Optical solitons of Zoomeron equation by newly 6-model expansion approach

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Research Article

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Optical solitons of Zoomeron equation by newly φ^6-model expansion approach

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Abstract

One of the equations describing incognito evolution, the nonlinear Zoomeron equation, is studied in this work. In a variety of physical circumstances, including laser physics, fluid dynamics and nonlinear optics, solitons with particular properties arise and the Zoomeron equation is a single example of one such situation. The method of φ6-model expansion allows for the explicit retrieval of a wide range of solution types, including kink-type solitons, these solitons are also called topological solitons in the context of water waves, their velocity does not depend on the wave amplitude, others are bright, singular, periodic, and combined singular soliton solutions. The outcomes of this research may improve the Zoomeron equation's nonlinear dynamical features. The method proposes a practical and effective approach for solving a large class of nonlinear partial differential equations. Interesting graphs are employed to explain and highlight the dynamical aspects of the results, all of the obtained results are put into the Zoomeron equation to show the accuracy of the results.

Keywords: Zoomeron equation, the φ^6-model expansion method, kink soliton, traveling wave solution, Jacobi elliptic solutions.

1 Introduction

The study of surfaces in geometry [1–5] and a wide range of issues in mechanics are where partial differential equations first appeared. Notable mathematicians from all over the world got actively interested in the study of a wide range of issues brought on by partial differential equations in the late 19th century. This study's main motivation was the fact that partial differential equations (PDEs) regularly appear in the mathematical analysis of a range of problems in science and engineering and describe many fundamental natural principles [6]. It is now incredibly beneficial to look for precise answers to the both nonlinear evolution equations and partial differential equations NLEEs using a variety of techniques, and there are numerous effective techniques, such the inverse scattering transform approach [7], the Homoclinic technique [8], the sinh-Gordon function method [9], the generalized exponential rational function method [10], the auxiliary equation method [11], An alternative method [12], the Bernoulli sub-equation function method [13, 14], the sub-equation analytical method [15], the modified sub-equation method [16], the auto-Bäcklund transformation method [17] and so on.

This study focuses on the nonlinear evolutionary Zoomeron equation. The basic Zoomeron equation was first developed by Calogero and Degasperis. When they investigated the one-dimensional Schrodinger equation and an extension of the well-known KdV equation to depict solitons that travel at different speeds and found a connection between their polarization effects and speed, they made significant progress in 1976. Two types of solitons were produced as a result, the first of which was an accelerated soliton that traveled from one side in the distant past to the opposite side with the same speed in the distant future. The second was a trapped soliton that oscillated repeatedly in the direction around a fixed point in space. The first was called Boomeron, while the second was called Trappon, which led to the derivation of the standard Zoomeron equation. [18].

The Zoomeron model, one of the incognito evolution equations, has been investigated via many direct methods. Among the significant ones are; Ghazala et al. used the modified auxiliary equation and Generalized projective Riccati equation method [18] to obtain solitary wave, dark peakon, bright and kink-type wave solution, the $\left(\frac{G'}{G}, \frac{1}{G}\right)$ −expansion approach [19] is utelized by Elsayed et al. to retrieved new solitary wave solutions, Tanki et al. examined the classical Lie point symmetries [20] of the model, four exact particular solutions are obtained including rational and periodic solutions using the $\left(\frac{\Phi(\xi)}{2}\right)$ −expansion method [21] by Jalil et al., the Yan's sine-cosine method and Wazwaz's sine-cosine method [22] are used by Hua to derive new exact traveling wave solutions, the new extended direct algebraic method [23] is used to obtain new analytical solutions by Wei et. al, the first integral method [24, 25], the hyperbolic trigonometric and rational function solutions are retrieved via the $\left(\frac{G'}{G}\right)$ −expansion approach [26] by Reza, the Lie point transformation method [27], Aminah used the sine-cosine function method [28] to construct the trigonometric wave solutions, Higazy et al. implement the extended simple equation method [29] to get solitary wave solutions, the Modified simple equation method and Exp-function method [30] are used to obtain dark, trigonometric and hyperbolic soliton solutions. The Zoomeron model is studied in this research using the newly developed φ^6–model expansion method [31–34]), which results in the restoration of optical solitary wave solutions.

The plan for this work is provided below. In Section 2, a presentation of the φ^6-model expansion method will be provided. The Zoomeron equation will be developed using the φ6-method in section 3 to provide novel traveling wave solutions to the equation. Additionally, the associated 3D, 2D, and density graphs clearly illustrate the physical structure of the traveling wave solution. In part 4, the soliton solutions' physical dynamics are examined, and in section 5, conclusions are reached.

2 Description of the method

According to [31–34], the steps involves for the φ^6-model expansion technique are given as:

Step-1: Assuming the nonlinear evolution equation (NLEE) for $Q = Q(x, y, t)$ is in the form.

$$H(Q, Q_x, Q_t, Q_{xx}, Q_{xt}, ...) = 0, \tag{1}$$

here H is a polynomial of $Q(x, t)$ which involves highest order partial derivatives and its nonlinear terms.

Step-2: By using the wave transformation

$$Q(x, t) = Q(\zeta), \qquad \zeta = x + cy - wt, \tag{2}$$

where w represents wave speed and Eq.(1) can be converted into the nonlinear ordinary differential equation shown below.

$$\Omega(Q, Q', QQ', Q'', ...) = 0, \tag{3}$$

where the derivatives with respect to ζ are shown by prime. **Step-3**: Suppose that the formal solution to Eq.(3) exists:

$$Q(\zeta) = \sum_{j=0}^{2M} \alpha_i A^j(\zeta), \tag{4}$$

M can be gotten using the balancing rule, $\alpha_j (j = 0, 1, 2, \ldots, M)$ are to be determined constants and $A(\zeta)$ satisfies the auxiliary NLODE;

$$A'^2(\zeta) = h_0 + h_2 A^2(\zeta) + h_4 A^4(\zeta) + h_6 A^6(\zeta), \tag{5}$$
$$A^{''}(\zeta) = h^2 A(\zeta) + 2h_4 A^3(\zeta) + 3h_6 A^5(\zeta),$$

here $h_j (j = 0, 2, 4, 6)$ are real constants that will be found later.

Step-4: It is known that the solution to Eq.(5) is given as;

$$U(\zeta) = \frac{P(\zeta)}{\sqrt{fP^2(\zeta) + g}}, \tag{6}$$

$P(\zeta)$ is the Jacobi elliptic equation solution, provided that $0 < fP^2(\zeta) + g$

$$P'^2(\zeta) = l_0 + l_2 P^2(\zeta) + l_4 P^4(\zeta), \tag{7}$$

where $l_j (j = 0, 2, 4)$ are unknown constants to be determined, g and f are given by

$$f = \frac{h_4(l_2 - h_2)}{(l_2 - h_2)^2 + 3l_0 l_4 - 2l_2(l_2 - h_2)}, \tag{8}$$
$$g = \frac{3l_0 h_4}{(l_2 - h_2)^2 + 3l_0 l_4 - 2l_2(l_2 - h_2)},$$

under the restricted condition

$$h_4^2(l_2 - h_2)[9l_0 l_4 - (l_2 - h_2)(2l_2 + h_2)] + 3h_6[-l_2^2 + h_2^2 + 3l_0 l_4]^2 = 0. \tag{9}$$

Step-5: The Jacobi elliptic solutions of Eq.(7) can be calculated when $0 < m < 1$, the exact solutions of Eq.(1) can be derived by substituting Eq.(6) and Eq.(7) into Eq.(4).

Function	$m \to 1$	$m \to 0$	Function	$m \to 1$	$k \to 0$
$sn(\zeta, m)$	$tanh(\zeta)$	$sin(\zeta)$	$ds(\zeta, m)$	$csch(\zeta)$	$csc(\zeta)$
$cn(\zeta, m)$	$sech(\zeta)$	$cos(\zeta)$	$sc(\zeta, m)$	$sinh(\zeta)$	$tan(\zeta)$
$dn(\zeta, m)$	$sech(\zeta)$	1	$sd(\zeta, m)$	$sinh(\zeta)$	$sin(\zeta)$
$ns(\zeta, m)$	$coth(\zeta)$	$csc(\zeta)$	$nc(\zeta, m)$	$cosh(\zeta)$	$sec(\zeta)$
$cs(\zeta, m)$	$csch(\zeta)$	$cot(\zeta)$	$cd(\zeta, m)$	1	$cos(\zeta)$

3 Application of the proposed method to the Zoomeron equation

The φ^6-model expansion method, which was explained in Part 3, will be used in this section to retrieve the exact solitary wave solutions of the nonlinear Zoomeron equation.

$$\left(\frac{Q_{xy}}{Q}\right)_{tt} - \left(\frac{Q_{xy}}{Q}\right)_{xx} + 2\left(Q^2\right)_{xt} = 0, \tag{10}$$

where the amplitude of the pertinent wave mode is Q and $Q = Q(x, y, t)$. Eq.(10) is reduced to the following ODE using the traveling wave transformation $Q(x, y, t) = Q(\zeta) = Q(x + cy - wt)$:

$$c\left(w^2 - 1\right) Q^{''} - 2wQ^3 - rQ = 0. \tag{11}$$

where a constant for integration is r, $M = 1$ is obtained from the balance principle between $Q^{''}$ and Q^3; as a result, the solution form can be expressed as

$$P(\zeta) = \alpha_0 + \alpha_1 A(\zeta) + \alpha_2 A^2(\zeta), \tag{12}$$

where α_0, α_1 and α_2 are constants to be determined.

We obtain the following algebraic equations by substituting Eq.(12) along with Eq.(5) into Eq.(11) and setting the coefficients of all powers of $A^j(\zeta)$, $j = 0, 1, \ldots, 6$ to be equal to zero

$$\begin{aligned}
&A^0(\zeta): -r\alpha_0 - 2w\alpha_0^3 - 2ch_0\alpha_2 + 2cw^2h_0\alpha_2 = 0,\\
&A^1(\zeta): -r\alpha_1 - ch_2\alpha_1 + cw^2h_2\alpha_1 - 6w\alpha_0^2\alpha_1 = 0,\\
&A^2(\zeta): -6w\alpha_0\alpha_1^2 - r\alpha_2 - 4ch_2\alpha_2 + 4cw^2h_2\alpha_2 - 6w\alpha_0^2\alpha_2 = 0,\\
&A^3(\zeta): -2ch_4\alpha_1 + 2cw^2h_4\alpha_1 - 2w\alpha_1^3 - 12w\alpha_0\alpha_1\alpha_2 = 0,\\
&A^4(\zeta): -6ch_4\alpha_2 + 6cw^2h_4\alpha_2 - 6w\alpha_1^2\alpha_2 - 6w\alpha_0\alpha_2^2 = 0,\\
&A^5(\zeta): -3ch_6\alpha_1 + 3cw^2h_6\alpha_1 - 6w\alpha_1\alpha_2^2 = 0,\\
&A^6(\zeta): -8ch_6\alpha_2 + 8cw^2h_6\alpha_2 - 2w\alpha_2^3 = 0.
\end{aligned} \tag{13}$$

the following solutions can be obtained after solving the above resulting system :

$$\alpha_0 = 0, \quad \alpha_2 = 0, \quad r = c\left(-1 + w^2\right)h_2, \tag{14}$$

$$h_4 = \frac{w\alpha_1^2}{c\left(-1 + w^2\right)}, \quad h_6 = 0.$$

the following exact solutions of Eq.(10) can be derived with the help of Eqs.(6), (12) and (14) along with the Jacobi elliptic functions in the above table

1. If $l_0 = 1$, $l_2 = -(1 + m^2)$, $l_4 = m^2$, $0 < m < 1$, then $P(\zeta) = sn(\zeta, m)$ or $P(\zeta) = cd(\zeta, m)$, and we have

$$Q_{1,0}(x,t) = \alpha_1\left(\frac{sn(\zeta,m)}{\sqrt{f\left(sn(\zeta,m)\right)^2 + g}}\right), \tag{15}$$

or

$$Q_{1,1}(x,t) = \alpha_1\left(\frac{cd(\zeta,m)}{\sqrt{f\left(cd(\zeta,m)\right)^2 + g}}\right), \tag{16}$$

such that $0 < b$, $\zeta = cy - tw + x$, and f and g in Eq. (8) are given by

$$\begin{aligned}
f &= \frac{\alpha_1^2 w\left(h_2 + m^2 + 1\right)}{c\left(w^2 - 1\right)\left(-h_2^2 + m^4 - m^2 + 1\right)},\\
g &= -\frac{3\alpha_1^2 w}{c\left(w^2 - 1\right)\left(-h_2^2 + m^4 - m^2 + 1\right)},
\end{aligned}$$

under the restriction condition

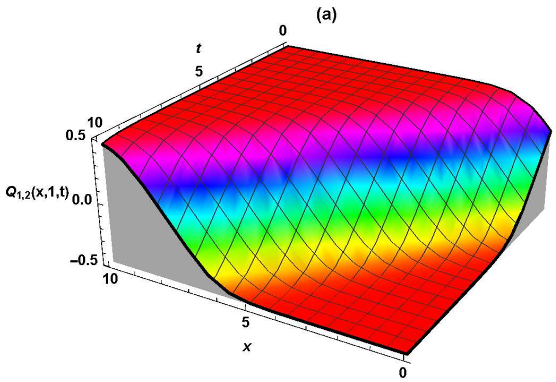
$$\frac{\alpha_1^4 w^2\left(-h_2 - m^2 - 1\right)\left(9m^2 - \left(-h_2 - m^2 - 1\right)\left(h_2 + 2\left(-m^2 - 1\right)\right)\right)}{c^2\left(w^2 - 1\right)^2} = 0.$$

If $m \to 1$, then the kink soliton is obtained

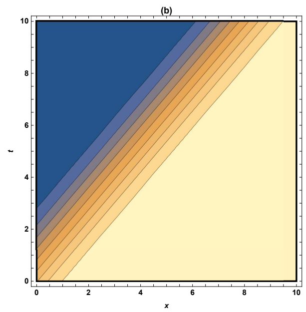
$$Q_{1,2}(x,t) = \frac{\alpha_1\tanh(\zeta)}{\sqrt{-\frac{\alpha_1^2 w\left((h_2+2)\tanh^2(\zeta) - 3\right)}{c\left(h_2^2 - 1\right)\left(w^2 - 1\right)}}}, \tag{17}$$

such that

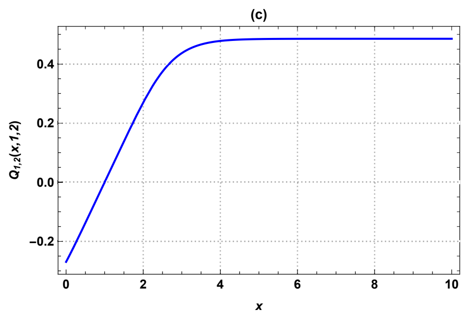
$$\frac{\alpha_1^4 w^2\left(-h_2 - 2\right)\left(9 - \left(-h_2 - 2\right)\left(h_2 - 4\right)\right)}{c^2\left(w^2 - 1\right)^2} = 0.$$



(a) $|Q_{1,2}|$



(b) $|Q_{1,2}|$



(c) $|Q_{1,2}|$

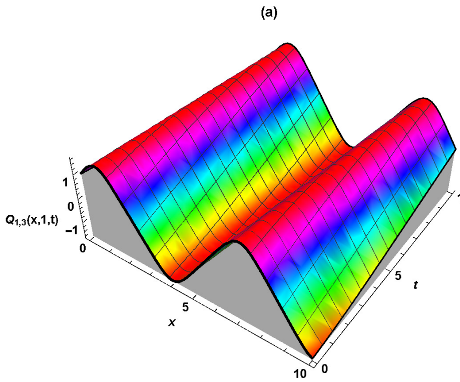
Figure 1: The numerical simulations corresponding to $|Q_{1,2}|$ given by Eq.(17), for $m = 1$; (a) is the 3D graphic, (b) is the 2D-contour graphic while (c) is the 2D graphic for $\alpha_1 = 0.1, h_2 = 0.03, c = 0.7, w = 0.85, y = 1, z = 1$.

If $m \to 0$, then the periodic solution is obtained

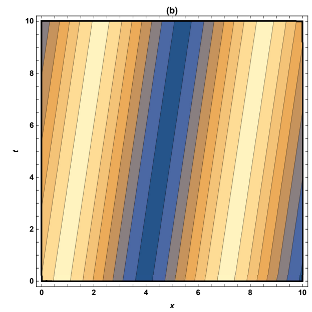
$$Q_{1,3}(x,t) = \frac{\alpha_1 \sin(\zeta)}{\sqrt{-\frac{\alpha_1^2 w(\sin^2(\zeta)+h_2\sin^2(\zeta)-3)}{c\left(h_2^2-1\right)(w^2-1)}}}, \tag{18}$$

such that

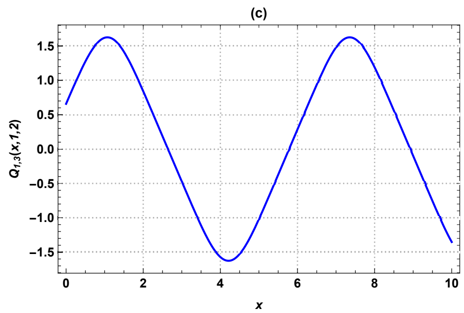
$$\frac{\alpha_1^4 w^2 \left(-h_2-1\right)\left(-\left(-h_2-1\right)\left(h_2-2\right)\right)}{c^2\left(w^2-1\right)^2} = 0.$$



(a) $|Q_{1,3}|$



(b) $|Q_{132}|$



(c) $|Q_{1,3}|$

Figure 2: The numerical simulations corresponding to $|Q_{1,3}|$ given by Eq.(18), for $m = 1$; (a) is the 3D graphic, (b) is the 2D-contour graphic while (c) is the 2D graphic for $\alpha_1 = 0.4, h_2 = 0.8, c = 0.15, w = 0.03, y = 1, z = 1$.

2. If $l_0 = 1 - m^2$, $l_2 = 2m^2 - 1$, $l_4 = -m^2$, $0 < m < 1$, then $P(\zeta) = cn(\zeta, m)$ therefore

$$Q_{2,1}(x,t) = \alpha_1 \left(\frac{cn(\zeta, m)}{\sqrt{f\,(cn(\zeta,m))^2 + g}} \right), \tag{19}$$

where f and g are determined by

$$f = \frac{\alpha_1^2 w \left(h_2 - 2m^2 + 1\right)}{c\,(w^2-1)\,(-h_2^2 + m^4 - m^2 + 1)},$$
$$g = \frac{3\alpha_1^2 \left(m^2 - 1\right) w}{c\,(w^2-1)\,(-h_2^2 + m^4 - m^2 + 1)},$$

under the constraint condition

$$\frac{\alpha_1^4 w^2 \left(-h_2 + 2m^2 - 1\right)\left(-\left(-h_2 + 2m^2 - 1\right)\left(h_2 + 2\left(2m^2 - 1\right)\right) - 9\left(1 - m^2\right)m^2\right)}{c^2\left(w^2 - 1\right)^2} = 0$$

If $m \to 1$, then the bright soliton is retrieved

$$Q_{2,2}(x,t) = \frac{\alpha_1 \operatorname{sech}(\zeta)}{\sqrt{-\frac{\alpha_1^2 w \operatorname{sech}^2(\zeta)}{c(h_2+1)(w^2-1)}}} \tag{20}$$

provided that

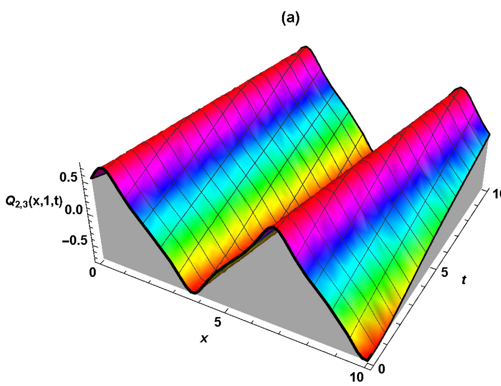
$$\frac{\alpha_1^4 w^2 \left(1-h_2\right)\left(-\left(1-h_2\right)\left(h_2+2\right)\right)}{c^2 \left(w^2-1\right)^2}=0.$$

If $m \to 0$, then the periodic solution is obtained

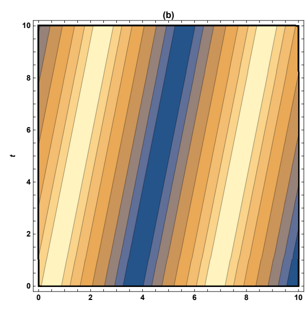
$$Q_{_{2,3}}\left(x,t\right)=\frac{\alpha_1 \cos(\zeta)}{\sqrt{-\frac{\alpha_1^2 w\left(\cos^2(\zeta)+h_2\cos^2(\zeta)-3\right)}{c\left(h_2^2-1\right)\left(w^2-1\right)}}}, \tag{21}$$

such that

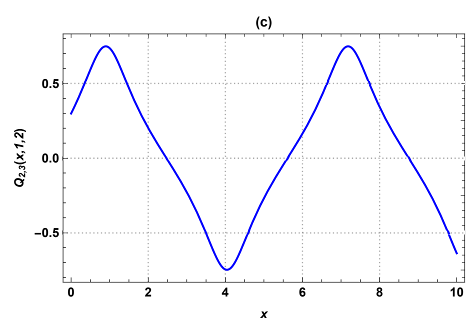
$$\frac{\alpha_1^4 w^2 \left(-h_2-1\right)\left(-\left(-h_2-1\right)\left(h_2-2\right)\right)}{c^2 \left(w^2-1\right)^2}=0.$$



(a) $|Q_{2,3}|$



(b) $|Q_{2,3}|$



(c) $|Q_{2,3}|$

Figure 3: The numerical simulations corresponding to $|Q_{2,3}|$ given by Eq.(21), for $m = 1$; (a) is the 3D graphic, (b) is the 2D-contour graphic while (c) is the 2D graphic for $\alpha_1 = 0.3, h_2 = 1.1, c = -0.5, w = 0.2, y = 1, z = 1$.

3. If $l_0 = m^2 - 1$, $l_2 = 2 - m^2$, $l_4 = -1$, $0 < m < 1$, then $P(\zeta) = dn(\zeta, m)$ which gives

$$Q_{_{3,1}}\left(x,t\right)=\alpha_1\left(\frac{dn(\zeta,m)}{\sqrt{f\left(dn(\zeta,m)\right)^2+g}}\right), \tag{22}$$

where f and g are determined by

$$f=\frac{\alpha_1^2 w\left(h_2+m^2-2\right)}{c\left(w^2-1\right)\left(-h_2^2+m^4-m^2+1\right)},$$
$$g=-\frac{3\alpha_1^2\left(m^2-1\right)w}{c\left(w^2-1\right)\left(-h_2^2+m^4-m^2+1\right)},$$

under the restriction condition

$$\frac{\alpha_1^4 w^2 \left(-h_2-m^2+2\right)\left(-\left(-h_2-m^2+2\right)\left(h_2+2\left(2-m^2\right)\right)-9\left(m^2-1\right)\right)}{c^2\left(w^2-1\right)^2}=0$$

If $m \to 1$, then the solitary solution is obtained

$$Q_{3,2}\left(x,t\right)=\frac{\alpha_1 \operatorname{sech}(\zeta)}{\sqrt{-\frac{\alpha_1^2 w \operatorname{sech}^2(\zeta)}{c(h_2+1)(w^2-1)}}}, \tag{23}$$

provided that

$$\frac{\alpha_1^4 w^2\left(1-h_2\right)\left(-\left(1-h_2\right)\left(h_2+2\right)\right)}{c^2\left(w^2-1\right)^2}=0.$$

If $m \to 0$, then the rational solution is obtained

$$Q_{3,3}\left(x,t\right)=\frac{\alpha_1}{\sqrt{-\frac{\alpha_1^2 w}{c(h_2-1)(w^2-1)}}}, \tag{24}$$

such that

$$\frac{\alpha_1^4 w^2\left(2-h_2\right)\left(9-\left(2-h_2\right)\left(h_2+4\right)\right)}{c^2\left(w^2-1\right)^2}=0.$$

4. If $l_0=m^2$, $l_2=-\left(1+m^2\right)$, $l_4=1$, $0<m<1$, then $P(\zeta)=ns(\zeta,m)$ or $P(\zeta)=dc(\zeta,m)$ then

$$Q_{4,0}\left(x,t\right)=\alpha_1\left(\frac{ns(\zeta,m)}{\sqrt{f\left(ns(\zeta,m)\right)^2+g}}\right), \tag{25}$$

or

$$Q_{4,1}\left(x,t\right)=\alpha_1\left(\frac{dc(\zeta,m)}{\sqrt{f\left(dc(\zeta,m)\right)^2+g}}\right), \tag{26}$$

where f and g are given by

$$f=\frac{\alpha_1^2 w\left(h_2+m^2+1\right)}{c\left(w^2-1\right)\left(-h_2^2+m^4-m^2+1\right)},$$
$$g=-\frac{3\alpha_1^2 m^2 w}{c\left(w^2-1\right)\left(-h_2^2+m^4-m^2+1\right)},$$

under the constraint condition

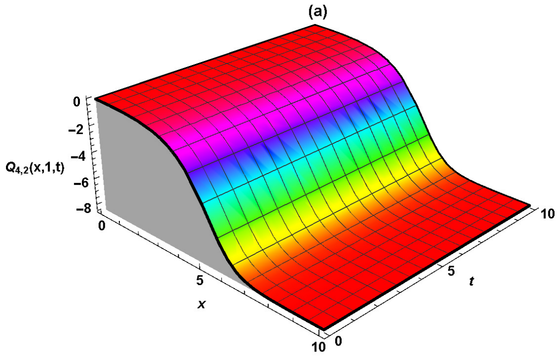
$$\frac{\alpha_1^4 w^2\left(-h_2-m^2-1\right)\left(9m^2-\left(-h_2-m^2-1\right)\left(h_2+2\left(-m^2-1\right)\right)\right)}{c^2\left(w^2-1\right)^2}=0.$$

If $m \to 1$, then the dark singular solution is obtained

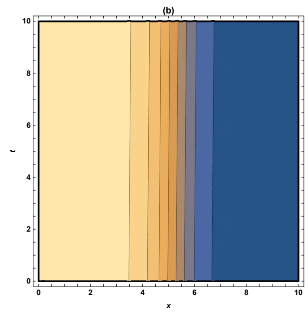
$$Q_{4,2}\left(x,t\right)=\frac{\alpha_1 \coth(\zeta)}{\sqrt{-\frac{\alpha_1^2 w\left(\left(h_2+2\right)\operatorname{csch}^2(\zeta)+h_2-1\right)}{c\left(h_2^2-1\right)(w^2-1)}}} \tag{27}$$

such that

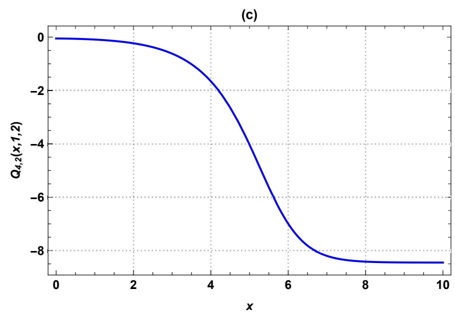
$$\frac{\alpha_1^4 w^2\left(-h_2-2\right)\left(9-\left(-h_2-2\right)\left(h_2-4\right)\right)}{c^2\left(w^2-1\right)^2}=0$$



(a) $|Q_{4,2}|$



(b) $|Q_{4,2}|$



(c) $|Q_{4,2}|$

Figure 4: The numerical simulations corresponding to $|Q_{4,2}|$ given by Eq.(27), for $m = 1$; (a) is the 3D graphic, (b) is the 2D-contour graphic while (c) is the 2D graphic for $\alpha_1 = -1.01, h_2 = 1.0001, c = 0.25, w = 0.007, y = 1, z = 1$.

If $m \to 0$, then the periodic solution is obtained

$$Q_{4,3}(x,t) = \frac{\alpha_1 \csc(\zeta)}{\sqrt{-\frac{\alpha_1^2 w \csc^2(\zeta)}{c(h_2-1)(w^2-1)}}}, \tag{28}$$

such that

$$\frac{\alpha_1^4 w^2 (-h_2 - 1)(-(-h_2-1)(h_2-2))}{c^2(w^2-1)^2} = 0.$$

5. If $l_0 = -m^2$, $l_2 = 2m^2 - 1$, $l_4 = 1 - m^2$, $0 < m < 1$, then $P(\zeta) = nc(\zeta, m)$ and we have

$$Q_{5,1}(x,t) = \alpha_1 \left(\frac{nc(\zeta,m)}{\sqrt{f\,(nc(\zeta,m))^2 + g}} \right), \tag{29}$$

where f and g are given by

$$f = \frac{\alpha_1^2 w \left(h_2 - 2m^2 + 1\right)}{c\,(w^2-1)\,(-h_2^2 + m^4 - m^2 + 1)},$$
$$g = \frac{3\alpha_1^2 m^2 w}{c\,(w^2-1)\,(-h_2^2 + m^4 - m^2 + 1)},$$

under the constraint condition

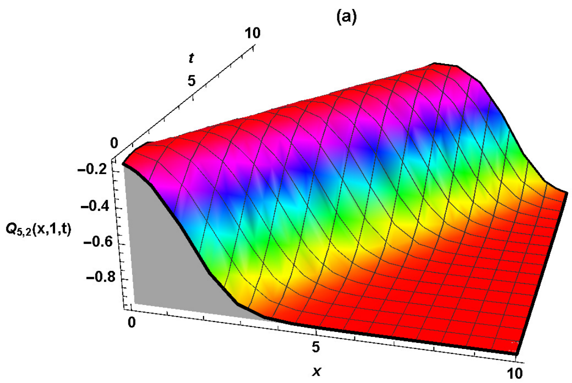
$$h_4^2\left(-1 + 2m^2 - h_2\right)\left[\left(-2 + m^2 + h_2\right)\left(1 + m^2 + h_2\right)\right] = 0.$$

If $m \to 1$, then the singular solitary wave solution is obtained

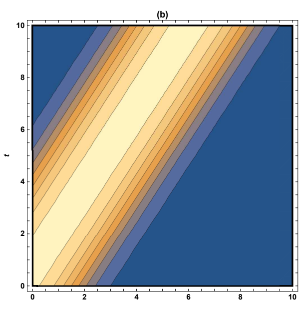
$$Q_{5,2}(x,t) = \frac{\alpha_1 \cosh(\zeta)}{\sqrt{\frac{\alpha_1^2 w\left(\cosh^2(\zeta) - h_2 \cosh^2(\zeta) - 3\right)}{c\left(h_2^2-1\right)\left(w^2-1\right)}}}, \tag{30}$$

such that

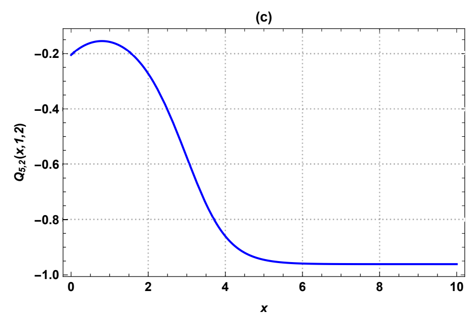
$$\frac{\alpha_1^4 w^2 \left(1-h_2\right)\left(-\left(1-h_2\right)\left(h_2+2\right)\right)}{c^2\left(w^2-1\right)^2} = 0.$$



(a) $|Q_{5,2}|$



(b) $|Q_{5,2}|$



(c) $|Q_{5,2}|$

Figure 5: The numerical simulations corresponding to $|Q_{5,2}|$ given by Eq.(30), for $m = 1$; (a) is the 3D graphic, (b) is the 2D-contour graphic while (c) is the 2D graphic for $\alpha_1 = -0.2, h_2 = 1.08, c = 0.5, w = 0.65, y = 1, z = 1$.

If $m \to 0$, then the periodic solution is obtained

$$Q_{5,3}(x,t) = \frac{\alpha_1 \sec(\zeta)}{\sqrt{-\frac{\alpha_1^2 w \sec^2(\zeta)}{c(h_2-1)(w^2-1)}}}, \tag{31}$$

such that

$$\frac{\alpha_1^4 w^2 \left(-h_2-1\right)\left(-\left(-h_2-1\right)\left(h_2-2\right)\right)}{c^2\left(w^2-1\right)^2} = 0.$$

6. If $l_0 = -1$, $l_2 = 2 - m^2$, $l_4 = -\left(1-m^2\right)$, $0 < m < 1$, then $P(\zeta) = nd(\zeta, m)$ and we have

$$Q_6(x,t) = \alpha_1 \left(\frac{nd(\zeta,m)}{\sqrt{f\left(nd(\zeta,m)\right)^2 + g}} \right), \tag{32}$$

where f and g are given by

$$f=\frac{\alpha_1^2 w\left(h_2+m^2-2\right)}{c\left(w^2-1\right)\left(-h_2^2+m^4-m^2+1\right)},$$
$$g=\frac{3\alpha_1^2 w}{c\left(w^2-1\right)\left(-h_2^2+m^4-m^2+1\right)},$$

under the constraint condition

$$\frac{\alpha_1^4 w^2\left(-h_2-m^2+2\right)\left(-\left(-h_2-m^2+2\right)\left(h_2+2\left(2-m^2\right)\right)-9\left(m^2-1\right)\right)}{c^2\left(w^2-1\right)^2}=0.$$

7. If $l_0=1$, $l_2=2-m^2$, $l_4=1-m^2$,$0<m<1$, then $P(\zeta)=sc(\zeta,m)$, and we have

$$Q_{7,1}(x,t)=\alpha_1\left(\frac{sc(\zeta,m)}{\sqrt{f\left(sc(\zeta,m)\right)^2+g}}\right), \tag{33}$$

where f and g are given by

$$f=\frac{\alpha_1^2 w\left(h_2+m^2-2\right)}{c\left(w^2-1\right)\left(-h_2^2+m^4-m^2+1\right)},$$
$$g=-\frac{3\alpha_1^2 w}{c\left(w^2-1\right)\left(-h_2^2+m^4-m^2+1\right)},$$

under the constraint condition

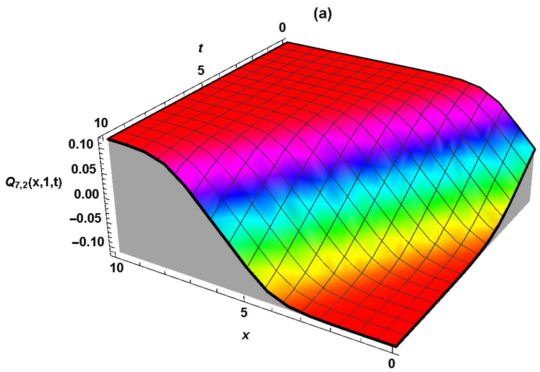
$$\frac{\alpha_1^4 w^2\left(-h_2-m^2+2\right)\left(9\left(1-m^2\right)-\left(-h_2-m^2+2\right)\left(h_2+2\left(2-m^2\right)\right)\right)}{c^2\left(w^2-1\right)^2}=0.$$

If $m\to 1$, then the kink soliton solution is obtained

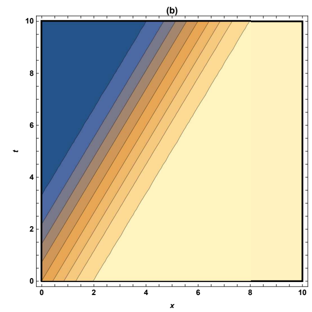
$$Q_{7,2}(x,t)=\frac{\alpha_1\sinh(\zeta)}{\sqrt{\frac{\alpha_1^2 w(\sinh^2(\zeta)-h_2\sinh^2(\zeta)+3)}{c\left(h_2^2-1\right)(w^2-1)}}}, \tag{34}$$

such that

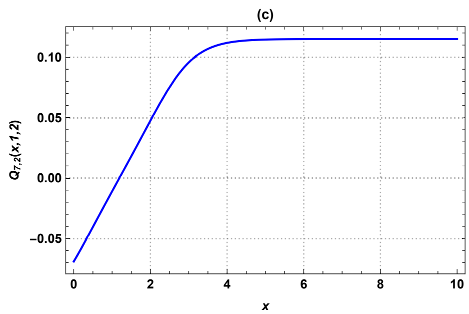
$$\frac{\alpha_1^4 w^2\left(1-h_2\right)\left(-\left(1-h_2\right)\left(h_2+2\right)\right)}{c^2\left(w^2-1\right)^2}.$$



(a) $|Q_{7,2}|$



(b) $|Q_{7,2}|$



(c) $|Q_{7,2}|$

Figure 6: The numerical simulations corresponding to $|Q_{7,2}|$ given by Eq.(34), for $m = 1$; (a) is the 3D graphic, (b) is the 2D-contour graphic while (c) is the 2D graphic for $\alpha_1 = 0.5, h_2 = 0.24, c = 0.01, w = 0.6, y = 1, z = 1$.

If $m \to 0$, then the periodic wave solution is obtained

$$Q_{7,3}(x,t) = \frac{\alpha_1 \tan(\zeta)}{\sqrt{\frac{\alpha_1^2 w(3-(h_2-2)\tan^2(\zeta))}{c\left(h_2^2-1\right)(w^2-1)}}}, \tag{35}$$

such that

$$\frac{\alpha_1^4 w^2 (2-h_2)(9-(2-h_2)(h_2+4))}{c^2 (w^2-1)^2}.$$

8. If $l_0 = 1$, $l_2 = 2m^2 - 1$, $l_4 = -m^2\left(1-m^2\right)$, $0 < m < 1$, then $P(\zeta) = sd(\zeta, m)$ and we have

$$Q_8(x,t) = \alpha_1 \left(\frac{sd(\zeta,m)}{\sqrt{f\,(sd(\zeta,m))^2 + g}} \right), \tag{36}$$

where f and g are given by

$$f = \frac{\alpha_1^2 w \left(h_2 - 2m^2 + 1\right)}{c\,(w^2-1)\,(-h_2^2 + m^4 - m^2 + 1)},$$
$$g = -\frac{3\alpha_1^2 w}{c\,(w^2-1)\,(-h_2^2 + m^4 - m^2 + 1)},$$

under the constraint condition

$$\frac{\alpha_1^4 w^2\left(-h_2+2m^2-1\right)\left(-\left(-h_2+2m^2-1\right)\left(h_2+2\left(2m^2-1\right)\right)-9\left(1-m^2\right)m^2\right)}{c^2\left(w^2-1\right)^2}=0.$$

9. If $l_0=1-m^2$, $l_2=2-m^2$, $l_4=1$, $0<m<1$, then $P(\zeta)=cs(\zeta,m)$ and we have

$$Q_{9,1}(x,t)=\alpha_1\left(\frac{cs(\zeta,m)}{\sqrt{f\left(cs(\zeta,m)\right)^2+g}}\right), \tag{37}$$

where f and g are given by

$$f=\frac{\alpha_1^2 w\left(h_2+m^2-2\right)}{c\left(w^2-1\right)\left(-h_2^2+m^4-m^2+1\right)},$$
$$g=\frac{3\alpha_1^2\left(m^2-1\right)w}{c\left(w^2-1\right)\left(-h_2^2+m^4-m^2+1\right)},$$

under the constraint condition

$$\frac{\alpha_1^4 w^2\left(-h_2-m^2+2\right)\left(9\left(1-m^2\right)-\left(-h_2-m^2+2\right)\left(h_2+2\left(2-m^2\right)\right)\right)}{c^2\left(w^2-1\right)^2}=0.$$

If $m\to 1$, then the singular soliton solution is obtained

$$Q_{9,2}(x,t)=\frac{\alpha_1\operatorname{csch}(\zeta)}{\sqrt{-\frac{\alpha_1^2 w\operatorname{csch}^2(\zeta)}{c(h_2+1)(w^2-1)}}}, \tag{38}$$

such that

$$\frac{\alpha_1^4 w^2\left(1-h_2\right)\left(-\left(1-h_2\right)\left(h_2+2\right)\right)}{c^2\left(w^2-1\right)^2}=0.$$

If $m\to 0$, then the periodic wave solution is obtained

$$Q_{9,3}(x,t)=\frac{\alpha_1\cot(\zeta)}{\sqrt{\frac{\alpha_1^2 w\left(2\cot^2(\zeta)-h_2\cot^2(\zeta)+3\right)}{c\left(h_2^2-1\right)(w^2-1)}}}, \tag{39}$$

such that

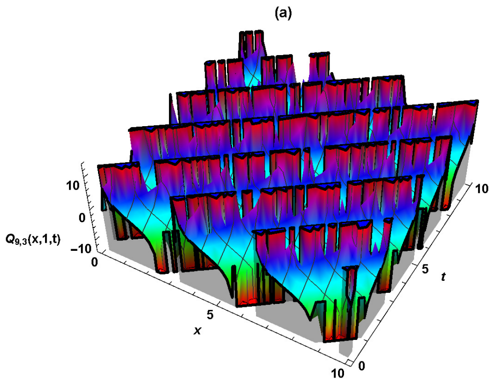
$$\frac{\alpha_1^4 w^2\left(2-h_2\right)\left(9-\left(2-h_2\right)\left(h_2+4\right)\right)}{c^2\left(w^2-1\right)^2}=0.$$

10. If $l_0=-m^2\left(1-m^2\right)$, $l_2=2m^2-1$, $l_4=1$, $0<m<1$, then $P(\zeta)=ds(\zeta,m)$ and we have

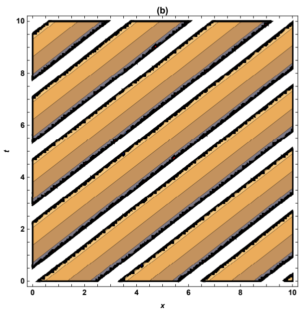
$$Q_{10}(x,t)=\alpha_1\left(\frac{ds(\zeta,m)}{\sqrt{f\left(ds(\zeta,m)\right)^2+g}}\right), \tag{40}$$

where f and g are given by

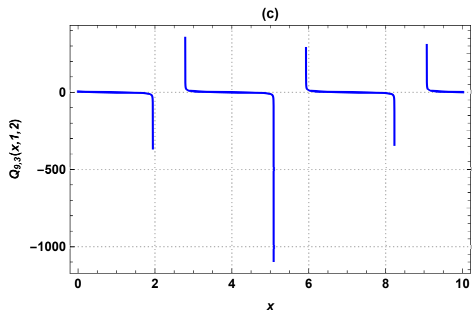
$$f=\frac{\alpha_1^2 w\left(h_2-2m^2+1\right)}{c\left(w^2-1\right)\left(-h_2^2+m^4-m^2+1\right)},$$
$$g=-\frac{3\alpha_1^2 m^2\left(m^2-1\right)w}{c\left(w^2-1\right)\left(-h_2^2+m^4-m^2+1\right)},$$



(a) $|Q_{9,3}|$



(b) $|Q_{9,3}|$



(c) $|Q_{9,3}|$

Figure 7: The numerical simulations corresponding to $|Q_{9,3}|$ given by Eq.(39), for $m = 1$; (a) is the 3D graphic, (b) is the 2D-contour graphic while (c) is the 2D graphic for $\alpha_1 = 3.5, h_2 = 2.6, c = 12.8, w = 1.3, y = 1, z = 1$.

under the constraint condition

$$\frac{\alpha_1^4 w^2 \left(-h_2 + 2m^2 - 1\right)\left(-\left(-h_2 + 2m^2 - 1\right)\left(h_2 + 2\left(2m^2 - 1\right)\right) - 9\left(1 - m^2\right)m^2\right)}{c^2\left(w^2 - 1\right)^2} = 0.$$

11. If $l_0 = \frac{1-m^2}{4}$, $l_2 = \frac{1+m^2}{2}$, $l_4 = \frac{1-m^2}{4}$, $0 < m < 1$, then $P(\zeta) = nc(\zeta, m) \pm sc(\zeta, m)$ or $P(\zeta) = \frac{cn(\zeta,m)}{1 \pm sn(\zeta,m)}$ and we have

$$Q_{11,0}(x,t) = \alpha_1 \left(\frac{nc(\zeta, m) \pm sc(\zeta, m)}{\sqrt{f\left(nc(\zeta, m) \pm sc(\zeta, m)\right)^2 + g}} \right), \tag{41}$$

or

$$Q_{11,1}(x,t) = \alpha_1 \left(\frac{\frac{cn(\zeta,m)}{1 \pm sn(\zeta,m)}}{\sqrt{f\left(\frac{cn(\zeta,m)}{1 \pm sn(\zeta,m)}\right)^2 + g}} \right), \tag{42}$$

where f and g are given by

$$f=-\frac{8\alpha_1^2 w\left(-2h_2+m^2+1\right)}{c\left(w^2-1\right)\left(-16h_2^2+m^4+14m^2+1\right)},$$
$$g=\frac{12\alpha_1^2\left(m^2-1\right)w}{c\left(w^2-1\right)\left(-16h_2^2+m^4+14m^2+1\right)},$$

under the constraint condition

$$\frac{\alpha_1^4 w^2\left(\frac{1}{2}\left(m^2+1\right)-h_2\right)\left(\frac{9}{16}\left(1-m^2\right)^2-\left(\frac{1}{2}\left(m^2+1\right)-h_2\right)\left(h_2+m^2+1\right)\right)}{c^2\left(w^2-1\right)^2}=0.$$

If $m\to 1$, then the combined singular soliton solution is obtained

$$Q_{11,2}(x,t)=\frac{\alpha_1(\sinh(\zeta)+\cosh(\zeta))}{\sqrt{-\frac{\alpha_1^2 w(\sinh(\zeta)+\cosh(\zeta))^2}{c(h_2+1)(w^2-1)}}}, \tag{43}$$

such that

$$\frac{\alpha_1^4 w^2\left(1-h_2\right)\left(-\left(1-h_2\right)\left(h_2+2\right)\right)}{c^2\left(w^2-1\right)^2}=0.$$

If $m\to 0$, then the combined periodic wave solutions

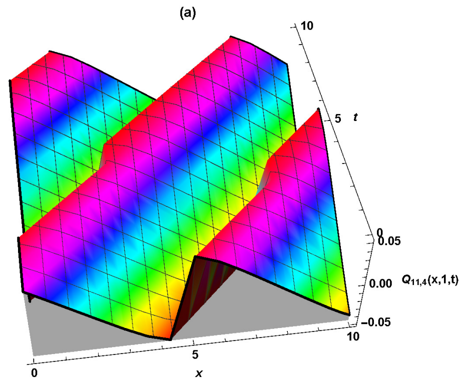
$$Q_{11,3}(x,t)=\frac{\alpha_1(\tan(\zeta)+\sec(\zeta))}{2\sqrt{\frac{\alpha_1^2 w(\sin(\zeta)+4h_2(\sin(\zeta)+1)-5)}{c\left(16h_2^2-1\right)(w^2-1)(\sin(\zeta)-1)}}}, \tag{44}$$

or

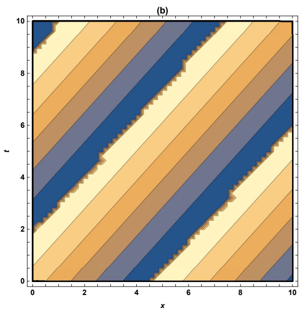
$$Q_{11,4}(x,t)=\frac{\alpha_1\cos(\zeta)}{2(\sin(\zeta)+1)\sqrt{\frac{\alpha_1^2 w(3(\sin(\zeta)+1)^2+2\cos^2(\zeta)-4h_2\cos^2(\zeta))}{c\left(16h_2^2-1\right)(w^2-1)(\sin(\zeta)+1)^2}}}, \tag{45}$$

are obtained, such that

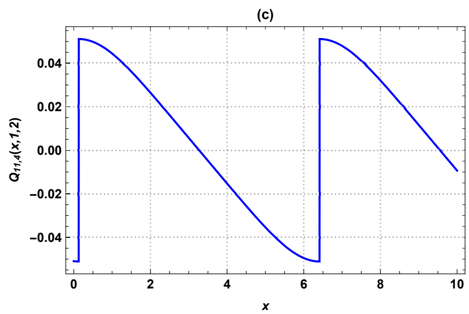
$$\frac{\alpha_1^4 w^2\left(\frac{1}{2}-h_2\right)\left(\frac{9}{16}-\left(\frac{1}{2}-h_2\right)\left(h_2+1\right)\right)}{c^2\left(w^2-1\right)^2}=0.$$



(a) $|Q_{11,4}|$



(b) $|Q_{11,4}|$



(c) $|Q_{11,4}|$

Figure 8: The numerical simulations corresponding to $|Q_{11,4}|$ given by Eq.(45), for $m = 1$; (a) is the 3D graphic, (b) is the 2D-contour graphic while (c) is the 2D graphic for $\alpha_1 = 0.5, h_2 = -0.005, c = 0.1, w = 0.9, y = 1, z = 1$.

12. If $l_0 = \frac{-\left(1-m^2\right)^2}{4}$, $l_2 = \frac{1+m^2}{2}$, $l_4 = \frac{-1}{4}$, $0 < m < 1$, then $P(\zeta) = mcn(\zeta, m) \pm dn(\zeta, m)$ and we have

$$Q_{12,1}\left(x,t\right) = \alpha_1 \left(\frac{mcn(\zeta,m) \pm dn(\zeta,m)}{\sqrt{f\left(mcn(\zeta,m) \pm dn(\zeta,m)\right)^2 + g}}\right), \tag{46}$$

where f and g are given by

$$f = -\frac{8\alpha_1^2 w\left(-2h_2 + m^2 + 1\right)}{c\left(w^2-1\right)\left(-16h_2^2 + m^4 + 14m^2 + 1\right)},$$
$$g = \frac{12\alpha_1^2\left(m^2-1\right)^2 w}{c\left(w^2-1\right)\left(-16h_2^2 + m^4 + 14m^2 + 1\right)},$$

under the constraint condition

$$\frac{\alpha_1^4 w^2\left(\frac{1}{2}\left(m^2+1\right) - h_2\right)\left(\frac{9}{16}\left(1-m^2\right)^2 - \left(\frac{1}{2}\left(m^2+1\right) - h_2\right)\left(h_2 + m^2 + 1\right)\right)}{c^2\left(w^2-1\right)^2} = 0.$$

If $m \to 1$, then the bright soliton solution is obtained

$$Q_{12,2}\left(x,t\right) = \frac{\alpha_1 \operatorname{sech}(\zeta)}{\sqrt{-\frac{\alpha_1^2 w \operatorname{sech}^2(\zeta)}{c(h_2+1)(w^2-1)}}} \tag{47}$$

such that

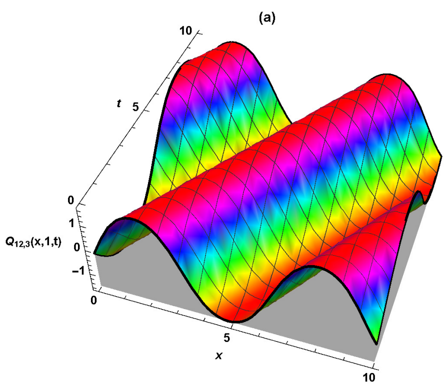
$$\frac{\alpha_1^4 w^2 \left(1-h_2\right)\left(-\left(1-h_2\right)\left(h_2+2\right)\right)}{c^2 \left(w^2-1\right)^2}=0.$$

If $m \to 0$, then the periodic wave solution is obtainedr

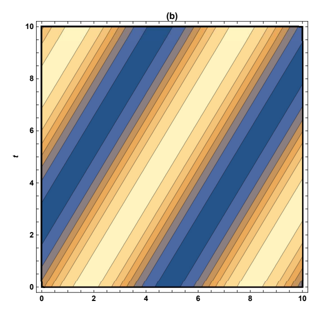
$$Q_{_{12,3}}\left(x,t\right)=\frac{\alpha_1 \cos(\zeta)}{2\sqrt{\frac{\alpha_1^2 w\left(\cos(2\zeta)-4h_2\cos^2(\zeta)-2\right)}{c\left(16h_2^2-1\right)\left(w^2-1\right)}}}, \tag{48}$$

such that

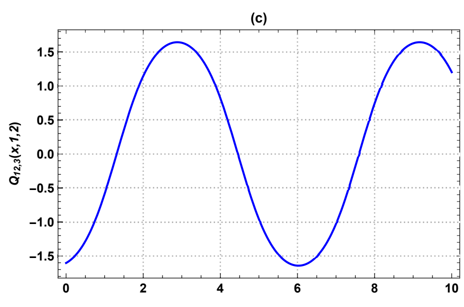
$$\frac{\alpha_1^4 w^2 \left(\frac{1}{2}-h_2\right)\left(\frac{9}{16}-\left(\frac{1}{2}-h_2\right)\left(h_2+1\right)\right)}{c^2 \left(w^2-1\right)^2}=0.$$



(a) $|Q_{12,3}|$



(b) $|Q_{12,3}|$



(c) $|Q_{12,3}|$

Figure 9: The numerical simulations corresponding to $|Q_{12,3}|$ given by Eq.(48), for $m = 1$; (a) is the 3D graphic, (b) is the 2D-contour graphic while (c) is the 2D graphic for $\alpha_1 = 0.01, h_2 = 0.8, c = 4.6, w = 0.6, y = 1, z = 1$.

13. If $l_0 = \frac{1}{4}$, $l_2 = \frac{1-2m^2}{2}$, $l_4 = \frac{1}{4}$, $0 < m < 1$, then $P(\zeta) = \frac{sn(\zeta,m)}{1 \pm cn(\zeta,m)}$ and we have

$$Q_{_{13,1}}\left(x,t\right)=\alpha_1\left(\frac{\frac{sn(\zeta,m)}{1\pm cn(\zeta,m)}}{\sqrt{f\left(\frac{sn(\zeta,m)}{1\pm cn(\zeta,m)}\right)^2+g}}\right), \tag{49}$$

where f and g are given by

$$f = \frac{8\alpha_1^2 w\left(2h_2+2m^2-1\right)}{c\left(w^2-1\right)\left(-16h_2^2+16m^4-16m^2+1\right)},$$
$$g = -\frac{12\alpha_1^2 w}{c\left(w^2-1\right)\left(-16h_2^2+16m^4-16m^2+1\right)},$$

under the constraint condition

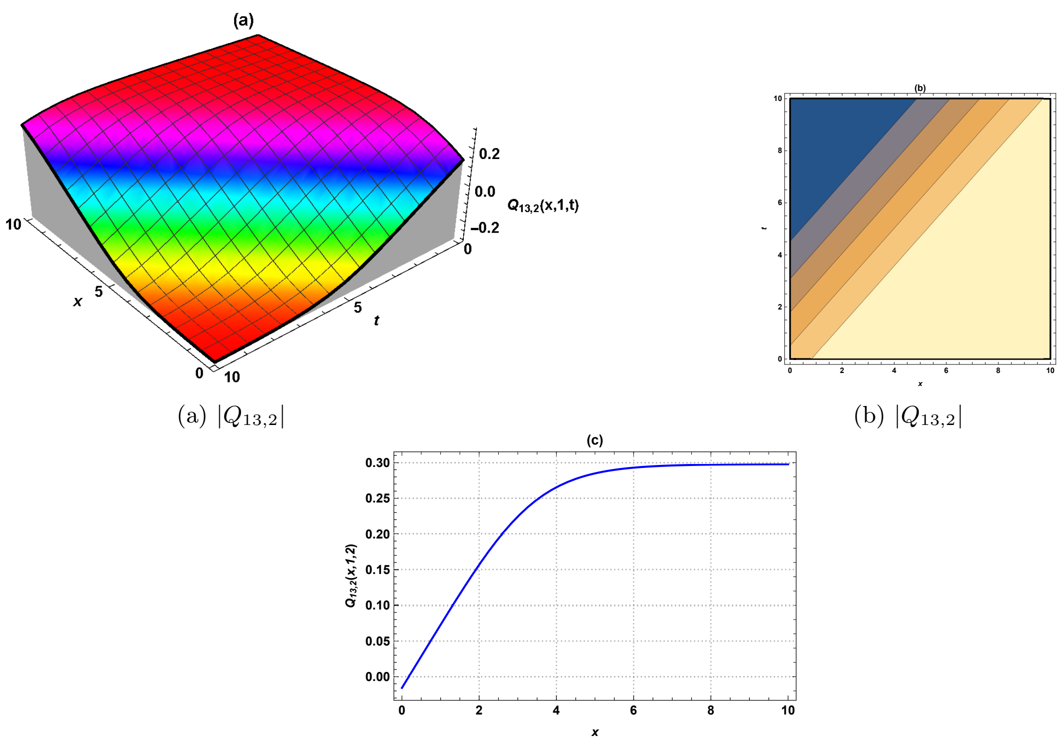
$$\frac{\alpha_1^4 w^2\left(\frac{1}{2}\left(1-2m^2\right)-h_2\right)\left(\frac{9}{16}-\left(\frac{1}{2}\left(1-2m^2\right)-h_2\right)\left(h_2-2m^2+1\right)\right)}{c^2\left(w^2-1\right)^2} = 0.$$

If $m \to 1$, then the kink soliton solution is obtained

$$Q_{13,2}(x,t) = \frac{\alpha_1 \tanh\left(\frac{\zeta}{2}\right)}{2\sqrt{\frac{\alpha_1^2 w(\cosh(\zeta)-4h_2(\cosh(\zeta)-1)+5)}{c\left(16h_2^2-1\right)(w^2-1)(\cosh(\zeta)+1)}}}, \tag{50}$$

such that

$$\frac{\alpha_1^4 w^2\left(-h_2-\frac{1}{2}\right)\left(\frac{9}{16}-\left(-h_2-\frac{1}{2}\right)\left(h_2-1\right)\right)}{c^2\left(w^2-1\right)^2} = 0.$$



(a) $|Q_{13,2}|$ (b) $|Q_{13,2}|$

(c) $|Q_{13,2}|$

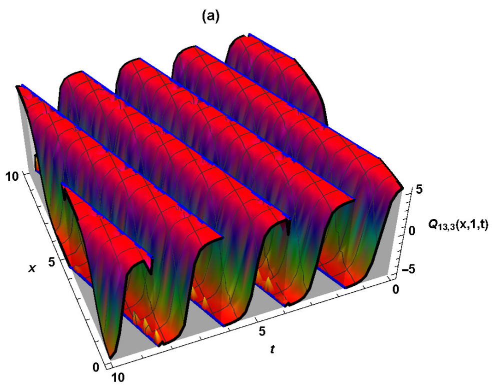
Figure 10: The numerical simulations corresponding to $|Q_{13,2}|$ given by Eq.(50), for $m = 1$; (a) is the 3D graphic, (b) is the 2D-contour graphic while (c) is the 2D graphic for $\alpha_1 = 2.5, h_2 = -0.015, c = 1.6, w = 0.889, y = 1, z = 1$.

If $m \to 0$, then the combined periodic wave solution is obtained

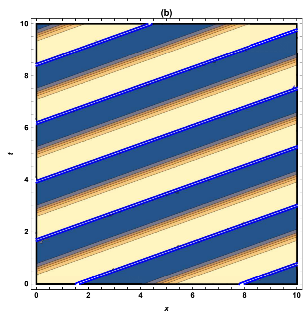
$$Q_{13,3}(x,t) = \frac{\alpha_1 \tan\left(\frac{\zeta}{2}\right)}{2\sqrt{\frac{\alpha_1^2 w(\cos(\zeta)+4h_2(\cos(\zeta)-1)+5)}{c\left(16h_2^2-1\right)(w^2-1)(\cos(\zeta)+1)}}}, \tag{51}$$

such that

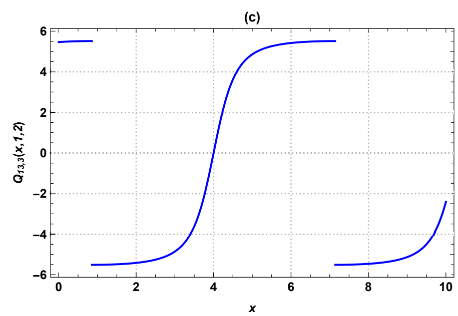
$$\frac{\alpha_1^4 w^2 \left(\frac{1}{2}-h_2\right)\left(\frac{9}{16}-\left(\frac{1}{2}-h_2\right)(h_2+1)\right)}{c^2\left(w^2-1\right)^2}=0.$$



(a) $|Q_{13,3}|$



(b) $|Q_{13,3}|$



(c) $|Q_{13,3}|$

Figure 11: The numerical simulations corresponding to $|Q_{13,3}|$ given by Eq.(51), for $m=1$; (a) is the 3D graphic, (b) is the 2D-contour graphic while (c) is the 2D graphic for $\alpha_1=0.01, h_2=-8.25, c=1.6, w=2.8, y=1, z=1$.

14. If $l_0=\frac{1}{4}$, $l_2=\frac{1+m^2}{2}$, $l_4=\frac{\left(1-m^2\right)^2}{4}$, $0<m<1$, then $P(\zeta)=\frac{sn(\zeta,m)}{cn(\zeta,m)\pm dn(\zeta,m)}$ and we have

$$Q_{14,1}(x,t)=\alpha_1\left(\frac{\frac{sn(\zeta,m)}{cn(\zeta,m)\pm dn(\zeta,m)}}{\sqrt{f\left(\frac{sn(\zeta,m)}{cn(\zeta,m)\pm dn(\zeta,m)}\right)^2+g}}\right), \tag{52}$$

where f and g are given by

$$f=-\frac{8\alpha_1^2 w\left(-2h_2+m^2+1\right)}{c\left(w^2-1\right)\left(-16h_2^2+m^4+14m^2+1\right)},$$
$$g=-\frac{12\alpha_1^2 w}{c\left(w^2-1\right)\left(-16h_2^2+m^4+14m^2+1\right)},$$

under the constraint condition

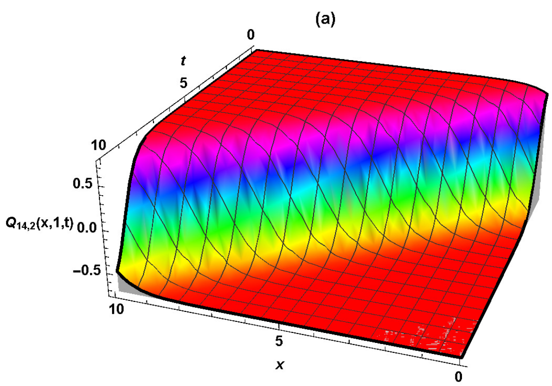
$$\frac{\alpha_1^4 w^2\left(\frac{1}{2}\left(m^2+1\right)-h_2\right)\left(\frac{9}{16}\left(1-m^2\right)^2-\left(\frac{1}{2}\left(m^2+1\right)-h_2\right)\left(h_2+m^2+1\right)\right)}{c^2\left(w^2-1\right)^2}=0.$$

If $m \to 1$, then the kink soliton solution is obtained

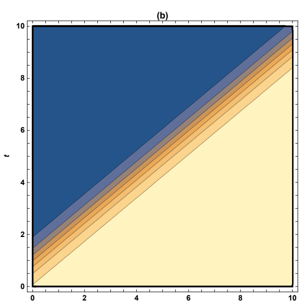
$$Q_{14,2}(x,t) = \frac{\alpha_1 \sinh(\zeta)}{\sqrt{\frac{\alpha_1^2 w\left(\sinh^2(\zeta) - h_2 \sinh^2(\zeta) + 3\right)}{c\left(h_2^2 - 1\right)\left(w^2 - 1\right)}}}, \tag{53}$$

such that

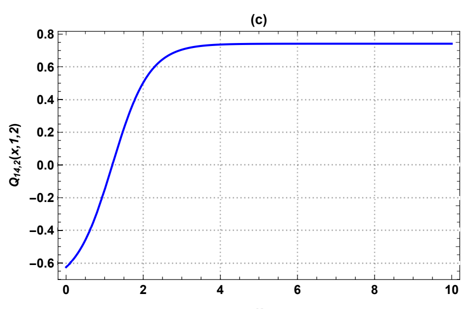
$$\frac{\alpha_1^4 w^2 \left(1 - h_2\right)\left(-\left(1 - h_2\right)\left(h_2 + 2\right)\right)}{c^2 \left(w^2 - 1\right)^2} = 0.$$



(a) $|Q_{14,2}|$



(b) $|Q_{14,2}|$



(c) $|Q_{14,2}|$

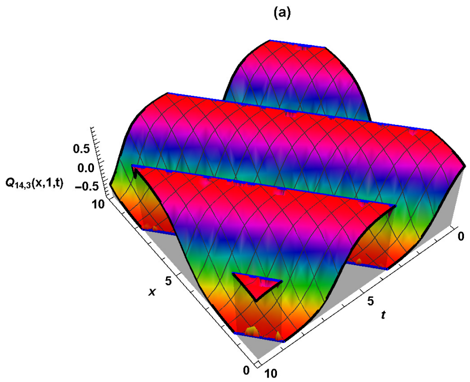
Figure 12: The numerical simulations corresponding to $|Q_{14,2}|$ given by Eq.(53), for $m = 1$; (a) is the 3D graphic, (b) is the 2D-contour graphic while (c) is the 2D graphic for $\alpha_1 = 0.01, h_2 = -2.25, c = 1.2, w = 1.2, y = 1, z = 1$.

If $m \to 0$, then the periodic wave is obtained

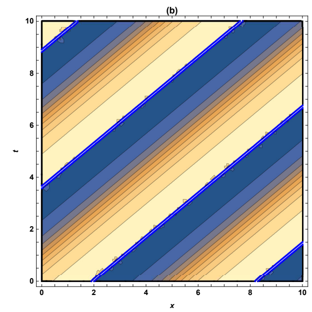
$$Q_{14,3}(x,t) = \frac{\alpha_1 \tan\left(\frac{\zeta}{2}\right)}{2\sqrt{\frac{\alpha_1^2 w\left(\cos(\zeta) + 4h_2(\cos(\zeta) - 1) + 5\right)}{c\left(16h_2^2 - 1\right)\left(w^2 - 1\right)\left(\cos(\zeta) + 1\right)}}}, \tag{54}$$

such that

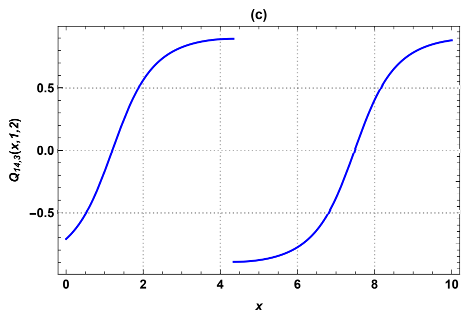
$$\frac{\alpha_1^4 w^2 \left(\frac{1}{2} - h_2\right)\left(\frac{9}{16} - \left(\frac{1}{2} - h_2\right)\left(h_2 + 1\right)\right)}{c^2 \left(w^2 - 1\right)^2} = 0.$$



(a) $|Q_{14,3}|$



(b) $|Q_{14,3}|$



(c) $|Q_{14,3}|$

Figure 13: The numerical simulations corresponding to $|Q_{14,3}|$ given by Eq.(54), for $m = 1$; (a) is the 3D graphic, (b) is the 2D-contour graphic while (c) is the 2D graphic for $\alpha_1 = 0.01, h_2 = -2.25, c = 1.2, w = 1.2, y = 1, z = 1$.

4 Physical interpretation and discussion of the obtained results

The plots of three-dimensional surfaces represent the moving patterns of the derived solutions. Some graphs, density plots and contours make it simple to see variations in 3D surface images. Contoured and density graphs are therefore plotted in accordance with every 3D form. Figure [1] shows an absolute 3D depiction of the kink soliton solution to Eq.(17) for the set of parameter values $\alpha_1 = 0.1, h_2 = 0.03, c = 0.7, w = 0.85, y = 1$ and $z = 1$, these solitons are also called topological solitons in the context of water waves, their velocity does not depend on the wave amplitude. For the selected set of values $\alpha_1 = 0.4, h_2 = 0.8, c = 0.15, w = 0.03, y = 1, z = 1$ and $\alpha_1 = 0.3, h_2 = 1.1, c = -0.5, w = 0.2, y = 1, z = 1$ for Eq.(18)and Eq.(21), the periodic soliton solutions are obtained in Figure [2] and Figure [3] respectively. Figure [4] represents the 3D soliton solution graph of Eq.(27) for the parameters $\alpha_1 = -1.01, h_2 = 1.0001, c = 0.25, w = 0.007, y = 1, z = 1$. Figure [5] shows the 3D soliton solution graph of Eq.(30) for the parameters $\alpha_1 = -0.2, h_2 = 1.08, c = 0.5, w = 0.65, y = 1, z = 1$. Figure [6] shows the graph of kink soliton soliton of Eq.(34) for the parameters $\alpha_1 = 0.5, h_2 = 0.24, c = 0.01, w = 0.6, y = 1, z = 1$. Figure [7] represents the 3D of a different periodic soliton solution of Eq.(39) for the parameters $\alpha_1 = 3.5, h_2 = 2.6, c = 12.8, w = 1.3, y = 1, z = 1$. Figure [8] and Figure [9] show the 3Ds of the combined periodic and periodic solutions of Eq.(45) for the parameters $\alpha_1 = 0.5, h_2 = -0.005, c = 0.1, w = 0.9, y = 1, z = 1$ and Eq.(48) for the parameters $\alpha_1 = 0.01, h_2 = 0.8, c = 4.6, w = 0.6, y = 1, z = 1$ respectively, while the Figure [10] shows the 3D of the kink soliton solution of Eq.(50) for the parameters $\alpha_1 = 2.5, h_2 = -0.015, c = 1.6, w = 0.889, y = 1, z = 1$. Figure [11] shows the 3D of the periodic solution of Eq.(51) for the parameters $\alpha_1 = 0.01, h_2 = -8.25, c =$

$1.6, w = 2.8, y = 1, z = 1$. Figure [12] shows the 3D of the kink soliton solution of Eq.(53) for the parameters $\alpha_1 = 0.01, h_2 = -2.25, c = 1.2, w = 1.2, y = 1, z = 1$. Figure [13] represents the 3D of the combined periodic solution of Eq.(54) for the parameters $\alpha_1 = 0.01, h_2 = -2.25, c = 1.2, w = 1.2, y = 1, z = 1$.

Our goal is to show the significance of the constructed outcomes and the paper's achievements. We are able to reach our desired goal by comparing our findings to those that have lately been published [18–30]. The following techniques have been used by researchers to retrieve solitons to the proposed nonlinear Zoomeron equation, Ghazala et al. used the modified auxiliary equation and Generalized projective Riccati equation method [18] to obtain solitary wave, dark peakon, bright and kink-type wave solution, the $\left(\frac{G'}{G}, \frac{1}{G}\right)$ −expansion approach [19] is utelized by Elsayed et al. to retrieved new solitary wave solutions, Tanki et al. examined the classical Lie point symmetries [20] of the model, four exact particular solutions are obtained including rational and periodic solutions using the $\left(\frac{\Phi(\xi)}{2}\right)$ −expansion method [21] by Jalil et al., the Yan's sine-cosine method and Wazwaz's sine-cosine method [22] are used by Hua to derive new exact traveling wave solutions, the new extended direct algebraic method [23] is used to obtain new analytical solutions by Wei et. al, the first integral method [24,25], the hyperbolic trigonometric and rational function solutions are retrieved via the $\left(\frac{G'}{G}\right)$ −expansion approach [26] by Reza, the Lie point transformation method [27], Aminah used the sine-cosine function method [28] to construct the trigonometric wave solutions, Higazy et al. implement the extended simple equation method [29] to get solitary wave solutions, the Modified simple equation method and Exp-function method [30] are used to obtain dark, trigonometric and hyperbolic soliton solutions. Trying to compare our results to those found in [18–30] reveals that nearly all of our answers are entirely distinct from theirs; nevertheless, by using certain unique values for the previously mentioned parameters, we can identify some solutions that are comparable to our solutions. The graphical presentation of the solution obtained by Ghazala et al. [18] in Figure [9] and our result illustrated in Figure [11] look the same and also the solution derived by Kamruzzaman et al. [30] in $Eq.(3.12)$ using the Modified simple equation method and Eq.(50) are almost the same but, our result is the general form of their result.

5 Conclusion

The nonlinear evolution Zoomeron equation is investigated in this work. Using the φ^6-model expansion approach, bright, kink, periodic, combined periodic and combined singular soliton solutions are explicitly retrieved. Singular soliton solutions are also regarded favorably. The graphics in Figures $1 - 13$ soliton solutions at any given time, which is important in the transmission of energy from one location to another. It is the internal dynamics of the traveling wave for various parameter values. We may conclude that the traveling wave behavior alters for different values of each. It is envisaged that the results presented in this study will help to improve the Zoomeron equation's nonlinear dynamical characteristics. The method proposes a practical and effective way for obtaining precise solutions to a wide variety of nonlinear partial differential equations.

Declarations

Ethical Approval

Not applicable as there is no research using either human or animal subjects on this topic.

Competing interests

The authors affirm that they have no financial or other conflicts of interest.

Authors' contributions

The authors did all of the current work for this paper. The first author solved the equation using the presented technique and wrote them down in the Latex while the second author designed the graphs, wrote the introduction, result and discussion and the conclusion part, then all the authors reviewed the paper together.

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Data sharing isn't applicable because the current study didn't produce any datasets to analyze.

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