Two Dimensional Volterra Integrodifferential Equation of Heat Conduction and Diffusion Problems and Its Numerical Solutions

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Two Dimensional Volterra Integrodifferential Equation of Heat Conduction and Diffusion Problems and Its Numerical Solutions

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Abstract: This paper deals with new equation in the form Two-dimensional Volterra integrodifferential equation (**TDVIDE**), which is related of the heat conduction and diffusion problems. Runge–Kutta method (**RKM**) and Bolck-by-Block method (**BBM**) have been applied to the problem. Numerical examples are presented, and their results are compared with the analytical solution to demonstrate the validity and applicability of the methods.

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1. INTRODUCTION

The numerical solution of convection-diffusion transport problems arises in many important applications in science and engineering. In [1], the author introduced methods for derivation of analytical and numerical solutions of heat diffusion in one-dimensional thin rod have investigated. Laura and others, in [2], studied a comparison of analytical and finite element results in two types of steady state diffusion problems: determination of solutions in domains of complicated boundary shape, and analysis diffusion-type problems in nonhomogeneous media. The authors, in [3], discussed a numerical method for one-dimensional convection-diffusion equation. The authors, in [4], discussed a numerical method for solving nonhomogeneous backward heat conduction problem. In addition, integral equations play an important role in many other sciences, such as contact problems in the theory of elasticity, vicodynamics fluid, and others [5-7]. Many different methods are used to obtain the solution of the Volterra integral equation. In [8] Linz, discussed the analytical and numerical methods of Volterra equation. In [9], Mirzaee and Rafei solved the nonlinear two-dimensional Volterra integral equations. In the references [10-15], the authors solved linear and nonlinear system of Volterra integral equations numerically. This paper describes the numerical solutions of (TDVIDE).

2. BASIC FORMULAS

In our work, we will study the problems in heat conduction and diffusion. Consider the formulas:

$$\hat{f}_s(w,v)=\sqrt{\frac{2}{\pi}}\int_0^\infty\int_0^\infty f(x,y)\sin wx\,\sin vy\,dx\,dy, \tag{1}$$

$$\hat{f}_c(w,v)=\sqrt{\frac{2}{\pi}}\int_0^\infty\int_0^\infty f(x,y)\cos wx\,\cos vy\,dx\,dy, \tag{2}$$

$$\hat{f}(w,v)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty\int_{-\infty}^\infty f(x,y)e^{iwx}e^{ivy}dx\,dy, \tag{3}$$

The corresponding inversion formulas then are

$$f(x,y)=\sqrt{\frac{2}{\pi}}\int_0^\infty\int_0^\infty \hat{f}_s(x,y)\sin wx\,\sin vy\,dw\,dv, \tag{4}$$

$$=\sqrt{\frac{2}{\pi}}\int_0^\infty\int_0^\infty \hat{f}_c(x,y)\cos wx\,\cos vy\,dw\,dv, \tag{5}$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty\int_{-\infty}^\infty \hat{f}(w,v)e^{-iwx}e^{-ivy}dw\,dv, \tag{6}$$

Consider the simple heat conduction equation

$$\frac{\partial^2 u}{\partial x^2}.\frac{\partial^2 u}{\partial y^2}=\frac{\partial u}{\partial t}.\frac{\partial u}{\partial s},\ 0\le x,y<\infty,\ t,s>0, \tag{7}$$

Subject to the conditions

$$u(x,0)=0, u(y,0)=0, 0\le x,y<\infty, \tag{8}$$

$$\frac{\partial u(0,t)}{\partial x}.\frac{\partial u(0,s)}{\partial y}=-g(t,s), t,s>0 \tag{9}$$

$$\lim_{x\to\infty}u(x,t)=0,\qquad \lim_{x\to\infty}\frac{\partial u(x,t)}{\partial x}=0$$

$$\lim_{y\to\infty}u(y,s)=0,\qquad \lim_{y\to\infty}\frac{\partial u(y,s)}{\partial y}=0$$

We apply the Fourier cosine transformation to equation (7), we obtain

$$\int_0^\infty\int_0^\infty\left(\frac{\partial^2 u(x,t)}{\partial x^2}.\frac{\partial^2 u(y,s)}{\partial y^2}\right)\cos wx\,\cos vy\,dx\,dy=\int_0^\infty\int_0^\infty\left(\frac{\partial u(x,t)}{\partial t}.\frac{\partial u(y,s)}{\partial s}\right)\cos wx\,\cos vy\,dx\,dy.$$

Then, we get:

$$g(t,s)-w^2\int_0^\infty\int_0^\infty u(x,t)u(y,s)\cos wx\,\cos vy\,dx\,dy=\frac{\partial}{\partial t}\frac{\partial}{\partial s}\int_0^\infty\int_0^\infty (u(x,t)u(v,y))\cos wx\,\cos vy\,dx\,dy,$$

$$\sqrt{\frac{\pi}{2}}\frac{\partial}{\partial t}\frac{\partial}{\partial s}\hat{u}_c(w,t)\hat{u}_c(v,s)=g(t,s)-\sqrt{\frac{\pi}{2}}w^2v^2\hat{u}_c(w,t)\hat{u}_c(\mathrm{v,s}) \tag{10}$$

So, from (8)

$$\hat{u}_c(w,0)=0, \hat{u}_c(v,0)=0 \tag{11}$$

Eq. (10) subject to the initial condition (11) has the solution

$$\hat{u}_c(w,t)=\sqrt{\frac{2}{\pi}}e^{-w^2t}\int_0^t g(s_1)e^{w^2s_1}ds_1,\ \hat{u}_c(v,s)=\sqrt{\frac{2}{\pi}}e^{-w^2s}\int_0^s g(s_2)e^{w^2s_2}ds_2 \tag{12}$$

Applying the inversion formula (5), we have

$$u(x,y;t,s)=\frac{2}{\pi}\int_0^\infty\int_0^\infty e^{-w^2t}e^{-v^2s}\int_0^t\int_0^s g(s_1)g(s_2)e^{w^2s_1}e^{v^2s_2}\cos wx\ \cos vy ds_1 ds_2 dw dv$$

$$=\frac{2}{\pi}\int_0^t\int_0^s g(s_1)g(s_2)\int_0^\infty\int_0^\infty e^{-w^2(t-s_1)}e^{-v^2(s-s_2)}\cos wx\ \cos vy dw dv ds_1 ds_2$$

So that

$$u(x,y;t;s)=\frac{1}{\sqrt{\pi}}\int_0^t\int_0^s g(s_1)g(s_2)(t-s_1)^{-1/2}(s-s_2)e^{-x^2/4(t-s_1)}e^{-y^2/4(s-s_2)}ds_1 \mathrm{ds}_2 \tag{13}$$

We must replace $g(t)$ with some function $G(u(0,t),t)$ and (13) becomes

$$u(x,y;t,s)=\frac{1}{\sqrt{\pi}}\int_0^t\int_0^s G(u(0,0;s_1;s_2);s_1,s_2)(t-s_1)^{-1/2}(s-s_2)^{-1/2}e^{-x^2/4(t-s_1)}e^{-y^2/4(s-s_2)}ds_1 \mathrm{ds}_2 \tag{14}$$

If we now use $U(t,s)$ for $u(0,0;t,s)$ and put x=0,y=0 in (14), we get:

$$U(t,s)=\frac{1}{\sqrt{\pi}}\int_0^t\int_0^s \frac{G(U(s_1,s_2);s_1,s_2)}{\sqrt{t-s_1}\sqrt{s-s_2}}ds_1 ds_2 \tag{15}$$

The relation between th $T(x,y;t,s)$, and the power produced $u(t,s)$, can be described by rather complicated set of integro-partial differential equations

$$\frac{d^2u(t,s)}{dtds}=\int_{-\infty}^\infty\int_{-\infty}^\infty \alpha(x,y)T(x,y;t,s)dxdy, \tag{16}$$

$$\frac{\partial^2 T(x,y;t,s)}{\partial t\partial s}=\frac{\partial^4 T(x,y;t,s)}{\partial x^2\partial y^2}+\eta(x,y)u(t,s), \tag{17}$$

For $-\infty<x,y<\infty, t,s>0.$ Additional conditions will be taken as

$$u(0,0)=0,$$
$$T(x,y;0,0)=f(x,y),$$
$$\lim_{x,y\to\pm\infty}T(x,y;t,s)=\lim_{x,y\to\pm\infty}\frac{\partial}{\partial x}\frac{\partial}{\partial y}T(x,y;t,s)=0$$

By applying the full Fourier transform to (17), we get

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s}\int_{-\infty}^\infty\int_{-\infty}^\infty T(x,y;t,s)e^{iwx}e^{ivy}dxdy=\int_{-\infty}^\infty\int_{-\infty}^\infty \frac{\partial^4 T(x,y;t,s)}{\partial x^2\partial y^2}e^{iwx}e^{ivy}dxdy+u(t,s)\int_{-\infty}^\infty\int_{-\infty}^\infty \eta(x,y)e^{iwx}e^{ivy}dxdy,$$

$$\frac{\partial}{\partial t}\frac{\partial}{\partial}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T(x,y;t,s)e^{iwx}e^{ivy}dxdy = -w^2v^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T(x,y;t,s)e^{iwx}e^{ivy}dxdy + u(t,s)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\eta(x,y)e^{iwx}e^{ivy}dxdy,$$

Therefore, the Fourier transform $\hat{T}(w,v;t,s)$ satisfies

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s}\hat{T}(w,v;t,s) = -w^2v^2\hat{T}(w,v;t,s) + u(t,s)\hat{\eta}(w,v), \tag{18}$$

With initial condition

$$\hat{T}(w,v;0,0) = \hat{f}(w,v).$$

The solution of this equation is

$$\hat{T}(w,v;t,s) = e^{-w^2t}e^{-v^2s}\{\hat{\eta}(w,v)\int_0^t\int_0^s u(s_1,s_2)e^{w^2s_1}e^{v^2s_2}ds_1ds_2 + \hat{f}(w,v)\}, \tag{19}$$

Which (6) can invert to give:

$$T(x,y;t,s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-iwx}e^{-ivy}e^{-w^2t}e^{-v^2s}\hat{\eta}(w,v)\int_0^t\int_0^s u(s_1,s_2)e^{w^2s_1}e^{v^2s_2}ds_1ds_2dwdv + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-iwx}e^{-ivy}e^{-w^2t}e^{-v^2s}\hat{f}(w,v)dwdv, \tag{20}$$

Substituting this into (16) and exchanging order of integration, we obtain

$$\frac{d^2u(t,s)}{dtds} = \frac{1}{\sqrt{2\pi}}\int_0^t\int_0^s u(s_1,s_2)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\alpha(x,y)e^{-iwx}e^{-ivy}e^{-w^2(t-s_1)}e^{-v^2(s-s_2)}\hat{\eta}(w,v)dxdydwdvds_1ds_2 +$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\alpha(x,y)e^{-iwx}e^{-ivy}e^{-w^2t}e^{-v^2s}\hat{f}(w,v)dxdydwdv, \tag{21}$$

Then, we get

$$\frac{d^2u(t,s)}{dtds} = \int_0^t\int_0^s k(t,s;s_1,s_2)u(s_1,s_2)ds_1ds_2 + g(t,s), \tag{22}$$

where

$$k(t,s,s_1,s_2) = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{\alpha}(-w,-v)\hat{\eta}(w,v)e^{-w^2(t-s_1)}e^{-v^2(s-s_2)}dwdv,$$

$$g(t,s) = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{\alpha}(-w,-v)\hat{f}(w,v)e^{-w^2t}e^{-v^2s}dwdv.$$

Under the condition $u(0,0) = u_{0,0}$.

Eq. (22) is a simple two-dimensional Volterra integrodifferential equation.

Integrating Eq. (22) twice, to obtain

$$u(t,s) = \int_0^t\int_0^s K(t,s,s_1,s_2)u(s_1,s_2)ds_1ds_2 + G(t,s), \tag{23}$$

Where,

$$K(t,s,s_1,s_2)=\int_0^{s_1}\int_0^{s_2}k(t,s;s_1,s_2)u(t,s)dtds, \quad G(t,s)=\int_0^{s_1}\int_0^{s_2}g(t,s)dtds$$

3. NUMERICAL METHODS

3.1 RKM

Consider the **VIE** (23) with continuous kernel.

$$u(t,s)-\lambda\int_0^t\int_0^s K(t,s,s_1,s_2)u(s_1,s_2)ds_1ds_2=G(t,s), \quad t,s\geq 0 \qquad (24)$$

We present a description of Pouzets derivation in the case when the equation (24) is linear. We first recall that a RKM for the initial value problem.

$$S(t,s)=W\left[t,s,S(t,s)\right], \quad s_1(0)=0, s_2(0)=0$$

Can be written as the sequence of operation defined by

$$\tilde{S}\left(u_{p_1}h,u_{p_2}h\right)=h\sum_{q_1=0}^{p_1-1}\sum_{q_2=0}^{p_2-1}A_{p_1q_1}A_{p_2q_2}W\left[u_{q_1}h,u_{q_2}h,\tilde{S}\left(u_{q_1}h,u_{q_2}h\right)\right],$$

$$p_1,p_2=1,2,\dots,m, s_1(0)=0, s_2(0)=0 \qquad (25)$$

Where the parameters $\left\{u_{q_1},u_{q_2}\right\}$ satisfy $0=u_0\leq u_1\leq\cdots\leq u_m=1$ and the weights satisfy:

$$\sum_{q_1=0}^{p_1-1}\sum_{q_2=0}^{p_2-1}A_{p_1q_1}A_{p_2q_2}=u_{p_1}u_{p_2}, \quad p_1,p_2=1,2,\dots,m \qquad (26)$$

For $p=4$ we get:

$$u_1=u_2=\frac{1}{2}, \quad u_3=u_4=1$$

$$A_{10}=A_{21}=\frac{1}{2}, \quad A_{20}=A_{30}=A_{31}=0, \quad A_{32}=1$$

$$A_{40}=A_{43}=\frac{1}{6}, \quad A_{41}=A_{42}=\frac{1}{3}$$

Suppose that:

$$K(t,s,s_1,s_2)=\sum_y\sum_x r_y(t)v_y(s_1)r_x(s)v_x(s_2) \qquad (27)$$

Then since:

$$u(t,s)=G(t,s)+\int_0^t\int_0^s\sum_y\sum_x r_y(t)v_y(s_1)r_x(s)v_x(s_2)u(s_1,s_2)\,\mathrm{d}s_1ds_2$$

$$u(t,s)=G(t,s)+\sum_y\sum_x r_y(t)r_x(s)\int_0^t\int_0^s v_y(s_1)v_x(s_2)u(s_1,s_2)ds_1ds_2$$

We have:

$$u(t,s)=G(t,s)+\sum_y\sum_x r_y(t)r_x(s)S_{y,x}(t,s),\quad t,s>0 \tag{28}$$

Where:

$$S_{y,x}(t,s)=\int_0^t\int_0^s v_y(s_1)v_x(s_2)u(s_1,s_2)\,\mathrm{d}s_1ds_2 \tag{29}$$

The derivation of the equation (29) we get:

$$S'_{y,x}(t,s)=v_y(t)v_x(s)u(t,s),\quad z_{y,x}(0,0)=0 \tag{30}$$

We apply the RK formula, to give:

$$\tilde{S}_{y,x}\left(u_{p_1,p_2}h\right)=h\sum_{q_1=0}^{p_1-1}\sum_{q_2=0}^{p_2-1}A_{p_1q_1}A_{p_2q_2}v_{y,x}\left(u_{q_1,q_2}h\right)u\left(u_{q_1}h,u_{q_2}h\right),\quad p_{1,2}=1,2,\ldots,m$$

Which leads to

$$\tilde{u}(u_{p_1,p_2}h)=G(u_{p_1,p_2}h)+\sum_y\sum_x \tilde{S}(u_{p_1,p_2}h)r_{y,x}(u_{p_1,p_2}h)$$

$$\tilde{u}(u_{p_1,p_2}h)=G(u_{p_1,p_2}h)+h\sum_y\sum_x r_{y,x}(u_{p_1,p_2}h)\sum_{q_1=0}^{p_1-1}\sum_{q_2=0}^{p_2-1}A_{p_1q_1}A_{p_2q_2}v_{y,x}(u_{p_1,p_2}h)\tilde{u}(u_{q_1,q_2}h)$$

$$\tilde{u}(u_{p_1,p_2}h)=G(u_{p_1,p_2}h)+h\sum_{q_1=0}^{p_1-1}\sum_{q_2=0}^{p_2-1}A_{p_1q_1}A_{p_2q_2}K\left[u_{p_1,p_2}h,u_{q_1,q_2}h\right]\tilde{u}(u_{q_1,q_2}h),\quad p_{1,2}=1,2,\ldots,m$$

Now we find $\tilde{u}(u_{p_1,p_2}h)$ for $j=1$ we get:

$$u(1)=G_1(t_{2,2})+\frac{h}{6}\left[K\left(t_2,t_1,s_2,s_1\right)p(1)+2K\left(t_2,t_{1+\frac{1}{2}},s_2,s_{1+\frac{1}{2}}\right)[q(1)+z(1)]+K\left(t_2,t_2,s_2,s_2\right)s(1)\right]$$

Where

$$p(1)=G(t_{1,1})$$

$$q(1)=G\left(t_{1+\frac{1}{2},1+\frac{1}{2}}\right)+\frac{h}{2}K\left(t_{1+\frac{1}{2}},t_1,s_{1+\frac{1}{2}},s_1\right)p(1)$$

$$z(1)=G\left(t_{1+\frac{1}{2},1+\frac{1}{2}}\right)+\frac{h}{2}K\left(t_{1+\frac{1}{2}},t_{1+\frac{1}{2}},s_{1+\frac{1}{2}},s_{1+\frac{1}{2}}\right)q(1)$$

$$s(1)=G(t_{2,2})+hK\left(t_2,t_2,s_2,s_2\right)z(1)$$

For $j = 2,3,\ldots,n-1$

$$u_j = G_j\left(t_{j+1,j+1}\right)+\frac{h}{6}\left[K\left(t_{j+1},t_j\right)p(j)+2K\left(t_{j+1},t_{j+\frac{1}{2}},s_{j+1},s_{j+\frac{1}{2}}\right)\left(q(j)+z(j)\right)+K\left(t_{j+1},t_{j+1},s_{j+1},s_{j+1}\right)s(j)\right] \quad (31)$$

Where

$$p(j) = u(j-1)$$

$$q(j) = G_j\left(t_{j+\frac{1}{2},j+\frac{1}{2}}\right)+\frac{h}{2}K\left(t_{j+\frac{1}{2}},t_j,s_{j+\frac{1}{2}},s_j\right)p(j)$$

$$z(j) = G_j\left(t_{j+\frac{1}{2},j+\frac{1}{2}}\right)+\frac{h}{2}K\left(t_{j+\frac{1}{2}},t_{j+\frac{1}{2}},s_{j+\frac{1}{2}},s_{j+\frac{1}{2}}\right)q(j)$$

$$s(j) = G_j\left(t_{j+1,j+1}\right)+hK\left(t_{j+1},t_{j+1},s_{j+1},s_{j+1}\right)z(j)$$

3.2 BBM

Let us assume that we have decided to use Simpson's rule as the numerical integration formula. If we knew $u_{1,1}$, then we could simply compute $u_{2,2}$ by

$$u_{2,2} = G(t_{2,2})+\frac{h}{3}\left\{K\left(t_2,t_0,s_2,s_0,u_{0,0}\right)+4K\left(t_2,t_1,s_2,s_1,u_{1,1}\right)+K\left(t_2,t_2,s_2,s_2,u_{2,2}\right)\right\} \quad (32)$$

Then use Simpson's rule with $u_{0,0}, u_{1/2,1/2}$, and $u_{1,1}$ to give

$$u_{1,1} = G(t_{1,1})+\frac{h}{6}\left\{K\left(t_1,t_0,s_1,s_0,u_{0,0}\right)+4K\left(t_1,t_{1/2},s_1,s_{1/2},u_{1/2,1/2}\right)+K\left(t_1,t_1,s_1,s_1,u_{1,1}\right)\right\} \quad (33)$$

That is, we replace $u_{1/2,1/2}$ by

$$u_{1/2,1/2} = \frac{3}{8}u_{0,0}+\frac{3}{4}u_{1,1}-\frac{1}{8}u_{2,2}$$

So we can compute $u_{1,1}$ by

$$u_{1,1} = G(t_{1,1})+\frac{h}{6}\left\{K\left(t_1,t_0,s_1,s_0,u_{0,0}\right)+4K\left(t_1,t_{1/2},s_1,s_{1/2},3/8u_{0,0}+3/4u_{1,1}-1/8u_{2,2}\right)+K\left(t_1,t_1,s_1,s_1,u_{1,1}\right)\right\} \quad (34)$$

The general process should now be clear; for $m = 0,1,2,\ldots$ we compute the approximate solution by

$$u_{2m+1,2m+1} = G\left(t_{2m+1,2m+1}\right)+h\sum_{i=0}^{2m}W_iK\left(t_{2m+1},t_i,s_{2m+1},s_i,u_{i,i}\right)+\frac{h}{6}\left\{K\left(t_{2m+1},t_{2m},s_{2m+1},s_{2m},u_{2m,2m}\right)\right.$$

$$\left.+4K\left(t_{2m+1},t_{2m+1/2},s_{2m+1},s_{2m+1/2}\,3/8u_{2m,2m}+3/4u_{2m+1+2m+1}-1/8u_{2m+2+2m+2}\right)+K\left(t_{2m+1},t_{2m+1},s_{2m+1},s_{2m+1}u_{2m+1,2m+1}\right)\right\} \quad (35)$$

$$u_{2m+2,2m+2} = G\left(t_{2m+2+2m+2}\right)+h\sum_{i=0}^{2m}W_iK\left(t_{2m+2},t_i,s_{2m+2},s_i,u_{i,i}\right)+\frac{h}{3}\left\{K\left(t_{2m+2},t_{2m},s_{2m+2},s_{2m},u_{2m,2m}\right)\right.$$

$$+4K\left(t_{2m+2},t_{2m+1},s_{2m+2},s_{2m+1},u_{2m+1,2m+1}\right)+K\left(t_{2m+2},t_{2m+2},s_{2m+2},s_{2m+2},u_{2m+2,2m+2}\right)\}$$

4. NUMERICAL RESULTS

We apply RKM and BBM to solve Eq. (22), we take $k(t,s;s_1,s_2)=t^2s^2.s_1s_2$, under the condition $u(0,0)=0$, and the exact solution $u(t,s)=t.s$. We divided the interval by N=20, 50 units. Since $0\le t\le T<\infty$, we choose the time T = 0.1, 0.3. **In tables (1-4):** ***Exact sol.→ exact solution,*** ***Appro. R.K.*** →approximate solution of **RKM**, ***Error. R.K.*** → the absolute error of **RKM**, ***Appro. B.B.***→approximate solution of **BBM**, ***Error. B.B.*** → the absolute error of **BBM**,

Table (1). The exact, approximate solutions, and the errors at ***T*=0.1, *N*=20**

x	*Exact sol.*	*Appro. R.K.*	*Error. R.K.*	*Appro. B.B.*	*Error. B.B.*
0	0	0	0	0	0
0.1	0.0010000	0.0014997	4.9997×10^{-4}	0.0009999	2.4161×10^{-8}
0.2	0.0040000	0.0049978	9.9978×10^{-4}	0.0039996	3.853×10^{-7}
0.3	0.0090000	0.0104922	1.4922×10^{-3}	0.0089980	1.9247×10^{-6}
0.4	0.0160000	0.0179805	1.9805×10^{-3}	0.0159941	5.8591×10^{-6}
0.5	0.0250000	0.0274608	2.4608×10^{-3}	0.0249868	1.3152×10^{-5}
0.6	0.0360000	0.0389329	2.9329×10^{-3}	0.0359770	2.2927×10^{-5}
0.7	0.0490000	0.0524020	3.4020×10^{-3}	0.0489707	2.9217×10^{-5}
0.8	0.0640000	0.0679851	3.8851×10^{-3}	0.0639848	1.5121×10^{-5}
0.9	0.0810000	0.0854199	4.4199×10^{-3}	0.0810568	5.682×10^{-5}
1	0.1000000	0.1048178	4.8178×10^{-3}	0.1000083	8.35×10^{-6}

Table (2). The exact, approximate solutions, and errors at ***T*=0.3, *N*=20**

x	*Exact sol.*	*Appro. R.K.*	*Error. R.K.*	*Appro. B.B.*	*Error. B.B.*
0	0	0	0	0	0
0.1	0.0030000	0.0049753	1.4975×10^{-3}	0.0029997	2.1745×10^{-7}
0.2	0.0120000	0.0014980	$2,9804\times10^{-3}$	0.0119965	3.4685×10^{-6}
0.3	0.0270000	0.0314308	4.4308×10^{-3}	0.0269826	1.7322×10^{-5}
0.4	0.0480000	0.0538289	5.8289×10^{-3}	0.0479472	5.2734×10^{-5}
0.5	0.0750000	0.0821593	7.1593×10^{-3}	0.0748816	1.1838×10^{-4}
0.6	0.1080000	0.1164275	8.4275×10^{-3}	0.1077935	2.0644×10^{-4}
0.7	0.1470000	0.1566904	9.6904×10^{-3}	0.1467367	2.6328×10^{-4}
0.8	0.1920000	0.2031120	1.1112×10^{-2}	0.1918634	1.3654×10^{-4}
0.9	0.2430000	0.2560541	1.3054×10^{-2}	0.2435136	5.1366×10^{-4}
1	0.3000000	0.3138329	1.3832×10^{-2}	0.3000744	7.4446×10^{-5}

Table (3). The exact, approximate solutions, and errors at ***T*=0.1, *N*=50**

x	***Exact sol.***	***Appro. R.K.***	***Error. R.K.***	***Appro. B.B.***	***Error. B.B.***
0	0	0	0	0	0
0.1	0.0010000	0.0011999	1.9990×10^{-4}	0.0009999	9.8647×10^{-9}
0.2	0.0040000	0.0043991	3.9917×10^{-4}	0.0039998	1.5735×10^{-7}
0.3	0.0090000	0.0095968	5.9698×10^{-4}	0.0089992	7.8607×10^{-7}
0.4	0.0160000	0.0167923	7.9237×10^{-4}	0.0159976	2.3948×10^{-6}
0.5	0.0250000	0.0259844	9.8448×10^{-4}	0.0249946	5.3855×10^{-6}
0.6	0.0360000	0.0371732	1.1732×10^{-3}	0.0359905	9.4288×10^{-6}
0.7	0.0490000	0.0503606	1.3606×10^{-3}	0.0489878	1.2163×10^{-5}
0.8	0.0640000	0.0655530	1.5530×10^{-3}	0.0639931	6.8630×10^{-6}
0.9	0.0810000	0.0827654	1.7654×10^{-3}	0.0810213	2.1380×10^{-5}
1	0.1000000	0.1019246	1.9246×10^{-3}	0.1000013	1.3344×10^{-6}

Table (4). The exact, approximate solutions, and errors at ***T*=0.3, *N*=50**

x	***Exact sol.***	***Appro. R.K.***	***Error. R.K.***	***Appro. B.B.***	***Error. B.B.***
0	0	0	0	0	0
0.1	0.0030000	0.0035991	5.9912×10^{-4}	0.0029999	8.8781×10^{-8}
0.2	0.0120000	0.0013192	1.1925×10^{-3}	0.0119985	1.4161×10^{-6}
0.3	0.0270000	0.0287733	1.7733×10^{-3}	0.0269929	7.0746×10^{-6}
0.4	0.0480000	0.0503338	2.3338×10^{-3}	0.0479784	2.1553×10^{-5}
0.5	0.0750000	0.0778692	2.8692×10^{-3}	0.0749515	4.8471×10^{-5}
0.6	0.1080000	0.1113838	3.3838×10^{-3}	0.1079151	8.4876×10^{-5}
0.7	0.1470000	0.1509044	3.9044×10^{-3}	0.1468904	1.0951×10^{-4}
0.8	0.1920000	0.1965019	4.5019×10^{-3}	0.1919381	6.1870×10^{-5}
0.9	0.2430000	0.2483267	5.3267×10^{-3}	0.2431926	1.9269×10^{-4}
1	0.3000000	0.3057405	5.7405×10^{-3}	0.3000111	1.1596×10^{-5}

5. THE CONCLUSION

This paper proposed an effective two numerical methods to obtain the solution of Two-dimensional Volterra integrodifferential equation (**TDVIDE**). For this purpose, RKM and BBM for solving Two-dimensional Volterra integrodifferential equation (**TDVIDE**), which is arising of the heat conduction and diffusion problems has been presented. The numerical examples showed the efficiency and accuracy of the introduced methods. From the previous results, we note that the values of absolute error are increasing when the time is increasing. In addition, the values of errors due to the BBM are less than the values of errors due to RKM. In addition, when the N is increasing the values of absolute error are decreasing.

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