Numerical simulations of stochastic biochemical oxygen demand equations via RBF method

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Research Article

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Abstract

In this paper, it is proposed a new approach to numerical simulations of stochastic biochemical oxygen demand equations. The analysis of biochemical oxygen demand is necessary in evaluating the effects of water pollution. Here, we consider radial basis functions. This approximation process can also be interpreted as a simple kind of neural network. Illustrative example is included to demonstrate the validity and applicability of the approach. The numerical experiments show that method perform well.

AMS subject classifications: Primary: 65C30, 60H35, 65C20, Secondary: 60H20, 92E99.

1 Introduction

White noise is a widespread and common phenomenon in many engineering, biological, economics and chemical models involving some form of error prediction. In these types of applications, the deterministic model is often already available and stochastic models are an active area of researches.

In the last decades, the use of mathematical models describing wastewater treatment is taken into consideration. Biochemical Oxygen Demand (BOD) is a measure of the dissolved oxygen consumed by microorganisms during the oxidation of reduced substances in waters and wastes. The analysis and forecast of BOD are vital and essential in assessing the effects of water pollution. The purpose of this paper is to extend Radial Basis Functions (RBFs) method on stochastic models for BOD in a more stochastic and therefore more realistic point of view. The final model will be used to predict the amount of BOD at any point on the stream [1–4].

The biochemical oxygen demand model presented in this paper is often used in water quality determination of river and estuarine systems and is an extension of the so-called StreeterPhelps model. In this model, the concentration levels of Carbonaceous Biochemical Oxygen Demand (CBOD), Dissolved Oxygen (DO), and Nitrogenous Biochemical Oxygen Demand (NBOD) are related to each other by a set of differential equations. Although the concentrations are time dependent, the purpose behind the model is often to monitor the concentration levels within a fixed volume of water, flowing downstream a river. An underlying assumption is that the velocity of the water flow is constant and thus time and distance are linearly related [5–7].

This paper establishes a direct method for solving system of stochastic differential equations via a set of RBFs. The approximation using RBF is an extremely powerful method to interpolation of functions. Sums of radial basis functions are typically used to approximate given functions. The RBF methodology was first introduced in 1971 and have been variously studied [8–10]. Recently, these functions have also been used in the numerical solution of stochastic equations, Also, they used in solving equation arisen in financial mathematics [11–14]. This method has been applied for solving different kinds of equations, problems and models such as PDEs, IEs and ODEs arisen in physics, biology and neural network etc.

The chapters of this paper are organized as follows. In next section, radial basis functions and the approach in solving stochastic Volterra integral equations are reviewed. In Section 3, it is introduced a model that we want to solve in this paper and our numerical findings are reported. In Section 4, the error of the RBFs method for the linear integral equations is discussed. Finally, Section 5 conclusions of this article are stated.

2 Rbf

The goal of this section is to recall notations and definitions of RBFs that are used for approximating the solutions of BOD equations. Also, applying the method for solving stochastic integral equations is described.

Definition 2.1

A function $\Phi(t): R^n \to R$ is called radial if there exists a one variable function $\varphi: (0, \infty) \to R$ such that $\Phi(t) = \varphi(\|$, where \left\| {\begin{array}{*{20}{c}} . \end{array}} \right\| is the Euclidean norm. A radial basis function \varphi (r)is a univariate continuous real valued function which depends on the distance from the origin (or any other fixed center point).

Definition 2.2

A set of of RBF depend to\varphi :(0,\infty) \to R is defined as follow.

{\varphi _i}(t)=\varphi (\left\| {t - {t_i}} \right\|),\begin{array}{*{20}{c}} {}&{i=1,2,...,N,} \end{array}

Where {\varphi _i}:{R^n} \to R and {t}_{i} for i=1,2,...,Nis the center of RBFs. Additionally,t, {t}_{i} are belong to {R}^{n}.

RBFs are mostly identified on the basis of smoothness. Some functions are infinitely smooth and some are piecewise smooth. Gaussian Function (GS), Multiquadric (MQ), Inverse Multi- quadric (IMQ) and Inverse quadric (IQ) are some example of infinitely smooth RBFs where as Thin Plate Spline (TPS) and Linear radial function (LR) are piecewise smooth RBFs. For infinitely smooth RBFs, there exists a free parameter called the shape parameter which controls the shape of RBF. The RBF become flat if shape parameter is closer to 0.

Let \left\{ {{t_1},{t_2},...,{t_N}} \right\} \subset {R^n} be a given set of distinct nodal points. To approximate a function y(t) using the radial function \Phi (t)=\varphi (\left\| t \right\|) we can give the following linear combination:

y(t) \cong {y_N}(t)=\sum\limits_{{j=1}}^{N} {{\lambda _j}} \varphi (\left\| {t - {t_j}} \right\|)

where the coefficients {\lambda _1},{\lambda _2},...,{\lambda _N} are determined by the interpolation conditions

$${y_N}({t_i})=y({t_i})={y_i},~~~~~~~~~~~~~~i=1,...,N. \quad (1)$$

Therefore, the solution of the interpolation problem based on the extended expansion(1) reduces to the solution of a system of linear equations of the matrix form

$$A\lambda =y,$$

where the pieces are given by $~~{A_{jk}}=\varphi \left({\left\| {{t_j} - {t_k}} \right\|} \right),j,k=1,2,...,N,\lambda ={\left[{{\lambda _1},...,{\lambda _N}} \right]^T}$ and $y={\left[{{y_1},...,{y_N}} \right]^T}$.

2.1 Description of Method on stochastic Volterra integral equation

The nonlinear stochastic Volterra integral equation takes the following form:

$$u(t)={u_0}(t)+\int\limits_{0}^{t} {{k_1}(s,{\text{ }}t)u(s)ds} +\int\limits_{0}^{t} {{k_2}(s,t)u(s)dB(s)} \begin{array}{*{20}{c}} {}&{} \end{array}~t \in ~[0,T], \quad (2)$$

where, $u\left(t\right)$, ${\text{u}}_{0}\left(t\right)$, ${k}_{1}(s, t)$ and ${k}_{2}(s, t)$,for $s, t \in [0, T)$,are the stochastic processes defined on the same probability space $({\Omega }, F, P)$ with a filtration $\{{F}_{t}, t \ge 0\}$ that is increasing and right continuous and ${F}_{0}$ contains all P-null sets.$u\left(t\right)$ is unknown random function and $B\left(t\right)$ is a standard Brownian motion defined on the probability space and $\int\limits_{0}^{t} {{k_2}(s,t)u(s)dB(s)}$ is the Itˆo integral.

Let’s approximate the function $u\left(t\right)$ in terms of radial basis functions,$\varphi \left(t\right)$, as follows

$$u(t) \simeq ~\sum\limits_{{i=1}}^{N} {{\lambda _i}\varphi (\parallel t - {t_i}\parallel)} ~~~~ \quad (3)$$

Then, from substituting Eq. (3) into Eq. (2) we have,

$$\sum\limits_{{i=1}}^{N} {} {\lambda _i}\varphi (\parallel t - {t_i}\parallel)={u_0}(t)+\int_{0}^{t} {{k_1}(s,t)~} \sum\limits_{{i=1}}^{N} {} {\lambda _i}\varphi (\parallel s - {t_i}\parallel)ds+$$

$$\int_{0}^{t} {} {k_2}(s,{\text{ }}t)\sum\limits_{{i=1}}^{N} {} {\lambda _i}\varphi (\parallel s - {t_i}\parallel)dB(s),\begin{array}{*{20}{c}} {}&{} \end{array}~t \in [0,T],~~~~~~~~~~~~~~~~~~~~~~~~~~\left(4 \right)$$

Substituting the collocation points $\begin{array}{*{20}{c}} {{t_j}} \end{array}_{{j=1}}^{N}$ into Eq. (4), we obtain:

$$\sum\limits_{{i=1}}^{N} {} {\lambda _i}\varphi (\parallel {t_j} - {t_i}\parallel)={u_0}({t_j})+\int_{0}^{{{t_j}}} {{k_1}(s,{t_j})~} \sum\limits_{{i=1}}^{N} {} {\lambda _i}\varphi (\parallel s - {t_i}\parallel)ds+$$

$$\int_{0}^{{{t_j}}} {} {k_2}(s,{\text{ }}{t_j})\sum\limits_{{i=1}}^{N} {} {\lambda _i}\varphi (\parallel s - {t_i}\parallel)dB(s),\begin{array}{*{20}{c}} {}&{} \end{array}~t \in [0,T],~~~j=1,...,N.~~~~~~~~~~~~~~~~~~\left(5 \right)$$

In above equation, we let $s=\frac{{t}_{j}}{2}x+\frac{{t}_{j}}{2}$. It reduces Eq. (5) to the following equation

$$\sum\limits_{{i=1}}^{N} {} {\lambda _i}\varphi (\parallel {t_j} - {t_i}\parallel)={u_0}({t_j})+\frac{{{t_j}}}{2}\int_{{ - 1}}^{1} {{k_1}(\frac{{{t_j}}}{2}x+\frac{{{t_j}}}{2},{t_j})~} \sum\limits_{{i=1}}^{N} {} {\lambda _i}\varphi (\parallel \frac{{{t_j}}}{2}x+\frac{{{t_j}}}{2} - {t_i}\parallel)dx+$$

$$\int_{0}^{{{t_j}}} {} {k_2}(s,{\text{ }}{t_j})\sum\limits_{{i=1}}^{N} {} {\lambda _i}\varphi (\parallel s - {t_i}\parallel)dB(s),\begin{array}{*{20}{c}} {}&{} \end{array}~t \in [0,T],~~~j=1,...,N.~~~~~~~~~~~~~~~~~~~~~~~~~\left(6 \right)$$

For computing or approximating the first integral, several methods can be used. Now, by applying Legendre-Gauss-Lobatto integration formula, we approximate the first integral of Eq. (6) as follow

$$\sum\limits_{{i=1}}^{N} {} {\lambda _i}\varphi (\parallel {t_j} - {t_i}\parallel)={u_0}({t_j})+\frac{{{t_j}}}{2}\sum\limits_{{k=1}}^{N} {{w_k}{k_1}(\frac{{{t_j}}}{2}{x_k}+\frac{{{t_j}}}{2},{t_j})} \sum\limits_{{i=1}}^{N} {} {\lambda _i}\varphi (\parallel \frac{{{t_j}}}{2}{x_k}+\frac{{{t_j}}}{2} - {t_i}\parallel)+$$

$$\int_{0}^{{{t_j}}} {} {k_2}(s,{\text{ }}{t_j})\sum\limits_{{i=1}}^{N} {} {\lambda _i}\varphi (\parallel s - {t_i}\parallel)dB(s),\begin{array}{*{20}{c}} {}&{} \end{array}~t \in [0,T],~~~j=1,...,N.~~~~~~~~~~~~~~~~~~~~~~~~~\left(7 \right)$$

where ${w}_{k}$, ${x}_{k}$, k = 1, ..., N.are weights and nodes of the integration rule respectively.

For approximating the Itˆo integral, let $0={s_0}<{s_1}<...<{s_m}=T$. Given a suitable function , the integral

$$\int\limits_{0}^{T} {f(t)dt} ,$$

may be approximated by the Riemann sum

$$\sum\limits_{{j=0}}^{{N - 1}} {f({s_j}})({s_{j+1}} - {s_j})$$
8

where the discrete points ${s_j}=j\delta t$ Indeed, the integral may be defined by taking the limit $\delta t \to 0$ in (8). In a similar way, we may consider a sum of the form

$$\sum\limits_{{j=0}}^{{N - 1}} {f({s_j}})(B({s_{j+1}}) - B({s_j})),$$

which, by analogy with (8), may be regarded as an approximation to a stochastic integral

$$\int\limits_{0}^{T} {f(t)dB(t)=} \sum\limits_{{j=0}}^{{N - 1}} {f({s_j}})(B({s_{j+1}}) - B({s_j})),$$
9

Here, we are integrating with respect to Brownian motion.

In Eq. (6) for Itˆo integral, let $0={s_0}<{s_1}<...<{s_m}={t_0}$. So, achieved

$$\sum\limits_{{i=1}}^{N} {} {\lambda _i}\varphi (\parallel {t_j} - {t_i}\parallel)={u_0}({t_j})+\frac{{{t_j}}}{2}\sum\limits_{{k=1}}^{N} {{w_k}{k_1}(\frac{{{t_j}}}{2}{x_k}+\frac{{{t_j}}}{2},{t_j})} \sum\limits_{{i=1}}^{N} {} {\lambda _i}\varphi (\parallel \frac{{{t_j}}}{2}{x_k}+\frac{{{t_j}}}{2} - {t_i}\parallel)+$$

$$\sum\limits_{{k=0}}^{{m - 1}} {} {k_2}({s_k},{\text{ }}{t_j})\sum\limits_{{i=1}}^{N} {} {\lambda _i}\varphi (\parallel {s_k} - {t_i}\parallel)\left[{B({s_{k+1}}) - B({s_k})} \right],\begin{array}{*{20}{c}} {}&{} \end{array}~t \in [0,T],~~~j=1,...,N.~~~~~~~~~~\left({10} \right)$$

We have a system of equations that can be solved by mathematical software for the unknowns vector λ. By computing that, unknown function $u\left(t\right)$ can be approximated.

3 Stochastic Biochemical Oxygen Demand Model

Biochemical oxygen demand (BOD) model is a model for the concentration of oxygen needed by aerobic microorganisms. The concentration is purposely for stabilization of waste water organic matters. The water quality in bodies of water is generally measured by BOD and oxygen concentration. Let b, o and n be the concentration levels, measured in mg/l, for Carbonaceous Biochemical Oxygen Demand (CBOD), Dissolved Oxygen (DO) and Nitrogenous Biochemical Oxygen Demand (NBOD) respectively [6, 15]. The CBOD, DO and NBOD are defined by the following differential equation

$$\begin{gathered} \frac{{db}}{{dt}}= - {k_b}b+{s_1}, \hfill \\ \frac{{do}}{{dt}}= - {k_c}b - {k_2}o - {k_n}n+s_{2}^{*}, \hfill \\ \frac{{dn}}{{dt}}= - {k_n}n+{s_3}, \hfill \\ \end{gathered}$$
11

Where, ${k_b}={k_c}+{k_3},s_{2}^{*}={k_2}{d_s}+{p_0} - {r_0}+{s_2}$, ${k_c}$is the reaction rate coefficient,${k_2}$is the reaeration rate coefficient, k_3 is the sedimentation and adsorption loss rate for CBO, ${k_n}$ is the decay rate of NBOD, p_0 is photosynthesis of oxygen, r_0 is the respiration of oxygen, d_s is the saturation concentration of oxygen, s_1 is the independent source for CBOD, s_2 is the independent source for OD, and s_3 is the independent source for NBOD. As for many environmental models, BOD is also subject to various uncertainties. These uncertainties can be incorporated into the model by adding white noise processes to each of the three equations of the model. For this equation assumes that we can model uncertaties in the source terms (s_1, s_2, s_3) by adding a random noise factor to each of three equations. The resulting linear SDE is [15]:

$$\begin{gathered} \frac{{db}}{{dt}}= - {k_b}b+{s_1}+{\sigma _b}d{B_1}, \hfill \\ \frac{{do}}{{dt}}= - {k_c}b - {k_2}o - {k_n}n+s_{2}^{*}+{\sigma _o}d{B_2}, \hfill \\ \frac{{dn}}{{dt}}= - {k_n}n+{s_3}+{\sigma _n}d{B_3}, \hfill \\ \end{gathered}$$
12

where $B_i(t)$ are independent standard Brownian motions and ${\sigma _b}$, ${\sigma _o}$, and ${\sigma _n}$ are diffusion coefficients in the noises of CBOD, OD and NBOD respectively and represent the intensities of $B_i(t)$, $i = 1, 2, 3$. Usually stochasticity is introduced into the model by making simple assumption about the random nature of coefficient. If, we can assume that the coefficients ${k_b}$, ${k_2}$ and ${k_n}$ have random nature, the equation can be written as follow:

$$\begin{gathered} \frac{{db}}{{dt}}= - {k_b}b+{s_1}+{\sigma _b}bd{B_1}, \hfill \\ \frac{{do}}{{dt}}= - {k_c}b - {k_2}o - {k_n}n+s_{2}^{*}+{\sigma _o}od{B_2}, \hfill \\ \frac{{dn}}{{dt}}= - {k_n}n+{s_3}+{\sigma _n}nd{B_3}, \hfill \\ \end{gathered}$$
13

for more information about this model see [6].

By integration of Eq. (13) the method presented in Section 2 is applied for the following system of equations:

$$\begin{gathered} b(t)={b_0}(t) - \int_{0}^{t} {{k_b}} b(s)ds+\int_{0}^{t} {{s_1}} ds+\int_{0}^{t} {{\sigma _b}bd{B_1}} (s), \hfill \\ o(t)={o_0}(t) - \int_{0}^{t} {{k_c}} b(s)ds - \int_{0}^{t} {{k_2}} o(s)ds - \int_{0}^{t} {{k_n}} n(s)ds+\int_{0}^{t} {s_{2}^{*}(s)} ds+\int_{0}^{t} {{\sigma _0}od{B_2}} (s), \hfill \\ n(t)={n_0}(t) - \int_{0}^{t} {{k_n}} n(s)ds+\int_{0}^{t} {{s_3}(s)} ds+\int_{0}^{t} {{\sigma _n}nd{B_3}} (s). \hfill \\ \end{gathered}$$
14

We approximate the functions based on RBF method and after solving the system, $b\left(t\right)$, $o\left(t\right)$ and $n\left(t\right)$ are approximated.

3.1 Numerical example

The parameter values used in the simulations are defined in the Table (1). In order to conform the results above, initial value $b\left(0\right) = 15$, $o\left(0\right) = 8.5$ and $n\left(0\right) = 5$ are chosen and parameters corresponding to Table (1). Then $b\left(t\right)$, $o\left(t\right)$ and $n\left(t\right)$ are shown in the Figs. 1.

4 Conclusion

Some system of stochastic differential equations cannot be solved analytically; in this article we present a new technique for solving them numerically. In this paper, we used Radial Basis Functions for approximating the solution of the BOD equations. Such nonlinear models occur in financial epidemiology, and biology. For solving Riemann integral part Legendre-Gauss-Lobatto integration formula was applied, then the problem was discretized. The proposed method reduces an integral equation to a system of equations. Implementation of our method is easy and accuracy of that has been verified by test examples. Efficiency of this method and good degree of accuracy was confirmed by a numerical example.

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Tables

Table 1: The parameter values used in simulations of BOD model.

n	k_c	k_2	k_3	k_n	σ_b	σ_o	σ_n	s_1	s_2^*	s_3
50	0.763	4.250	0.254	0.978	1.5	1.5	1	3	42.03	0

Figures

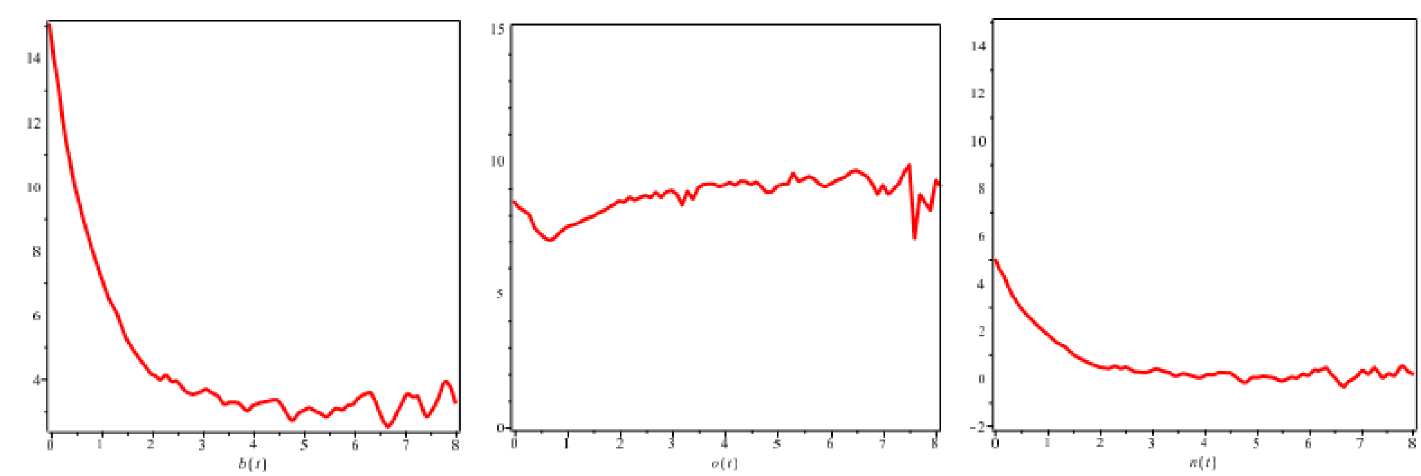

Figure 1

The trajectory of the approximate solution by RBF method for simulating the BOD example