzy metric space, and let $x = \{x_{jkl}\}$ be a triple sequence whose terms are in X . Then, for $\theta_3 = \theta_{r,s,t}$ lacunary triple sequence, the following conditions are equivalent:

- (i) x is sts_{θ_3} -Cauchy.
- (ii) there exists an increasing index sequence $K = \{(j, k, l) : j, k, l = 1, 2, \dots\} \subset \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ such that $\delta_{\theta_3}(K) = 1$ and the subsequence $\{x_{jkl}\}_{(j,k,l) \in K}$ is a Cauchy sequence with respect to fuzzy metric M .

Now, we introduce the lacunary statistically complete fuzzy metric space using the lacunary statistical Cauchy triple sequence defined above.

Definition 12. The fuzzy metric space $(X, M, *)$ is called triple lacunary statistically complete if every lacunary statistically Cauchy triple sequence in X lacunary statistically converges to a point in X .

In such a case we say that $(X, M, *)$ is sts_{θ_3} -complete.

Theorem 6. *Let $(X, M, *)$ be a fuzzy metric space. If X is sts_{θ_3} -complete, then it is complete.*

Proof. Let $\{x_{jkl}\}$ be a Cauchy sequence in X . Let $\alpha \in (0, 1)$ and $t > 0$. Then there exists $\alpha_1 \in (0, \alpha)$ such that $(1 - \alpha_1) * (1 - \alpha_1) > 1 - \alpha$. Therefore there exists $N \in \mathbb{N}$ such that

$$M(x_{jkl}, x_{pqr}, t) > 1 - \alpha_1$$

whenever $j \geq p \geq N$, $k \geq q \geq N$, $l \geq r \geq N$. Then $\{x_{jkl}\}$ is sts_{θ_3} -Cauchy by the above Corollary 1. Since X is sts_{θ_3} -complete, there exists an $L \in X$ such that $\{x_{jkl}\}$ lacunary statistically converges to L . Now, we show that $\{x_{jkl}\}$ converges to this L in X . Thus, by Theorem 3, there exists an increasing index sequence $K = \{(j, k, l) : j, k, l = 1, 2, \dots\} \subset \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ such that $\delta_{\theta_3}(K) = 1$ and the subsequence $\{x_{jkl}\}_{(j,k,l) \in K}$ converges to L . Then there exists $(p, q, r) \in K$ with $n_0 \geq N$ such that

$$M(x_{pqr}, L, \frac{t}{2}) > 1 - \alpha_1 \text{ for all } p > n_0, q > n_0, r > n_0.$$

Hence

$$\begin{aligned} M(x_{jkl}, L, t) &\geq M(x_{jkl}, x_{pqr}, \frac{t}{2}) * M(x_{pqr}, L, \frac{t}{2}) \\ &\geq (1 - \alpha_1) * (1 - \alpha_1) > 1 - \alpha \end{aligned}$$

for all $j \geq n_0 \geq N$, $k \geq n_0 \geq N$, $l \geq n_0 \geq N$. That is $\{x_{jkl}\}$ converges to L , therefore X is complete. This completes the proof. $\square$

5. CONCLUDING REMARKS

In this paper, we have dealt with the notion of the lacunary statistical convergence and lacunary statistical Cauchy of triple sequence in fuzzy metric space. However, further investigation in these aspects is required. We also introduce the concept of triple lacunary statistical completeness and prove some of the basic properties.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

Ethical approval This article does not contain any studies with human participants & animals performed by any of the authors.

Informed Consent Informed consent was obtained from all individual participants included in the study.

The Authorship contributions Conceptualization: Mehmet Gürdal, Ekrem Savaş; Investigation: Mehmet Gürdal, Ekrem Savaş; Methodology: Mehmet Gürdal, Ekrem Savaş; Writing –original draft: Mehmet Gürdal, Ekrem Savaş; Writing – review & editing: Mehmet Gürdal, Ekrem Savaş

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