Multiple-soliton and Periodic Solutions to Space-time Fractional Whitham-Broer-Kaup Equations

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Multiple-soliton and periodic solutions to space-time fractional Whitham-Broer-Kaup equations

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ABSTRACT

In this paper, the space-time fractional Whitham-Broer-Kaup equations are investigated. By means of new fractional scaling transformations, the fractional nonlinear system of different time and space orders is transformed to the integer one. The multiple solitary solutions and periodic solutions are obtained respectively. All those solutions are given exactly by introducing new scaling transformations, which makes our study is unique and different from most existing studies. It is expected that exact solutions for nonlinear wave system of fractional order can be handled in the similar way.

Introduction

Recently, fractional-order differential equations¹ have been received much attention because they have been argued to be more appropriate than traditional integer-order ones to describe nonlinear phenomena in real worlds [2–5]. Many different analytical and numerical methods have been employed to capture solutions of fractional differential equations, such as the homotopy analysis method⁶, the reproducing kernel Hilbert space method⁷, the iterative Adomian decomposition method⁸, the Hermite wavelet method⁹, the implicit collocation method¹⁰, the finite element method¹¹, the B-spline wavelet operational method¹², the generalized fractional-order Legendre function method¹³, the boundary element method¹⁴.

On the other hand, many efforts have been done towards finding solutions of shallow water equations either analytically or numerically. For example, Garner *et al.*¹⁵ presented multiple solitary solutions to the well-known Korteweg-de Vries equation. Date *et al.*¹⁶ gave quasi-periodic solutions to the orthogonal Kadomtsev-Petviashvili equations. Wang¹⁷ obtained solitary solutions for the variant Boussinesq equations. Fan and Zhang¹⁸ exhibited exact solutions of the Whitham-Broer-Kaup equations in shallow water. Naranmandula and Chen¹⁹ further showed multiple soliton solutions of the Whitham-Broer-Kaup equations. Wang *et al.*²⁰ captured a class of novel exact solutions of periodic, solitary, and kink types of the Whitham-Broer-Kaup equations.

Very recently, Many researchers made efforts to investigate different types of water wave equations of fractional order including the time-fractional Whitham-Broer-Kaup equations⁸, the time-fractional nonlinear Schrödinger equation [21–22], the stochastic fractional nonlinear Schrödinger equation²³, the space-time fractional Whitham-Broer-Kaup equations²⁴, the time fractional KdV-Burgers-Kuramoto equation²⁵, the space-time fractional KdV equation²⁶, the Boussinesq equations with fractional diffusion²⁷, the time-fractional generalized Boussinesq equation²⁸, the fractional potential Kadomtsev-Petviashvili equation [29–30], and the space-time fractional Sharma-Tasso-Olver equation³⁰.

In most aforementioned studies, researchers aimed to obtain solutions by solving the fractional differential equations directly. However, Guo and Zhou²⁴ introduced a fractional scaling transformation to convert the fractional differential equations into the integral ones. As a result, solution techniques for regular wave equations can be used to solve fractional ones straightforwardly. In this paper, the space-time fractional Whitham-Broer-Kaup equations are to be examined. The multiple soliton solutions and travelling solutions will be captured. By means of the fractional scaling transformations, the fractional differential equations are transformed to integer ones. Then the homogeneous balance method¹⁹ and the first time integration method²⁴ are employed to give exact solutions to the fractional nonlinear systems, respectively.

Fractional calculus and governing equations

The modified Riemann-Liouville fractional calculus³¹ is defined as

$$D_x^\mu F(x) = \begin{cases} \frac{1}{\Gamma(1-\mu)} \int_0^x (x-\zeta)^{-\mu-1} [F(\zeta) - F(0)] d\zeta, & \mu < 0, \\ \frac{1}{\Gamma(1-\mu)} \int_0^x (x-\zeta)^{-\mu} [F(\zeta) - F(0)] d\zeta, & 0 < \mu < 1, \\ [F^{\mu-n}(x)]^{(n)}, & n \leq \mu \leq n+1, n \geq 1, \end{cases} \quad (1)$$

where μ is a decimal, Γ denotes the gamma function.

According to the definitions of the fractional calculus (1), the following properties are held:

$$\begin{aligned} D_x^\mu x^\gamma &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\mu)} x^{\gamma-\mu}, \\ D_x^\mu [F(x)G(x)] &= g(x)[D_x^\mu F(x)] + F(x)[D_x^\mu G(x)], \\ D_x^\mu F[G(x)] &= F'_G[G(x)]D_x^\mu G(x) = (D_G^\mu F[G(x)])(G'(x))^\mu, \\ D_x^\mu [F(x)G(x)] &= G(x)[D_x^\mu F(x)] + F(x)[D_x^\mu G(x)], \end{aligned} \quad (2)$$

where $F(x)$ and $G(x)$ are arbitrary functions, $\gamma > -1$ is a constant.

The governing equations describe the space-time fractional Whitham-Broer-Kaup equations are written by

$$D_t^\alpha u + uD_x^\beta u + D_x^\beta h + c_2 D_x^{2\beta} u = 0, \quad (3)$$

$$D_t^\alpha h + D_x^\beta (uh) - c_2 D_x^{2\beta} h + c_1 D_x^{3\beta} u = 0, \quad (4)$$

where u and h are, respectively, the horizontal velocity and the height deviating from the equilibrium position of the liquid, c_1 and c_2 are the coefficients associated with the different diffusion capabilities of the medium. α and β are the fractional numbers that satisfy $0 < \alpha < 1$ and $0 < 3\beta < 1$.

Multiple solitary solutions

By means of the following fractional scaling transformations

$$X = \frac{x^\beta}{\Gamma(1+\beta)}, \quad T = \frac{t^\alpha}{\Gamma(1+\alpha)}, \quad u(x,t) = U(X,T), \quad h(x,t) = H(X,T), \quad (5)$$

the governing equations (3) and (4) are converted to

$$U_T + UU_X + H_X + c_2 U_{XX} = 0, \quad (6)$$

$$H_T + (UH)_X - c_2 H_{XX} + c_1 U_{XXX} = 0. \quad (7)$$

The homogeneous balance method¹⁹ is used to give multiple solitary solutions to Eqs.(6) and (7). With this technique, we define

$$U(X,T) = a + f'(\Omega)\Omega_X, \quad H(X,T) = b + s''(\Omega)\Omega_X^2 + s'(\Omega)\Omega_{XX}, \quad (8)$$

where $f(\Omega)$ and $s(\Omega)$ are intermediate transition functions, the prime denotes the derivation to Ω , and Ω is a function of X, T .

Substituting the transformations (8) into Eqs.(6) and (7), we obtain

$$\begin{aligned} f''\Omega_T\Omega_X + af''\Omega_X^2 + (f'f'' + c_2f''' + s''')\Omega_X^3 + f'\Omega_{XT} + af'\Omega_{XX} \\ + (f'^2 + 3c_2f'' + 3s'')\Omega_X\Omega_{XX} + (c_2f' + s')\Omega_{XXX} = 0, \end{aligned} \quad (9)$$

$$\begin{aligned} bf''\Omega_X^2 + s'''\Omega_T\Omega_X^2 + as'''\Omega_X^3 + (f''s'' + f's''' + c_1f'''' - c_2s''''')\Omega_X^4 \\ + 2s''\Omega_X\Omega_{XT} + bf'\Omega_{XX} + s''\Omega_T\Omega_{XX} + 3as''\Omega_X\Omega_{XX} \\ + (s'f'' + 4f's'' + 6c_1f''' - 6c_2s''')\Omega_X^2\Omega_{XX} + (f's' + 3c_1f'' - 3c_2s'')\Omega_X^2\Omega_{XX} \\ + s'\Omega_{XXT} + as'\Omega_{XXX} + (f's' + 4c_1f'' - 4c_2s'')\Omega_X\Omega_{XXX} \\ + (c_1f' - c_2s')\Omega_{XXXX} = 0. \end{aligned} \quad (10)$$

To seek solutions of Eqs.(9) and (10), it is required that the coefficients of Ω and its derivatives for certain orders have to be equal to zero. Particularly, we are able to obtain the following particular solutions, by enforcing the coefficients of Ω_X^3 for Eq.(9) and Ω_X^4 for Eq.(10) to be equal to zero, as

$$f(\Omega) = -2(c_0 - c_2)\ln\Omega, \quad s(\Omega) = 2c_0(c_0 - c_2)\ln\Omega, \quad (11)$$

where $c_0 = c_2 + \sqrt{c_1 + c_2^2}$.

From Eq.(11), we are readily to obtain the following correlations

$$\begin{aligned} s' &= -c_0 f', & s'' &= -c_0 f'', & s''' &= -c_0 f''', & f'^2 &= 2(c_0 - c_2) f'', \\ f' s' &= -2c_0(c_0 - c_2) f'', & f' s'' &= f'' s' = -c_0(c_0 - c_2) f'''. \end{aligned} \quad (12)$$

Substituting the correlations (12) into Eqs.(9) and (10), and then enforcing the coefficients of f''' , f'' and f' to be equal to zero, we are able to obtain the following core equation

$$\Omega_T + a\Omega_X + (c_2 - c_0)\Omega_{XX} = 0, \quad (13)$$

whose solution is

$$\Omega(X, T) = 1 + \sum_{j=1}^N \exp [k_j X + (k_j^2 \sqrt{c_1 + c_2^2} - a k_j) T], \quad (14)$$

in which N denotes the quantities of solitary waves, k_j is the wave number of the k th solitary wave. Here $b = 0$ is taken so that the consistence of the coefficients is satisfied.

We therefore obtain the solutions of Eqs.(3) and (4), by taking account of the scaling transformations denoted in (5), as

$$u(x, t) = a - 2\sqrt{c_1 + c_2^2} \left\{ \frac{\sum_{j=1}^N [k_j \exp(\xi_j)]}{1 + \sum_{j=1}^N \exp(\xi_j)} \right\}, \quad (15)$$

$$h(x, t) = 2\sqrt{c_1 + c_2^2} \left(c_2 + \sqrt{c_1 + c_2^2} \right) \left\{ \frac{\sum_{j=1}^N [k_j^2 \exp(\xi_j)]}{1 + \sum_{j=1}^N \exp(\xi_j)} - \frac{\left[\sum_{j=1}^N k_j \exp(\xi_j) \right]^2}{\left[1 + \sum_{j=1}^N \exp(\xi_j) \right]^2} \right\}, \quad (16)$$

where $\xi_j = \frac{k_j}{\Gamma(1+\beta)} x^\beta + \frac{k_j^2 \sqrt{c_1 + c_2^2} - a k_j}{\Gamma(1+\alpha)} t^\alpha$.

Periodic solutions

To obtain the travelling wave solution of fractional order, we introduce the following scaling transformations

$$\eta = \frac{x^\beta}{\Gamma(1+\beta)} - \frac{\hat{\omega} t^\alpha}{\Gamma(1+\alpha)}, \quad u(x, t) = \hat{U}(\eta), \quad h(x, t) = \hat{H}(\eta). \quad (17)$$

Substituting the scaling transformations (17) into Eqs.(3) and (4), we obtain

$$-\hat{\omega} \hat{U}' + \hat{U} \hat{U}' + \hat{H}' + c_2 \hat{U}'' = 0, \quad (18)$$

$$-\hat{\omega} \hat{H}' + (\hat{U} \hat{H})' - c_2 \hat{H}'' + c_1 \hat{U}''' = 0. \quad (19)$$

Integrating Eqs.(18) and (19) with η once time, we obtain

$$-\hat{\omega} \hat{U} + \frac{1}{2} \hat{U}^2 + \hat{H} + c_2 \hat{U}' = P_1, \quad (20)$$

$$-\hat{\omega} \hat{H} + \hat{U} \hat{H} - c_2 \hat{H}' + c_1 \hat{U}'' = P_2, \quad (21)$$

where P_1 and P_2 are integral constants.

The travelling solutions of Eqs.(18) and (19) will be determined using the first integration method, which is briefly depicted below and can be found in Refs.²⁴ for more details.

Incorporating Eq.(21) with Eq.(20) by eliminating $\hat{H}(\eta)$ and its derivatives, we obtain

$$(P_1 - \hat{\omega}^2) \hat{U} + \frac{3}{2} \hat{\omega} \hat{U}^2 - \frac{1}{2} \hat{U}^3 + (c_1 + c_2^2) \hat{U}'' = \hat{\omega} P_1 + P_2. \quad (22)$$

Let $\hat{X} = \hat{U}$, and $\hat{Y} = \hat{X}'$, Eq.(22) is transformed to

$$\hat{X}' = \hat{Y}, \quad (23)$$

$$\hat{Y}' = k_1 \hat{U}^3 - k_2 \hat{U}^2 - k_3 \hat{U} - k_4, \quad (24)$$

where

$$k_1 = \frac{1}{2(c_1 + c_2^2)}, \quad k_2 = \frac{3\hat{\omega}}{2(c_1 + c_2^2)}, \quad k_3 = \frac{P_1 - \hat{\omega}^2}{(c_1 + c_2^2)}, \quad k_4 = -\frac{\hat{\omega}P_1 + P_2}{(c_1 + c_2^2)}.$$

Let

$$Q(\hat{X}, \hat{Y}) = \sum_{j=0}^m \hat{a}_j(\hat{X}) \hat{Y}^j, \quad (25)$$

where $\hat{a}_j(\hat{X})$ ($j = 0, 1, \dots, m$) are polynomials on $\hat{X}$, satisfying $\hat{a}_j(\hat{X}) \neq 0$. Here it holds

$$Q(\hat{X}(\eta), \hat{Y}(\eta)) = \frac{\partial Q}{\partial \eta}(\hat{X}(\eta), \hat{Y}(\eta)) = 0, \quad (26)$$

where

$$\frac{dQ}{d\eta} = (\hat{g}(\hat{X}) + \hat{h}(\hat{X})\hat{Y}) \sum_{j=0}^m \hat{a}_j(\hat{X}) \hat{Y}^j, \quad (27)$$

in which $\hat{g}(\hat{X})$ and $\hat{h}(\hat{X})$ are unknown functions to be determined.

For $m = 1$, taking account of the solution expansions (25) and (27) and the correlation (26), we obtain

$$\begin{aligned} & \hat{a}_1(X)Y' + \hat{a}_0'(X)Y + \hat{a}_1'(X)Y^2 = \\ & \hat{g}(X)\hat{a}_0(X) + [\hat{h}(X)\hat{a}_0(X) + \hat{g}(X)\hat{a}_1(X)]Y + \hat{h}(X)\hat{a}_1(X)Y^2. \end{aligned} \quad (28)$$

Balancing the coefficients of different order of Y in above equation, after a series of simplification and solving, we obtain

$$\hat{h}(\hat{X}) = 0, \quad \hat{g}(\hat{X}) = \hat{A}_1\hat{X} + \hat{A}_0, \quad \hat{a}_1(\hat{X}) = 1, \quad \hat{a}_0(\hat{X}) = \frac{1}{2}\hat{A}_1\hat{X}^2 + \hat{A}_0\hat{X} + \hat{C}_0, \quad (29)$$

where

$$\hat{A}_1 = \pm \frac{1}{\sqrt{c_1 + c_2^2}}, \quad \hat{A}_0 = \mp \frac{\omega}{\sqrt{c_1 + c_2^2}}, \quad \hat{C}_0 = \mp \frac{P_1}{\sqrt{c_1 + c_2^2}}. \quad (30)$$

Substituting the expressions denoted in Eq.(29) into Eq.(26), with consideration of Eqs.(23) and (24), we obtain

$$\hat{X}' = \pm \frac{1}{2\sqrt{c_1 + c_2^2}} [-(\hat{X} - \hat{\omega})^2 + \hat{\omega}^2 + 2P_1]. \quad (31)$$

Clearly, Eq.(31) is a Riccati-type equation. As addressed by Ma and Fuchssteiner³², the standard Riccati equation of the form $\hat{v}_\eta = \gamma(1 - \hat{v}^2)$ admits the general solution

$$\hat{v}(\eta) = \frac{\hat{A} - \hat{B} \exp(-2\hat{\gamma}\eta)}{\hat{A} + \hat{B} \exp(-2\hat{\gamma}\eta)} = \begin{cases} 1 & \text{for } \hat{B} = 0, \\ -1 & \text{for } \hat{A} = 0, \\ \tanh \left[\hat{\gamma}\eta - \frac{1}{2} \ln \left(\frac{\hat{B}}{\hat{A}} \right) \right] & \text{for } \hat{A}\hat{B} > 0, \\ \coth \left[\hat{\gamma}\eta - \frac{1}{2} \ln \left(-\frac{\hat{B}}{\hat{A}} \right) \right] & \text{for } \hat{A}\hat{B} < 0, \end{cases} \quad (32)$$

where $\hat{A}$ and $\hat{B}$ are arbitrary constants satisfying $\hat{A}^2 + \hat{B}^2 \neq 0$.

Therefore, the solution of Eq.(31) is obtained in different cases, after taking account of Eq.(32) and travelling variables (17), as shown below.

For the case $\hat{\omega}^2 + 2P_1 = 0$, its solution reads

$$u(x, t) = \frac{\pm 2\sqrt{c_1 + c_2^2}}{\frac{x^\beta}{\Gamma(1+\beta)} - \frac{\hat{\omega}t^\alpha}{\Gamma(1+\alpha)} \pm 2\sqrt{c_1 + c_2^2}\eta_0} + \hat{\omega} \quad (33)$$

where η_0 is a constant.

For the case $\hat{\omega}^2 + 2P_1 > 0$, its solution is

$$u(x, t) = \begin{cases} \sqrt{\hat{\omega}^2 + 2P_1} + \hat{\omega}, \\ -\sqrt{\hat{\omega}^2 + 2P_1} + \hat{\omega}, \\ \sqrt{\hat{\omega}^2 + 2P_1} \tanh \left\{ \pm \frac{\sqrt{\hat{\omega}^2 + 2P_1}}{2\sqrt{c_1 + c_2^2}} \left[\frac{x^\beta}{\Gamma(1+\beta)} - \frac{\hat{\omega}t^\alpha}{\Gamma(1+\alpha)} \right] - \frac{1}{2} \ln \eta_0^+ \right\} + \hat{\omega}, \\ \sqrt{\hat{\omega}^2 + 2P_1} \coth \left\{ \pm \frac{\sqrt{\hat{\omega}^2 + 2P_1}}{2\sqrt{c_1 + c_2^2}} \left[\frac{x^\beta}{\Gamma(1+\beta)} - \frac{\hat{\omega}t^\alpha}{\Gamma(1+\alpha)} \right] - \frac{1}{2} \ln (-\eta_0^-) \right\} + \hat{\omega}, \end{cases} \quad (34)$$

where η_0^+ and η_0^- are, respectively, the positive and negative constants.

For the case $\hat{\omega}^2 + 2P_1 < 0$, its solution is given as

$$u(x, t) = \begin{cases} \sqrt{-(\hat{\omega}^2 + 2P_1)} + \hat{\omega}, \\ -\sqrt{-(\hat{\omega}^2 + 2P_1)} + \hat{\omega}, \\ \sqrt{-(\hat{\omega}^2 + 2P_1)} \tanh \left\{ \pm \frac{\sqrt{-(\hat{\omega}^2 + 2P_1)}}{2\sqrt{c_1 + c_2^2}} \left[\frac{x^\beta}{\Gamma(1+\beta)} - \frac{\hat{\omega}t^\alpha}{\Gamma(1+\alpha)} \right] - \frac{1}{2} \ln \eta_0^+ \right\} + \hat{\omega}, \\ \sqrt{-(\hat{\omega}^2 + 2P_1)} \coth \left\{ \pm \frac{\sqrt{-(\hat{\omega}^2 + 2P_1)}}{2\sqrt{c_1 + c_2^2}} \left[\frac{x^\beta}{\Gamma(1+\beta)} - \frac{\hat{\omega}t^\alpha}{\Gamma(1+\alpha)} \right] - \frac{1}{2} \ln (-\eta_0^-) \right\} + \hat{\omega}. \end{cases} \quad (35)$$

Here $h(x, t)$ can be obtained using Eq.(20) and travelling transformation (17), which is not shown here.

For other values of m , solution is obtained via similar procedures. For example, when $m = 2$, we finally obtain the Riccati-type equation of the form

$$\hat{X}' = \pm \frac{1}{2\sqrt{c_1 + c_2^2}} (-\hat{X}^2 + 2\hat{\omega}_0\hat{X} + 2P_1), \quad (36)$$

whose solution is the exactly same as that of the case $m = 1$.

Conclusion

The space-time fractional Whitham-Broer-Kaup equations have been investigated. By means of fractional scaling transformations, the fractional nonlinear system is transformed to the integer one. The multiple solitary solutions and periodic solutions are obtained respectively. In summary, the novel aspects of the paper are:

- The Whitham-Broer-Kaup equations of different space and time fractional orders are studied.
- The novel scaling transforms are introduced.
- The exact multiple solitons solutions are obtained.
- The exact periodic solutions are presented.

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Author contributions

H.X wrote the whole manuscript text and performed the mathematical modeling. J.F.C. revised and corresponded the paper. Both authors reviewed the manuscript.

Competing interests

The authors declare no competing interests.

Additional information

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